ALMOST PERIODIC TRANSFORMATION GROUPS*

BY

DEANE MONTGOMERY

1. In view of the recent work on topological groups it is natural to consider the situation which arises when such groups act as transformation groups on various types of spaces. Such a study is begun here from the point of view of almost periodic transformation groups, the definition of which is suggested by von Neumann's paper on almost periodic functions in a group.† Compact topological transformation groups are a special case of almost periodic transformation groups, at least for a rather wide class of spaces. The paper concerns itself chiefly with the nature of the minimal closed invariant sets of such groups. There are some results for general spaces but the main results are for Euclidean spaces and more particularly for three-dimensional Euclidean spaces. One of the most interesting theorems states that if a compact one-dimensional group acts on three-space in such a way that its orbits are uniformly bounded in diameter, then every point of the space is fixed under the group, so that if such a group is to act in a non-trivial manner the diameters of its orbits must be unbounded. Under some restrictions a similar theorem is proved for one-parameter almost periodic groups. Furthermore it is shown that for this latter class of groups, many of the orbits must actually be simple closed curves if they have one-dimensional closures.

2. The group considered here will be denoted by G. It will be subjected to various conditions as the occasion demands but it will always be Abelian. In case it is the group of real numbers, it will be spoken of as a one-parameter group; in case it is the real numbers reduced modulo one, it will be spoken of as the circle group. The space on which the group acts will be denoted by R. It will be specialized in various ways, but in any case it will always be a locally compact metric space. If x and y are two points of R, the distance between them will be denoted by $d(x, y)$.

DEFINITION 1. *The group G is said to be a transformation group (t.g.) of R if the following conditions are satisfied:*

(1) *for every g in G there is a homeomorphism of R into itself denoted by $x^1 = g(x)$;*

* Presented to the Society, February 20, 1937; received by the editors January 13, 1936.

† These Transactions, vol. 36 (1934), p. 445. Cameron has defined almost periodic transformations. These Transactions, vol. 36 (1934), p. 276.

(2) *if* $g_3=g_1+g_2$, *then* $g_3(x)=g_1[g_2(x)]$.

DEFINITION 2.* *If G is a t.g. of R, G is said to be an almost periodic t.g. (a.p.t.g.) if for every a in R there is a neighborhood $U(a)$ having the following property*:

For any $e>0$ there is a finite set of elements $g_1, g_2, \cdots, g_n$ in G which are such that for any g in G there is an i such that $d[g(x), g_i(x)]<e$ for all x in $G[U(a)]$.†

No use is made of a topology in G in either of these two definitions so that in them G may or may not be topological.

DEFINITION 3. *If G is a topological group and a t.g. of R, then G is said to be a topological t.g. of R provided that $g(x)$ is continuous simultaneously in x and g.*

A t.g. may of course be both topological and a.p. If R is locally compact, a compact topological t.g. is an a.p.t.g.

DEFINITION 4. *The t.g. G is said to be regular‡ if for every positive e and every a in R there is a d such that if $d(x, a)<d$, then $d[g(x), g(a)]<e$ for all g in G.*

THEOREM 1. *If G is an a.p.t.g. it is regular.*

Let e be any positive number, and let a be any point of R; let $U(a)$ be the neighborhood of Definition 2, and let $g_1, g_2, \cdots, g_n$ be the finite set of elements associated with $e/3$. Since $g_i(x)$ is a homeomorphism, we may choose $U^1(a)$ within $U(a)$ in such a way that if x is in $U^1(a)$ then

$$d[g_i(a), g_i(x)] < e/3 \qquad (i = 1, 2, \cdots, n).$$

If g is any element of G, there is an i such that

$$d[g(a), g_i(a)] < e/3$$

and for any x in $U^1(a)$

$$d[g(x), g_i(x)] < e/3.$$

From these three inequalities the definition of regularity is seen to be satisfied.

The set $O_a=\sum_g g(a)$ is called the orbit of a.

THEOREM 2. *If G is a regular t.g., then $\overline{O}_a$ is the minimal closed invariant set including a, and if b is in $\overline{O}_a$, then $\overline{O}_a=\overline{O}_b$.*

* von Neumann, loc. cit.

† Following the usual convention this symbol is used for the set of points in R of the form $g(x)$ when g is in G and x is in $U(a)$.

‡ This concept is due to Kerékjártó. See Acta Litterarum ac Scientiarum, Szeged, vol. 6, p. 235.

The set $\bar{O}_a$ is the minimal closed set including O_a so that for the proof of the first part of the theorem it is only necessary to prove that $\bar{O}_a$ is invariant. Any point p in $\bar{O}_a$ is the limit of a set of points $p_n = g_n(a)$.† But then $g(p)$ is the limit of $g(p_n) = (g+g_n)(a)$ and hence for every g, $g(p)$ is in $\bar{O}_a$ which proves that $\bar{O}_a$ is invariant. It will now be shown that if b is in $\bar{O}_a$ then $\bar{O}_a = \bar{O}_b$. It is clear that $\bar{O}_b$ is in $\bar{O}_a$. To prove that $\bar{O}_a$ is in $\bar{O}_b$ it will be shown that a is in $\bar{O}_b$. Let a positive number e be given, and choose d so that if $d(x, b) < d$, then $d[g(x), g(b)] < e$ for all g. Since b is in $\bar{O}_a$ there is a g such that $d[g(a), b] < d$. Therefore $d[a, -g(b)] < e$; that is, there are points of O_b arbitrarily near a, which is sufficient for the proof.

The following theorem is due in essence to Kerékjártó‡ who stated it for the group of integers and a special R. Only a sketch of the proof will be given.

THEOREM 3. *If G is a regular t.g. of a complete metric space R, then $\bar{O}_x$ for any x is homeomorphic to a topological Abelian group.*

The set $\bar{O}_x$ is a metric space and in order to prove the theorem it must be shown that the points of $\bar{O}_x$ may be considered as the elements of a group in such a way that $a+b$ and $-a$ are continuous operations. It will first be shown how $a+b$ may be defined for any a and b in $\bar{O}_x$. Let A and B be small closed neighborhoods of a and b in $\bar{O}_x$, and let S and T contain those elements s and t of G such that $s(x)$ is in A and $t(x)$ is in B. Now let C be the set of points of the form $(s+t)(x)$, where s is in S and t is in T. It will be shown that the diameter of C may be made arbitrarily small by choosing A and B to have sufficiently small diameters. The point $(s_1 - s_2)(x)$ is near x if A is sufficiently small, for since $s_1(x)$ is near a, $(-s_2+s_1)(x)$ is near $-s_2(a)$ by regularity and $-s_2(a)$ is near x by regularity. Similarly $(t_1 - t_2)(x)$ is near x. From these facts we see, because of regularity, that $(s_1 - s_2 + t_1 - t_2)(x)$ is near x. Hence, again by regularity, points of the type $(s_1+t_1)(x)$ form a set of small diameter (because they are near $(s_2+t_2)(x)$ for fixed s_2 and t_2). As the diameters of A and B approach zero, the diameter of the set C therefore approaches zero and since R is complete, C must shrink toward a limit point which is defined to be $a+b$. This operation is clearly commutative and associative, and the process of defining it shows that it is also continuous simultaneously in a and b. In the above notation let D be all points of the form $-s(x)$. The diameter of D approaches zero with that of A, and the point approached by the shrinking set D is defined to be $-a$. This element satisfies the group postulates for an inverse and is continuous. The theorem is therefore completely demonstrated.

† This is meant to include the case where for all but a finite number of n's, $p_n = p$.

‡ Acta Litterarum ac Scientiarum, Szeged, vol. 7, p. 76.

THEOREM 4. *If G is an a.p.t.g. of a complete metric space then $\overline{O}_a$ is homeomorphic to a compact topological group.*

By Theorem 3, $\overline{O}_a$ is a topological group and it is a compact set because it is closed and conditionally compact.†

If G is a compact topological t.g. of R, then those elements of G, say $H(a)$, which leave a fixed form a compact subgroup of G, and O_a which in this case equals $\overline{O}_a$ is in reality the difference group $G-H(a)$. It is true therefore that the dimension of G is equal to the sum of the dimensions of O_a and $H(a)$.‡

3. If G is an a.p.t.g. of R, let F denote all points of R which are fixed under the action of every element of G, and let M be the set of moving points, that is, the set of points which are moved by some element of G. The set F is closed and hence M is open. Under the action of G, R is divided into a family of non-overlapping sets, the minimal closed invariant sets. This is a continuous family, for from Definition 2 it follows that if $x_n \rightarrow x$, then $\overline{O}_{x_n} \rightarrow \overline{O}_x$. Consider a set of elements R^* in one-to-one correspondence with the minimal closed invariant sets of R. If x^* is the element of R^* corresponding to $\overline{O}_x$, let $L(x) = x^*$, this being the definition of the function $L(x)$. It is clearly single valued, but in general its inverse is multiple valued. The set R^* may be made a metric space called the auxiliary space by the definition

$$d(x^*, y^*) = d[L^{-1}(x^*), L^{-1}(y^*)].\S$$

The function $L(x)$ now is a continuous function. It is an inner transformation of the space R into the space R^*; that is, it takes open sets into open sets.

We shall now be concerned with the case in which R is R_n, Euclidean space of n dimensions. If G is a t.g. of R_n a point may be added to R_n to make it $\overline{R}_n$ the n-sphere, and G may be extended to a t.g. of $\overline{R}_n$ by simply requiring that this point be fixed under all elements of G. If G is topological, it remains so by this definition; whether or not a group retains its a.p. character by this extension need not concern us. In any case the function $L(x)$ remains continuous in the extended case for an a.p.t.g. The space $\overline{R}_n^*$ is compact, connected, and locally connected.

4. We prove the following theorem.

THEOREM 5. *If G is an a.p.t.g. of R_n and if there is a number K such that for every a in R_n* diam $(O_a) < K$, *then $\overline{R}_n^*$ is cyclic.*‖

† See von Neumann, loc. cit.

‡ van Kampen, American Journal of Mathematics, vol. 58 (1936), p. 178.

§ For the definition of the distance between two closed sets see Hausdorff, *Mengenlehre*.

‖ For the definition of cyclic sets see Alexandroff, Mathematische Annalen, vol. 106, p. 218 and p. 223.

Let x be any point of R_n with coordinates $(x_1, \cdots, x_n)$, and let the coordinates of $g(x)$ be denoted by $[g(x)]_i$. For each fixed i and x, $[g(x)]_i$ is a real valued a.p. function defined on the group G. This follows from the definitions of a.p.t.g. and a.p. functions. This function has a mean† which will be denoted by $F_i(x)$. The n functions $F_i(x)$ determine a point denoted by $F(x)$. Hence we have defined a transformation of the space R_n into itself. By the properties of the mean this function is constant on an orbit O_a. The continuity of the function or transformation can be seen in the following way. Let a positive e and a point a be given. Choose d so that if $d(a, x) < d$ then $d[g(a), g(x)] < e$; this last relation may also be written as $| [g(a)]_i - [g(x)]_i| < e(n)^{-1/2}$. But

$$|F_i(a) - F_i(x)| = | M_g[g(a)]_i - M_g[g(x)]_i| \leqq M_g| [g(a)]_i - [g(x)]_i| \leqq M_g e(n)^{-1/2} = e(n)^{-1/2},$$

and therefore $d[F(a), F(x)] < e$. Because $F(x)$ is constant on O_a and continuous it must be constant on $\overline{O}_a$. The properties of the mean also imply that $d[x, F(x)] < K(n)^{1/2}$.

If the point at infinity P_∞ be added to R_n to form the space $\overline{R}_n$, $F(x)$ may be extended by defining $F(P_\infty) = P_\infty$, and the resulting transformation of $\overline{R}_n$ into itself is continuous. The transformation of $\overline{R}_n$ is also essential;‡ this is because the degree of F is one, and we see this from the fact that no point can move to its diametrically opposite point since in R_n $d[x, F(x)] < K(n)^{1/2}$.

We can define a transformation $h(x^*)$ from $\overline{R}_n^*$ to $\overline{R}_n$ in this way: for any point x^* let $h(x^*) = F[L^{-1}(x^*)]$. Since the transformation F takes $L^{-1}(x^*)$ into a single point, $h(x^*)$ is single valued and furthermore it is continuous. Clearly

$$h[L(x)] = F(x).$$

The transformation $h(x^*)$ must be an essential transformation of $\overline{R}_n^*$ into $\overline{R}_n$. If this were not the case there would be a function $h(x^*, t)$ defined for every x^* in $\overline{R}^*$ and for every t in the interval $0 \leqq t \leqq 1$, this function being such that $h(x^*, 0) = h(x^*)$ and $h(x^*, 1)$ is constant. But then

$$k(x, t) = h[L(x), t]$$

is a deformation of $F(x)$ having properties similar to the ones just mentioned for $h(x^*, t)$, and this implies that $F(x)$ is not essential. From this contradiction we conclude that $h(x^*)$ is essential and hence that $\overline{R}_n^*$ is cyclic; that is, it contains an n-dimensional power cycle which is not homologous to zero.

---

† For the existence and properties of the mean see von Neumann, loc. cit.

‡ See the previously cited paper of Alexandroff.

COROLLARY. *If G is an a.p.t.g. of R_n and if $\overline{R}_n^*$ is not cyclic, then the diameters of the orbits in R_n are unbounded.*

The theorem below can be applied more easily than Theorem 5 to certain special cases which will be considered later. The set $L(M)$ is denoted by M^*, $L(F)$ by F^*.

THEOREM 6. *If G is an a.p.t.g. of R_n whose orbits are uniformly bounded, and if the dimension of M^* is less than n, then every point of R_n is fixed under G.*

The set F^* is homeomorphic to F, and if we assume that the theorem is false and that the set M is not vacuous, then F^* is homeomorphic to a proper part of the sphere $\overline{R}_n$. The set F^* itself therefore cannot be cyclic. Since M is at most $(n-1)$-dimensional, F^*+M^* cannot contain a power cycle homologous to zero. This can be seen from the Phragmén-Brouwer† theorem. Therefore F^*+M^* is not cyclic, but this contradicts Theorem 5, so that we must conclude that M is vacuous and that the theorem is true.

THEOREM 7. *If G is an a.p.t.g. of R_n, then no minimal closed invariant set is n-dimensional and at least one such set is of dimension less than $(n-1)$.*

If a minimal closed invariant set were n-dimensional, it would have to contain an interior point of R_n‡ and the family of minimal closed invariant sets could not be a continuous family in this case.

Suppose now that every minimal closed invariant set is $(n-1)$-dimensional. If $\overline{O}_a$ is $(n-1)$-dimensional, it goes by an ϵ-transformation into a set containing an $(n-1)$-dimensional torus;§ and therefore $\overline{O}_a$ must separate R_n.‖ The point $L(\overline{O}_a)$ must therefore separate $\overline{R}_n^*$. Hence every point except possibly P_∞^* is a cut point of $\overline{R}_n^*$. But since $\overline{R}_n^*$ must have at least two non-cut points,¶ we see that at least one minimal closed invariant set is less than $(n-1)$-dimensional.

5. For the application of these theorems to certain special cases a few preliminary facts are necessary and these will now be discussed. Let G be an a.p.t.g. of R, and let A be a subgroup of G. It can be verified that A is also an a.p.t.g. of R and the minimal closed invariant sets (under A) in R form a continuous family. Denote the space formed by this continuous family of sets by R_A and the continuous transformation taking R into R_A by L_A. If $G-A=B$,

† Alexandroff, loc. cit., p. 186. See also p. 218.

‡ Menger, *Dimensionstheorie*, p. 244.

§ Pontrjagin, Annals of Mathematics, vol. 35 (1934), p. 386.

‖ Alexandroff, Annals of Mathematics, vol. 30 (1929), p. 148.

¶ R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, p. 45.

then B can be considered in a very natural way as a t.g. of R_A. If $x^* = L_A(x)$ and if $\bar{b}$ is an element of B corresponding to the coset $b+A$, the following definition is made:

$$\bar{b}(x^*) = L_A[b(x)].$$

This is a unique definition and by it the transformations $\bar{b}(x^*)$ are homeomorphisms of R satisfying the conditions for a t.g.

THEOREM 8. *Under the conditions given above B is an a.p.t.g. of R_A.*

Only the a.p. character of B remains to be proved. Let a^* be any point of R_A; then $L_A^{-1}(a^*)$ is a minimal closed invariant set in R which will be denoted by $\bar{O}_a(A)$, the point a being selected as any point of the set. Since G is a.p. there is a neighborhood $U(a)$ having the property that for every positive d there is a finite set $g_1, g_2, \cdots, g_n$ such that for any g in G there is an i such that $d[g(x), g_i(x)] < d$ for all x in $G[U(a)]$. Now assume that d is fixed and that $g_1, g_2, \cdots, g_n$ is the corresponding finite set. Now let $L_A U(a) = U(a^*)$. Let $\bar{g}_i$ be the finite set of elements of B corresponding to the cosets $g_i + A$. Let $\bar{g}$ be any element of B corresponding to the coset $g+A$, and consider the distance $d[\bar{g}(x^*), \bar{g}_i(x^*)]$, where x^* is any element of $U(a^*)$. Remembering that $\bar{g}(x^*) = L_A[g(x)]$ and that $\bar{g}_i(x^*) = L_A[g_i(x)]$ as well as the local compactness of R, we see that d may be so chosen that the distance in question is less than any e specified in advance. This is sufficient to prove the theorem.

If K^* is a closed set in an auxiliary space, the set K is said to be a true section of $L^{-1}(K^*)$ provided (1) that it is closed, and (2) that it contains precisely one point on $L^{-1}(x^*)$ for every x^* in K^*.† Bearing in mind the notation of the preceding theorem, we let $L_B(R_A) = R_{AB}$. This space R_{AB} is homeomorphic to R_G which we ordinarily denote by R^*.

LEMMA 1. *Let K^* be a closed set in $R^*(R_{AB})$. If K_A is a true section of $L_B^{-1}(K^*)$, and if K is a true section of $L_A^{-1}(K_A)$, then K is a true section of $L^{-1}(K^*)$.*

This lemma, the proof of which will be left to the reader, provides an opportunity to obtain true sections in complicated cases from true sections for simple cases.

The following lemma due to Zippin will also be useful in this connection.‡

LEMMA 2. *Let $F(t)$ be a continuous function which is defined on a linear*

† Montgomery and Zippin, these Transactions, vol. 40 (1936), p. 24.

‡ Zippin stated this theorem to the author in conversation, but he has not as yet published his proof. The author has found an independent proof. Added in proof. On March 27, 1937, G. T. Whyburn read a paper containing a proof of this theorem. His paper will appear in the Duke Mathematical Journal.

interval $a \leqq t \leqq b$ and whose values are disjoined closed zero-dimensional sets in a compact metric space. Then there is a continuous point valued function $f(t)$ defined on $a \leqq t \leqq b$ in such a way that $f(t)$ is in $F(t)$. This may be chosen so that if c is any point in the interval, $f(c)$ is any desired point of $F(c)$.

This is equivalent to saying that there is a true section of the continuous family of sets which are the functional values of $F(t)$.

THEOREM 9. *If G is a one-parameter topological a.p.t.g. of R whose minimal closed invariant sets are at most one-dimensional, then for any arc a^*b^* in R^* there is a true section ab in $L^{-1}(a^*b^*)$.*

To begin with assume that a^*b^* is in M^*. It will first be shown that if c^* is any point of a^*b^*, then there is a subarc e^*f^* including c^* on its interior which is such that if c is any point of $L^{-1}(c^*)$ there is a true section ef in $L^{-1}(e^*f^*)$ containing c.

In order to prove this, note that the family of curves filling $L^{-1}(ab)$ is a regular family so that Whitney's results on local cross sections† may be applied. Let c be a definite point of $L^{-1}(c^*)$, and let S be a cross section of the family of curves through c (Whitney, loc. cit.). Now $\bar{O}_c \cdot S$ must be zero-dimensional for if it were one-dimensional, then $\bar{O}_c$ would be at least two-dimensional. Hence there is in S an open set O including c which has no points of $\bar{O}_c$ on its boundary. There must be a small arc e^*f^* including c^* in its interior which is such that for any point x^* in e^*f^*, $[L^{-1}(x^*)]\cdot O$ is a closed zero-dimensional set and $L^{-1}(x^*)$ does not intersect the boundary of O. The family of sets $[L^{-1}(x^*)]\cdot O$ for all x^* in e^*f^* is a continuous family of zero-dimensional sets, and hence by Lemma 2 there is an arc ef which is a true section and which goes through c.

By the Heine-Borel theorem there are a finite number of arcs of the same type as e^*f^* which cover a^*b^*. Hence a^*b^* may be divided into a finite number of subarcs $a^*a_1^*$, $a_1^*a_2^*$, $\cdots$, $a_n^*b^*$ which are such that $L^{-1}(a^*a_1^*)$ has a true section through any point of $L^{-1}(a^*)$, $L^{-1}(a_1^*a_2^*)$ has a true section through any point of $L^{-1}(a_1^*)$, and so on. Now choose any true section aa_1 of $L^{-1}(a^*a_1^*)$. Choose a true section of $L^{-1}(a_1^*a_2^*)$ which begins at a_1, call it a_1a_2. Then choose a true section of $L^{-1}(a_2^*a_3^*)$ which begins at a_2, call it a_2a_3. Proceeding in this way there is obtained a finite set of true sections whose sum is a true section of $L^{-1}(a^*b^*)$. The proof of the theorem is now complete for the case where a^*b^* is in M^*.

Suppose now that a^*b^* is anywhere in R^* so that part of it may be in F^*. Let $K^* = a^*b^*$. The set $L^{-1}(K^* \cdot F^*)$ is its own true section. The set $K^* \cdot M^*$

---

† Annals of Mathematics, vol. 34 (1933), p. 244.

may be represented as the sum of the interiors of a countable set of intervals c^*d^*, where c^* and d^* are in F^*. The interior of any interval c^*d^* may be represented as the sum of a doubly infinite set of adjacent intervals $c_i^*c_{i+1}^*$, where i runs over all positive and negative integers. For each i the set $L^{-1}(c_i^*c_{i+1}^*)$ has a true section c_ic_{i+1}, where either c_i or c_{i+1} can be chosen arbitrarily in $L^{-1}(c_i^*)$ or $L^{-1}(c_{i+1}^*)$ respectively. We may therefore assume, as our notation has anticipated, that the last point of c_ic_{i+1} coincides with the first point of $c_{i+1}c_{i+2}$. Let $L^{-1}(c^*)=c$ and $L^{-1}(d^*)=d$. Then the set $\sum c_ic_{i+1}+c+d$ is a true section of $L^{-1}(c^*d^*)$; because of the a.p. character of G, the sum converges properly toward c and d. Since this same process can be carried out in each interval, it is clear that there is a true section of $L^{-1}(a^*b^*)$ and the proof of the theorem is now complete.

COROLLARY. *Let G be the circle group, and let G be a topological t.g. of R. Then if K^* is an arc in R^*, there is a true section of $L^{-1}(K^*)$.*

This is a special case of the preceding theorem.

THEOREM 10. *Let G be a connected compact one-dimensional t.g. of R. Then if K^* is an arc in R^*, there is a true section K of $L^{-1}(K^*)$.*

In this case G contains a compact zero-dimensional subset A such that $G-A$ is the circle group. Hence the theorem is an immediate consequence of the preceding corollary and Lemmas 1 and 2.

LEMMA 3. *If G is a compact one-dimensional topological group, there is in G an arc C including the zero of G as an interior point and which has a certain group property as follows: There is an open subset V of C including zero and such that if u and v are in V, then $u+v$ is in C.*

This is contained in a result of Pontrjagin.†

THEOREM 11. *Let G be a compact connected one-dimensional t.g. of R. Then if K^* is an arc in M^*, $L^{-1}(K^*)$ is two-dimensional.*

There is an arc ab which is a true section of $L^{-1}(K^*)$ by Theorem 10. Let C be the arc and V the subset of C of the preceding lemma. The "group" C may be assumed to be locally isomorphic to a part of the real number group, and we will speak as if the elements of C were real numbers. If V is chosen sufficiently small, then for every v different from 0 in V, $v(a)\neq a$; for if there were a set of elements v_n with zero as a limit and such that $v_n(a)=a$, then for every element v in V, $v(a)=a$. In this case the set $H(a)$ of elements of G leaving a fixed is one-dimensional, and therefore the orbit is zero-dimensional which means in this case that it is a point. But this is a contradiction, and

† Loc. cit., p. 387.

we conclude that our original statement must be true. A similar statement is true for all elements x in K. There is in fact a V so small that for every v different from zero in V and every x in K, $v(x) \neq x$. If this were not true there would be a set of elements v_n approaching zero and a set of elements x_n in K which may be assumed to approach x_0 such that $v_n(x_n) = x_n$. Assume that V is so chosen that if v is in V, $v(x_0) \neq x_0$. Now let v be any element in V. For any n, $v = k_n v_n + u_n$, where k is an integer and $|u_n| < |v_n|$. We know that $v(x_n)$ approaches $v(x_0)$, but $v(x_n) = (k_n v_n + u_n)(x_n) = u_n(x_n)$. Hence $v(x_0) = x_0$, and we have reached a contradiction from which we conclude that a V of the type stated exists. Assume that V consists of all elements of the type $|v| < r$ where r is a positive number. If $|v_1| < r/2$ and $|v_2| < r/2$ and $v_1 \neq v_2$ then $v_1(x) = v_2(x)$, for if $v_1(x) = v_2(x)$, then $(v_1 - v_2)(x) = x$ and $|v_1 - v_2| < r$.

Consider all elements of the form $v(x)$ for x in K and v in V. This set is homeomorphic to the product of an open interval and a closed interval and is therefore two-dimensional. We have now proved a slightly stronger result than the one stated in the theorem, namely, that the arc ab is actually imbedded in a two-cell which is in $L^{-1}(a^*b^*)$.

6. We have the following theorem:

THEOREM 12. *If G is a one-dimensional connected compact t.g. of R_3, then M^* is at most two-dimensional.*

If M^* were three-dimensional, it would contain a compact three-dimensional set and hence a compact three-dimensional Cantorian manifold.† In fact there would have to be a point a^* in M^* such that an arbitrarily small three-dimensional Cantorian manifold contained it. On the other hand, M^* contains no cut points because the inverse of a point is one-dimensional, and a^* must be an interior point of an arc which is in M^*.‡ This arc separates M^* locally at a^* because the inverse of the arc contains a two-cell and therefore separates R_3 locally. These two facts contradict each other, and it may therefore be concluded that M^* is at most two-dimensional.

THEOREM 13. *If G is a one-dimensional connected compact t.g. of R_3 whose orbits are uniformly bounded, then G leaves every point of R_3 fixed.*

This is an immediate consequence of Theorems 6 and 12.

THEOREM 14. *Let G be a one-parameter topological a.p.t.g. of R_3 whose minimal closed invariant sets are one-dimensional. If the orbits are uniformly bounded, then every point of R_3 is fixed under G.*

It is necessary to prove that M^* is at most two-dimensional. If a^*b^* is

† Alexandroff, Mathematische Annalen, vol. 106, p. 214 and p. 217.

‡ R. L. Moore, loc. cit., p. 148.

an arc in M^* and ab is a true section of $L^{-1}(a^*b^*)$, it can be shown as in Theorem 11 that ab is imbedded in a two-cell in $L^{-1}(a^*b^*)$ and therefore the dimensionality of M^* may be verified in the same way as in Theorem 12.

It would be interesting to remove the restriction on the dimensionality of the minimal closed invariant sets in this theorem.

THEOREM 15. *If G is a one-parameter topological a.p.t.g. of R_3, and if the minimal closed invariant sets are at most one-dimensional, then the points whose orbits are points or simple closed curves are everywhere dense.*

In order to prove the theorem we need only concern ourselves with the moving points, for the fixed points clearly lie on orbits of the type described. Let a^* be any point in M^*, and let e^*f^* be an arc in M^* containing a^* as an interior point. The point a^* may be accessible from $M^*-e^*f^*$, but if not there is in any case a point b^* arbitrarily near a^* which is accessible from $M^*-e^*f^*$. It will be shown that either $L^{-1}(b^*)$ is a simple closed curve or that it has a simple closed curve as an orbit near it. Let ef be a true section of $L^{-1}(e^*f^*)$. Let b^*c^* be an arc in M^* which has only the point b^* in common with e^*f^*, and let bc be a true section of $L^{-1}(b^*c^*)$ with b in the set ef. Let H be the two-cell including ef, the existence of which was proved in Theorem 14.

Our definition of almost periodicity is equivalent in the case under consideration to the more familiar one,† and we can therefore conclude that ef and bc are so chosen that there is a set of real numbers g_n approaching infinity such that for any point x in $H+bc$, $d[g_n(x), x]<1/n$. If $g_n(H)$ intersects H for a large value of n, some point of H is on a periodic orbit, and if this is true no matter how small e^*f^* is, b is arbitrarily near a periodic orbit. This case may therefore be dismissed, and we turn to the case where for all n sufficiently large H and $g_n(H)$ do not intersect. Then a small open set U may be chosen with b as an interior point in such a way that H separates U into two connected parts U_1 and U_2. Assume that bc lies in $\overline{U}_2$. Choose n so large that $g_n(H)$ is very near H and $g_n(bc)$ is very near bc. How $g_n(H)$ must cut either U_1 or U_2, and we may assume that it is U_2; for if it cut U_1 it would only be necessary to interchange the parts played by H and $g_n(H)$. Now $g_n(H)$ must separate b from c in U_2, and hence $g_n(H)$ must intersect bc at an interior point, which contradicts our choice of b^*c^* and bc. This case having led us to a contradiction we conclude that the theorem is true.

† Bochner, Acta Mathematica, vol. 61, p. 154.

SMITH COLLEGE,
NORTHAMPTON, MASS.