SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

III. EXISTENCE THEOREMS FOR NONREGULAR PROBLEMS*

BY

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1. Outline of the method of proof. This note is the third of a series, the first and second of which have already appeared in these Transactions.†

In the present paper we shall establish an existence theorem for single-integral problems of the calculus of variations without requiring quasi-regularity. In order to help the reader to find his way through the many details of this proof, we first present a heuristic outline of the method used.

Given an integral $\mathcal{F}(C) = \int F(z, \dot{z}) dt$, where $z = (z^1, \dots, z^q)$, we seek to find a rectifiable curve (in q -dimensional space) joining two fixed points z_1, z_2 for which this integral is least. It is always possible to find a minimizing sequence $\{C_n\}$; that is, a sequence for which $\mathcal{F}(C_n)$ tends to the greatest lower bound μ of $\mathcal{F}(C)$ on the class of curves under consideration. Moreover, it is easy to find hypotheses on $\mathcal{F}(C)$ which will ensure that a minimizing sequence exist which has a curve of accumulation C_0 . That is, the curves C_n have representations $z = z_n(t)$, ($0 \leq t \leq 1$), for which $z_n(t)$ tends uniformly to a limit function $z_0(t)$. If now we have some method of selecting the C_n so that $z'_n(t)$ tends almost everywhere to $z'_0(t)$, then $\mathcal{F}(C_n) \rightarrow \mathcal{F}(C_0)$, and the curve C_0 is the one sought.

Suppose that a curve $C: z = z(t)$, ($t_1 \leq t \leq t_2$), in q -dimensional space minimizes an integral $\mathcal{F}(C)$ in the class of all curves of class‡ D' joining two fixed points z_1 and z_2 . As is well known, the equation

$$(1.1) \quad F_i(z(t), z'(t)) = \int_{t_1}^t F_{z^i}(z(t), z'(t)) + c_i$$

holds. Also, at each corner $z(t_0)$ of C the relations

$$(1.2) \quad F_i(z(t_0), z'(t_0 - 0)) = F_i(z(t_0), z'(t_0 + 0))$$

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‡ A curve C is of class D' if it has a representation $z = z(t)$, ($t_1 \leq t \leq t_2$), such that the functions $z^i(t)$ are continuous and the interval $[t_1, t_2]$ contains a finite number of points between which $z'(t)$ is non-vanishing and uniformly continuous. (At these points $z'(t)$ may be $(0, \dots, 0)$ or may fail to exist.)

and*

$$(1.3) \quad \Omega(z(t_0), z'(t_0 - 0), z'(t_0 + 0)) \leq 0$$

hold, where

$$(1.4) \quad \Omega(z, p, r) = p^\alpha F_{z^\alpha}(z, r) - r^\alpha F_{z^\alpha}(z, p).$$

If now we choose a sequence of polygons Π_n joining z_1 to z_2 and such that $\mathcal{F}(\Pi_n)$ tends to the lower bound μ of $\mathcal{F}(C)$ on the class of curves under consideration, we can not immediately make any statement similar to (1.1) or (1.3) for Π_n . However, if the number of vertices of Π_n is s_n , and Π_n actually minimizes $\mathcal{F}(C)$ in the class of all polygons joining z_1 and z_2 and having not more than s_n vertices, then it is to be expected that relations (1.1), (1.2), and (1.3) hold in an approximate sense for Π_n .

We now define an *approach set* at z to be an aggregate of vectors p such that

$$(1.5) \quad F_i(z, p_1) - F_i(z, p_2) = 0, \quad i = 1, \dots, q,$$

for each pair p_1, p_2 of vectors of the set. In this terminology, equation (1.2) states that $z'(t_0 - 0)$ and $z'(t_0 + 0)$ belong to an approach set at $z(t_0)$. Suppose then that the polygons Π_n have been chosen as above so that Π_n minimizes $\mathcal{F}(C)$ in the class of polygons which have not more than s_n vertices and which join z_1 to z_2 . We suppose that these are represented by functions $z = z_n(t)$, $(0 \leq t \leq 1)$, which are piecewise linear on $[0, 1]$. Also we suppose that we have already chosen a convergent sequence, so that $z_n(t)$ tends to a limit function $z_0(t)$ uniformly on $[0, 1]$. If we fix on any number t_0 in $[0, 1]$, it is possible to choose a subsequence $\{\Pi_m\}$ of the sequence $\{\Pi_n\}$ in such a way that the vectors $z'_m(t_0 + 0)$ tend to a limit p_0 . Whenever m is large and t is near t_0 , the point $z_m(t)$ is near $z_0(t_0)$ and the vector $z'_m(t_0 + 0)$ is near p_0 . If now equation (1.1) holds, at least approximately, for Π_m , the value of $F_i(z_m(t), z'_m(t))$ can change only a small amount on a short arc of Π_m . For by (1.1) the change in $F_i(z_m(t), z'_m(t))$ is equal to the integral of F_{z^i} over a short interval. Hence for all t near t_0 the values of the $F_i(z_m(t), z'_m(t))$ are nearly equal to the values of $F_i(z_m(t_0), z'_m(t_0))$, and these in turn are nearly equal to $F_i(z_0(t_0), p_0)$. Thus for all large m and for all t near t_0 the equations

$$F_i(z_0(t_0), z'_m(t)) = F_i(z_0(t_0), p_0), \quad i = 0, 1, \dots, q,$$

are almost true, and we may expect that there is an approach set A such that $z'_m(t)$ is in or near A whenever m is large and $t - t_0$ small.

Suppose now that each such approach set contains only a finite number

* Cf. I, Theorem 1.

of unit vectors, which can be set in an order $p_1, p_2, \dots, p_k$ such that

$$(1.6) \quad \Omega(z_0(t_0), p_i, p_i) > 0, \quad j > i.$$

Now (1.3) must hold approximately on Π_m , and for t near t_0 and m large the directions $z'_m(t)/|z'_m(t)|$ are each near some p_j ; hence by (1.3) and (1.6) we find that a side with direction near p_j cannot precede one with direction near p_i if $j > i$. Thus the arc of Π_m near $z_0(t_0)$ can be split into subarcs, the first of which consists of sides with directions near p_1 , the second of sides with directions near p_2 , and so on. That is, each such arc is almost a line segment. So is its limit arc; and for line segments l_n tending to a limit l_0 it is clear that $\mathcal{F}(l_n) \rightarrow \mathcal{F}(l_0)$. Thus our point t_0 is in an interval along which the integrand $F(z_n, \dot{z}_n)$ converges, with arbitrarily small error, to $F(z_0, \dot{z}_0)$.

This argument, applied to all t_0 , would yield $\mathcal{F}(\Pi_m) \rightarrow \mathcal{F}(C_0)$. Since $\mathcal{F}(\Pi_m) \rightarrow \mu$, we have $\mathcal{F}(C_0) = \mu$, and C_0 is the curve sought.

In the following pages the argument just suggested will be generalized and made rigorous.

2. Choice of a minimizing sequence. We now suppose that we are given an integral $\mathcal{F}(C)$ in parametric form, and seek the minimum of $\mathcal{F}(C)$ in the class of all rectifiable curves C joining two given points z_1 and z_2 . We suppose that $\mathcal{F}(C)$ satisfies the following condition:

(2.1) *For every constant M there is a number L_M such that all curves C joining z_1 and z_2 and giving $\mathcal{F}(C)$ a value less than or equal to M have lengths not greater than L_M .*

For example, (2.1) is satisfied if there is a number $c > 0$ such that

$$F(z, z') \geq c |z'| / (1 + |z|).$$

We shall reserve the word "vertex" for points on a polygon at which successive sides join; that is, the initial and final points will not be called vertices.

LEMMA 1. *Let $\mathcal{F}(C)$ satisfy (2.1). Let K_s be the class of all polygons joining a point z_1 to a point z_2 and having not more than s vertices. Then for every s the class K_s contains a polygon which minimizes $\mathcal{F}(C)$ on the class K_s .*

Let μ_s be the greatest lower bound of $\mathcal{F}(C)$ on the class K_s . We establish a correspondence between the polygons of K_s and the points $(\zeta^1, \dots, \zeta^{sq})$ of sq -dimensional space by writing the coordinates of the first vertex, then those of the second, and so on.* By (2.1), there is a number R such that if any coordinate of the point ζ representing a polygon Π exceeds R in absolute value, then $\mathcal{F}(\Pi) > \mu + 1$. So the minimum of $\mathcal{F}(\Pi)$ on the class K_s is to be

* We may suppose that every polygon of K_s has s vertices, because if it has less we can insert points of division in the sides and increase the number of vertices up to s .

sought among the images of the set $|\zeta| \leq Rsq$. This is bounded and closed, and on it $\mathcal{F}(\Pi)$ is continuous; hence the minimum is assumed.

Now we introduce the notation:

(2.2) *K is the class of all rectifiable curves joining the two distinct fixed points z_1 and z_2 , and μ is the greatest lower bound of $\mathcal{F}(C)$ on the class K.*

Then we have the following lemma:

LEMMA 2. *If $\mathcal{F}(C)$ satisfies (2.1), there exists a sequence of polygons Π_n : $z = z_n(t)$, ($0 \leq t \leq 1$), in the class K with the properties:*

(a) $\mathcal{F}(\Pi_n) \rightarrow \mu$.

(b) s_n being the number of vertices of Π_n , and K_{s_n} being the subclass of K consisting of polygons of not more than s_n vertices, Π_n minimizes $\mathcal{F}(C)$ on the class K_{s_n} .

(c) There is a constant L such that $|\dot{z}_n(t)| \leq L$ for all n and all t in $[0, 1]$.

(d) $z_n(t)$ converges uniformly to a limit function $z_0(t)$, ($0 \leq t \leq 1$).

Let $\{C_n\}$ be a sequence of curves of K for which $\mathcal{F}(C_n) \rightarrow \mu$. For each fixed n we inscribe in C_n a sequence of polygons $\{\Pi_k^*\}$ with sides tending to 0. It is well known that under these conditions $\mathcal{F}(\Pi_k^*)$ tends to $\mathcal{F}(C_n)$. Hence we can find a particular k for which $\mathcal{F}(\Pi_k^*) < \mathcal{F}(C_n) + 1/n$.

Let s_k^* be the number of vertices of Π_k^* . By Lemma 1, there is a polygon Π_n of s_n vertices, where $s_n \leq s_k^*$, which minimizes $\mathcal{F}(C)$ on the class K_{s_n} . Therefore $\mathcal{F}(\Pi_n) \leq \mathcal{F}(\Pi_k^*) < \mathcal{F}(C) + 1/n$. Since Π_n minimizes $\mathcal{F}(C)$ in the class K_{s_n} and is in the class $K_{s_n} \subset K_{s_k^*}$, it is clear that Π_n minimizes $\mathcal{F}(C)$ on K_{s_n} . Also,

$$\limsup_{n \rightarrow \infty} \mathcal{F}(\Pi_n) \leq \lim_{n \rightarrow \infty} \mathcal{F}(C_n) = \mu.$$

But Π_n is in K; so $\mathcal{F}(\Pi_n) \geq \mu$, and $\liminf_{n \rightarrow \infty} \mathcal{F}(\Pi_n) \geq \mu$. Thus (a) and (b) are established.

There is no loss of generality in supposing $\mathcal{F}(\Pi_n) < \mu + 1$ for all n . Hence, by hypothesis (2.1), there is an L such that each Π_n has length $\mathcal{L}(\Pi_n)$ less than L . On Π_n we introduce the parameter $t = s/\mathcal{L}(\Pi_n)$, where s is arc length. Then Π_n has a representation $z = z_n(t)$, ($0 \leq t \leq 1$), with $|\dot{z}_n(t)| \leq \mathcal{L}(C_n) < L$, and (c) is satisfied.

Finally, the $z_n(t)$ all satisfy a Lipschitz condition with constant L ; so by Ascoli's theorem there is a subsequence (which we may without loss of generality suppose to be the whole sequence) for which $z_n(t)$ converges uniformly to a limit function $z_0(t)$. This completes the proof of the lemma.

Remark. If a sequence $\{\Pi_n\}$ satisfies (a), (b), (c), and (d) of Lemma 2, so does every subsequence of $\{\Pi_n\}$.

Before stating the next lemma we introduce a definition.

(2.3) If $[\alpha, \beta]$ is an interval having points in common with $[0, 1]$ and $t_{n,1}, t_{n,2}, \dots, t_{n,k}$ are (in that order) the values of t in $[\alpha, \beta]$ which define vertices of Π_n , then

$$\theta(n, \alpha, \beta) \equiv t_{n,1}, \quad \Theta(n, \alpha, \beta) \equiv t_{n,k}.$$

If no value of t in $[\alpha, \beta]$ defines a vertex of Π_n , we define

$$(2.4) \quad \theta(n, \alpha, \beta) = \Theta(n, \alpha, \beta) = (\alpha + \beta)/2.$$

Then we have the following lemma:

LEMMA 3. If $\{\Pi_n\}$ is the sequence of Lemma 2, then for every $\epsilon > 0$ and every t_0 in $[0, 1]$ there is a $\delta > 0$ such that

$$(2.5) \quad |F_i(z_0(t_0), z'_n(t')) - F_i(z_0(t_0), z'_n(t''))| \leq \epsilon, \quad i = 1, \dots, q,$$

if t_0, t' , and t'' are all in the (open) interval $(\theta(n, t_0 - \delta, t_0 + \delta), \Theta(n, t_0 - \delta, t_0 + \delta))$ and $z_n(t')$ and $z_n(t'')$ are not vertices† of Π_n .

We shall suppose, to be specific, that $t' < t''$. Let $t_1, t_2, \dots, t_k$ define successive vertices of Π_n such that $t_1 < t' < t_2$ and $t_{k-1} < t'' < t_k$. By the definition of θ and Θ we know that

$$(2.6) \quad t_0 - \delta < t_1 < t_k < t_0 + \delta.$$

From Π_n we now form the polygon $\Pi_n^j(\tau)$ by displacing each of the vertices $z(t_2), \dots, z(t_{k-1})$ by the amount $(0, \dots, \tau, 0, \dots, 0)$, the τ being in the j th place. Because of the minimizing property of Π_n we have

$$(2.7) \quad \left. \frac{d}{d\tau} \mathcal{F}(\Pi_n^j(\tau)) \right|_{\tau=0} = 0.$$

But if we apply formula (2.8) of I to the successive sides of $\Pi_n^j(\tau)$ defined by $t_1 \leq t \leq t_2, \dots, t_{k-1} \leq t \leq t_k$, and then add, we obtain

$$(2.8) \quad \begin{aligned} \left. \frac{d}{d\tau} \mathcal{F}(\Pi_n^j(\tau)) \right|_{\tau=0} &= F_j(z_n(\bar{t}), z'_n(\bar{t})) - F_j(z_n(\bar{t}), z'_n(\bar{t})) \\ &+ \int_{t_1}^{t_2} F_{zi}(z_n, \dot{z}_n) \frac{t - t_1}{t_2 - t_1} dt + \int_{t_2}^{t_{k-1}} F_{zi}(z_n, \dot{z}_n) dt \\ &+ \int_{t_{k-1}}^{t_k} F_{zi}(z_n, \dot{z}_n) \frac{t_k - t}{t_k - t_{k-1}} dt, \end{aligned}$$

† It is easy to see that this restriction is essentially no restriction, for if t' defines a vertex, we first write (2.5) with t' replaced by $t' + h$, $h > 0$, and then let $h \rightarrow 0$. For small h the value of $z'(t' + h)$ is constantly equal to $z'(t' + 0)$; whence we find (2.5) with t' replaced by $t' + 0$. Similarly, we could replace t' by $t' - 0$, and analogously for t'' .

where $t_1 < \bar{t} < t_2$ and $t_{k-1} < \bar{t} < t_k$. All the curves Π_n lie in a bounded closed subset of the space, and all $|\dot{z}_n|$ are less than L . Hence the functions $F_{z_i}(z_n, \dot{z}_n)$ are bounded, say less than N in absolute value. Then the sum of the three integrals on the right is at most $N(t_k - t_1)$ in absolute value, so that

$$(2.9) \quad 0 = F_j(z_n(\bar{t}), z'_n(\bar{t})) - F_j(z_n(\bar{t}), z'_n(\tilde{t})) + \theta_j,$$

where $|\theta_j| \leq N(t_k - t_{k-1}) \leq 2N\delta$.

On the interval (t_1, t_2) the derivative $z'_n(t)$ is constantly equal to $z'_n(t')$, and on (t_{k-1}, t_k) it is constantly equal to $z'_n(t'')$. Hence from (2.9) we obtain

$$(2.10) \quad \begin{aligned} & |F_j(z_n(t'), z'_n(t')) - F_j(z_n(t''), z'_n(t''))| \\ & \leq |F_j(z_n(t'), z'_n(t')) - F_j(z_n(\bar{t}), z'_n(t'))| \\ & \quad + |F_j(z_n(t''), z'_n(t'')) - F_j(z_n(\bar{t}), z'_n(t''))| + |\theta_j|. \end{aligned}$$

Because of the continuity of F_j , each of the first two terms on the right is less than $\epsilon/3$ if $|z_n(\bar{t}) - z_n(t')|$ and $|z_n(\bar{t}) - z_n(t'')|$ are less than a certain $\gamma > 0$. This is surely the case if $|\bar{t} - t'|$ and $|\bar{t} - t''|$ are less than γ/L , which again is surely true if $2\delta < \gamma/L$. Also, if $\delta < \epsilon/6N$, the term $|\theta_j|$ is at most $2N\delta < \epsilon/3$. Hence, if δ is any number less than the smaller of $\gamma/2L$ and $\epsilon/6N$, inequality (2.10) implies (2.5).

Remark. From the last sentence of the proof it is obvious that δ can be chosen as near to 0 as desired.

3. Directions of the sides of the minimizing sequence. The next lemma will explain the name "approach set" given to the sets satisfying (1.5).

LEMMA 4. *Let $\{\Pi_n\}$ be a sequence of polygons satisfying (a), (b), (c), and (d) of Lemma 2, and let $z_1 \neq z_2$. Let t_0 be interior to $(0, 1)$. Then there is a subsequence $\{\Pi_m\}$ of $\{\Pi_n\}$ and an approach set A at $z_0(t_0)$ such that for every $\epsilon > 0$ there is a $\delta > 0$ and an m_0 such that $z'_m(t)$ is in the ϵ -neighborhood of A whenever $z_m(t)$ is not a vertex, $m > m_0$, and*

$$(3.1) \quad \theta(m, t_0 - \delta, t_0 + \delta) < t < \Theta(m, t_0 - \delta, t_0 + \delta).$$

If there is a $\delta > 0$ and a subsequence $\{\Pi_m\}$ such that not more than one t in $(t_0 - \delta, t_0 + \delta)$ defines a vertex of Π_m , then the conclusion holds vacuously, for then $\theta(m, t_0 - \delta, t_0 + \delta) = \Theta(m, t_0 - \delta, t_0 + \delta)$, and (3.1) is not satisfied by any t .

Otherwise, for each integer m there are infinitely many n_m such that the interval

$$(3.2) \quad (\theta(n_m, t_0 - m^{-1}, t_0 + m^{-1}), \Theta(n_m, t_0 - m^{-1}, t_0 + m^{-1}))$$

has length greater than zero. We can suppose n_m so chosen that $n_1 < n_2 < \dots$.

In each interval (3.2) we choose a number t_{n_m} which does not define a vertex of Π_{n_m} . The vectors $z'_{n_m}(t_{n_m})$ all have lengths less than or equal to L ; hence they have an accumulation vector p_0 . We select a subsequence $\{m\}$ of the sequence $\{n_m\}$ such that $z'_m(t_m) \rightarrow p_0$.

Now let A be the approach set at $z_0(t_0)$ which contains p_0 , and let U be the set of all vectors u whose distance from A is less than ϵ . Then U is clearly open. We are to show that $z'_m(t)$ is in U if t and m satisfy the conditions of our lemma. If V is the set of all vectors v of length $|v| \leq L$ which are not in U , then V is bounded and closed and has no point in common with A . Hence the function

$$(3.3) \quad \sum_{i=1}^q |F_i(z_0(t_0), p_0) - F_i(z_0(t_0), v)|$$

is positive for all v in V . Being continuous† on V , it is greater than a positive number 3γ .

Now

$$(3.4) \quad \begin{aligned} & \sum_{i=1}^q |F_i(z_0(t_0), z'_m(t)) - F_i(z_0(t_0), p_0)| \\ & \leq \sum_{i=1}^q |F_i(z_0(t_0), z'_m(t)) - F_i(z_m(t), z'_m(t))| \\ & \quad + \sum_{i=1}^q |F_i(z_m(t), z'_m(t)) - F_i(z_m(t_m), z'_m(t_m))| \\ & \quad + \sum_{i=1}^q |F_i(z_m(t_m), z'_m(t_m)) - F_i(z_0(t_0), p_0)|. \end{aligned}$$

Here, as in Lemma 3, the parameter t is $s/\mathcal{L}(\Pi_m)$, so that $|z'_m(t)| = \mathcal{L}(\Pi_m) \geq |z_2 - z_1| > 0$. Thus the arguments in the first term on the right have a bounded closed range and $z'_m(t)$ is bounded from 0. Hence the first term is less than γ if $|z_0(t_0) - z_m(t)|$ is smaller than a certain number κ . Since $z_m(t)$ tends uniformly to $z_0(t)$ and $z_0(t)$ is continuous, this is true if δ is small enough and m is greater than a certain m_0 .

In the last term on the right, $z'_m(t_m)$ tends to p_0 and $z_m(t_m)$ to $z_0(t_0)$; so this term is less than γ if m is greater than a certain m_2 .

If δ is small enough and m is greater than a certain m_3 , then by Lemma 3 the third term is less than γ whenever t does not define a vertex of Π_m and (3.1) holds. Hence if δ is small enough and $m > m_0 \equiv \max(m_1, m_2, m_3)$, then for all t satisfying the requirements of our lemma we have

† The only point at which $F_i(z_0(t_0), v)$ is discontinuous is $v=0$, which is a limit point of A and therefore not in V .

$$\sum_{i=1}^q |F_i(z_0(t_0), z_m'(t)) - F_i(z_0(t_0), p_0)| < 3\gamma.$$

But if $z_m'(t)$ were in V , this expression (see (3.3)) would exceed 3γ . So $z_m'(t)$ is not in V . Also, it satisfies the condition $|z_m'(t)| \leq L$. Hence it must belong to U , and the lemma is established.

Remark. Again it is clear that the δ can be chosen as small as desired.

4. Proof of the principal theorem. We now impose a new hypothesis on the integral $\mathcal{F}(C)$. We shall make the following supposition:

(4.1) *For every z , each approach set A at z is the sum of a finite number of convex sets $A_1, \dots, A_k$, which can be so ordered that*

$$\Omega(z, p_i, p_j) < 0$$

if p_i is in A_i and p_j in A_j , ($i < j$).

Our principal theorem is the following:

THEOREM 1. *If $F(z, z')$ satisfies conditions (2.1) and (4.1), then in the class K of all rectifiable curves joining two distinct points z_1 and z_2 there is a curve which minimizes the integral*

$$\mathcal{F}(C) = \int_C F(z, \dot{z}) dt.$$

We select a sequence of polygons Π_n and a curve C_0 satisfying the conclusions of Lemma 2, and we define

$$(4.2) \quad \phi_n(t) = \int_0^t F(z_n, \dot{z}_n) dt, \quad 0 \leq t \leq 1; n = 0, 1, 2, \dots$$

Since $|\dot{z}_n(t)| \leq L$, the integrands are bounded, and the ϕ_n all satisfy the same Lipschitz condition. Hence by Ascoli's theorem we can select a subsequence (we suppose it to be the whole sequence) such that $\phi_n(t)$ tends uniformly to a limit function $\phi(t)$. By (a) of Lemma 2,

$$(4.3) \quad \phi(1) = \lim_{n \rightarrow \infty} \phi_n(1) = \lim_{n \rightarrow \infty} \mathcal{F}(\Pi_n) = \mu.$$

Since

$$(4.4) \quad \phi_0(1) = \mathcal{F}(C_0),$$

we must show then that

$$(4.5) \quad \phi_0(1) = \phi(1) = \mu.$$

We shall in fact show that

$$(4.6) \quad \phi_0(t) = \phi(t), \quad 0 \leq t \leq 1.$$

Let t_0 be a point interior to $(0, 1)$ at which $\phi'(t)$ and $\phi'_0(t)$ are both defined. We shall first show

$$(4.7) \quad \phi'(t_0) = \phi'_0(t_0).$$

This will be established if we can prove the following statement:

(4.8) *For every positive number γ there is a $t^* > t_0$ such that if $t_0 < \rho < \sigma < t^*$, then*

$$(4.9) \quad |[\phi(\sigma) - \phi(\rho)] - [\phi_0(\sigma) - \phi_0(\rho)]| \leq \gamma(\sigma - \rho).$$

For if (4.8) holds, then by continuity (4.9) holds with ρ replaced by t_0 . Dividing by $\sigma - t_0$ and letting σ tend to t_0 we obtain

$$|\phi'(t_0) - \phi'_0(t_0)| \leq \gamma.$$

Since γ is an arbitrary positive number, this can be true only if (4.7) holds. We therefore take γ to be a positive number and proceed to establish (4.8).

Let us first dispose of the relatively simple case in which there is a subsequence $\{\Pi_h\}$ and a $\delta > 0$ such that either (a) $\lim_{h \rightarrow \infty} \theta(h, t_0 - \delta, t_0 + \delta) > t_0$ or (b) $\lim_{h \rightarrow \infty} \Theta(h, t_0 - \delta, t_0 + \delta) \leq t_0$. If (a) holds, we denote $\lim \theta$ by t^* ; if (b) holds, we let t^* equal $t_0 + \delta$. Let ρ, σ be numbers such that $t_0 < \rho < \sigma < t^*$. Then if h is large enough, the interval $[\rho, \sigma]$ is contained in case (a) in $[t_0 - \delta, \theta(h, t_0 - \delta, t_0 + \delta)]$, and in case (b) in $[\Theta(h, t_0 - \delta, t_0 + \delta), t_0 + \delta]$. In either case, $z_h(t)$ is linear on $[\rho, \sigma]$ for all large h . The same is then true of its limit $z_0(t)$, and it follows at once that $z'_h(t)$ tends to $z'_0(t)$ uniformly on $[\rho, \sigma]$. Therefore

$$\begin{aligned} \phi(\sigma) - \phi(\rho) &= \lim_{h \rightarrow \infty} \{\phi_h(\sigma) - \phi_h(\rho)\} = \lim_{h \rightarrow \infty} \int_{\rho}^{\sigma} F(z_h, z'_h) dt \\ &= \int_{\rho}^{\sigma} F(z_0, z'_0) dt = \phi_0(\sigma) - \phi_0(\rho); \end{aligned}$$

and (4.8) is established for all positive γ at once.

Now we return to the general case. By Lemma 4, there is an approach set A at $z_0(t_0)$ with the properties there specified. If p_0 is any fixed vector in A , then for every vector p in A the equations

$$F_i(z_0(t_0), p) = F_i(z_0(t_0), p_0), \quad i = 1, \dots, q,$$

hold, by definition. If we denote the right-hand members of these equations by $l_1, \dots, l_q$, respectively, then on multiplying by p^i and summing we find, for all p in A , that

$$F(z_0(t_0), p) = l_{\alpha} p^{\alpha}.$$

The set of vectors p in A with $|p| \leq L$ is bounded and closed, and the function F is continuous. Therefore there is a $\lambda > 0$ such that

$$(4.10) \quad |F(z, p) - l_\alpha p^\alpha| < \gamma/2$$

if $|z - z_0(t_0)| < \lambda$ and p is in the λ -neighborhood U_λ of the intersection of A with the sphere $|p| \leq L$.

The set A is the sum of convex sets $A_1, \dots, A_k$. Let us denote by $U_{i,\epsilon}$ the ϵ -neighborhood of the intersection of A_i with the sphere $|p| \leq L$. Then $U_\epsilon = U_{1,\epsilon} + \dots + U_{k,\epsilon}$. By hypothesis, the A_i have the property that

$$(4.11) \quad \Omega(z_0(t_0), p_i, p_j) < 0$$

if p_i is in A_i , p_j is in A_j , and $i < j$. Therefore on the bounded closed set of arguments (z, p_i, p_j) which satisfy the conditions $z = z_0(t_0)$, p_i in A_i , p_j in A_j , ($i < j$), $|z_2 - z_1| \leq |p_i| \leq L$, $|z_2 - z_1| \leq |p_j| \leq L$ this function has a negative upper bound -2κ . Since it is continuous, we find that there is an $\eta > 0$ such that

$$(4.12) \quad \Omega(z, p_i, p_j) < -\kappa$$

whenever p_i is in $U_{i,\eta}$, p_j is in $U_{j,\eta}$, ($i < j$), $|z_2 - z_1| \leq |p_i| \leq L$, $|z_2 - z_1| \leq |p_j| \leq L$, and $|z - z_0(t_0)| < 3\eta$.

Now define

$$(4.13) \quad \epsilon = \min(\eta, \lambda).$$

We use this for the ϵ in Lemma 4, and choose (see the remark after the lemma) a value for δ which is less than $\epsilon/2(L+1)$. Furthermore, from the sequence $\{\Pi_m\}$ of that lemma we discard those polygons (finite in number) for which the inequality

$$(4.14) \quad |z_m(t) - z_0(t)| < \epsilon/2, \quad 0 \leq t \leq 1,$$

fails to hold. Then, by Lemma 4, together with (4.10) and (4.13), we find:

(4.15) *For all functions $z_m(t)$ and all t such that*

$$\theta(m, t_0 - \delta, t_0 + \delta) < t < \Theta(m, t_0 - \delta, t_0 + \delta),$$

the inequality

$$|F(z_m(t), \dot{z}_m(t)) - l_\alpha \dot{z}_m^\alpha(t)| < \gamma/2$$

holds, and $z_m'(t)$ is in one of the sets $U_{1,\epsilon}, \dots, U_{k,\epsilon}$ if $z_m'(t)$ is defined.

Let t_1, t_2, t_3 define three consecutive vertices of Π_m , and let the inequalities

$$\theta(m, t_0 - \delta, t_0 + \delta) \leq t_1 < t_2 < t_3 \leq \Theta(m, t_0 - \delta, t_0 + \delta)$$

hold. The sides of Π_m corresponding to $[t_1, t_2]$ and $[t_2, t_3]$ have directions

$z_m'(t_1+0), z_m'(t_2+0)$. By (4.15) these belong respectively to neighborhoods $U_{i,\epsilon}, U_{j,\epsilon}$; and the inequalities

$$|z_2 - z_1| \leq |z_m'(t)| \leq L$$

always hold. We now show that i is less than j . Suppose the contrary. If we interchange these two sides of Π_m , we obtain a new polygon Π_m^* , and by formula (3.19) of I

$$(4.16) \quad \mathcal{F}(\Pi_m^*) - \mathcal{F}(\Pi_m) = -|z_m(t_3) - z_m(t_2)| |z_m(t_2) - z_m(t_1)| \cdot \Omega(\bar{z}, z_m'(t_1+0), z_m'(t_2+0)),$$

where $\bar{z}$ is in the parallelogram determined by the two sides. However, $|t_1 - t_0| < \delta < \epsilon/2L$; so $|z_m(t_1) - z_m(t_0)| < L \cdot \epsilon/2L = \epsilon/2$, while by (4.14), $|z_m(t_0) - z_0(t_0)| < \epsilon/2$. Hence $|z_m(t_1) - z_0(t_0)| < \epsilon$. The sides of this parallelogram have lengths less than $|t_3 - t_1|L < 2\delta L < \epsilon$. Thus the whole parallelogram lies in the 3ϵ -neighborhood of $z_0(t_0)$, and in particular $|\bar{z} - z_0(t_0)| < 3\epsilon \leq 3\eta$, by (4.13). Therefore by (4.12) and (4.13) the factor Ω in (4.16) is positive,[†] and $\mathcal{F}(\Pi_m^*) < \mathcal{F}(\Pi_m)$. This contradicts the minimizing property of Π_m , and establishes the inequality $i < j$.

We have therefore established, as a sort of addition to (4.15), that for t between θ and Θ no side of Π_m whose direction is in a $U_{j,\epsilon}$ is followed by one whose direction is in a $U_{i,\epsilon}$ with $i < j$. Let us now define

$$(4.17) \quad t_{m,0} = \theta(m, t_0 - \delta, t_0 + \delta), \quad t_{m,k} = \Theta(m, t_0 - \delta, t_0 + \delta).$$

By the previous remark, there are numbers $t_{m,i}$ defining vertices of Π_m such that

$$t_{m,0} \leq t_{m,1} \leq \cdots \leq t_{m,k},$$

and if $t_{m,i-1} < t < t_{m,i}$ and $z_m'(t)$ is defined, then $z_m'(t)$ is in $U_{j,\epsilon}$. We now select a subsequence $\{\Pi_h\}$ of $\{\Pi_m\}$ such that for each $j=0, 1, \cdots, k$ the limit

$$\tau_j = \lim_{h \rightarrow \infty} t_{h,j}$$

exists. We may suppose $\tau_0 \leq t_0 < \tau_k$, for otherwise we are back to the simpler case first considered.

There is a first j such that $\tau_j > t_0$; hence $\tau_{j-1} \leq t_0 < \tau_j$. Let ρ, σ be numbers such that $t_0 < \rho < \sigma < \tau_j$. For all large h the inequalities $\tau_{h,j-1} < \rho < \sigma < \tau_{h,j}$ hold, so that $z_h(t)$ is in $U_{j,\epsilon}$ if $\rho \leq t \leq \sigma$. The set $U_{j,\epsilon}$ is the ϵ -neighborhood of the

[†] Recall that the interchange of p_i and p_j in (4.12) changes the sign of Ω .

intersection of two convex sets, A_j and the sphere $|p| \leq L$; so $U_{j,\epsilon}$ is convex. By Jensen's inequality (in geometric form), if $\rho \leq t < t' \leq \sigma$, the vector

$$\frac{1}{t' - t} \int_t^{t'} \dot{z}_h(t) dt = \frac{z_h(t') - z_h(t)}{t' - t}$$

is in the closure $\bar{U}_{j,\epsilon}$. Letting $h \rightarrow \infty$, we find that the limit

$$\frac{z_0(t') - z_0(t)}{t' - t}$$

is also in $\bar{U}_{j,\epsilon}$. Now let $t' \rightarrow t$; we find that the vector $z'_0(t)$ is in $\bar{U}_{j,\epsilon}$ if it is defined. Since the zero vector is in $U_{j,\epsilon}$, we see that in any case $\dot{z}_0(t)$ is in $\bar{U}_{j,\epsilon}$ for $\rho \leq t < \sigma$.

Turning now to (4.10), we therefore obtain

$$|F(z_0(t), \dot{z}_0(t)) - l_\alpha \dot{z}_0^\alpha(t)| \leq \gamma/2$$

if $\rho \leq t < \sigma$. Hence

$$(4.18) \quad |\phi_0(\sigma) - \phi_0(\rho) - l_\alpha \{z_0^\alpha(\sigma) - z_0^\alpha(\rho)\}| = \left| \int_\rho^\sigma \{F(z_0, \dot{z}_0) - l_\alpha \dot{z}_0^\alpha\} dt \right| \leq \gamma(\sigma - \rho)/2.$$

On the other hand, by (4.15)

$$(4.19) \quad |\phi_h(\sigma) - \phi_h(\rho) - l_\alpha \{z_h^\alpha(\sigma) - z_h^\alpha(\rho)\}| = \left| \int_\rho^\sigma \{F(z_h, \dot{z}_h) - l_\alpha \dot{z}_h^\alpha\} dt \right| < \gamma(\sigma - \rho)/2.$$

If we here let $h \rightarrow \infty$, we obtain

$$(4.20) \quad |\phi(\sigma) - \phi(\rho) - l_\alpha \{z_0^\alpha(\sigma) - z_0^\alpha(\rho)\}| \leq \gamma(\sigma - \rho)/2.$$

From (4.18) and (4.20) we at once obtain (4.9). Thus statement (4.8), and with it (4.7), is established.

Since $\phi(0) = \phi_0(0) = 0$ and (4.7) holds for almost all t , and since the functions ϕ and ϕ_0 are both absolutely continuous, it follows that $\phi(t) \equiv \phi_0(t)$, so that (4.5) is established. Therefore the curve $z = z_0(t)$ is the minimizing curve sought.

5. Example. If $F(z, z')$ satisfies (2.1) and is quasi-regular† (see (6.2)), every approach set A at every point z_0 consists of a single convex set; so (4.1) is trivially satisfied. Hence for such integrands there is always a minimizing

† Necessarily positive quasi-regular, for if it is negative quasi-regular and not linear, (2.1) cannot hold.

curve for $\mathcal{Y}(C)$ in the class K of all curves joining two given points z_1 and z_2 . This is, of course, a special case of a known theorem.

An example of an essentially different type is $\int F dt$ where

$$F(x, y, x', y') = (x'^2 + y'^2)^{1/2} + y^2[4(x'^2 + y'^2)^{1/2} - (x'^2 + 8y'^2)^{1/2}].$$

This integrand is not positive quasi-regular, for if $|y| > 1/2$, the $\mathcal{E}$ -function can be negative. Given a set (x, y, p, q) with $p^2 + q^2 = 1$, we seek to determine all $(\bar{p}, \bar{q})$ with $\bar{p}^2 + \bar{q}^2 = 1$ such that

$$(5.1) \quad F_{x'}(x, y, \bar{p}, \bar{q}) = F_{x'}(x, y, p, q), \quad F_{y'}(x, y, \bar{p}, \bar{q}) = F_{y'}(x, y, p, q).$$

We can write (5.1) in the form

$$(5.2) \quad \begin{aligned} p[1 + 4y^2 - y^2(8 - 7p^2)^{-1/2}] &= \bar{p}[1 + 4y^2 - y^2(8 - 7\bar{p}^2)^{-1/2}], \\ q[1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2}] &= \bar{q}[1 + 4y^2 - 8y^2(1 + 7\bar{q}^2)^{-1/2}]. \end{aligned}$$

The function in the left-hand member of the first of these equations is easily seen to have a positive derivative with respect to p for $|p| \leq 1$; so the only solution of the first equation is $\bar{p} = p$. Since $p^2 + q^2 = \bar{p}^2 + \bar{q}^2 = 1$, this implies $\bar{q} = \pm q$. Hence we substitute $-q$ for $\bar{q}$ in the second equation of (5.2). If it is not satisfied, then the entire approach set at (x, y) containing (p, q) is the set of multiples (kp, kq) , ($k > 0$). If it is satisfied, then

$$1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2} = 0$$

or

$$(5.3) \quad q = \pm \left[\frac{1}{7} \left\{ \left(\frac{8y^2}{1 + 4y^2} \right)^2 - 1 \right\} \right]^{1/2}.$$

If q has the value (5.3), then A consists of the positive multiples of (p, q) and of $(p, -q)$, which are distinct if $q \neq 0$. Equation (5.3) can hold with $q \neq 0$ only for $|y| > 1/2$, and it is easy to see that $|q| < (3/7)^{1/2}$.

In any case, the approach set at (x, y) containing (p, q) consists either of a half-line or of two half-lines; and a half-line is a convex set. If the set consists of a single half-line, (4.1) clearly holds. Otherwise, let A consist of the multiples of (p, q) and of $(p, -q)$. If $y > 0$, we define A_1 to be the multiples of $(p, -|q|)$ and A_2 to be the multiples of $(p, |q|)$; if $y < 0$, we interchange the definitions of A_1 and A_2 . Then each vector in A_1 has the form $(p_1, q_1) = (k_1 p, -k_1 \operatorname{sgn} y |q|)$, ($k_1 > 0$), and each vector in A_2 has the form $(p_2, q_2) = (k_2 p, k_2 \operatorname{sgn} y |q|)$, ($k_2 > 0$). Hence

$$\begin{aligned} \Omega_F(x, y; p_1, q_1; p_2, q_2) &= 2y \{ q_1 [4(p_2^2 + q_2^2)^{1/2} - (p_2^2 + 8q_2^2)^{1/2}] \\ &\quad - q_2 [4(p_1^2 + q_1^2)^{1/2} + (p_1^2 + 8q_1^2)^{1/2}] \}. \end{aligned}$$

The factors in square brackets are positive, and yq_1 and $-yq_2$ are negative (q_1 and q_2 are not zero, since by hypothesis A contains more than one unit vector). Hence the left member is negative, and (4.1) holds.

6. Generalization to sets S with boundary points. A slight generalization of Theorem 1 can be established at once. We need not assume that $z_2 \neq z_1$. The only use made of this hypothesis was to ensure that $|z'_n(t)|$ had a positive lower bound. If $z_2 = z_1$, we distinguish two cases.

Case I. The $|z'_n(t)|$ have a positive lower bound. In this case the preceding proof applies without change.

Case II. The $\liminf |z'_n(t)| = 0$. Since $|z'_n(t)| = \mathcal{L}(\Pi_n)$ for almost all t , this implies that for a subsequence $\{\Pi_m\}$ of the minimizing sequence $\{\Pi_n\}$ we have $\mathcal{L}(\Pi_m) = 0$. Then $\mu = \lim \mathcal{Y}(\Pi_n) = 0$; so the degenerate curve consisting of the single point z_1 is the curve sought.

Less trivial is the generalization which allows us to study integrands not defined for all z . Before stating this theorem it is desirable to introduce some definitions and establish a lemma.

(6.1) *The function $F(z, z')$ is positive (negative) regular the point z_0 if† $u^\alpha F_{\alpha\beta}(z_0, p)u^\beta > 0$ (< 0) for all pairs u, p of orthogonal unit vectors.*

(6.2) *The function $F(z, z')$ is positive (negative) quasi-regular at z_0 if $u^\alpha F_{\alpha\beta}(z_0, p)u^\beta \geq 0$ (≤ 0) for all pairs u, p of orthogonal unit vectors.*

We use the abbreviations p.q.r., n.q.r. for positive quasi-regular, negative quasi-regular, respectively. It is well known that $F(z, z')$ is p.q.r. (n.q.r.) at z_0 if and only if $\mathcal{E}(z_0, p, r) \geq 0$ (≤ 0) for all $p \neq 0$ and all r . Although we do not use the concept in this note, we shall also make the following definition:

(6.3) *The function $F(z, z')$ is p.q.r. normal (n.q.r. normal) at z_0 if $\mathcal{E}(z_0, p, r) > 0$ (< 0) whenever $p \neq 0$ and $r \neq kp, k \geq 0$.*

With this terminology we state the following lemma:

LEMMA 5. *If $\phi'_0(t_0)$ and $\phi'(t_0)$ are defined, and $F(z, z')$ is p.q.r. at $z_0(t_0)$, then $\phi'(t_0) \geq \phi'_0(t_0)$.*

Let ϵ be an arbitrary positive number, and let $F^\epsilon(z, z') = F(z, z') + \epsilon|z'|$. By an elementary computation, we find that if u and p are orthogonal unit vectors, then

$$(6.4) \quad u^\alpha F_{\alpha\beta}^\epsilon(z, p)u^\beta = u^\alpha F_{\alpha\beta}(z, p)u^\beta + \epsilon.$$

The first term on the right is nonnegative at $z = z_0(t_0)$, by hypothesis. Hence the left-hand member is positive for all z of the set S in a neighborhood U

† $F_{ij}(z, z')$ means $F_{s^i, s^j}(z, z')$.

of $z_0(t_0)$. If h is a sufficiently small positive number, less than $\dagger 1 - t_0$, then for all large n the arcs $z = z_n(t)$ and $z = z_0(t)$, ($t_0 \leq t \leq t_0 + h$), are in U , and by a known semicontinuity theorem

$$(6.5) \quad \liminf \int_{t_0}^{t_0+h} F^*(z_n, \dot{z}_n) dt \geq \int_{t_0}^{t_0+h} F^*(z_0, \dot{z}_0) dt.$$

Recalling the definition of F^* and the inequality $|\dot{z}_n| \leq L$, we deduce that

$$(6.6) \quad \liminf [\phi_n(t_0 + h) - \phi_n(t_0)] + \epsilon h L \geq \phi_0(t_0 + h) - \phi_0(t_0).$$

That is,

$$(6.7) \quad \phi(t_0 + h) - \phi(t_0) + \epsilon h L \geq \phi_0(t_0 + h) - \phi_0(t_0).$$

If we divide by h and let h tend to 0, we obtain

$$(6.8) \quad \phi'(t_0) + \epsilon L \geq \phi'_0(t_0).$$

But ϵ is an arbitrary positive number; so (6.8) implies

$$(6.9) \quad \phi'(t_0) \geq \phi'_0(t_0),$$

which was to be proved.

Our next theorem is the following:

THEOREM 2. *Let S be a closed point set in q -dimensional space. If $F(z, z')$ satisfies condition (2.1), and condition (4.1) holds at every interior point of S , while $F(z, z')$ is positive quasi-regular at each boundary point of S , then in the class K there is a curve which minimizes $\mathcal{F}(C)$.*

Choose first a minimizing sequence C_n^* of curves $z = z_n^*(t)$, ($0 \leq t \leq 1$). There is a set of values of t open relative to $[0, 1]$ for which $z_n^*(t)$ has distance greater than $1/n$ from the boundary of S . This open set consists of a finite or denumerable number of intervals, open relative to $[0, 1]$. Only a finite number h_n of these define arcs $C_{n,j}^*$, ($j=1, \dots, h_n$), of C_n^* of length greater than $1/n$. These arcs we replace by polygonal arcs $\Pi_{n,j}^*$ having the same end points and such that $|\mathcal{F}(\Pi_{n,j}^*) - \mathcal{F}(C_{n,j}^*)| < 1/nh_n$, ($j=1, \dots, h_n$). If we denote by C_n' the curve obtained by replacing the arcs $C_{n,j}^*$ by the polygonal arcs $\Pi_{n,j}^*$, then $|\mathcal{F}(C_n') - \mathcal{F}(C_n^*)| < 1/n$; so $\lim \mathcal{F}(C_n') = \mu$. By a minor modification of Lemma 1, each arc $\Pi_{n,j}^*$ can be replaced by an arc $\Pi_{n,j}$ having a number $s_{n,j}$ of vertices which is not greater than the number of vertices of $\Pi_{n,j}^*$ and which minimizes $\mathcal{F}(C)$ in the class of polygons joining the ends of $\Pi_{n,j}^*$ and having not more than $s_{n,j}$ vertices. Let C_n be the curve obtained by replacing the arcs $\Pi_{n,j}^*$ by the arcs $\Pi_{n,j}$. Then $\lim \mathcal{F}(C_n) = \mu$.

$\dagger$ If $t_0=1$, we take h negative; modifications are obvious.

Now let t_0 be a point of E . If $z_0(t_0)$ is interior to S , there is a neighborhood of $z_0(t_0)$ on which the C_n are polygonal if n is large. All the discussion leading up to equation (4.7) remains valid, for Lemmas 3 and 4 are purely local. So in this case (6.9) is valid. If $z_0(t_0)$ is a boundary point of S , then by Lemma 5 inequality (6.9) holds. Integrating, we obtain

$$(6.10) \quad \mu = \phi(1) \geq \phi_0(1) = \mathcal{J}(C_0).$$

But C_0 is in K ; so $\mathcal{J}(C_0) \geq \mu$. This, with (6.10), proves $\mathcal{J}(C_0) = \mu$, and the theorem is established.

7. Geometric interpretation of approach sets. For fixed z , let us construct the graph in (u, p) -space of the function $u = F(z, p)$. As is well known, this is a conical hypersurface with vertex at the origin. Let A be an approach set at z . If p_1 and p_2 are both in A , then

$$(7.1) \quad F_i(z, p_1) = F_i(z, p_2), \quad i = 1, \dots, q.$$

But $u = p^\alpha F_\alpha(z, p_i)$ is the equation of the hyperplane tangent to $u = F(z, p)$ at p_i , ($j = 1, 2$). Therefore equation (7.1) shows that this same hyperplane is tangent to the surface $u = F(z, p)$ at all points of A .

Conversely, let $u = l_\alpha p^\alpha$ be a hyperplane tangent to the hypersurface $u = F(z, p)$ for all p in a set A . Then if p is in A , the partial derivatives $F_i(z, p)$ must be the same as the partial derivatives l_i of the osculating function $l_\alpha p^\alpha$. Therefore $F_i(z, p) = l_i$ for all p in A , and A is an approach set at z .

If $F(z, p) > 0$ for $|p| > 0$, this interpretation can be formulated somewhat differently. In the preceding paragraphs set $u = 1$. The hyperplane $u = 1$ intersects $u = F(z, p)$ in a $(q-1)$ -dimensional hypersurface, and the hyperplane $u = l_\alpha p^\alpha$ is tangent to $F(z, p) = 1$ if and only if $1 = l_\alpha p^\alpha$ is tangent to $F(z, p) = 1$. Thus if $q = 2$ and $F(z, p) > 0$ for $|p| > 0$, we can find all approach sets at z by constructing the curve $F(z, p) = 1$ and finding all the points (p^1, p^2) at which an arbitrary line $a_1 p^1 + a_2 p^2 + a_3 = 0$ is tangent to the curve.

This interpretation suggests that it might be more descriptive to replace the name "approach set" by "isotangential set."

8. $\mathcal{E}$ -admissibility of approach sets; a more general existence theorem. The geometric interpretation in §7 suggests the following line of reasoning. If $z = z(t)$ minimizes $\mathcal{J}(C)$, then for each t_0 the surface $u = F(z(t_0), p)$ never sinks below the plane tangent to it at $p = z'(t_0)$. The analytical statement is $\mathcal{E}(z(t_0), z'(t_0), p) \geq 0$ for all p . The definition of approach sets A was suggested by the Weierstrass-Erdman corner condition. Might it not be true that only those approach sets A are of importance which have the property that the hypersurface $u = F(z, p)$ never sinks below the hyperplane tangent to it at

the points p of A ? We shall give this property of the set A the name " $\mathcal{E}$ -admissibility." Stated analytically the definition is as follows:

(8.1) *If A is an approach set at z , it is $\mathcal{E}$ -admissible if $\mathcal{E}(z, p_0, p) \geq 0$ for all p_0 in A and all p .*

It would make no difference here if we replaced the words "all p_0 in A " by the words "some one p_0 in A ." For if p_0, p_1 are any two vectors in A , then for all p the equation

$$(8.2) \quad \begin{aligned} \mathcal{E}(z, p_0, p) &= F(z, p) - p^\alpha F_\alpha(z, p_0) = F(z, p) - p^\alpha F_\alpha(z, p_1) \\ &= \mathcal{E}(z, p_1, p) \end{aligned}$$

holds.

The conjecture that approach sets which are not $\mathcal{E}$ -admissible can be disregarded is indeed true, and we shall establish it in Theorem 3 below. However, the proof is rather complicated. It does not seem possible to rest upon the minimizing property of each Π_n , as we have done before. For in any direct analogue of the proof of the Weierstrass condition polygons are introduced which have new vertices, and Π_n does not necessarily minimize $\mathcal{J}(C)$ in the class of polygons with not more than $s_n + 1$ vertices. The proof which we shall give is therefore based on the property of the sequence as a whole that $\lim \mathcal{J}(\Pi_n) = \mu$.

THEOREM 3. *Theorems 1 and 2 remain valid if in hypothesis (4.1) the words "each approach set" are replaced by "each $\mathcal{E}$ -admissible approach set."*

In the proof of Theorems 1 and 2 the approach set A entered by way of Lemma 4, and then only in case there was no subsequence $\{\Pi_h\}$ and $\delta > 0$ for which either (a) $\lim_{h \rightarrow \infty} \theta(h, t_0 - \delta, t_0 + \delta) > t_0$ or (b) $\lim_{h \rightarrow \infty} \Theta(h, t_0 - \delta, t_0 + \delta) \leq t_0$. For if either (a) or (b) held, the proof was relatively simple and did not involve any approach set A . We may suppose then that there is no subsequence $\{\Pi_h\}$ and no $\delta > 0$ for which either (a) or (b) holds. If we can then show that the approach set A in Lemma 4 is necessarily $\mathcal{E}$ -admissible, our proof is complete.

In Lemma 4 there is no loss of generality in assuming that the common value of the partial derivatives $F_i(z_0(t_0), p)$ for all p in A is zero. For let us replace $F(z, p)$ by $F(z, p) - p^\alpha F_\alpha(z_0(t_0), p_0)$, where p_0 is in A . All the curves C of the family K join z_1 to z_2 ; so $\mathcal{J}(C)$ changes by $(z_2^\alpha - z_1^\alpha) F_\alpha(z_0(t_0), p_0)$, independent of C . Therefore the minimizing properties of the Π_n are unchanged by the alteration of $F(z, p)$. The statement that A is not $\mathcal{E}$ -admissible assumes the form that there is a p_1 such that $F(z_0(t_0), p_1) < 0$. Because of the homogeneity of F , we may suppose that p_1 is a unit vector, and we write

$$F(z_0(t_0), p_1) = -\gamma < 0.$$

By the continuity of F , there is an $\epsilon > 0$ such that

$$(8.3) \quad F(z, p_1) < -6\gamma$$

if $|z - z_0(t_0)| < 4\epsilon$. Furthermore, $F_i(z_0(t_0), p) = 0$ if p is in A ; hence we may suppose that ϵ has been chosen small enough so that the following statement holds:

(8.4) *The vector $(F_1(z, p), \dots, F_q(z, p))$ has length less than γ if $|z - z_0(t_0)| < 4\epsilon$, $|z_2 - z_1| \leq |p| \leq 2L + 1$, and p is in the 3ϵ -neighborhood of A .*

Clearly we may suppose $\epsilon < \min [1, |z_2 - z_1|]$.

On the bounded closed set $|z - z_0(t_0)| \leq 1$, $|p| \leq 2L + 1$, the functions $F_i(z, p)$ are continuous; hence we have the following statement:

(8.5) *There is an $M > 0$ such that the vector $(F_1(z, p), \dots, F_q(z, p))$ has length less than M if $|z - z_0(t_0)| \leq 1$ and $|p| \leq 2L + 1$.*

Now from the subsequence $\{\Pi_m\}$ of Lemma 4 we choose a subsequence $\{\Pi_h\}$ with the following properties:

(8.6) $|z_h(t) - z_0(t)| < \epsilon$ for $0 \leq t \leq 1$ and all h .

(8.7) The limits $t_1 \equiv \lim_{h \rightarrow \infty} \theta(h, t_0 - \delta, t_0 + \delta)$ and $t_2 \equiv \lim_{h \rightarrow \infty} \Theta(h, t_0 - \delta, t_0 + \delta)$ exist, where δ is prescribed by Lemma 4.

(As previously remarked, we need consider only the case $t_1 \leq t_0 < t_2$.)

(8.8) δ is chosen less than ϵ/L .

(8.9) For all h the following inequality holds:

$$\Theta(h, t_0 - \delta, t_0 + \delta) - \theta(h, t_0 - \delta, t_0 + \delta) > (t_2 - t_1)/2.$$

Each of the conditions (8.6), (8.7), (8.9) is readily satisfied by appropriate choice of a subsequence; for (8.6) and (8.9) we need only reject a finite number of terms.

Let $d_{h,1}, \dots, d_{h,n(h)}$ be a sequence of points with the following properties:

(8.10) $d_{h,1} < d_{h,2} < \dots$

(8.11) $|d_{h,j+1} - d_{h,j}| < \gamma/M$ for $j = 1, \dots, n(h) - 1$.

(8.12) $d_{h,1} = \theta(h, t_0 - \delta, t_0 + \delta)$, $d_{h,n(h)} = \Theta(h, t_0 - \delta, t_0 + \delta)$.

(8.13) Every t between $\theta(h, t_0 - \delta, t_0 + \delta)$ and $\Theta(h, t_0 - \delta, t_0 + \delta)$ which defines a vertex of Π_h is included among the $d_{h,j}$. (Thus $z_h(t)$ is linear on $d_{h,j} \leq t \leq d_{h,j+1}$.)

Let $m_{h,j}$ be the mid-point of the interval $[d_{h,j}, d_{h,j+1}]$.

On each interval $[d_{h,j}, d_{h,j+1}]$ we shall replace the line segment $z = z_h(t)$ belonging to Π_h by a polygon of two sides having the same beginning and the same end as the line segment. To keep the notation from becoming too complicated, we temporarily replace the symbols $d_{h,j}$, $m_{h,j}$, $d_{h,j+1}$, z_h by d_1 , m , d_2 , z ,

respectively. For $0 \leq \tau \leq \epsilon$ we define $\Pi(\tau)$ to be the two-sided polygon $z = z(t, \tau)$, where

$$(8.14) \quad \begin{aligned} z(t, \tau) &= z(d_1) + (t - d_1)\tau p_1, & d_1 \leq t < m, \\ z(t, \tau) &= z(2t - d_2) + (d_2 - t)\tau p_1, & m \leq t \leq d_2. \end{aligned}$$

We readily verify that $z(t, \tau)$ is continuous on $[d_1, d_2]$ and that

$$(8.15) \quad z(d_1, \tau) \equiv z(d_1), \quad z(d_2, \tau) \equiv z(d_2).$$

Also it is easy to show that

$$(8.16) \quad \int_m^d F(z(t, 0), z'(t, 0)) dt = \int_{d_1}^{d_2} F(z(t), z'(t)) dt,$$

where the prime denotes the derivative with respect to t .

The polygon $\Pi(\tau)$ may be regarded as arising from $\Pi(0)$ by displacing the vertex $z(m, 0)$ by the amount $(d_2 - d_1)\tau p_1/2$. Hence we may calculate $\mathcal{F}'(\Pi(\tau))$ by applying (2.8) of I to the sides $[d_1, m]$, $[m, d_2]$ and adding. We obtain

$$(8.17) \quad \begin{aligned} \frac{d}{d\tau} \mathcal{F}(\Pi(\tau)) &= \frac{1}{2}(d_2 - d_1)F_\alpha(z(\bar{t}, \tau), p_1)p_1^\alpha \\ &\quad - \frac{1}{2}(d_2 - d_1)F_\alpha(z(\bar{t}, \tau), z'(\bar{t}, \tau))p_1^\alpha \\ &\quad + \int_{d_1}^{d_2} F_{z^\alpha}(z(t, \tau), \dot{z}(t, \tau))p_1^\alpha |m - t| dt, \end{aligned}$$

where $d_1 < \bar{t} < m$ and $m < \bar{t} < d_2$. Now we must prove the following statements:

$$(8.18) \text{ If } d_1 \leq t \leq d_2 \text{ and } 0 \leq \tau \leq \epsilon, \text{ then } |z(t, \tau) - z_0(t_0)| < 3\epsilon.$$

$$(8.19) \text{ If } m < t < d_2 \text{ and } 0 \leq \tau \leq \epsilon, \text{ then } z'(t, \tau) \text{ is in the } \epsilon\text{-neighborhood of } A, \text{ and } |z_2 - z_1| \leq |z'(t, \tau)| \leq 2L + 1.$$

We establish (8.19) first. By (8.14), we have

$$z'(t, \tau) = 2z'(2t - d_2) - \tau p_1.$$

But by our choice of parameter t , we have $|z_2 - z_1| \leq z'(2t - d_2) \leq L$, while $|\tau p_1| \leq \epsilon < \min [|z_2 - z_1|, 1]$. Hence $|z_2 - z_1| < |z'(t, \tau)| < 2L + 1$. Also, by Lemma 4 the vector $z'(2t - d_2)$ is in the ϵ -neighborhood of A ; whence $z'(t, \tau)$ is in the 3ϵ -neighborhood of A . Thus (8.19) is established.

By (c) of Lemma 2, $|\dot{z}(t)| \leq L$. Hence if $t_0 - \delta \leq t \leq t_0 + \delta$, then (recalling (8.8)) we have

$$|z(t) - z(t_0)| \leq L\delta < \epsilon.$$

By (8.6), $|z(t_0) - z_0(t_0)| < \epsilon$. Combining these inequalities, we find that both

$z(d_1)$ and $z(d_2)$ are in the 2ϵ -neighborhood of $z_0(t_0)$. The distance from $z(d_1) \equiv z(d_1, \tau)$ to $z(m, \tau)$ is $|(m - d_1)\tau p_1| < \epsilon$. So all three vertices $z(d_1, \tau)$, $z(m, \tau)$, and $z(d_2, \tau)$ are in the 3ϵ -neighborhood of $z_0(t_0)$, and (8.18) is proved.

Thus the arguments in (8.17) satisfy the requirements laid down in (8.3), (8.4), and (8.5), and we have from (8.17) and (8.11)

$$(8.20) \quad \begin{aligned} \frac{d}{d\tau} \mathcal{F}(\Pi(\tau)) &\leq -3\gamma(d_2 - d_1) + \frac{1}{2}\gamma(d_2 - d_1) + \int_{d_1}^{d_2} M |m - t| dt \\ &< -2\gamma(d_2 - d_1) + M(d_2 - d_1)^2 < -\gamma(d_2 - d_1). \end{aligned}$$

If we integrate from $\tau=0$ to $\tau=\epsilon$, we obtain (using (8.16))

$$(8.21) \quad \mathcal{F}(\Pi(\epsilon)) < \mathcal{F}(\Pi(0)) - \gamma\epsilon(d_2 - d_1) = \mathcal{F}(\Pi) - \gamma\epsilon(d_2 - d_1).$$

In the foregoing paragraphs $[d_1, d_2]$ was any one of the intervals $[d_{h,j}, d_{h,j+1}]$. Let the construction be carried out on each of these intervals. The arc $z = z_h(t)$, $d_{h,1} \leq t \leq d_{h,n(h)}$ is thereby replaced by a polygonal arc $z = z_h(t, \epsilon)$ having the same ends and satisfying the relation

$$\int_{d_{h,1}}^{d_{h,n(h)}} F(z_h(t, \epsilon), \dot{z}_h(t, \epsilon)) dt < \int_{d_{h,1}}^{d_{h,n(h)}} F(z_h, \dot{z}_h) dt - \sum_{j=1}^{n(h)-1} \gamma\epsilon(d_{h,j+1} - d_{h,j}).$$

If we extend the range of definition of $z_h(t, \epsilon)$ by setting it equal to $z_h(t)$ for $0 \leq t < d_{h,1}$ and for $d_{h,n(h)} < t \leq 1$, we obtain a polygon $\Pi_h(\epsilon)$ joining z_1 to z_2 and satisfying (by (8.21) and (8.9))

$$\begin{aligned} \mathcal{F}(\Pi_h(\epsilon)) &< \mathcal{F}(\Pi_h) - \sum_{j=1}^{n(h)-1} \gamma\epsilon(d_{h,j+1} - d_{h,j}) = \mathcal{F}(\Pi_h) - \gamma\epsilon(d_{h,n(h)} - d_{h,1}) \\ &< \mathcal{F}(\Pi_h) - \gamma\epsilon(t_2 - t_1)/2. \end{aligned}$$

Therefore

$$\limsup_{h \rightarrow \infty} \mathcal{F}(\Pi_h(\epsilon)) \leq \lim_{h \rightarrow \infty} \mathcal{F}(\Pi_h) - \gamma\epsilon(t_2 - t_1)/2 = \mu - \frac{1}{2}\gamma\epsilon(t_2 - t_1) < \mu.$$

This is impossible by the definition of μ , and our theorem is established.

It is evident that the restriction $z_2 \neq z_1$ can be removed here just as it was in §6. We thus arrive at the following theorem, which includes the three preceding ones as special cases:

THEOREM 4. *Let $F(z, z')$ be defined and satisfy the usual continuity and homogeneity conditions* for all z in a closed set S and all z' . Let $F(z, z')$ satisfy the following conditions:*

- (i) *At every boundary point z of S , $F(z, z')$ is positive quasi-regular.*
- (ii) *At every interior point z of S , every $\mathcal{E}$ -admissible approach set A is*

* As specified in I.

the sum of a finite number of convex sets $A_1, \dots, A_k$ such that $\Omega(z, p_i, p_j) < 0$ if p_i is in A_i , p_j in A_j , $i < j$.

Let condition (2.1) be satisfied. Then for every two points z_1, z_2 of S the class K of all rectifiable curves lying in S and joining z_2 to z_1 either is empty or contains a minimizing curve for $\mathcal{F}(C)$.

9. Geometric interpretation of $\mathcal{E}$ -admissibility. The geometric interpretation of the property of $\mathcal{E}$ -admissibility has already been given at the beginning of §8. If A is an approach set at z , it is $\mathcal{E}$ -admissible if and only if the surface $u = F(z, p)$ never sinks below the hyperplane tangent to it at the points (u, p) with p in A . The alternative geometric interpretation of approach sets also carries with it an interpretation of $\mathcal{E}$ -admissibility. Let the hyperplane (in p -space) $a_\alpha p^\alpha + b = 0$ be tangent to the hypersurface $F(z, p) = 1$ at all the points p of the surface which belong to A . This hyperplane divides p -space into two half-spaces. The half-space which contains the origin necessarily contains other points of $F(z, p) = 1$. The set A is then $\mathcal{E}$ -admissible if and only if the entire hypersurface $F = 1$ lies in this same half-space.

It follows that if we form the least convex body Q containing $F(z, p) = 1$, then the points p of the hypersurface which are not on the boundary of Q cannot belong to any $\mathcal{E}$ -admissible approach set. In particular, if $q = 2$, the points p_0 of $F(z, p) = 1$ on the boundary of Q either lie in a line segment belonging to the boundary of Q or they do not. In the second case the approach set A containing p_0 consists of the half-line from the origin through p_0 . In the first case, let B be the set of all points p at which the line segment touches $F(z, p) = 1$. The (maximal) approach set at z containing p_0 then consists of all the half-lines from the origin through the points of B .

Consider, for example, the function $F(x, y, x', y')$ studied in §5. If $|y| \leq 1/2$, the curve $F = 1$ is convex. If $|y| > 1/2$, the curve is dumbbell-shaped, curved inwards near its intersections with the x' -axis. It is then evident from the graph that there are just two approach sets which consist of more than one half-line, and these are determined by the points of contact with the two lines $x' = \text{const.}$ tangent to the curve at the points at which x' is respectively greatest and least. In §5 we showed rather more than this. We showed that these are the *only* approach sets other than half-lines; that besides these there are no other approach sets other than half-lines, whether $\mathcal{E}$ -admissible or not. But Theorem 3 shows that the simpler conclusion reached here is sufficient to show the existence of minimizing curves for $\mathcal{F}(C)$.