GENERALIZED INTEGRALS AND DIFFERENTIAL EQUATIONS*

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Introduction. The idea of the following considerations can best be explained in the simplest case of an integral

$$\int a(x, f(x))db(x, f(x))$$

in which a and b are continuously differentiable functions of two variables, $f(x)$ a continuously differentiable function. The routine estimate of $\int adb$ gives bounds depending either on $\int |df(x)|$ or on the maximum of $|df/dx|$ in the interval of integration. It is, however, possible, as shown in Theorem 1, to give a bound that is entirely independent of the derivative of $f(x)$, and, consequently, to define, by a limiting process, $\int adb$, even in the case where $f(x)$ not only has no derivative, but is no longer continuous, provided $f(x)$ belongs to Baire's first class. The same observation holds for a great number of functionals of $f(x)$ whose construction depends on the derivative of $f(x)$, but for which bounds can be found nevertheless without reference to df/dx . In this paper we are concerned mainly with ordinary differential equations (Theorems 2-3') and systems of hyperbolic equations in two independent variables (Theorem 6 and corollaries) whose treatment is based on a detailed study of the double integral (33).

1. **Simple integrals.** The theory of the Lebesgue-Stieltjes integral contains the following statement: If in a closed interval J the function $\beta_0(x)$ is monotone and the sequence of continuous functions $\alpha_\mu(x)$ is uniformly bounded and tends to a limit function $\alpha_0(x)$, then the Stieltjes integral $\int \alpha_\mu(x)d\beta_0(x)$ tends to the Lebesgue-Stieltjes integral $\int \alpha_0(x)d\beta_0(x)$ as $\mu \rightarrow \infty$. If, furthermore, a sequence of functions $\beta_\mu(x)$ of bounded variation tends to $\beta_0(x)$ as $\mu \rightarrow \infty$ so that the total variation of the difference $\beta_\mu(x) - \beta_0(x)$ tends to zero, then

$$\begin{aligned} \limsup \left| \int \alpha_\mu(x)d\beta_\mu(x) - \int \alpha_0(x)d\beta_0(x) \right| \\ \leq \limsup \left\{ \max |\alpha_\mu(x)| \cdot \int |d(\beta_\mu - \beta_0)| \right\} \rightarrow 0, \end{aligned}$$

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which relation leads to the following lemma:

LEMMA 1. *If in the interval $X_0 \leq x \leq X_1$ the continuous functions $\alpha_\mu(x)$, $\mu = 1, 2, \dots$, are uniformly bounded and tend to $\alpha_0(x)$ as $\mu \rightarrow \infty$, and if the functions $\beta_\mu(x)$ of bounded variation tend to a function $\beta_0(x)$ of bounded variation while the total variation of $\beta_\mu(x) - \beta_0(x)$ tends to zero as $\mu \rightarrow \infty$, then*

$$\lim_{\mu \rightarrow \infty} \int_{X_0}^x \alpha_\mu(x) d\beta_\mu(x) \text{ exists and } = \int_{X_0}^x \alpha_0(x) d\beta_0(x),$$

where the integral on the right is the Lebesgue-Stieltjes integral.

We proceed to introduce a new notion of integral which is essentially different from the Lebesgue-Stieltjes integral and is based upon the following theorem:

THEOREM 1. *In the interval J ($X_0 \leq x \leq X_1$) there are given a function $f(x)$ satisfying the inequality $|f(x) - f(X_0)| < F$, a sequence $\{f_\mu(x)\}$ of continuously differentiable functions with $|f_\mu(x) - f(X_0)| < F$ such that $f_\mu(x) \rightarrow f(x)$ as $\mu \rightarrow \infty$, and n continuous functions $g_1(x), \dots, g_n(x)$ with bounded total variations $T[g_1], \dots, T[g_n]$. Denote by $f^0, g_1^0, \dots, g_n^0$ the values of $f(X_0), g_1(X_0), \dots, g_n(X_0)$ and by $\gamma_1, \dots, \gamma_n$ upper bounds of $|g_1(x) - g_1^0|, \dots, |g_n(x) - g_n^0|$ in J . Let $\{g_{1\mu}(x)\}, \dots, \{g_{n\mu}(x)\}$ be n sequences of continuous functions of bounded variation, tending to $g_1(x), \dots, g_n(x)$, respectively, as $\mu \rightarrow \infty$ while the total variations of the differences $T[g_{1\mu} - g_1], \dots, T[g_{n\mu} - g_n]$ tend to zero and $|g_{i\mu}(x) - g_i^0| < \gamma_i$ for all x in J , $i = 1, 2, \dots, n$ and $\mu = 1, 2, \dots$. Suppose that in the $(n+2)$ -dimensional domain*

$$D: \begin{aligned} |z - f^0| &\leq F, \quad X_0 \leq x \leq X_1, \\ |y_1 - g_1^0| &\leq \gamma_1, \dots, |y_n - g_n^0| \leq \gamma_n \end{aligned}$$

the functions $a(z, x, y_1, \dots, y_n)$ and $b(z, x, y_1, \dots, y_n)$ are continuously differentiable. Then the Stieltjes integral

$$(1) \quad S_\mu(x) = \int_{X_0}^x a(f_\mu(x), x, g_{1\mu}(x), \dots, g_{n\mu}(x)) db(f_\mu(x), x, g_{1\mu}(x), \dots, g_{n\mu}(x))$$

tends to a limit $L(x)$ as $\mu \rightarrow \infty$.

Remarks. It is clear that any function $f(x)$ with $|f(x) - f^0| < F$ which is a limit of continuous functions may be considered as a limit of continuously differentiable functions $f_\mu(x)$ with $|f_\mu(x) - f^0| < F$. Moreover, the limit $L(x)$ is independent of the approximating sequences $f_\mu(x)$ and $g_{i\mu}(x)$. For the statement of Theorem 1 implies the existence of a limit no matter which sequences are used and hence any two sequences may be considered as subsequences of

a third one containing both. Consequently we may state the following as a definition:

DEFINITION.

$$L(x) = \int_{x_0}^x a(f(x), x, g_1(x), \dots, g_n(x)) db(f(x), x, g_1(x), \dots, g_n(x)).$$

In order to prove Theorem 1, we first assume that $b(z, x, y_1, \dots, y_n)$ has continuous derivatives of second order of the mixed type. We determine a function $A(z, x, y_1, \dots, y_n)$ in D as the solution of the differential equation

$$\frac{\partial A}{\partial z} = a \frac{\partial b}{\partial z}$$

with the initial condition $A(f^0, x, y_1, \dots, y_n) = 0$. We obtain

$$\begin{aligned} A(z, x, y_1, \dots, y_n) \\ = \int_{f^0}^z a(z', x, y_1, \dots, y_n) b_z(z', x, y_1, \dots, y_n) dz', \end{aligned}$$

where we may differentiate with respect to $x, y_1, \dots, y_n$ under the integral sign. Thus we find

$$\begin{aligned} adb &= ab_z dz + ab_x dx + \sum_{i=1}^n ab_{y_i} dy_i \\ &= dA + (ab_x - A_x) dx + \sum (ab_{y_i} - A_{y_i}) dy_i, \end{aligned}$$

$$\begin{aligned} A_x(z, x, y_1, \dots, y_n) &= \int_{f^0}^z (ab_z)_x dz' \\ &= \int_{f^0}^z [(ab_z)_x + (a_x b_z - a_z b_x)] dz' \\ &= a(z, x, y_1, \dots) b_x(z, x, y_1, \dots) \\ &\quad - a(f^0, x, y_1, \dots) b_x(f^0, x, y_1, \dots) + \int_{f^0}^z (a_x b_z - a_z b_x) dz' \end{aligned}$$

and

$$\begin{aligned} A_{y_i}(z, x, y_1, \dots, y_n) &= a(z, x, y_1, \dots) b_{y_i}(z, x, y_1, \dots) \\ &\quad - a(f^0, x, y_1, \dots) b_{y_i}(f^0, x, y_1, \dots) \\ &\quad + \int_{f^0}^z (a_{y_i} b_z - a_z b_{y_i}) dz'. \end{aligned}$$

Hence

$$\begin{aligned}
 S_\mu(x) &\equiv \int_{X_0}^x a(f_\mu(x'), x', g_{i\mu}(x')) db(f_\mu(x'), x', g_{i\mu}(x')) \\
 &= A(f_\mu(x), x, g_{i\mu}(x)) + \int_{X_0}^x a(f^0, x', g_{i\mu}(x')) b_x(f^0, x', g_{i\mu}(x')) dx \\
 (2) \quad &+ \sum_{i=1}^n \int_{X_0}^x a(f^0, x', g_{1\mu}(x'), \dots) b_{y_i}(f^0, x', g_{1\mu}(x'), \dots) dg_{i\mu}(x') \\
 &+ \int_{X_0}^x dx' \int_{f^0}^{f_\mu(x')} [a_z(z', x', g_{1\mu}(x'), \dots) b_x(z', x', g_{1\mu}(x'), \dots) \\
 &\quad - a_x b_z(\dots)] dz' \\
 &+ \sum_i \int_{X_0}^x dg_{i\mu}(x') \int_{f^0}^{f_\mu(x')} [a_z(z', x', g_{1\mu}(x'), \dots) b_{y_i} - a_{y_i} b_z] dz'.
 \end{aligned}$$

This formula, derived under the assumption that b has continuous second derivatives of mixed type, still holds under the conditions of Theorem 1. For any b which is continuously differentiable in D may be uniformly approximated by a polynomial such that its first derivatives uniformly approximate those of b . Introducing the approximations instead of b into (2) and passing to the limit we obtain again the formula (2) as both sides of (2) involve only first derivatives of b and the passage to the limit under the integral signs is legitimate in view of the uniform convergence of the derivatives of the polynomials to those of b .

In the right-hand member of (2) we can effect the passage $\mu \rightarrow \infty$ by simply cancelling all reference to μ . This may be seen as follows. An integral $\int_{f^0}^z a_z(z', x, y_1, \dots, y_n) b_{y_i} dz'$, for instance, is continuous in $z, x, y_1, \dots, y_n$. Thus $\int_{f^0}^{f_\mu(x)} a(z', x, g_{1\mu}(x), \dots) b_{y_i} dz'$ is a continuous function of x , bounded as $\mu \rightarrow \infty$, and converging as $\mu \rightarrow \infty$. Now the convergence of

$$\int_{X_0}^x dg_{i\mu}(x') \int_{f^0}^{f_\mu(x')} a_z(z', x', g_{1\mu}(x'), \dots) b_{y_i} dz'$$

follows from Lemma 1.

Thus Theorem 1 is proved.

From (2) we have the following estimate:

$$\begin{aligned}
 |L(x)| &\leq MN \left\{ |f(x) - f^0| + \sum_{i=1}^n T_{X_0}^x[g_i] + |x - X_0| \right\} \\
 &\quad + 2N^2F \left(\sum_i^n T_{X_0}^x[g_i] + |x - X_0| \right),
 \end{aligned}$$

where M is an upper bound for $|a|$ and $|b|$, N an upper bound for the moduli of the first derivatives of a and b in D .

Remark. We have, for instance, for every admissible $f(x)$

$$L(x) \equiv \int_0^x f(x') df(x') = \frac{1}{2}(f^2(x) - f^2(0)),$$

which leads to $L(1) = \frac{1}{2}$ for $f(x) = 0$ if $0 \leq x < 1$, $f(1) = 1$. The Lebesgue-Stieltjes integral, however, would be 1.

2. Ordinary differential equations. We may now prove the following theorem:

THEOREM 2. *Let the functions $f(x)$, $g_1(x)$, $\dots$, $g_n(x)$ be continuously differentiable in $0 \leq x \leq X$ and $f(0) = f^0$, $g_1(0) = \dots = g_n(0) = 0$. Denote by F an upper bound of $|f(x)|$ and by $G_1, G_2, \dots, G_n$ the total variations of $g_1(x), g_2(x), \dots, g_n(x)$ in $[0, X]$. Suppose, for $\epsilon > 0$, that in the domain*

$$D_{\epsilon}: \quad |z| \leq F, \quad |y_i| \leq G_i, \quad |u| < \epsilon + 3FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i$$

the functions $a_0, a_1, \dots, a_n$ are continuous functions of z, y_i, u satisfying the inequalities $|a_k| \leq M_k$, $k=0, 1, \dots, n$, and that a_0 has continuous derivatives of first order, bounded in absolute value by N . Then the solution $u(x)$ of the equation

$$(E) \quad \begin{aligned} du(x) &= a_0(f(x), g_1(x), \dots, g_n(x), u(x)) df(x) \\ &+ \sum_{i=1}^n a_i(f(x), g_1(x), \dots, g_n(x), u(x)) dg_i(x) \end{aligned}$$

with $u(0) = 0$ can be extended over the whole interval $0 \leq x \leq X$. It satisfies, moreover, the inequality

$$|u(x)| \leq 2FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

Let us solve the auxiliary partial equation for $A(z, y_1, \dots, y_n, u)$

$$(3) \quad \frac{\partial A}{\partial z} + \frac{\partial A}{\partial u} \cdot a_0 = a_0, \quad 1 - \frac{\partial A}{\partial u} \neq 0$$

under the initial condition

$$(4) \quad A(z, y_i, u) = 0 \text{ for } z = 0.$$

The characteristic equations of (3) are

$$(5) \quad dz:du:dA = 1:a_0:a_0.$$

Consider the family C of curves satisfying the differential equation $du/dz = a_0$ and passing through any point P of the domain

$$D_{2,\epsilon}: \quad |z| \leq F, \quad |y_i| \leq G_i, \quad |u| < \epsilon + 2FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

Since a_0 has continuous derivatives of first order throughout $D_{2,\epsilon}$ and is bounded by M_0 , there exists one and only one curve through an arbitrary point P , and the corresponding value $\bar{u}$ of u for $z=0$ lies within the range

$$|\bar{u}| < \epsilon + 3FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

Conversely, an arbitrary curve of C is uniquely determined by the quantities $\bar{u}, y_1, \dots, y_n$, and a point of the curve is determined by giving in addition the corresponding value of z . On writing $u = u(z, y_i, \bar{u})$, we find bounds for the derivatives of u with respect to the arguments $\bar{u}, y_1, \dots, y_n, z$. We have, in fact,

$$\frac{d}{dz} \frac{\partial u}{\partial \bar{u}} = \frac{\partial a_0}{\partial u} \frac{\partial u}{\partial \bar{u}} \quad \text{with} \quad \frac{\partial u}{\partial \bar{u}} = 1 \quad \text{for} \quad z = 0,$$

hence

$$e^{-N|z|} \leq \frac{\partial u}{\partial \bar{u}} \leq e^{N|z|}.$$

Similarly

$$\frac{d}{dz} \frac{\partial u}{\partial y_i} = \frac{\partial a_0}{\partial y_i} + \frac{\partial a_0}{\partial u} \frac{\partial u}{\partial y_i} \quad \text{with} \quad \frac{\partial u}{\partial y_i} = 0 \quad \text{for} \quad z = 0,$$

hence

$$\left| \frac{\partial u}{\partial y_i} \right| \leq e^{N|z|} - 1.$$

On introducing the new variables $z, y_1, \dots, y_n, u$ instead of $z, y_1, \dots, y_n, \bar{u}$ we find throughout $D_{2,\epsilon}$

$$(6) \quad \frac{\partial(z, y_i, u)}{\partial(z, y_i, \bar{u})} = \frac{\partial u}{\partial \bar{u}}, \quad \frac{\partial(z, y_i, \bar{u})}{\partial(z, y_i, u)} = \frac{\partial \bar{u}}{\partial u} = \left(\frac{\partial u}{\partial \bar{u}} \right)^{-1}, \quad e^{-|z|N} \leq \frac{\partial \bar{u}}{\partial u} \leq e^{|z|N},$$

$$\frac{\partial \bar{u}}{\partial y_i} = - \frac{\partial u}{\partial y_i} \bigg/ \frac{\partial u}{\partial \bar{u}}, \quad \left| \frac{\partial \bar{u}}{\partial y_i} \right| \leq e^{|z|N} (e^{|z|N} - 1).$$

Since $\bar{u}$ is constant along any curve of C , we have

$$\frac{\partial \bar{u}}{\partial z} + a_0 \frac{\partial \bar{u}}{\partial u} = 0, \quad \left| \frac{\partial \bar{u}}{\partial z} \right| \leq M_0 e^{N|z|}.$$

Now put, throughout $D_{2,\epsilon}$,

$$(7) \quad A(z, y_i, u) = u - \bar{u}.$$

Evidently A satisfies (3) and (4). Furthermore we have

$$(8) \quad \begin{aligned} |A| &\leq |z| M_0, & \left| \frac{\partial A}{\partial y_i} \right| &\leq e^{|z|N} (e^{|z|N} - 1), \\ \left| \frac{\partial A}{\partial z} \right| &\leq M_0 e^{N|z|}, & e^{-|z|N} &\leq \left| 1 - \frac{\partial A}{\partial u} \right| \leq e^{|z|N}. \end{aligned}$$

Returning to the ordinary differential equation (E), we remark that the conditions of our theorem allow us to write (E) in the form

$$\frac{du}{dx} = \phi(x, u)$$

with $\phi(x, u)$ continuous in the rectangle R determined by

$$0 \leq x \leq X, \quad |u| \leq \frac{\epsilon}{2} + 2FM_0 + H, \quad H = \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

The fundamental existence theorem for differential equations shows that a solution through $[x=0, u(0)=0]$ always may be continued until it reaches the boundary of R . Hence a solution which cannot be continued across a certain point x_1 with $0 \leq x_1 < X$ may be assumed to exist for $0 \leq x \leq x_1$ and to satisfy the condition

$$|u(x_1)| = \frac{\epsilon}{2} + 2FM_0 + H.$$

Thus our theorem is proved as soon as we show the following property of $u(x)$. If, in the interval $0 \leq x \leq x_1$, the solution $u(x)$ of (E) satisfies the inequality

$$|u(x)| \leq \frac{\epsilon}{2} + 2FM_0 + H,$$

it satisfies the stronger inequality

$$|u(x)| \leq 2FM_0 + H.$$

Indeed, by (E) we have, since $z=f(x)$, $y_i=g_i(x)$, $u=u(x)$ stay in $D_{2,\epsilon}$,

$$\begin{aligned}
 d(u - A) &= (a_0 - A_z)df(x) + \sum_{i=1}^n (a_i - A_{y_i})dg_i(x) - A_u du \\
 (9) \qquad &= (a_0 - A_z - A_u a_0)df(x) + \sum_{i=1}^n (a_i(1 - A_u) - A_{y_i})dg_i(x),
 \end{aligned}$$

with $z=f(x)$, $y_i=g_i(x)$ and $u(0)=0$. Hence we conclude from (3) and (8)

$$(10) \qquad |u(x) - A(f(x), g_i(x), u(x))| \leq FM_0 + H,$$

and, by (8),

$$(11) \qquad |u(x)| \leq 2FM_0 + H.$$

Remarks. The function $u(z, y_i, \bar{u})$ is continuous. This may be expressed by the statement: If the quantities $z, y_i, u - A(z, y_i, u)$ tend to limiting values which belong to the domain $|z| \leq F, |y_i| \leq G_i, |u - A(z, y_i, u)| \leq H + FM_0$, then u itself tends to a limiting value which in absolute value does not exceed $2FM_0 + H$.

Any function satisfying a Lipschitz condition of exponent 1 in z, y_i, u satisfies also a Lipschitz condition of exponent 1 in the variables $z, y_i, u - A$. This follows from (8), for we have

$$(12) \qquad |u_1 - A(z, y_i, u_1) - u_2 + A(z, y_i, u_2)| \geq |u_2 - u_1| e^{-FN}.$$

THEOREM 3. Assume that in the interval $0 \leq x \leq X$

- (i) the functions $f_\mu(x)$, $\mu = 1, 2, \dots$, are continuously differentiable, $|f_\mu(x)| < F$, and $f_\mu(x)$ converges to a function $f(x)$ as $\mu \rightarrow \infty$;
- (ii) the functions $g_{i\mu}(x), \dots, g_{n\mu}(x)$, $\mu = 1, 2, \dots$, are continuously differentiable, $g_{i\mu}(x) \rightarrow g_i(x)$ as $\mu \rightarrow \infty$, $i = 1, 2, \dots, n$, where $g_i(x)$ is continuous, and the total variations of the differences $T[g_{i\mu} - g_i]$ tend to zero as $\mu \rightarrow \infty$; furthermore $g_{i\mu}(0) = 0$ and $T[g_{i\mu}] \leq G_i$ for all i and μ ;
- (iii) the functions $a_0, a_1, \dots, a_n$ are defined in

$$|z| \leq F, \quad |y_i| \leq G_i, \quad |u| < \epsilon + 3FM_0 + H,$$

$$D_{3,\epsilon}: \qquad H = \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN}) G_i, \qquad (\epsilon > 0),$$

and we have in $D_{3,\epsilon}$

$$(13) \qquad |a_0| \leq M_0, \quad |a_1| \leq M_1, \dots, \quad |a_n| \leq M_n.$$

Furthermore, $a_0, a_1, \dots, a_n$ have continuous derivatives of first order in $D_{3,\epsilon}$, and those of a_0 are bounded in absolute value by N and satisfy a Lipschitz condition of exponent 1.

Then the solution of

$$(E_\mu) \quad du_\mu(x) = a_0(f_\mu(x), g_{i\mu}(x), u_\mu(x))df_\mu(x) + \sum_{i=1}^n a_i dg_{i\mu}(x), \quad u_\mu(0) \equiv 0,$$

exists for $0 \leq x \leq X$, satisfies

$$(14) \quad |u_\mu(x)| \leq 2FM_0 + H,$$

and tends to a limit function $u(x)$ as $\mu \rightarrow \infty$. $u(x)$ is said to be the solution of (E) for the initial condition $u(0) = 0$.

From Theorem 2 we conclude the existence of $u_\mu(x)$ and the inequality (14). The classical statement about uniqueness of the solution of the initial problem may, incidentally, be used to establish the uniqueness of $u_\mu(x)$. On putting

$$B_\mu(x) = u_\mu(x) - A(f_\mu(x), g_{i\mu}(x), u_\mu(x)),$$

we conclude from (9) and (8)

$$(15) \quad |B_\mu(x_1) - B_\mu(x_2)| \leq \sum_{i=1}^n (M_i e^{FN} + e^{2FN} - e^{FN}) \int_{x_1}^{x_2} |dg_{i\mu}(x)|.$$

Now a theorem by Adams and Clarkson† shows that the total variation between any two points x_1 and x_2 , of $g_{i\mu}(x)$ tends uniformly to that of $g_i(x)$ on account of the continuity of $g_i(x)$, and of the assumption (iii) that $T[g_{i\mu} - g_i] \rightarrow 0$, $g_{i\mu} \rightarrow g_i$. Thus (15) establishes equicontinuity for $B_\mu(x)$, while (14) gives boundedness. Hence, by Ascoli's theorem, we may select a subsequence $B_{\mu'}(x)$ tending uniformly to a function $B^*(x)$. From the remark on page 444 we conclude that also the corresponding subsequence of $u_\mu(x)$, say $u_{\mu'}(x)$, converges to a function $u^*(x)$. $B^*(x)$ satisfies the following integral equation

$$(16) \quad B^*(x) = \int_0^x \sum_{i=1}^n (a_i(1 - A_u) - A_{y_i}) dg_i(x) - A(f^0, 0, 0, \dots)$$

in which the expressions in a_i and A are to be considered as functions of z, y_i, u with $z = f(x)$, $y_i = g_i(x)$, $u - A = B^*(x)$. This follows from Lemma 1 and (9).

Any two subsequences of $B^*(x)$ converge to the same limit. In the opposite case we would have two functions $B^*(x)$ and $B^{**}(x)$, both satisfying (16). In view of (iii) the coefficients $a_0, a_1, \dots, a_n$ admit of continuous derivatives with respect to z, y_i, u , whence also with respect to $z, y_i, u - A$, and thus satisfy a Lipschitz condition of exponent 1 in these variables, in the closed domain $|z| \leq F, |y_i| \leq G_i, |u - A| \leq FM_0 + H$. On account of (3) and (4), the

† C. R. Adams and J. A. Clarkson, *On convergence in variation*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 413-417.

derivatives A_v and A_u satisfy a Lipschitz condition with respect to z, y_i, u , whence with respect to $z, y_i, u - A$. Therefore we conclude from (16)

$$(17) \quad |B^*(x) - B^{**}(x)| \leq K \sum_i \int_0^x |B^*(x) - B^{**}(x)| |dg_i(x)|,$$

where K is a certain constant. On iterating (17) m times we easily find

$$(18) \quad \max_{0 \leq x \leq X} |B^*(x) - B^{**}(x)| \leq K^m \max_{0 \leq x \leq X} |B^*(x) - B^{**}(x)| \left(\sum_{i=1}^n G_i \right)^m / m!$$

which gives $B^*(x) - B^{**}(x) = 0$ as $M \rightarrow \infty$.

Now the uniqueness of $B^*(x)$ implies, by the remark on page 444, the uniqueness of $u^*(x)$, which in turn justifies defining $u(x) = u^*(x)$ as the solution of (E) for the initial condition $u(0) = 0$.

Remark. The assumptions of Theorem 3 state bounds for the functions $a_0, a_1, \dots, a_n(z, y_1, \dots, y_n, u)$ holding in a domain that depends on these same bounds. One may ask for a formulation of the theorem such that a statement results for *any* functions $a_0, a_1, \dots, a_n$, defined in an *arbitrary* neighborhood of the origin. Therefore we observe that there always exists a sub-neighborhood of the form $D_{\delta, \epsilon}$, provided the constants $F, G_1, \dots, G_n$, ϵ can be decreased sufficiently. Since the functions $g_{1\mu}(x), \dots, g_{n\mu}(x)$ are continuous, their total variations are also continuous and may be shown† to converge *uniformly* to those of $g_1(x), \dots, g_n(x)$. Thus, omitting at most a finite number of values of μ and taking the upper end of the x -interval sufficiently small, makes it possible to choose $G_1, G_2, \dots, G_n$ arbitrarily small. Whence we conclude the following theorem:

THEOREM 3'. *Suppose that, in a neighborhood of the origin of a $(z, y_1, \dots, y_n, u)$ -space the functions $a_0, a_1, \dots, a_n$ are continuously differentiable and the derivatives of a_0 satisfy a Lipschitz condition of exponent 1. Assume that n continuously differentiable functions $g_{1\mu}(x), \dots, g_{n\mu}(x)$ are defined in an interval I ($0 \leq x \leq X$), that they tend to continuous functions $g_1(x), \dots, g_n(x)$, and the total variations $T[g_{i\mu} - g_i] \rightarrow 0$ as $\mu \rightarrow \infty$, and that $g_{i\mu}(0) = 0$. Assume furthermore that continuously differentiable functions $f_\mu(x)$ converge to a function $f(x)$ in I and that for all μ we have $|f_\mu(x)| < F$. Then the solution $u_\mu(x)$ of (E_μ) exists in a sufficiently small interval $0 \leq x \leq X'$ which does not depend on μ , and converges there to a limit function $u(x)$ provided F is sufficiently small.*

The existence Theorems 3 and 3' can be supplemented by a study of the manner in which the solution $u(x)$ depends on a parameter α on which the known functions in (E) may be supposed to depend. Usual methods of proving

† See Adams and Clarkson, loc. cit.

the continuity of $u(x)$, considered as a function of x and α , from that of the known functions could be carried through with only slight modifications.

Instead of (E) a system of differential equations of the form

$$(19) \quad du_h(x) = a_{0h}(f(x), g_i(x), u_i(x))df(x) + \sum_{i=1}^n a_{ih}dg_i(x), \quad u_h(0) = 0,$$

where $h=1, 2, \dots, m$, may be studied, and the existence of a solution $u_h(x)$, $h=1, 2, \dots, m$, can be concluded by a method analogous to that used in the proofs of Theorems 2 and 3. In view of the similarity of the procedure we shall not carry out these generalizations.

3. Double integrals. We denote by $T[\alpha, \beta]$ a triangle bounded by the line $\alpha=\beta$ of an (α, β) -plane and the parallels to the axes through (α, β) . Similarly $t[f, g]$ designates a triangle of an (f, g) -plane, bounded by $f=g$ and the parallels to the f - and g -axes through (f, g) . The elements of area $d\alpha d\beta$ and $df dg$ are to be counted positive. By $f(\alpha)$ and $g(\beta)$ we understand continuously differentiable mappings of the α -axis on the f -axis and of the β -axis on the g -axis, which are, but for the elements used, identical with each other; $f(\alpha)=g(\beta)$ if $\alpha=\beta$. The range of the four variables α, β, f, g is the domain D with origin as center

$$(D) \quad |\alpha| \leq \omega, \quad |\beta| \leq \omega, \quad |f| \leq \omega, \quad |g| \leq \omega,$$

and the function $f(\alpha)$ (and consequently $g(\beta)$) is such that for α and β satisfying (D) the point $(\alpha, f(\alpha), \beta, g(\beta))$ belongs to D . Furthermore there are defined in D three functions a, b, c of α, f, β, g having continuous derivatives up to the fourth order.

We introduce three functions X, Y, Z in D by the relations

$$(20) \quad X_{fg} = ca_f b_g,$$

$$(21) \quad Y_f = ca_f b_\beta, \quad Z_g = ca_\alpha b_g,$$

and the initial conditions

$$\left. \begin{aligned} (22) \quad X &= X_f = X_g = 0, \\ (23) \quad Y &= 0, \\ Z &= 0, \end{aligned} \right\} \quad \text{if } f = g.$$

We find

$$X(\alpha, f, \beta, g) = \iint_{t[f, g]} ca_f b_g(\alpha, f', \beta, g') df' dg'.$$

Here, as in all integrals that follow, care has been taken to indicate the argu-

ments of the integrand at least in one of the factors of the integrand, to denote the variables of integration by a prime and to denote by subscripts the partial derivatives, while we reserve the symbols $d/d\alpha$ and $d/d\beta$ for total derivatives with respect to these variables.

We are going to study the function

$$\begin{aligned}
 I(\alpha, f, \beta, g) &= X(\alpha, f, \beta, g) - \int_{\beta}^{\alpha} (X_{\alpha}(\alpha', f(\alpha'), \beta, g) - Z) d\alpha' \\
 (24) \quad &+ \int_{\beta}^{\alpha} (X_{\beta}(\alpha, f, \beta', g(\beta')) - Y) d\beta' \\
 &+ \iint_{T[\alpha, \beta]} (-X_{\alpha\beta} + Z_{\beta} + Y_{\alpha} - ca_{\alpha}b_{\beta}(\alpha', f(\alpha'), \beta', g(\beta'))) d\alpha' d\beta'.
 \end{aligned}$$

In order to abbreviate as much as possible, we write $A \sim B$ if $A - B$ is expressible as a polynomial in a, b, c , their first partial derivatives and their second partial derivatives of the type $\partial^2/\partial\alpha\partial\beta$, $\partial^2/\partial\alpha\partial g$, $\partial^2/\partial\beta\partial f$, $\partial^2/\partial f\partial g$. We write $A \cong B$ if $A - B$ is expressible as an integral over a function which itself is ~ 0 .

Thus we find

$$(25) \quad I_f = X_f + \int_{\beta}^{\alpha} (X_{\beta f}(\alpha, f, \beta', g(\beta')) - Y_f) d\beta',$$

$$(26) \quad I_g = X_g - \int_{\beta}^{\alpha} (X_{\alpha g}(\alpha', f(\alpha'), \beta, g) - Z_g) d\alpha',$$

$$\begin{aligned}
 I_{\alpha} &= X_{\alpha}(\alpha, f, \beta, g) - X_{\alpha}(\alpha, f(\alpha), \beta, g) + Z(\alpha, f(\alpha), \beta, g) \\
 &+ X_{\beta}(\alpha, f, \alpha, f(\alpha)) - Y(\alpha, f, \alpha, f(\alpha)) \\
 (27) \quad &+ \int_{\beta}^{\alpha} (-X_{\alpha\beta} + Z_{\beta} + Y_{\alpha} - ca_{\alpha}b_{\beta}(\alpha, f(\alpha), \beta', g(\beta'))) d\beta', \\
 &+ \int_{\beta}^{\alpha} (X_{\alpha\beta}(\alpha, f, \beta', g(\beta')) - Y_{\alpha}) d\beta',
 \end{aligned}$$

$$\begin{aligned}
 I_{\beta} &= X_{\beta}(\alpha, f, \beta, g) - X_{\beta}(\alpha, f, \beta, g(\beta)) + Y(\alpha, f, \beta, g(\beta)) \\
 &+ X_{\alpha}(\beta, g(\beta), \beta, g) - Z(\beta, g(\beta), \beta, g) \\
 (28) \quad &- \int_{\beta}^{\alpha} (X_{\alpha\beta}(\alpha', f(\alpha'), \beta, g) - Z_{\beta}) d\alpha' \\
 &- \int_{\beta}^{\alpha} (-X_{\alpha\beta} + Z_{\beta} + Y_{\alpha} - ca_{\alpha}b_{\beta}(\alpha', f(\alpha'), \beta, g(\beta))) d\alpha',
 \end{aligned}$$

$$\begin{aligned}
 I_{f\theta} &= ca_f b_\theta, \\
 I_{f\beta} &= X_{f\beta}(\alpha, f, \beta, g) - X_{f\beta}(\alpha, f, \beta, g(\beta)) + Y_f(\alpha, f, \beta, g(\beta)), \\
 (29) \quad I_{\theta\alpha} &= X_{\theta\alpha}(\alpha, f, \beta, g) - X_{\theta\alpha}(\alpha, f(\alpha), \beta, g) + Z_\theta(\alpha, f(\alpha), \beta, g), \\
 I_{\alpha\beta} &= X_{\alpha\beta}(\alpha, f, \beta, g) - X_{\alpha\beta}(\alpha, f(\alpha), \beta, g) + Z_\beta(\alpha, f(\alpha), \beta, g) \\
 &\quad - Z_\beta(\alpha, f(\alpha), \beta, g(\beta)) + X_{\alpha\beta}(\alpha, f(\alpha), \beta, g(\beta)) - X_{\alpha\beta}(\alpha, f, \beta, g(\beta)) \\
 &\quad + Y_\alpha(\alpha, f, \beta, g(\beta)) - Y_\alpha(\alpha, f(\alpha), \beta, g(\beta)) + ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)).
 \end{aligned}$$

Hence

$$\begin{aligned}
 I_{f\beta} &= \int_{\theta(\beta)}^{\theta} (X_{f\beta\theta} - Y_{f\theta}(\alpha, f, \beta, g')) dg' + Y_f(\alpha, f, \beta, g) \\
 &= \int_{\theta(\beta)}^{\theta} [(ca_f b_\theta)_\beta - (ca_f b_\beta)_\theta] dg' + ca_f b_\beta \\
 &= \int_{\theta(\beta)}^{\theta} [(ca_f)_\beta b_\theta - (ca_f)_\theta b_\beta] dg' + ca_f b_\beta, \\
 (30) \quad I_{f\beta} &\cong ca_f b_\beta.
 \end{aligned}$$

Similarly

$$(31) \quad I_{\theta\alpha} \cong ca_\alpha b_\theta.$$

Moreover,

$$\begin{aligned}
 X_{\alpha\beta}(\alpha, f, \beta, g) - X_{\alpha\beta}(\alpha, f(\alpha), \beta, g) + X_{\alpha\beta}(\alpha, f(\alpha), \beta, g(\beta)) \\
 - X_{\alpha\beta}(\alpha, f, \beta, g(\beta)) = \int_{f(\alpha)}^f \int_{\theta(\beta)}^{\theta} X_{\alpha\beta f\theta}(\alpha, f', \beta, g') df' dg'.
 \end{aligned}$$

But

$$\begin{aligned}
 X_{\alpha\beta f\theta} &= (ca_f b_\theta)_{\alpha\beta} = (c_\alpha a_f b_\theta + ca_f a_\alpha b_\theta + ca_f b_{\theta\alpha})_\beta \\
 &\sim c_\alpha a_f b_{\theta\beta} + c_\beta a_\alpha b_\theta + ca_f a_\beta b_\theta + ca_f a_\beta b_{\theta\beta} + ca_f b_{\theta\alpha\beta} \\
 &\sim (c_\alpha a_f b_\beta)_\theta + (c_\beta a_\alpha b_\theta)_f + (ca_{\alpha\beta} b_\theta)_f + (ca_f b_{\alpha\beta})_\theta + ca_\alpha b_\beta b_\theta, \\
 ca_\alpha b_\beta b_\theta &= (ca_\alpha b_\beta)_f - c_f a_\alpha b_\beta - ca_\alpha b_{\theta\beta f} \\
 &\sim (ca_\alpha b_\beta)_{\theta f} - [(ca_\alpha)_\theta b_\beta]_f - (c_f a_\alpha b_\beta)_\theta - (ca_\alpha b_\beta f)_\theta.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \int_{f(\alpha)}^f \int_{\theta(\beta)}^{\theta} X_{\alpha\beta f\theta}(\alpha, f', \beta, g') df' dg' &\cong ca_\alpha b_\beta(\alpha, f, \beta, g) - ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g) \\
 &\quad + ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)) - ca_\alpha b_\beta(\alpha, f, \beta, g(\beta)).
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 Z_{\beta}(\alpha, f(\alpha), \beta, g) - Z_{\beta}(\alpha, f(\alpha), \beta, g(\beta)) &= \int_{\sigma(\beta)}^{\sigma} Z_{\beta\sigma}(\alpha, f(\alpha), \beta, g') dg' \\
 &= \int_{\sigma(\beta)}^{\sigma} [ca_{\alpha}b_{\sigma}(\alpha, f(\alpha), \beta, g')]_{\beta} dg' \\
 &\cong \int_{\sigma(\beta)}^{\sigma} [ca_{\alpha}b_{\beta}(\alpha, f(\alpha), \beta, g')]_{\sigma} dg' \\
 &\cong ca_{\alpha}b_{\beta}(\alpha, f(\alpha), \beta, g) - ca_{\alpha}b_{\beta}(\alpha, f(\alpha), \beta, g(\beta)).
 \end{aligned}$$

Similarly,

$$Y_{\alpha}(\alpha, f, \beta, g(\beta)) - Y_{\alpha}(\alpha, f(\alpha), \beta, g(\beta)) \cong ca_{\alpha}b_{\beta}(\alpha, f, \beta, g(\beta)) - ca_{\alpha}b_{\beta}(\alpha, f(\alpha), \beta, g(\beta)).$$

Finally

$$(32) \quad I_{\alpha\beta}(\alpha, f, \beta, g) \cong ca_{\alpha}b_{\beta}(\alpha, f, \beta, g).$$

In view of (29), we find

$$\begin{aligned}
 \frac{d^2 I}{d\alpha d\beta}(\alpha, f(\alpha), \beta, g(\beta)) &= ca_{\alpha}b_{\beta} + ca_{\beta}b_{\beta} \frac{df(\alpha)}{d\alpha} + ca_{\alpha}b_{\sigma} \frac{dg(\beta)}{d\beta} + ca_{\beta}b_{\sigma} \frac{df(\alpha)}{d\alpha} \frac{dg(\beta)}{d\beta} \\
 &= c \frac{da(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha} \frac{db(\alpha, f(\alpha), \beta, g(\beta))}{d\beta}
 \end{aligned}$$

and

$$I(\alpha, f(\alpha), \beta, g(\beta)) = 0, \quad \frac{dI}{d\alpha} = 0, \quad \frac{dI}{d\beta} = 0,$$

for $\alpha = \beta$. Thus

$$(33) \quad I(\alpha, f(\alpha), \beta, g(\beta)) = - \int \int_{T[\alpha, \beta]} c \frac{da}{d\alpha'} \frac{db}{d\beta'} (\alpha', f(\alpha'), \beta', g(\beta')) d\alpha' d\beta'.$$

In order to transform $I_f(\alpha, f, \beta, g)$ we calculate

$$\begin{aligned}
 \int_{\beta}^{\alpha} X_{\beta f}(\alpha, f, \beta', g(\beta')) d\beta' &= \int_{\beta}^{\alpha} d\beta' \left[\int_f^{\sigma(\beta')} X_{\beta f \sigma}(\alpha, f, \beta', g') dg' + X_{\beta f}(\alpha, f, \beta', f) \right] \\
 &= \int_{\beta}^{\alpha} d\beta' \int_f^{\sigma(\beta')} (ca_{\beta}b_{\sigma}(\alpha, f, \beta', g'))_{\beta} dg' \\
 &\cong \int_{\beta}^{\alpha} d\beta' \int_f^{\sigma(\beta')} (ca_{\beta}b_{\beta}(\alpha, f, \beta', g'))_{\sigma} dg' \\
 &\cong \int_{\beta}^{\alpha} d\beta' [ca_{\beta}b_{\beta}(\alpha, f, \beta', g(\beta')) - ca_{\beta}b_{\beta}(\alpha, f, \beta', f)]
 \end{aligned}$$

since, by (22), $X_{\beta f}(\alpha, f, \beta', f)$ vanishes.

Consequently

$$(34) \quad I_f(\alpha, f, \beta, g) \cong 0.$$

Similarly

$$(35) \quad I_g(\alpha, f, \beta, g) \cong 0.$$

Also we have

$$\begin{aligned} X_\alpha(\alpha, f, \beta, g) &= - \int \int_{t[f, g]} (ca_f b_g(\alpha, f', \beta, g'))_\alpha df' dg' \\ &\cong - \int \int_{t[f, g]} (ca_\alpha b_g(\alpha, f', \beta, g'))_f df' dg' \\ &\cong - \int ca_\alpha b_g(\alpha, f', \beta, g') dg', \end{aligned}$$

where the integration is to be extended over the boundary of $t[f, g]$; whence

$$X_\alpha(\alpha, f, \beta, g) \cong 0, \quad X_\beta(\alpha, f, \beta, g) \cong 0.$$

Thus

$$(36) \quad I_\alpha(\alpha, f, \alpha, g) \cong 0,$$

as obviously $Y \cong 0, Z \cong 0$.

On writing

$$I_\alpha(\alpha, f, \beta, g) = I_\alpha(\alpha, f, \alpha, g) + \int_\alpha^\beta I_{\alpha\beta}(\alpha, f, \beta', g) d\beta',$$

we find by (36) and (32)

$$(37) \quad I_\alpha(\alpha, f, \beta, g) \cong 0.$$

Similarly,

$$(38) \quad I_\beta(\alpha, f, \beta, g) \cong 0.$$

Finally

$$(39) \quad \begin{aligned} I(\alpha, f, \beta, g) &= I(\alpha, f, \alpha, g) + \int_\alpha^\beta I_\beta(\alpha, f, \beta', g) d\beta', \\ I(\alpha, f, \beta, g) &\cong 0. \end{aligned}$$

The formulas (29), (30), (31), (32), (34), (35), (37), (38), and (39) prove that $I(\alpha, f, \beta, g)$, its first derivatives and its second derivatives of the type

$\partial^2/\partial\alpha\partial\beta$, $\partial^2/\partial\alpha\partial g$, $\partial^2/\partial\beta\partial f$, $\partial^2/\partial f\partial g$ may be expressed in terms of a , b , c , their first and second derivatives of the same type, and integrals over products of such functions.

Henceforth the definition (24) of I is to be replaced by the explicit formula whose abbreviated equivalent is (39), and which retains sense in the case that a , b , c admit only of continuous first derivatives and of continuous second derivatives of the indicated type. If we uniformly approximate a , b , c and said derivatives by polynomials in α , f , β , g and their respective derivatives, we may easily see that all of the formulas (29)–(32), (34), (35), (37)–(39) remain valid under the new assumptions and that $I(\alpha, f, \beta, g)$ still retains continuous first and second derivatives of said type.

Moreover a study of the dependence of I on the function $f(\alpha)$ shows that convergence of a sequence of continuously differentiable functions $f_\mu(\alpha)$ to a limit function $f_0(\alpha)$ implies the convergence of the corresponding functionals $I_\mu(\alpha, f, \beta, g)$ to a limit functional $I_0(\alpha, f, \beta, g)$, and uniform convergence of $f_\mu(\alpha)$ to $f_0(\alpha)$ entails uniform convergence of I_μ to I_0 . In fact, $f_\mu(\alpha)$ appears in the definition of $I_\mu(\alpha, f, \beta, g)$ only in limits of integration with respect to f' or g' , which implies the convergence mentioned of I_μ to I .

From the formulas (29)–(32), (34), (35), (37)–(39) we can derive estimates for $I(\alpha, f, \beta, g)$ and its derivatives. Suppose first that in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leq \omega$

$$|a|, |b|, |c| \leq K, \quad |a_\alpha|, \dots, |c_g| \leq K', \quad |a_{\alpha\beta}|, \dots, |c_{fg}| \leq K''.$$

The terms suppressed in the above formulas by the use of the symbol $\cong$ are simple, double, triple, and quadruple integrals of polynomials of third degree in a , b , c , a_α , $\dots$, c_{fg} with ranges of integration, respectively, $\leq \rho\omega$, $\rho\omega^2$, $\rho\omega^3$, $\rho\omega^4$, where ρ denotes a sufficiently large number, for instance, 64. On the other hand, any one of the polynomials to be integrated is numerically smaller than a suitable polynomial in K , K' , K'' of third degree with positive coefficients. Hence there exists a polynomial p with positive coefficients and of third degree in K , K' , K'' , such that for any one of formulas (29), $\dots$, (39) the terms suppressed by the symbol $\cong$ are numerically less than or equal to

$$p(K, K', K'')(\omega + \omega^2 + \omega^3 + \omega^4).$$

We apply these estimates to the case where a , b , c depend only indirectly on α , f , β , g and are functions of $\psi_1, \dots, \psi_n(\alpha, f, \beta, g)$ having third derivatives with respect to $\psi_1, \dots, \psi_n$. We assume that in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leq \omega$, the following quantities exist and satisfy

$$(40) \quad |\psi_1|, \dots, |\psi_n| \leq k,$$

$$(41) \quad |\psi_{1\alpha}|, \dots, |\psi_{n\theta}| \leq k', \quad |\psi_{1\alpha\beta}|, \dots, |\psi_{n\theta f}| \leq k'', \quad (k, k', k'' > 0),$$

and that for ψ_i satisfying (40)

$$(42) \quad \begin{aligned} &|a|, |b|, |c| \leq L, \\ &\left| \frac{\partial a}{\partial \psi_i} \right|, \left| \frac{\partial b}{\partial \psi_i} \right|, \left| \frac{\partial c}{\partial \psi_i} \right| \leq L', \\ &\left| \frac{\partial^2 a}{\partial \psi_i \partial \psi_j} \right|, \dots, \left| \frac{\partial^2 c}{\partial \psi_i \partial \psi_j} \right| \leq L'', \\ &\left| \frac{\partial^3 a}{\partial \psi_i \partial \psi_j \partial \psi_l} \right|, \dots, \left| \frac{\partial^3 c}{\partial \psi_i \partial \psi_j \partial \psi_l} \right| \leq L'''. \end{aligned}$$

We then write

$$I(\alpha, f, \beta, g) = I \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi \end{smallmatrix} \right).$$

In order to utilize the estimates found, it is legitimate to replace K by L , K' by $nL'k'$, K'' by $nLk'' + n^2L''k'^2$. Thus from (29), $\dots$, (39) we obtain

$$(43) \quad \begin{aligned} &\left| I \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi \end{smallmatrix} \right) \right|, |I_\alpha|, \dots, |I_g| \\ &\leq (\omega + \omega^2 + \omega^3 + \omega^4)q(L, L', L'', k, k', k'') \\ &|I_{\alpha\beta}|, |I_{\alpha g}|, |I_{\beta f}|, |I_{fg}| \\ &\leq n^2LL'^2k'^2 + (\omega + \omega^2 + \omega^3 + \omega^4)q(L, \dots, k''), \end{aligned}$$

where q is a suitable polynomial with positive coefficients.

We next state bounds for the difference between

$$I \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi \end{smallmatrix} \right) \quad \text{and} \quad I \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi' \end{smallmatrix} \right)$$

and its derivatives, assuming ψ'_i to be another system of continuously differentiable functions of α, f, β, g which satisfy the same inequalities (40), (41) as the ψ_i themselves do. For the sake of simplicity we suppose $\omega < \Omega$, with $\Omega > 0$ and denote by u and v

$$\begin{aligned} u &= \sum_{i=1}^n \max |\psi_i - \psi'_i| + \sum_{i=1}^n \max |\psi_{i\alpha} - \psi'_{i\alpha}| + \dots + \sum_{i=1}^n \max |\psi_{ig} - \psi'_{ig}|, \\ v &= \sum_{i=1}^n \max |\psi_{i\alpha\beta} - \psi'_{i\alpha\beta}| + \sum_{i=1}^n \max |\psi_{i\alpha g} - \psi'_{i\alpha g}| \\ &\quad + \sum_{i=1}^n \max |\psi_{i\beta f} - \psi'_{i\beta f}| + \sum_{i=1}^n \max |\psi_{ifg} - \psi'_{ifg}|, \end{aligned}$$

the maxima to be taken for the domain $|\alpha|, |f|, |\beta|, |g| \leq \omega$.

$$I_{i,jl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi \end{matrix} \right) \quad \text{the functional} \quad I \left(\begin{matrix} \alpha, f, \beta, g \\ \psi \end{matrix} \right)$$

formed for

$$a = \psi_j, \quad b = \psi_l, \quad c = c_{ijl}.$$

THEOREM 4. Denote by $\omega > 0$ a number $< \Omega$ and by D the domain $|\alpha|, |f|, |\beta|, |g| \leq \omega$. Suppose c_{ijl} to be continuously differentiable up to the third order with respect to its arguments $\psi_1, \dots, \psi_n$, and that for ψ_i in (40), $a = \psi_j, b = \psi_l, c = c_{ijl}$ the relations (42) hold. Suppose furthermore that in D the functions $\psi_i^0(\alpha, f, \beta, g)$ are continuously differentiable and admit of continuous second derivatives of the type mentioned such that

$$(46) \quad |\psi_i^0(\alpha, f, \beta, g)| \leq k/2,$$

$$(47) \quad |\psi_{ia}^0|, \dots, |\psi_{io}^0| \leq k'/2,$$

$$(48) \quad |\psi_{ia\beta}^0|, |\psi_{ia\beta}^0|, |\psi_{i\beta f}^0|, |\psi_{i\beta f}^0| \leq k''/3.$$

Then the system

$$(49) \quad \psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^n I_{ijl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi \end{matrix} \right), \quad (i = 1, 2, \dots, n),$$

has a solution $\psi_i(\alpha, f, \beta, g)$ existing and uniquely determined in $|\alpha|, |f|, |\beta|, |g| \leq \omega'$, where $\omega' > 0$ is a number $\leq \omega$ that may be determined with the aid of $\Omega, k, k', k'', L, L', L'', L'''$ only.* $\psi_{ia}(\alpha, f, \beta, g), \dots, \psi_{io}(\alpha, f, \beta, g)$ exist, are continuous and in absolute value $\leq k'$, and $\psi_{ia\beta}(\alpha, f, \beta, g), \dots, \psi_{i\beta f}(\alpha, f, \beta, g)$ exist, are continuous and in absolute value $\leq \max(k'', 12n^5 L L'^2 k'^2)$.

We start the proof by increasing, if necessary, k'' so as to satisfy the inequality

$$4n^5 L L'^2 k'^2 < k''/3.$$

We use successive approximations:

$$\psi_i^1(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{k,l=1}^n I_{ikl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi^0 \end{matrix} \right),$$

and generally for $m \geq 0$

$$(49.1) \quad \psi_i^{m+1}(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{k,l=1}^n I_{ikl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi^m \end{matrix} \right).$$

* In particular ω' does not depend on the function $f(\alpha)$ used in the definition of I_{ijl} .

Denoting, as before, by C a constant depending only on $\Omega, k, k', k'', L, L', L'',$ by ∂ a generic first derivative with respect to $\alpha, f, \beta,$ or $g,$ and by ∂^2 a generic derivative of type $\partial^2/\partial\alpha\partial\beta, \partial^2/\partial\alpha\partial g, \partial^2/\partial\beta\partial f, \partial^2/\partial f\partial g,$ we set, for $m > 0,$

$$\begin{aligned} u_m &= \sum_{i,j,l=1}^n \max \left| I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^m \end{smallmatrix} \right) - I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^{m-1} \end{smallmatrix} \right) \right| \\ &\quad + \sum_{i,j,l=1}^n \sum_{\partial} \max \left| \partial(I_{ijl}(\psi^m) - I_{ijl}(\psi^{m-1})) \right|, \\ v_m &= \sum_{i,j,l=1}^n \sum_{\partial^2} \max \left| \partial^2 \left(I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^m \end{smallmatrix} \right) - I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^{m-1} \end{smallmatrix} \right) \right) \right|, \end{aligned}$$

where the maxima are to be taken in $|\alpha|, |f|, |\beta|, |g| \leq \omega'$ with $\omega' < \Omega$ to be determined later. On putting

$$\begin{aligned} u_0 &= \sum_{i,j,l=1}^n \max \left| I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^0 \end{smallmatrix} \right) \right| + \sum_{i,j,l=1}^n \sum_{\partial} \max \left| \partial I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^0 \end{smallmatrix} \right) \right|, \\ v_0 &= \sum_{i,j,l=1}^n \sum_{\partial^2} \max \left| \partial^2 I_{ijl} \left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^0 \end{smallmatrix} \right) \right|, \end{aligned}$$

we find, by (43),

$$(50) \quad u_0 \leq C\omega',$$

$$(51) \quad v_0 \leq 4n^5 LL'^2 k'^2 + C\omega',$$

and, for u_m and v_m the recursion formulas, in view of (44.1) and (44.2),

$$(52.1) \quad u_{m+1} \leq C\omega'(u_m + v_m), \quad (m = 0, 1, 2, \dots)$$

$$(52.2) \quad v_{m+1} \leq Cu_m + C\omega'v_m,$$

provided, however, that we can choose $\omega' > 0$ so as to make sure the existence of *all* successive approximations $|\alpha|, |f|, |\beta|, |g| \leq \omega'$ in the common domain D' ($|\alpha|, |f|, |\beta|, |g| \leq \omega'$). Now determine $\omega' < \Omega$ so small that

$$\begin{aligned} 1 - C\omega' &> 0, \quad (1 - C\omega')^2 - C^2\omega' > 0, \\ (53) \quad U &\equiv \frac{C\omega'(1 - C\omega') + (4n^5 LL'^2 k'^2 + C\omega')C\omega'}{(1 - C\omega')^2 - C^2\omega'} \leq \frac{\min(k, k')}{2}, \\ V &\equiv \frac{C^2\omega' + (1 - C\omega')(4n^5 LL'^2 k'^2 + C\omega')}{(1 - C\omega')^2 - C^2\omega'} \leq \frac{2k''}{3}. \end{aligned}$$

Note that

$$(1 + U + V)C\omega' \leq U$$

and

$$4n^5 LL'^2 k'^2 + C\omega' + CU + C\omega'V \leq V.$$

Now the conditions (46), (47), and (48) permit the construction of

$$I_{ikl} \begin{pmatrix} \alpha, f, \beta, g \\ \psi^0 \end{pmatrix}$$

in D , hence a fortiori in D' ($|\alpha|, |f|, |\beta|, |g| \leq \omega'$), and we certainly have, by (53),

$$u_0 \leq U,$$

$$v_0 \leq V.$$

Suppose that we could construct, throughout D' , the m th approximation $\psi_i^m(\alpha, f, \beta, g)$ and that

$$(54) \quad \sum_0^m u_i \leq U,$$

$$(55) \quad \sum_0^m v_i \leq V.$$

We are then able to prove that we can construct the $(m+1)$ st approximation and that

$$\sum_0^{m+1} u_i \leq U, \quad \sum_0^{m+1} v_i \leq V.$$

In fact, we have, in view of (54) and (55), (46), (47), and (48),

$$(56) \quad \begin{aligned} |\psi_i^{m+1}(\alpha, f, \beta, g)| &\leq |\psi_i^0(\alpha, f, \beta, g)| + \sum_0^m u_i \\ &\leq |\psi_i^0(\alpha, f, \beta, g)| + U \leq k, \end{aligned}$$

$$(57) \quad |\partial \psi_i^{m+1}(\alpha, f, \beta, g)| \leq |\partial \psi_i^0(\alpha, f, \beta, g)| + \sum_0^m u_i \leq k',$$

$$(58) \quad |\partial^2 \psi_i^{m+1}(\alpha, f, \beta, g)| \leq |\partial^2 \psi_i^0(\alpha, f, \beta, g)| + \sum_0^m v_i \leq k'',$$

and by (52),

$$\sum_0^{m+1} u_i \leq u_0 + C\omega' \left(\sum_0^m u_i + \sum_0^m v_i \right) \leq C\omega'(1 + U + V) \leq U,$$

$$\sum_0^{m+1} v_i \leq v_0 + C \sum_0^m u_i + C\omega' \sum_0^m v_i \leq 4n^5 LL'^2 k'^2 + C\omega' + CU + C\omega'V \leq V.$$

Thus (54) and (55) hold for all $m \geq 0$, and we conclude the uniform convergence of $\psi_i^m(\alpha, f, \beta, g)$, $\partial\psi_i^m$, $\partial^2\psi_i^m$ to limit functions $\psi_i(\alpha, f, \beta, g)$ and their corresponding derivatives $\partial\psi_i$, $\partial^2\psi_i$. The continuity relation (45) finally proves

$$I\left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi^m \end{smallmatrix}\right) \rightarrow I\left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi \end{smallmatrix}\right).$$

Hence by passage to the limit in (49.1) we obtain (49).

The uniqueness follows similarly from the relations analogous to (44.1) and (44.2),

$$(59.1) \quad u \leq C\omega'(u + v)$$

$$(59.2) \quad v \leq Cu + C\omega'v,$$

which yield $u(1 - C\omega') \leq C\omega'v \leq C\omega' \cdot Cu/(1 - C\omega')$, $u((1 - C\omega')^2 - C^2\omega') \leq 0$, $u = 0$, $v = 0$, with

$$\begin{aligned} u &= \sum_{i,j,l=1}^n \max \left| I_{ijl}\left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi \end{smallmatrix}\right) - I_{ijl}\left(\begin{smallmatrix} \alpha, f, \beta, g \\ \psi' \end{smallmatrix}\right) \right| \\ &\quad + \sum_{i,j,l} \sum_{\partial} \max | \partial(I_{ijl}(\psi) - I_{ijl}(\psi')) |, \\ v &= \sum_{i,j,l} \sum_{\partial^2} \max | \partial^2(I_{ijl}(\psi) - I_{ijl}(\psi')) |, \end{aligned}$$

$\psi_i(\alpha, f, \beta, g)$ and $\psi'_i(\alpha, f, \beta, g)$ being solutions of (49).

COROLLARY 1. *If, in Theorem 4, $f(\alpha)$, $\psi_1^0(\alpha, f, \beta, g)$, $\dots$, $\psi_n^0(\alpha, f, \beta, g)$, $\partial\psi_1^0(\alpha, f, \beta, g)$, $\dots$, $\partial\psi_n^0(\alpha, f, \beta, g)$, $\partial^2\psi_1^0(\alpha, f, \beta, g)$, $\dots$, $\partial^2\psi_n^0(\alpha, f, \beta, g)$, and $c_{ijl}(\psi_1, \dots, \psi_n)$ and its derivatives up to the third order depend on a parameter μ and converge uniformly as $\mu \rightarrow \infty$, then $\psi_1(\alpha, f, \beta, g)$, $\dots$, $\psi_n(\alpha, f, \beta, g)$, $\partial\psi_1$, $\dots$, $\partial\psi_n$, $\partial^2\psi_1$, $\dots$, $\partial^2\psi_n$ converge uniformly, as $\mu \rightarrow \infty$, in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leq \omega'' \leq \omega'$, where ω'' depends only on Ω , k , k' , k'' , L , L' , L'' .*

Denote by Δ the operation of taking the difference for two sufficiently large values of μ , and put, in the successive approximations of the proof of Theorem 4,

$$\begin{aligned} u_m &= \sum_{i=1}^n | \Delta\psi_i^m(\alpha, f, \beta, g) | + \sum_{i=1, \partial}^n | \Delta\partial\psi_i^m |, \\ v_m &= \sum_{i=1, \partial^2}^n | \Delta\partial^2\psi_i^m(\alpha, f, \beta, g) |. \end{aligned}$$

Observing that $f(\alpha)$ enters in the functional I only as a limit of integration, as has been remarked earlier, we may use (45) and find, with some

$C = C(\Omega, k, k', k'', L, L', L'', L''')$ and a new and smaller value ω'' of ω' , satisfying (53) with the new C :

$$u_m \leq C(u_{m-1} + v_{m-1} + \epsilon)\omega'' + C\epsilon + u_0,$$

$$v_m \leq Cu_{m-1} + Cv_{m-1}\omega'' + C\epsilon + v_0,$$

$$u_0 \leq \epsilon, \quad v_0 \leq \epsilon.$$

Hence, for $m \rightarrow \infty$, $\lim u_m = u$, $\lim v_m = v$

$$u \leq C(u + v + \epsilon)\omega'' + C\epsilon + u_0,$$

$$v \leq Cu + Cv\omega'' + C\epsilon + v_0,$$

$$u(1 - C\omega'') \leq (C + 1)\epsilon + \epsilon C\omega'' + C\omega''v,$$

$$C\omega''v \leq \frac{C\omega''}{1 - C\omega''}Cu + \frac{(C + 1)\epsilon C\omega''}{1 - C\omega''}.$$

Hence u and v are $\leq C'\epsilon$, with $C'(C, \omega'')$, which proves the corollary.

COROLLARY 2. If, in Theorem 4, $f(\alpha), \psi_i^0(\alpha, f, \beta, g), C_{i;l}$ depend on a parameter μ , and if $f(\alpha)$ converges uniformly as $\mu \rightarrow \mu_0$, then there exists a subsequence of μ , such that $\psi_i(\alpha, f, \beta, g)$ and also $\psi_i(\alpha, f(\alpha), \beta, g(\beta))$ converge uniformly.

For by (57), $|\partial\psi_i(\alpha, f, \beta, g)| \leq k'$ and hence the $\psi_i(\alpha, f, \beta, g)$ are equicontinuous and bounded, in view of (56). Hence there exists a uniformly convergent subsequence, and the corollary follows. From (33) we conclude under the hypothesis of the theorem, that

$$(60) \quad \frac{d^2}{d\alpha d\beta} \psi_i(\alpha, f(\alpha), \beta, g(\beta)) = \frac{d^2}{d\alpha d\beta} \psi_i^0(\alpha, f(\alpha), \beta, g(\beta)) + \sum_{j,l=1}^n c_{i;jl}(\psi) \frac{d\psi_j}{d\alpha} \frac{d\psi_l}{d\beta},$$

and, for $\alpha = \beta, f = f(\alpha), g = f(\alpha)$

$$\psi_i = \psi_i^0, \quad \partial\psi_i = \partial\psi_i^0.$$

In the application we intend to make of Theorem 4 and its corollaries, the values of the constants such as $\Omega, k, k', k'', L, L', L'', L'''$ are of no importance. What matters, however, is their existence and their interdependence. Therefore, we are led to use the following terminology: we call a function *bounded* if its absolute value is bounded by a positive number irrespective of the values of its arguments and possible other parameters; we call, in a theorem, a quantity *relatively bounded* if its absolute value can be bounded by a positive number which depends only on other bounds *previously* introduced in the theorem; and we use the same term, in a proof, as meaning *limitable by bounds*, either assumed by the hypotheses of the theorem, or previously introduced in the course of the same proof.

Thus Theorem 4 and Corollary 2 may be formulated as follows:

THEOREM 5. *Suppose $c_{i;l}$ and its derivatives up to the third order with respect to its arguments $\psi_1, \dots, \psi_n$ bounded for bounded values of ψ_i , and assume that $\psi_i = \psi_i^0(\alpha, f, \beta, g)$, $i = 1, \dots, n$, has derivatives $\partial\psi_i^0, \partial^2\psi_i^0$ which are continuous and bounded when α, f, β, g are bounded. Then the system*

$$\psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^n I_{i;jl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi \end{matrix} \right), \quad (i = 1, 2, \dots, n),$$

has a solution $\psi_i(\alpha, f, \beta, g)$, continuous together with $\partial\psi_i$ and $\partial^2\psi_i$. This solution exists, is uniquely determined and is relatively bounded together with the derivatives $\partial\psi_i, \partial^2\psi_i$ for relatively bounded α, f, β, g . If, in addition, $f(\alpha), \psi_i^0(\alpha, f, \beta, g)$ and $c_{i;l}$ and its derivatives up to the third order depend on a parameter μ , and if $f(\alpha)$ converges uniformly as $\mu \rightarrow \mu_0$, then there exists a subsequence of μ such that the corresponding functions $\psi_i(\alpha, f, \beta, g)$ and $\psi_i(\alpha, f(\alpha), \beta, g(\beta))$ converge uniformly for relatively bounded values of α, f, β, g .

4. Hyperbolic systems. The results of the preceding section may be used for a study of Cauchy's problem* for the system

$$(61) \quad \begin{aligned} \sum_{j=1}^n a_{ij}(\phi_1, \dots, \phi_n) \frac{\partial \phi_j(\alpha, \beta)}{\partial \alpha} &= 0, & i = 1, 2, \dots, m < n, \\ \sum_{j=1}^n a_{ij}(\phi_1, \dots, \phi_n) \frac{\partial \phi_j(\alpha, \beta)}{\partial \beta} &= 0, & i = m + 1, \dots, n, \end{aligned}$$

in which a_{ij} and its partial derivatives up to the fourth order as well as the reciprocal value of the determinant $|a_{ij}|$ are bounded for bounded values of $\phi_1, \dots, \phi_n$. The initial line is a bounded neighborhood of the origin on the line $\alpha = \beta$, and on it the unknown functions $\phi_1(\alpha, \beta), \dots, \phi_n(\alpha, \beta)$ assume relatively bounded values $\xi_1(\alpha), \dots, \xi_n(\alpha)$ which are continuously differentiable.

In view of the applications we subject the $\xi_1(\alpha), \dots, \xi_n(\alpha)$ to the following condition:

Condition ϑ . $\xi_1(\alpha), \dots, \xi_n(\alpha)$ depend continuously on α , and there exists a transformation

$$\zeta_i(\alpha) = \sum_{j=1}^n \xi_j(\alpha) \gamma_{ji}, \quad \xi_j(\alpha) = \sum_{i=1}^n \Gamma_{ji} \zeta_i(\alpha),$$

* A study of this Cauchy problem with a view to enlarging the class of admissible initial conditions was undertaken by Margaret Gurney in her dissertation, Brown University, 1935 (unpublished).

with constant γ_{ij} of determinant ± 1 such that the derivatives of $\zeta_2(\alpha)$, $\zeta_3(\alpha)$, $\dots$, $\zeta_n(\alpha)$ are bounded.

Then there exists a relatively bounded solution $\phi_1(\alpha, \beta), \dots, \phi_n(\alpha, \beta)$ of (61) in a relatively bounded (α, β) -neighborhood of the origin, assuming the given initial values, continuous in α, β , and continuously differentiable with respect to α and β .

It should be noticed that the essential content of the above statement lies in the fact that the derivative of $\zeta_1(\alpha)$ has no influence on the determination of the domain of existence.

The idea of the proof is to construct instead of functions $\phi_i(\alpha, \beta)$ other functions ψ_i of four arguments α, f, β, g which reduce to the solution of the initial problem in question for $f = \zeta_1(\alpha), g = \zeta_1(\beta)$. In order to conform with the terminology formerly introduced, we henceforth shall identify $\zeta_1(\alpha)$ with $f(\alpha)$.

We try to satisfy the following conditions for functions $\psi_i^0(\alpha, f, \beta, g)$:

$$\begin{aligned} \text{(i)} \quad & \frac{d^2 \psi_i^0}{d\alpha d\beta}(\alpha, f(\alpha), \beta, g(\beta)) = 0, \\ \text{(ii)} \quad & \psi_i^0(\alpha, f(\alpha), \alpha, f(\alpha)) = \xi_i(\alpha), \\ & \sum_{j=1}^n a_{ij}(\xi_1(\alpha), \dots, \xi_n(\alpha)) \left[\psi_{i\alpha}^0(\alpha, f(\alpha), \alpha, f(\alpha)) \right. \\ & \quad \left. + \psi_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha)) \frac{df(\alpha)}{d\alpha} \right] = 0, \quad i \leq m, \\ \text{(iii)} \quad & \sum_{j=1}^n a_{ij}(\xi_1(\alpha), \dots, \xi_n(\alpha)) \left[\psi_{i\beta}^0(\alpha, f(\alpha), \alpha, f(\alpha)) \right. \\ & \quad \left. + \psi_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha)) \frac{df(\alpha)}{d\alpha} \right] = 0, \quad i > m. \end{aligned}$$

We first introduce $\psi_i^0(\alpha, f, \alpha, f)$ by

$$(62) \quad \psi_i^0(\alpha, f, \alpha, f) = \sum_{k=2}^n \Gamma_{ik} \zeta_k(\alpha) + \Gamma_{i1} f, \quad i = 1, 2, \dots, n,$$

which yields

$$(63) \quad \psi_{i\beta}^0(\alpha, f, \alpha, f) + \psi_{i\alpha}^0(\alpha, f, \alpha, f) = \Gamma_{i1}.$$

To determine $\psi_{i\beta}^0$ and $\psi_{i\alpha}^0$ we set up the system

$$\begin{aligned} (64) \quad & \sum_{k=1}^n a_{ik}(\psi^0(\alpha, f, \alpha, f)) \psi_{k\beta}^0(\alpha, f, \alpha, f) = 0, \quad i = 1, \dots, m, \\ & \sum_{k=1}^n a_{ik}(\psi^0(\alpha, f, \alpha, f)) \psi_{k\alpha}^0(\alpha, f, \alpha, f) = 0, \quad i = m+1, \dots, n, \end{aligned}$$

which together with (63), in view of the boundedness of $|a_{ik}|^{-1}$, determines $\psi_{ig}^0(\alpha, f, \alpha, f)$ and $\psi_{i\beta}^0(\alpha, f, \alpha, f)$ as analytic functions of $a_{ik}(\psi^0(\alpha, f, \alpha, f))$ and thus as relatively bounded functions with relatively bounded and continuous total derivatives with respect to α and f .

From (ii)

$$\psi_{i\alpha}^0(\alpha, f, \alpha, f) + \psi_{i\beta}^0(\alpha, f, \alpha, f) = \sum_{k=2}^n \Gamma_{ik} \frac{d\zeta_k(\alpha)}{d\alpha}, \quad i = 1, 2, \dots, n,$$

and by (64) and (iii)

$$(65) \quad \begin{aligned} \sum_{k=1}^n a_{ik}(\xi_1(\alpha), \dots, \xi_n(\alpha)) \psi_{i\alpha}^0(\alpha, f(\alpha), \alpha, f(\alpha)) &= 0, & i \leq m; \\ \sum_{k=1}^n a_{ik} \psi_{i\beta}^0(\alpha, f(\alpha), \alpha, f(\alpha)) &= 0, & i > m, \end{aligned}$$

which determine $\psi_{i\alpha}^0(\alpha, f(\alpha), \alpha, f(\alpha))$ and $\psi_{i\beta}^0(\alpha, f(\alpha), \alpha, f(\alpha))$ as continuous and relatively bounded functions of α for relatively bounded α .

We now put

$$A_i(\alpha, f) = \int_0^f \psi_{ig}^0(\alpha, f', \alpha, f') df'.$$

Obviously, $A_i(\alpha, f)$ has continuous and relatively bounded derivatives with respect to f and α .

Finally put

$$(66) \quad \begin{aligned} \Psi_i^0(\alpha, f, \beta, g) &= \psi_i^0(\alpha, f, \alpha, f) + A_i(\beta, g) - A_i(\alpha, f) \\ &\quad - \int_{\beta}^{\alpha} [\psi_{i\beta}^0(\alpha', f(\alpha'), \alpha', f(\alpha')) - A_{i\alpha}(\alpha', f(\alpha'))] d\alpha'. \end{aligned}$$

The reader will easily verify that the function $\Psi_i^0(\alpha, f, \beta, g)$, as defined by (66), has the following properties:

$$\begin{aligned} \Psi_i^0(\alpha, f, \alpha, f) &= \psi_i^0(\alpha, f, \alpha, f), \\ \Psi_{ig}^0(\alpha, f, \alpha, f) &= \psi_{ig}^0(\alpha, f, \alpha, f), \\ \Psi_{if}^0(\alpha, f, \alpha, f) &= \frac{d}{df} \psi_i^0(\alpha, f, \alpha, f) - \frac{dA_i(\alpha, f)}{df} = \psi_{if}^0(\alpha, f, \alpha, f), \\ \Psi_{i\beta}^0(\alpha, f(\alpha), \alpha, f(\alpha)) &= \psi_{i\beta}^0(\alpha, f(\alpha), \alpha, f(\alpha)), \\ \Psi_{i\alpha}^0(\alpha, f(\alpha), \alpha, f(\alpha)) &= \psi_{i\alpha}^0(\alpha, f(\alpha), \alpha, f(\alpha)). \end{aligned}$$

Hence we are justified in considering $\Psi_i^0(\alpha, f, \beta, g)$ as an extension of those elements of the unknown function $\psi_i^0(\alpha, f, \beta, g)$ which were used in the construction of $\Psi_i^0(\alpha, f, \beta, g)$, and we write $\Psi_i^0(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g)$. We have, moreover, $\partial^2 \psi_i^0 = 0$ so that (i) is true. From (62) we obtain (ii). Formulas (64) and (65) give (iii).

Evidently, $|\psi_i^0(\alpha, f, \beta, g)|$ is less than an arbitrary positive number ϵ if the bounds of the initial data $\xi_1(\alpha), \dots, \xi_n(\alpha)$ are sufficiently small and if α, f, β, g are relatively bounded.

On differentiating the first m equations of (61) with respect to β , and the last $(n-m)$ equations with respect to α and solving with respect to the mixed derivatives, we obtain a system of the form

$$(67) \quad \frac{\partial^2 \phi_i(\alpha, \beta)}{\partial \alpha \partial \beta} = \sum_{j,l} c_{ijl}(\phi_1, \dots, \phi_n) \frac{\partial \phi_j(\alpha, \beta)}{\partial \alpha} \frac{\partial \phi_l(\alpha, \beta)}{\partial \beta},$$

where the $c_{ijl}(\phi_1, \dots, \phi_n)$ have bounded derivatives up to the third order for bounded $\phi_1, \dots, \phi_n$. Replacing ϕ by ψ , we solve

$$\psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^n I_{ijl} \left(\begin{matrix} \alpha, f, \beta, g \\ \psi \end{matrix} \right)$$

with the aid of Theorem 5. By (60) and (i) we have

$$\begin{aligned} & \frac{d^2 \psi_i(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha d\beta} \\ &= \sum_{j,l} c_{ijl}(\psi(\alpha, f(\alpha), \beta, g(\beta))) \frac{d\psi_j(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha} \frac{d\psi_l(\alpha, f(\alpha), \beta, g(\beta))}{d\beta}. \end{aligned}$$

In view of (ii), $\phi_i(\alpha, \beta) = \psi_i(\alpha, f(\alpha), \beta, g(\beta))$ assumes the given initial values, satisfies (61) on $\alpha = \beta$, and has continuous derivatives with respect to α and β and continuous mixed second derivatives with respect to α and β .

A conclusion, familiar in the theory of hyperbolic equations shows that equations (61) are satisfied identically in α and β .

Thus we have established the following theorem:

THEOREM 6. *If in (61) a_{ij} and its partial derivatives up to the fourth order and the reciprocal value of the determinant $|a_{ij}|$ are bounded for bounded values of $\phi_1, \phi_2, \dots, \phi_n$, and if the initial values of $\phi_i(\alpha, \beta)$ on $\alpha = \beta$ are relatively bounded in a bounded neighborhood of $\alpha = 0$ ($= \beta$) and satisfy condition $\mathfrak{D}$, then Cauchy's problem has a solution existing for all relatively bounded α, β . This solution has continuous derivatives with respect to α and β . If the initial values and the a_{ij} depend on a parameter μ and converge uniformly as $\mu \rightarrow \mu_0$, then there exists a subsequence of μ , for which the corresponding solutions $\phi_i(\alpha, \beta)$ converge uniformly.*

COROLLARY 1. *The solution of Theorem 6 is unique.**

Let $|\alpha| \leq A$, $|\beta| \leq B$ be the common domain D of existence of two solutions of our initial problem. Denote by $u, v, w(\tau)$

$$u(\tau) = \sum_{i=1}^n \max |\Delta \phi_i|, \quad v(\tau) = \sum_i \max \left| \Delta \frac{\partial \phi_i}{\partial \alpha} \right|,$$

$$w(\tau) = \sum_i \max \left| \Delta \frac{\partial \phi_i}{\partial \beta} \right|,$$

where the operator Δ indicates the difference of the expression following Δ for the two solutions, and the maximum is to be taken on that segment of the line $\tau = |\alpha - \beta|$ which is contained in D . By (67) we have for a suitable constant K

$$u(\tau) \leq \int_0^\tau (v + w) |d\tau|, \quad u(0) = 0,$$

$$v(\tau) \leq K \int_0^\tau (u + v + w) |d\tau|, \quad v(0) = 0,$$

$$w(\tau) \leq K \int_0^\tau (u + v + w) |d\tau|, \quad w(0) = 0.$$

Hence

$$u + v + w \leq (2K + 1) \int_0^\tau (u + v + w) |d\tau|,$$

and by the well known iteration $u \equiv v \equiv w \equiv 0$.

By reasoning very similar to the preceding it may be shown that the dependence of the initial data on a parameter such that the initial data of ϕ_i and those of $\partial \phi_i / \partial \alpha$, $\partial \phi_i / \partial \beta$ satisfy a Lipschitz condition of exponent 1 in the parameter implies a Lipschitz condition of exponent 1 in the solution. Furthermore, passing to the limit from difference quotient to derivative with respect to the parameter we obtain the following corollary:

COROLLARY 2. *If, in Theorem 6, the initial data of ϕ_i and those of $\partial \phi_i / \partial \alpha$, $\partial \phi_i / \partial \beta$ are continuously differentiable with respect to a parameter, the solution and its first derivatives with respect to α and β are also continuously differentiable with respect to the parameter, continuity being understood with respect to the parameter and variables.*

* Cf. Hadamard, *Leçons sur le Problème de Cauchy*, Paris, 1932, pp. 488–501.