THE GEOMETRY OF FIELDS OF LINEAL ELEMENTS*

BY

JOHN DE CICCO

1. **Introduction.** We shall begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A *turn* T_α converts each element into one having the same point and making a fixed angle α with the original direction. By a *slide* S_k, the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together form a continuous group of three-parameters which we call the *whirl group* W_3. The group of whirls is isomorphic to the group of rigid motions M_3. These two three-parameter groups are commutative and together generate a continuous group of six-parameters which we term the *whirl-motion group* G_6. In preceding papers (see the bibliography at the end of this paper), Kasner and the author developed the geometry of this group G_6. In this paper, which is a continuation of the paper by the author *The differential geometry of series of lineal elements*, these Transactions, vol. 46 (1939), pp. 348–361, we shall give the differential geometry of fields of lineal elements with respect to the whirl-motion group G_6.

A set of ∞^1 elements is called a *series*; this includes a union (curve or point) as a special case. A collection of ∞^2 elements is termed a *field*, which of course corresponds to a differential equation of the first order, $F(x, y, y')=0$. The totality of ∞^3 elements of the plane is called the *opulence* (as defined by Kasner).

In the earlier paper, we considered the tangent turbines, the osculating flat fields, and the osculating limaçon (circular) series of a given series S. We defined the *curvature* κ and the *torsion* τ of any series. The curvature $\bar{\kappa}$ and the torsion τ of a series $\bar{S}$ conjugate to a given series S are given by the formulas $\bar{\kappa}=\kappa/\tau$, $\bar{\tau}=1/\tau$. We proved the fundamental result that *any two general (equiparallel) series which have their curvatures and torsions the same functions of the angle u (arc length s) are equivalent under the whirl-motion group G_6*. This result establishes the intrinsic equations of any series in the geometry of the whirl-motion group G_6.

In the present paper, we shall derive the analogues of some of the classic theorems for a surface in a euclidean three-dimensional space. In particular,

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we shall consider the Meusnier, the Euler, the Joachimsthal, and the Beltrami-Enneper theorems for a field in the geometry of the whirl-motion group G_6. The theory of geodesic series (minimum curvature) will be developed. We shall define the *gaussian curvature* K of a field. Finally the theory of conjugate fields will be considered.

2. **The tangent turbines of a series.** A series which consists of ∞^1 nonparallel (parallel) elements is called a *general* (*equiparallel*) *series*. A general series is given by $v=v(u)$, $w=w(u)$, whereas an equiparallel series is given by $u=c$, $w=w(v)$, where c is a constant. A general series possesses a point-union and a line-union, whereas an equiparallel series possesses only a point-union.

A *turbine* is the series which is obtained by applying a turn T_α to the elements of an oriented circle (the outer circle). It is *nonlinear* or *linear* according as this base circle is not or is a straight line.

A nonlinear turbine is a general series. Its point-union is a circle (the outer circle), and its line-union is also a circle (the inner circle). These two circles are concentric, and their common center is called the *center* of the turbine.

From the preceding remarks, we find that a nonlinear turbine may be constructed by applying a slide S_s to the elements of an oriented circle (the inner circle). Thus the equations of a nonlinear turbine are

$$(1)\qquad \begin{aligned} v &= a \cos u + b \sin u + r, \\ w &= - a \sin u + b \cos u + s, \end{aligned}$$

where (a, b) are the cartesian coordinates of the center, r is the radius of the inner circle, and s is the constant distance of the slide S_s. We call $T(a, b, r, s)$ a set of nonlinear turbine coordinates.

A linear turbine is an equiparallel series whose base curve is a straight line. The equations of a linear turbine are

$$(2)\qquad u = U - \omega, \qquad v \cos \omega + w \sin \omega = V,$$

where (U, V) are the hessian coordinates of the base line and ω is the constant angle of the turn T_ω. We call $T(U, V, \omega)$ a set of linear turbine coordinates. Obviously

$$T(U, V, \omega) = T(U + \pi, - V, \omega + \pi).$$

The *angle* $u_2 - u_1$ between any two elements is the angle between their lines. Two elements are *parallel* or *supplementary* (*antiparallel*) according as the angle between them is 0 or π. The *distance* $[(v_2-v_1)^2+(w_2-w_1)^2]^{1/2}$ between two *parallel* elements is the distance between their points.

Two parallel elements are on a unique linear turbine. Two nonparallel elements are contained in a unique nonlinear turbine T. The center of T is the intersection between the perpendicular bisector of the segment determined by the points of E_1 and E_2, and the angle bisector of the angle determined by the oriented lines of E_1 and E_2.

Two series S_1 and S_2 are said to be *tangent* (or to have contact of the first order) at a common element E if they have two (but not three) consecutive elements in common at E. The two series S_1 and S_2 are said to be *osculating* (or to have contact of the second order) at E if they have three (but not four) consecutive elements in common at E.

If a one-parameter family of series has the property that consecutive series have a common element, the family is called a *set of enveloping series*. The locus of intersection of consecutive series is termed the *envelope*. It is easy to prove that *any series S_t of a set of enveloping series is tangent to the envelope S at any one of their common elements.*

In the remainder of the paper, an accent will always mean total differentiation with respect to u unless otherwise specified.

THEOREM 1. *The tangent turbines of a general series are the ∞^1 nonlinear turbines whose parameter values are*

$$(3)\qquad \begin{aligned} a &= -v' \sin u - w' \cos u, & b &= v' \cos u - w' \sin u, \\ r &= v + w', & s &= -v' + w. \end{aligned}$$

On the other hand, the tangent turbines of an equiparallel series S are the ∞^1 linear turbines all of which possess the common direction of S and whose base lines are tangent to the base curve of S.

It may be observed that two series S_1 and S_2 are tangent at a common element E if and only if they have the same tangent turbine at E.

3. **Conjugate series.** Two turbines T and $\overline{T}$ are said to be *conjugate* if they have the same circle as point-locus and the elements of the two turbines are symmetrically related to the elements of the circle. Two series S and $\overline{S}$ are said to be *conjugate* if there exists a one-to-one correspondence between their elements in such a way that the tangent turbines of the two series at the corresponding elements are conjugate turbines.

The conjugate turbines $\overline{T}_1$ and $\overline{T}_2$ of two given turbines T_1 and T_2 (not both linear) do or do not possess a common element according as T_1 and T_2 do or do not possess a common element. The conjugate turbines of two intersecting linear turbines never possess a common element.

THEOREM 2. *For any general series S, there always exists one and only one conjugate series $\overline{S}$ which either consists of one element or is a general series. This* *series $\bar{S}$ is given by the equations*

$$\cos \bar{u} = \frac{-a'r' + b's'}{a'^2 + b'^2}, \qquad \sin \bar{u} = \frac{-a's' - b'r'}{a'^2 + b'^2}, \tag{4}$$

$$\bar{v} = a \cos \bar{u} + b \sin \bar{u} + r, \qquad \bar{w} = -a \sin \bar{u} + b \cos \bar{u} - s,$$

where (a, b, r, s) are the parameter values of the tangent turbines of S.

If an equiparallel series S is not a turbine, then there is no series conjugate to it.

4. **The osculating flat fields of a series.** A *nonlinear flat field* consists of the ∞^2 elements cocircular with a given element, called the *central element.* The equation of a nonlinear flat field Π is

$$w = (v - \bar{v}) \cot (u - \bar{u})/2 - \bar{w}, \tag{5}$$

where $(\bar{u}, \bar{v}, \bar{w})$ are the hessian coordinates of the central element $\bar{G}$ of Π. We call $\Pi(\bar{u}, \bar{v}, \bar{w})$ a set of nonlinear flat field coordinates.

A *linear flat field* is the set of ∞^2 elements on ∞^1 parallel straight lines. Any linear field is given by $u = \text{const.}$

The invariants between two nonlinear flat fields are identical with those between their central elements.

In a given flat field, there are ∞^2 turbines. The turbines which are contained in a nonlinear flat field Π are those whose conjugate turbines possess the central element $\bar{G}$ of Π. The turbines which are contained in a linear flat field Π are the linear turbines which have the common direction of Π.

Three parallel elements determine a unique linear flat field. Three elements which are not all parallel and which do not lie on one turbine determine a unique nonlinear flat field Π. The central element $\bar{G}$ of Π is the single intersection of the conjugate turbines of the three turbines which pass through these elements.

Two elements of a flat field Π determine a turbine which lies entirely in Π. Two flat fields (not both linear) intersect in a turbine. Two linear flat fields have no common elements.

The flat field which has three consecutive elements in common with a series S at an element E of S is called the *osculating flat field* of S at E.

THEOREM 3. *The osculating flat fields of a general series S are the nonlinear flat fields whose central elements are the elements of the series $\bar{S}$ conjugate to S.*

If $\bar{S}$ consists of only one element $\bar{G}$, then S is contained in the nonlinear flat field whose central element is $\bar{G}$. In this case, we shall say that S is a *coflat series.*

Any equiparallel series S has one and only one osculating flat field, namely, the linear flat field in which it is contained.

5. **The osculating limaçon series of a general series.** Let T be a nonlinear turbine, let $\overline{G}$ be a fixed element on the conjugate turbine $\overline{T}$ of T, and let γ be a real number. Let O be the point of $\overline{G}$, and let P be the point of any element E of T. On the line (OP), let us select the points P_i $(i=1, 2)$ such that the distance $d(P, P_i)=2\gamma$. Let E_i be the element whose point is P_i and whose direction is that of E. By this construction, to each element E of T there are associated two parallel elements E_1 and E_2. The totality of these elements E_1, E_2 is called a *limaçon series* with *central turbine* T and *radius* γ.

Upon letting C and D denote

$$C = -2\gamma \sin \bar{u}/2, \qquad D = 2\gamma \cos \bar{u}/2, \tag{6}$$

we find that the equations of a limaçon series are

$$\begin{aligned} v &= A \cos u + B \sin u + C \cos u/2 + D \sin u/2 + R, \\ w &= -A \sin u + B \cos u - C \sin u/2 + D \cos u/2 + S, \end{aligned} \tag{7}$$

where (A, B, R, S) are the parameters of the central turbine T, $\bar{u}$ is the normal angle of the fixed element $\overline{G}$, and γ is the radius of the limaçon series. We call $L(A, B, C, D, R, S)$ a set of limaçon series coordinates. Obviously

$$L(A, B, C, D, R, S) = L(A, B, -C, -D, R, S).$$

A limaçon series L is contained in the nonlinear flat field Π whose central element is $\overline{G}$. The centers of the tangent turbines of L, which of course are in Π, are on a circle with center (A, B) and radius γ. We call this the *associated circle* of L. A limaçon series is uniquely determined by its flat field and its associated circle.

Three elements no two of which are parallel and which do not all lie on one turbine determine *four* limaçon series. Three elements only two of which are parallel determine *two* limaçon series. The flat field Π of these limaçon series is the one determined by the given elements. Their associated circles are those which are tangent to all three of the angle bisectors of the angles formed by each of the oriented lines of these elements with the oriented line of the central element $\overline{G}$ of Π. In the first case, there are four circles, namely, the inscribed and escribed circles of the triangle formed by these lines. In the second case, there are only two circles since two of these three lines are parallel.

THEOREM 4. *The osculating limaçon series of a general series S are those whose parameter values are*

$$
(8) \quad \begin{aligned}
A &= a + 2r' \sin u + 2s' \cos u, & B &= b - 2r' \cos u + 2s' \sin u, \\
C &= -4r' \sin u/2 - 4s' \cos u/2, & D &= 4r' \cos u/2 - 4s' \sin u/2, \\
R &= r + 2s', & S &= s - 2r',
\end{aligned}
$$

where (a, b, r, s) are the parameters of the tangent turbines of S.

The envelope of the central turbines of the osculating limaçon series of a general series S is called the *series of curvature* of S. This series is given by

$$
(9) \qquad U = u + \pi, \qquad V = v + 2w', \qquad W = -2v' + w.
$$

THEOREM 5. *The tangent turbines and the central turbines of any general series S have in common the series of curvature of S.*

6. **The osculating circular series of an equiparallel series.** An equiparallel series whose point-union is a circle with center (A, B) and radius γ is called a *circular series* with *center* (A, B) and *radius* γ.

The osculating circular series of an equiparallel series S are those which possess the common direction of S and whose circles are the osculating circles of the point-union of S.

7. **The curvature and torsion of a general series.** The *curvature* κ at an element E of a general series S is defined by the formula

$$
(10) \qquad \kappa = (r'^2 + s'^2)^{1/2},
$$

where (a, b, r, s) are the parameters of the tangent turbine of S at E.

The quantity κ is one half of the radius of the osculating limaçon series L of S at E, and also it is one half of the distance between the centers of the tangent and central turbines of S at E.

The *torsion* τ at an element E of a general series S is defined by the formula

$$
(11) \qquad \tau = d\bar{u}/du,
$$

where u and $\bar{u}$ are the normal angles of the element E of S and the element $\overline{E}$ which is the central element of the osculating flat field of S at E.

The torsion τ at an element E of a general series S is the rate of change of the angle of the osculating flat field per unit radian measure of the angle of the element E.

8. **The curvature of an equiparallel series.** The *curvature* $\kappa = 1/\gamma$ at an element E of an equiparallel series S is defined by the formula

$$
(12) \qquad \kappa = \frac{1}{\gamma} = \frac{w''}{(1 + w'^2)^{3/2}},
$$

where the accent denotes differentiation with respect to v.

The quantity $\gamma = 1/\kappa$ is the radius of the osculating circular series of S at E.

The *torsion* τ of an equiparallel series is taken to be zero.

9. **The osculating spherical fields of a general series.** Let E denote an element, and let Π denote a nonlinear flat field with central element $\overline{G}$. On the oriented line of E, construct the element G which is in Π. Let l be the line connecting the points of G and $\overline{G}$. The perpendicular distance between the point of E and the line l is said to be the *distance* between E and Π.

The set of ∞^2 elements $E(u, v, w)$ which are at a constant distance ρ from a fixed nonlinear flat field $\Pi(\overline{U}, \overline{V}, \overline{W})$ is called a *spherical field* Σ. We term Π the *central flat field* and ρ the *radius* of Σ. The equation of Σ is

$$w = (v - \overline{V}) \cot (u - \overline{U})/2 + \rho \csc (u - \overline{U})/2 - \overline{W}. \tag{13}$$

We call $\Sigma(\overline{U}, \overline{V}, \overline{W}, \rho)$ a set of spherical field coordinates. Obviously

$$\Sigma(\overline{U}, \overline{V}, \overline{W}, \rho) = \Sigma(\overline{U}, \overline{V}, \overline{W}, -\rho).$$

The integral curves of Σ are given in hessian line coordinates by the equation

$$\begin{aligned} v = & -\rho[\cos (u - \overline{U})/2 + \sin^2 (u - \overline{U})/2 \log \cot (u - \overline{U})/4] \\ & - C \cos (u - \overline{U}) + \overline{W} \sin (u - \overline{U}) + \overline{V} + C, \end{aligned} \tag{14}$$

where C is an arbitrary constant. If $\rho = 0$, then Σ becomes its central flat field Π, and its integral curves are the ∞^1 circles which contain the central element $\overline{G}$ of Π. Otherwise if $\rho \neq 0$, the integral curves are transcendental.

Let C be the circular series whose center is the point of the central element $\overline{G}$ of the central flat field Π of the spherical field Σ, whose radius is the radius ρ of Σ, and whose direction is that of $\overline{G}$. We call C the *associated circular series* of Σ. Obviously a spherical field Σ is uniquely determined by its associated circular series C.

The only turbines in a spherical field Σ are the ∞^1 linear turbines whose conjugates are tangent to the associated circular series C of Σ. These are the equiparallel series of Σ. Thus a spherical field Σ contains no nonlinear turbines.

There are ∞^3 limaçon series in a spherical field Σ. Their central turbines are contained in the central flat field Π of Σ. If ρ is the radius of Σ, γ is the radius of any one of these limaçon series L, and α is the angle between Π and the flat field of L, then

$$\rho = 2\gamma \sin \alpha/2. \tag{15}$$

A least limaçon series of a spherical field Σ is any limaçon series of Σ

which has either one of the two equivalent properties: (1) its radius γ is one half the radius ρ of Σ, or (2) its flat field is supplementary or antiparallel to the central flat field Π of Σ. There are ∞^2 least limaçon series in a spherical field Σ.

A spherical field Σ intersects a nonlinear flat field Π_1 which is not parallel to the central flat field Π of Σ in a single limaçon series. If Π_1 and Π are parallel but not identical, then Σ intersects Π_1 in two linear turbines (which may be coincident or imaginary). If Π_1 and Π are identical, then Σ and $\Pi=\Pi_1$ have no common elements. A spherical field intersects a linear flat field in two linear turbines.

Two spherical fields Σ_1 and Σ_2 whose central flat fields Π_1 and Π_2 are not parallel intersect in two limaçon series. If Π_1 and Π_2 are parallel but not identical, then Σ_1 and Σ_2 intersect in four linear turbines (two of which may be coincident, or two or all four of which may be imaginary). If Π_1 and Π_2 are identical, then Σ_1 and Σ_2 have no common elements.

Four elements, at most two of which are parallel and which do not all lie in one flat field, determine *eight* spherical fields. Let us denote the four elements by E_1, E_2, E_3, E_4, where E_3 and E_4 are the two possible parallel ones. Now E_1, E_2, E_j $(j=3, 4)$ determine four limaçon series. The associated circles of these are the inscribed and escribed circles of a triangle T_j which has one vertex at the center O of the turbine determined by E_1 and E_2. Let L_{j1} and L_{j2} denote the two limaçon series whose associated circles are the inscribed circle and the escribed circle opposite the vertex O of T_j. Let L_{j3} and L_{j4} denote the remaining two limaçon series. The central turbines of L_{31}, L_{32}, L_{41}, L_{42} will have a common element F_1, and hence will determine four nonlinear flat fields Π_i $(i=1, 2, 3, 4)$. Similarly, the central turbines of L_{33}, L_{34}, L_{43}, L_{44} will have a common element $F_2 \neq F_1$, and hence will determine four new nonlinear flat fields Π_i $(i=5, 6, 7, 8)$. These eight flat fields Π_i $(i=1, \cdots, 8)$ are the central flat fields of our eight spherical fields Σ_i. The radius ρ_i of Σ_i is the distance between Π_i and any one of the four given elements.

The four elements E_1, E_2, E_3, E_4 such that only E_1, E_2, E_3 are parallel to each other but are not all on one turbine, determine *six* spherical fields. Construct the linear turbine T_1 determined by E_i and E_j $(i, j=1, 2, 3)$. Let Π be the linear flat field which contains the conjugate turbine $\overline{T}_1$ of T_1. Construct the two linear turbines $\overline{T}_2$ and $\overline{T}_3$ contained in Π such that their conjugate turbines T_2 and T_3 contain the remaining two elements. Our spherical fields are those whose associated circular series are tangent to $\overline{T}_1$, $\overline{T}_2$, and $\overline{T}_3$.

The spherical field Σ which has four consecutive elements in common with a general series S at a given element E of S is called the *osculating spherical field* of S at E.

THEOREM 6. *The parameter values of the* ∞^1 *osculating spherical fields of a general series S are*

$$\overline{U} = u - 2 \text{ arc tan } R'/S',$$
$$(16)\qquad \overline{V} = A \cos \overline{U} + B \sin \overline{U} + R, \qquad \overline{W} = -A \sin \overline{U} + B \cos \overline{U} - S,$$
$$\rho = \frac{4(r'R' + s'S')}{(R'^2 + S'^2)^{1/2}},$$

where (a, b, r, s) *and* (A, B, R, S) *are the parameters of the tangent and central turbines of S.*

Let v and w in (13) be replaced by functions of u. Upon differentiating this result three times with respect to u and simplifying, we obtain

$$(17)\qquad \begin{aligned} (v - \overline{V}) \sin (u - \overline{U})/2 + (w + \overline{W}) \cos (u - \overline{U})/2 \\ = 2v' \cos (u - \overline{U})/2 - 2w' \sin (u - \overline{U})/2, \\ -\rho/4 = r' \sin (u - \overline{U})/2 + s' \cos (u - \overline{U})/2, \\ 0 = R' \cos (u - \overline{U})/2 - S' \sin (u - \overline{U})/2. \end{aligned}$$

The second and last of these equations give the values of $\overline{U}$ and ρ of (16). Upon replacing ρ in (13) by the value given in the second of these equations, and then solving this result and the first of the above equations for $\overline{V}$ and $\overline{W}$, we find their values to be those of (16). This completes the proof of Theorem 6.

An immediate consequence of Theorem 6 is

THEOREM 7. *The central flat fields of the osculating spherical fields of a general series S are the osculating flat fields of the series of curvature of S (the envelope of the central turbines of S).*

If α denotes the angle between the osculating flat fields of a general series S and its series of curvature, we find from (4) and (16)

$$(18)\qquad \sin \alpha/2 = \frac{r'R' + s'S'}{(r'^2 + s'^2)^{1/2}(R'^2 + S'^2)^{1/2}} = \frac{\rho}{2\gamma},$$

where ρ and γ are the radii of the osculating spherical field Σ and the osculating limaçon series L at an element E of S. Since the central turbine of L is also in the osculating flat field of the series of curvature of S, we find that the following result holds.

THEOREM 8. *The osculating limaçon series L at an element E of a general series S is the intersection of the osculating flat field* Π *and the osculating spherical field* Σ *of S at E.*

From (4), (8), (10), and (11), we find

(19) $$r' + is' = \kappa e^{i(\bar{u}-u)/2}, \qquad R' + iS' = (\kappa\tau - 2i\kappa')e^{i(\bar{u}-u)/2}.$$

Substituting these into the last of equations (16), we obtain

THEOREM 9. *The radius of spherical curvature ρ in terms of the curvature κ and the torsion τ of the general series S is*

(20) $$\rho = \frac{4\kappa^2\tau}{(\kappa^2\tau^2 + 4\kappa'^2)^{1/2}}.$$

Upon substituting the last of equations (19) into the formulas for the curvature and torsion of the series of curvature of a general series S, we obtain the theorem which follows.

THEOREM 10. *The curvature κ_1 and the torsion τ_1 of the series of curvature of a general series S in terms of the curvature κ and the torsion τ of S are*

(21) $$\kappa_1 = (\kappa^2\tau^2 + 4\kappa'^2)^{1/2}, \qquad \tau_1 = \frac{\kappa^2\tau^3 + 4\tau(2\kappa'^2 - \kappa\kappa'') + 4\kappa\kappa'\tau'}{\kappa^2\tau^2 + 4\kappa'^2}.$$

Differentiating (20) with respect to u, we find

THEOREM 11. *The derivative ρ' of the radius of spherical curvature ρ with respect to u is*

(22) $$\rho' = 4\kappa\kappa'\tau_1/\kappa_1.$$

It may be that the osculating spherical fields of a general series S consist of only one spherical field, namely, the one in which it is contained. In that case, we shall say that S is *cospherical*. From the preceding theorem, we deduce

THEOREM 12. *A general series S is cospherical if and only if its series of curvature is coflat.*

10. **The tangent flat fields of a field.** A set of ∞^2 elements of the plane is called a *field*. We shall omit from consideration the linear flat fields. That is, whenever we speak of a field, we shall understand it to be *not* a linear flat field. It is always possible to find a whirl-motion transformation such that any field F is given by $w = w(u, v)$.

Let $v = v(u)$, $w = w(u, v)$ be a general series S contained in the field F. Its tangent turbines are given by the parameter values

(23) $$\begin{aligned} a &= -w_u \cos u - (\sin u + w_v \cos u)v', \\ b &= -w_u \sin u + (\cos u - w_v \sin u)v', \\ r &= v + w_u + v'w_v, \qquad s = w - v'. \end{aligned}$$

From these equations, we conclude that the following proposition is true.

THEOREM 13. *The tangent turbines of all the series, contained in a field F and passing through an element E of F, constructed at E are contained in a nonlinear flat field.*

The nonlinear flat field of Theorem 13 is called the *tangent flat field* of F at E. Its central element $\bar{E}(\bar{u}, \bar{v}, \bar{w})$ is given by

$$(24)\qquad \begin{aligned} \cos(\bar{u} - u) &= -\frac{1 - w_v^2}{1 + w_v^2}, & \sin(\bar{u} - u) &= -\frac{2w_v}{1 + w_v^2}, \\ \bar{v} &= v + \frac{2w_u}{1 + w_v^2}, & \bar{w} &= -w - \frac{2w_u w_v}{1 + w_v^2}. \end{aligned}$$

We call $\bar{E}$ the *conjugate element* of E with respect to F.

Two fields F_1 and F_2 are said to be *tangent* at a common element E if they have the same tangent flat field at E.

As an application of the above, we find that the tangent flat fields of a spherical field Σ consist of the ∞^1 flat fields whose central elements are those of the associated circular series of Σ.

11. **One-parameter families of fields.** The equation

$$(25)\qquad w = w(u, v, a)$$

defines a one-parameter family of fields. The series of intersection of any two consecutive fields of this family is called a *characteristic*. The locus of all the characteristics is a field, called the *envelope* of the family. The equations

$$(26)\qquad w = w(u, v, a), \qquad w_a(u, v, a) = 0$$

for each a represents a characteristic of the family. When we eliminate a from the above equations, the result is the equation of the envelope.

It may be easily proved by the preceding equations that *the envelope is tangent to each member of the family at all elements of its characteristic.*

The series of intersection of consecutive characteristics of a one-parameter family of fields is called the *edge of regression*. The eliminants with respect to a of the equations

$$(27)\qquad w = w(u, v, a), \qquad w_a(u, v, a) = 0, \qquad w_{aa}(u, v, a) = 0$$

give the equations of the edge of regression.

We may prove by (26) and (27) that *the edge of regression is tangent to any characteristic at a common element.*

12. **Developable fields.** The envelope of ∞^1 nonlinear flat fields is called a *developable field F*. The series S formed by the central elements of these ∞^1

tangent flat fields of F is called the *associated series* of F. The characteristics of F are turbines. These are called the *generators* of F.

Since each flat field is tangent to the envelope along its characteristic, it follows that *the tangent flat field to a developable field F is the same at all elements of a generator.*

An *umbilical field F* is a developable field whose associated series S is an equiparallel series. Thus a spherical field Σ is an umbilical field whose associated series is a circular series. The generators of an umbilical field F are linear turbines. These are the conjugates of the tangent turbines of its associated equiparallel series S. The edge of regression of F does *not* exist. The equation of any umbilical field F is

$$w = v \cot (u - a)/2 + b(u), \tag{28}$$

where a is a constant.

A developable field F is said to be *general* if its associated series S is a general series. The generators of a general developable field F are nonlinear turbines. These are the tangent turbines of the edge of regression R. Since consecutive generators are the consecutive tangent turbines of R at an element E of R, the osculating flat field of R at E is that flat field of the family which contains these generators. But this flat field is tangent to the developable. Hence *the osculating flat field at any element E of the edge of regression R of a general developable field F is the tangent flat field of F at E.* We find from this that *the edge of regression R and the associated series S of a general developable field F are conjugate series.*

The necessary and sufficient condition that a field F: $w=w(u, v)$ be developable is that its conjugate elements $\overline{E}(\bar{u}, \bar{v}, \bar{w})$ of (24) consist of at most ∞^1 elements. These will then form the associated series S of F. Hence upon setting the three jacobians of the three functions $\bar{u}$, $\bar{v}$, $\bar{w}$ of (u, v) equal to zero, we obtain

THEOREM 14. *A field F: $w=w(u, v)$ is a developable field if and only if*

$$(1 + w_v^2 + 2w_{uv})^2 - 4w_{vv}(w_{uu} + w_u w_v) = 0. \tag{29}$$

A field F: $w=w(u, v)$ is a general developable field if and only if the function w of (u, v) satisfies the above equation and $w_{vv} \neq 0$.

13. **Conjugate fields.** The conjugate elements $\overline{E}(\bar{u}, \bar{v}, \bar{w})$ of (24) of a field F: $w=w(u, v)$ form a field $\overline{F}$: $\bar{w}=\bar{w}(\bar{u}, \bar{v})$ if and only if F is nondevelopable, or if and only if the function w of (u, v) does not satisfy (29). This field $\overline{F}$: $\bar{w}=\bar{w}(\bar{u}, \bar{v})$ is termed the *conjugate field* of F. The equation of $\overline{F}$ is the eliminant with respect to u and v of the equations (24).

Since the conjugate field $\bar{F}$ consists of the central elements of the tangent flat fields of a nondevelopable field F (and conversely), it follows that the tangent turbines of $\bar{F}$ are the conjugates of the tangent turbines of F (and conversely). Hence for the special case when the tangent turbines are circles (the self-conjugate turbines), we deduce the following result.

THEOREM 15. *Two fields F and $\bar{F}$ are conjugate if and only if their integral curves possess the same osculating circles.*

14. **The gaussian curvature of a field.** The series of intersection between a nonlinear flat field and a given field F is called a *flat section* of F. If F is *not* a flat field, there are ∞^3 flat sections in F. There pass ∞^2 flat sections of F through any element E. Finally there are ∞^1 flat sections of F which contain a given element E and which possess a fixed tangent turbine at E.

Let S_1 be any general series contained in a field F. There is a unique flat section S which osculates S_1 at a given element E of S_1. This flat section S is the intersection between the field F and the osculating flat field of S_1 at E. The two series S_1 and S will have the same tangent turbine, the same osculating flat field, the same osculating limaçon series, and the same curvature at E. Thus in order to study the curvatures and the osculating limaçon series of any general series contained in a field F, it is necessary merely to study those of any flat section of F.

Next we shall seek to obtain the curvature κ of any flat section S of a field F: $w=w(u, v)$ at any element E of S in terms of the angle β between the flat field of S and the tangent flat field of F at E. Upon eliminating v'' from the values of r' and s', the first derivatives with respect to u of the last two parameters r and s of the tangent turbine of S at E, we find

$$r' + w_v s' = (w_{uu} + w_u w_v) + (1 + w_v^2 + 2w_{uv})v' + w_{vv}v'^2. \tag{30}$$

We see from (4) and (24) that the angle β satisfies the equation

$$\frac{r'}{s'} = \frac{\sin \beta/2 - w_v \cos \beta/2}{\cos \beta/2 + w_v \sin \beta/2}. \tag{31}$$

Solving the preceding two equations for r' and s', and then substituting these results into the curvature formula, we find that the value of the curvature κ of a flat section S of a field F at any element E of S in terms of the angle β between the flat field of S and the tangent flat field of F at E is

$$\kappa = \frac{(w_{uu} + w_u w_v) + (1 + w_v^2 + 2w_{uv})v' + w_{vv}v'^2}{(1 + w_v^2)^{1/2} \sin \beta/2}. \tag{32}$$

When the angle $\beta=\pi$, we shall call the flat section a *supplementary section* and its curvature at E the *supplementary curvature* κ_s. By the preceding equation, the value of the supplementary curvature κ_s is

$$\kappa_s = \frac{(w_{uu} + w_u w_v) + (1 + w_v^2 + 2w_{uv})v' + w_{vv}v'^2}{(1 + w_v^2)^{1/2}}. \tag{33}$$

By the above two equations, we obtain the following analogue of Meusnier's theorem in the geometry of the whirl-motion group G_6.

THEOREM 16. *Let κ_s and κ be the curvatures of a supplementary section and any other flat section which have the same tangent turbine at a common element E of a field F. If α denotes the angle between these two flat sections, then*

$$\kappa_s = \kappa \cos \alpha/2. \tag{34}$$

The above result shows that a supplementary section possesses the least curvature of all the flat sections of a field F which pass through a given element E in a given tangent turbine direction.

A field F which is generated by a one-parameter family of turbines such that consecutive turbines of the family do not lie in a flat field is termed a *ruled field.* We shall say that a ruled field F is *general* or *special* according as the turbines of the family are nonlinear or linear. A special ruled field F is given by either $w_{vv}=0$, or $w=vm(u)+b(u)$, which is not of the form (28).

A field F: $w=w(u, v)$ is called a *general field* if it is neither a special ruled field nor an umbilical field. Thus F is a general field if and only if $w_{vv}\neq 0$. Of course, the general ruled and the general developable fields are all examples of general fields.

THEOREM 17. *At any element E of a general field F, there is one and only one extremal (maximum or minimum) supplementary curvature κ_0. It is given by the formula*

$$\kappa_0 = \frac{-(1 + w_v^2 + 2w_{uv})^2 + 4w_{vv}(w_{uu} + w_u w_v)}{4w_{vv}(1 + w_v^2)^{1/2}}. \tag{35}$$

For upon completing the square of the quadratic expression in v' of (33), we find that (33) may be written in the form

$$\kappa_s = \frac{-(1 + w_v^2 + 2w_{uv})^2 + 4w_{vv}(w_{uu} + w_u w_v)}{4w_{vv}(1 + w_v^2)^{1/2}} + \frac{[(1 + w_v^2 + 2w_{uv}) + 2w_{vv}v']^2}{4w_{vv}(1 + w_v^2)^{1/2}}. \tag{36}$$

This will be a maximum or a minimum with respect to v' only when the squared bracket is zero. The remaining part of the above expression will give us the value of κ_0 of (35). Theorem 17 is completely proved.

Through any element E of a field F, there passes a unique equiparallel series S contained in F. The curvature of S at E is called the *equiparallel curvature* λ of F at E. It is given by

$$\lambda = \frac{w_{vv}}{(1 + w_v^2)^{3/2}} \cdot \tag{37}$$

By means of (35) and (37), we find that (33) or (36) may be written in the form

$$\kappa_s = \kappa_0 + \lambda(1 + w_v^2)\left[v' + \frac{1 + w_v^2 + 2w_{uv}}{2w_{vv}}\right]^2. \tag{38}$$

If δ denotes the distance between the centers of the tangent turbines of (23) of the extremal supplementary section and any supplementary section, we obtain the following analogue of Euler's theorem.

THEOREM 18. *Let κ_0 be the extremal supplementary curvature and λ the equiparallel curvature at an element E of a general field F. If δ is the distance between the centers of the tangent turbines of the extremal supplementary section and any supplementary section whose curvature at E is κ_s, then*

$$\kappa_s = \kappa_0 + \lambda\delta^2. \tag{39}$$

The *gaussian curvature* K of any field F at any element E of F is given by the formula

$$K = \frac{(1 + w_v^2 + 2w_{uv})^2 - 4w_{vv}(w_{uu} + w_u w_v)}{(1 + w_v^2)^2} \cdot \tag{40}$$

We note that K is zero if and only if F is developable.

THEOREM 19. *The gaussian curvature K at any element E of a general field F is minus four times the product of the extremal supplementary curvature κ_0 and the equiparallel curvature λ at E. That is*

$$K = -4\kappa_0\lambda. \tag{41}$$

By relations (23), (33), and (40), we now deduce the following proposition.

THEOREM 20. *If δ is the distance between the centers of the tangent turbines of any two supplementary sections whose curvatures are κ_s and κ_s' at a common element E of a special ruled field F, then*

$$\kappa_s = \kappa_s' + K^{1/2}\delta. \tag{42}$$

For an umbilical field F of (28), all the supplementary curvatures at an element E of F are equal, and they have the common value

$$\kappa_s = \frac{w_{uu} + w_u w_v}{(1 + w_v^2)^{1/2}} = b'' \sin (u - a)/2 + b' \cos (u - a)/2. \tag{43}$$

Through any element E of a general field F, there is a single tangent turbine direction which gives the extremal supplementary curvature κ_0. This is called the *principal direction* of F at E. Any general series S of a general field F such that the tangent turbine direction of any element E of S is a principal direction is called a *principal series*. The differential equation of all principal series of a general field F is

$$2w_{vv}v' + (1 + w_v^2 + 2w_{uv}) = 0. \tag{44}$$

Since the equiparallel curvature λ at an element E of any field F is a sort of extremal curvature, the equiparallel series of F may be considered to be principal series. Thus a general field F possesses $2\infty^1$ principal series, namely, (1) the ∞^1 general series which satisfy (44), and (2) the ∞^1 equiparallel series. A special ruled field F has only ∞^1 principal series, the equiparallel series of F. Finally all ∞^∞ series of an umbilical field F are principal series.

The angle α between the two tangent flat fields at a common element E of two fields $F: w = f(u, v)$ and $G: w = g(u, v)$ is defined to be the angle between F and G at E. By (24), the derivative of this angle α with respect to u is

$$\alpha' = \frac{(1 + f_v^2 + 2f_{uv}) + 2f_{vv}v'}{1 + f_v^2} - \frac{(1 + g_v^2 + 2g_{uv}) + 2g_{vv}v'}{1 + g_v^2}. \tag{45}$$

We deduce from this the following analogue of Joachimsthal's theorem.

THEOREM 21. *If the series of intersection of two fields is a principal series on both, the fields cut at a constant angle. Conversely, if two fields cut at a constant angle, and the series of intersection is a principal series on one, then it is a principal series on the other.*

15. **The gaussian curvature of the conjugate field.** From (24), we find that the partial derivatives of the first and second orders with respect to $\bar{u}$ and $\bar{v}$ of the function $\bar{w} = \bar{w}(\bar{u}, \bar{v})$ which defines the field $\bar{F}$ conjugate to the field $F: w = w(u, v)$ are

$$
(46)\qquad
\begin{aligned}
&\frac{\partial \bar w}{\partial \bar u} = -\,w_u, \qquad \frac{\partial \bar w}{\partial \bar v} = -\,w_v,\\
&\frac{\partial^2 \bar w}{\partial \bar u^2} = \frac{-\,w_{uu}(1 + w_v^2)^2 + 4w_u w_v(w_{uu}w_{vv} - w_{uv}^2)}{(1 + w_v^2 + 2w_{uv})^2 - 4w_{vv}(w_{uu} + w_u w_v)},\\
&\frac{\partial^2 \bar w}{\partial \bar u \partial \bar v} = \frac{-\,w_{uv}(1 + w_v^2)^2 + 2(1 + w_v^2)(w_{uu}w_{vv} - w_{uv}^2)}{(1 + w_v^2 + 2w_{uv})^2 - 4w_{vv}(w_{uu} + w_u w_v)},\\
&\frac{\partial^2 \bar w}{\partial \bar v^2} = \frac{-\,w_{vv}(1 + w_v^2)^2}{(1 + w_v^2 + 2w_{uv})^2 - 4w_{vv}(w_{uu} + w_u w_v)}.
\end{aligned}
$$

By these equations, we deduce the following result.

THEOREM 22. *The product of the gaussian curvatures K and $\bar K$ at conjugate elements E and $\bar E$ of two conjugate fields F and $\bar F$ is unity. That is*

$$
(47)\qquad K\bar K = 1.
$$

The series S of a field F and the series $\bar S$ of the field $\bar F$ conjugate to F are said to be conjugate with respect to F or $\bar F$ if they are corresponding series under the transformation (24).

For any two series S and $\bar S$ conjugate with respect to two conjugate general fields F or $\bar F$, we obtain the following relation:

$$
(48)\qquad 1 + \bar w_{\bar v}^2 + 2\bar w_{\bar u\bar v} + 2\bar w_{\bar v\bar v}\,\frac{d\bar v}{d\bar u} = \frac{(1 + w_v^2)^2}{1 + w_v^2 + 2w_{uv} + 2w_{vv}v'}.
$$

By substituting this into the supplementary curvature formula of $\bar F$, we prove

THEOREM 23. *Let E be any element of a general nondevelopable field F and $\bar E$ its conjugate element of the general field $\bar F$ conjugate to F. The corresponding supplementary curvatures κ_s and $\bar\kappa_s$, the extremal supplementary curvatures κ_0 and $\bar\kappa_0$, and the equiparallel curvatures λ and $\bar\lambda$ of F and $\bar F$ at E and $\bar E$ are related by the formulas*

$$
(49)\qquad \bar\kappa_s = \frac{\kappa_s}{4\lambda(\kappa_s - \kappa_0)}, \qquad \bar\kappa_0 = \frac{1}{4\lambda}, \qquad \bar\lambda = \frac{1}{4\kappa_0}.
$$

For any two series S and $\bar S$ conjugate with respect to two conjugate special ruled fields F or $\bar F$, we find the following relation (since $w_{vv} = 0$)

$$
(50)\qquad \frac{d\bar v}{d\bar u} = v' + \frac{2(1 + w_v^2)w_{uu} - 4w_u w_v w_{uv}}{(1 + w_v^2)(1 + w_v^2 + 2w_{uv})}.
$$

Upon substituting this into the supplementary curvature formula of $\bar{F}$, we obtain the following result.

THEOREM 24. *Let E be any element of a special ruled field F and $\bar{E}$ its conjugate element of the special ruled field $\bar{F}$ conjugate to F. The corresponding supplementary curvatures κ_s and $\bar{\kappa}_s$ of F and $\bar{F}$ at E and $\bar{E}$ are related by the formula*

$$\bar{\kappa}_s = \frac{\kappa_s}{K}. \tag{51}$$

The principal series of two conjugate fields F and $\bar{F}$ are conjugate with respect to F or $\bar{F}$. A principal general series of one of these two general fields corresponds by the transformation (24) to an equiparallel series of the other. Otherwise the equiparallel series of these two special ruled fields correspond to each other under the transformation (24).

16. **The osculating limaçon series of a field.** Upon solving the equations (30) and (31) for r' and s' and making use of the supplementary curvature formula (33), we find that their values are

$$r' = \frac{\kappa_s(\sin \beta/2 - w_v \cos \beta/2)}{(1 + w_v^2)^{1/2} \sin \beta/2}, \qquad s' = \frac{\kappa_s(\cos \beta/2 + w_v \sin \beta/2)}{(1 + w_v^2)^{1/2} \sin \beta/2}. \tag{52}$$

Substituting these values into (8), we see that the parameters of the osculating limaçon series L of any flat section S of a field $F: w = w(u, v)$ at an element E of F are

$$\begin{aligned}
A &= a + \frac{2\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [\cos (\beta/2 - u) + w_v \sin (\beta/2 - u)], \\
B &= b + \frac{2\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [- \sin (\beta/2 - u) + w_v \cos (\beta/2 - u)], \\
C &= - \frac{4\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [\cos (\beta - u)/2 + w_v \sin (\beta - u)/2], \\
D &= \frac{4\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [\sin (\beta - u)/2 - w_v \cos (\beta - u)/2], \\
R &= r + \frac{2\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [\cos \beta/2 + w_v \sin \beta/2], \\
S &= s - \frac{2\kappa_s}{(1 + w_v^2)^{1/2} \sin \beta/2} [\sin \beta/2 - w_v \cos \beta/2],
\end{aligned} \tag{53}$$

where (a, b, r, s) are the parameters of the tangent turbine T of (23) of S at E,

κ_s is the supplementary curvature at E of the supplementary section of F which is tangent to S at E, and β is the angle between the flat field of S and the tangent flat field of F at E.

Next let us consider all the ∞^1 flat sections of the field F which pass through the element E of F and which possess the same tangent turbine T at E. In the formulas (53) for the ∞^1 osculating limaçon series of these flat sections at E, we observe that *only* the angle β is variable. The ∞^1 central turbines of these osculating limaçon series all contain the element E_1 of the tangent turbine T which is supplementary (antiparallel) to E. By (53), it may be proved after some calculation that these central turbines all are contained in the flat field Π whose central element $\overline{G}(U, V, W)$ is given by

$$\cos(U - u) = -\frac{1 - w_v^2}{1 + w_v^2}, \qquad \sin(U - u) = -\frac{2w_v}{1 + w_v^2},$$
(54)
$$V = v + \frac{2w_u}{1 + w_v^2} + \frac{4\kappa_s w_v}{(1 + w_v^2)^{1/2}}, \qquad W = -w - \frac{2w_u w_v}{1 + w_v^2} + \frac{4\kappa_s}{(1 + w_v^2)^{1/2}}.$$

From these equations and from Meusnier's Theorem 16 (formula 34), we deduce

THEOREM 25. *Let us consider the ∞^1 flat sections of a field F which pass through an element E of F and which possess the same tangent turbine T at E. The ∞^1 osculating limaçon series of these flat sections at E generate a spherical field Σ.*

Let E_1 be the element on the tangent turbine T which is supplementary (antiparallel) to the fixed element E. Let T_S be the linear turbine whose direction is that of the central element $\overline{E}$ of the tangent flat field of F at E, and whose base line joins the points of $\overline{E}$ and E_1. The central element $\overline{G}$ of the central flat field Π of the spherical field Σ of Theorem 25 is on the linear turbine T_S and the distance of $\overline{G}$ from $\overline{E}$ is $4\kappa_s$. The radius ρ of Σ is also $4\kappa_s$.

If we vary the tangent turbine direction T of Theorem 25, there will result ∞^1 spherical fields. The ∞^1 central elements of their central flat fields will generate the linear turbine T_S.

17. **The geodesic series of a field.** A series S of a field F is termed a *geodesic series* if its curvature at any element E of S does not exceed the curvature at E of any other series of F which is tangent to S at E. By setting the partial derivative with respect to v'' of the curvature κ of any general series of a field F equal to zero, and solving the result for v'', we find that the differential equation of all the geodesic series of a field F: $w = w(u, v)$ is

(55) $$v'' = \frac{w_u - w_v w_{uu} - 2w_v w_{uv} v' - w_v w_{vv} v'^2}{1 + w_v^2}.$$

There are ∞^2 geodesic series, all general, in a given field F. The ∞^1 equiparallel series of a field F are also considered to be geodesic series. The geodesic series of a spherical field are its ∞^2 least limaçon series, together with its ∞^1 linear turbines. The geodesic series of a flat field are its ∞^2 turbines.

The curvature κ of any geodesic series S of a field F at any element E of S is equal to the supplementary curvature κ_s of the supplementary section of F which is tangent to S at E.

To calculate the torsion τ of a geodesic series S, we proceed as follows. The derivatives r' and s' with respect to u of the last two parameters r and s of the tangent turbine T of S at any element E of S are given by (52) where $\beta=\pi$. Thus $s'/r'=w_v$. By this and (4), we find that the normal angle $\bar{u}$ of the osculating flat field of S at E is

(56) $$\bar{u} = u + 2 \text{ arc tan } w_v.$$

Differentiating this with respect to u, we see that the torsion τ of any geodesic series S at any element E of S is

(57) $$\tau = \frac{1 + w_v^2 + 2w_{uv} + 2w_{vv}v'}{1 + w_v^2}.$$

THEOREM 26. *The torsion τ of a geodesic series S of a field F at an element E of S is zero if and only if S is tangent to a principal series of F at E. The necessary and sufficient condition that a geodesic series be a principal series is that it be coflat.*

By equation (57), we obtain the following two results.

THEOREM 27. *Let λ be the equiparallel curvature of a general field F at an element E of F. Let δ be the distance between the centers of the tangent turbines of the extremal supplementary section and any geodesic series S through E. The torsion τ of S is*

(58) $$\tau = 2\lambda\delta.$$

THEOREM 28. *The torsion τ of any geodesic series S of a special ruled or an umbilical field F at any element E of S is*

(59) $$\tau = K^{1/2}.$$

To define the *geodesic curvature* κ_g of any series S of a field F at any element E of S, we proceed as follows. Let S_g be the geodesic series tangent to S

at E. Let E_s be the element on S which makes an angle Δu with E, and let E_g be the element on S_g which makes the same angle Δu with E. At E_s and E_g, construct the two tangent turbines T_s and T_g of S and S_g. Let $\Delta\delta$ be the distance between the centers of T_s and T_g. Then

$$\kappa_g = \frac{d\delta}{du} = \lim_{\Delta u \to 0} \frac{\Delta\delta}{\Delta u} \tag{60}$$

is defined to be the geodesic curvature of S at E.

It is found that $\kappa_g = (1+w_v^2)^{1/2}(v''-v_g'')$, where v_g'' belongs to the geodesic series S_g. From this, it follows that the geodesic curvature κ_g of any series S of a field F: $w=w(u, v)$ at any element E of S is

$$\kappa_g = \frac{(1 + w_v^2)v'' - (w_u - w_v w_{uu}) + 2w_v w_{uv} v' + w_v w_{vv} v'^2}{(1 + w_v^2)^{1/2}} . \tag{61}$$

By means of (33), (52), and (61), we obtain the result which follows.

THEOREM 29. *Let β be the angle between the osculating flat field of a general series S of a field F and the tangent flat field of F at an element E of S. Let κ be the curvature and κ_g the geodesic curvature of S at E. Let κ_s be the supplementary curvature of the supplementary section which is tangent to S at E. Then*

$$\kappa_s = \kappa \sin \beta/2 = - \kappa_g \tan \beta/2, \quad \kappa_g = - \kappa \cos \beta/2, \quad \kappa_s^2 + \kappa_g^2 = \kappa^2 . \tag{62}$$

18. **The asymptotic series of a field.** Reciprocal directions at an element E of a field F may be defined as follows. Let G be an element in F adjacent to E. Let ER be the turbine of intersection of the tangent flat fields of F at E and G. As G tends to coincidence with E, the limiting tangent turbine directions of EG and ER are said to be *reciprocal* at E.

The necessary and sufficient condition that the tangent turbine directions dv/du and $\delta u/\delta v$ be reciprocal are

$$2w_{vv} \frac{\delta v}{\delta u} \frac{dv}{du} + (1 + w_v^2 + 2w_{uv}) \left(\frac{\delta v}{\delta u} + \frac{dv}{du}\right) + 2(w_{uu} + w_u w_v) = 0. \tag{63}$$

THEOREM 30. *Let T_0 be the principal tangent turbine (the principal direction), and let T and T_1 be any other two tangent turbines at an element E of a general field F. Let δ and δ_1 be the distances of the centers of T and T_1 from that of T_0. The directions of T and T_1 are reciprocal if and only if*

$$\delta\delta_1 = - \kappa_0/\lambda, \tag{64}$$

where κ_0 is the extremal supplementary curvature and λ is the equiparallel curvature of F at E.

THEOREM 31. *The two tangent turbine directions T and T_1 at an element E of a special ruled field F are reciprocal if and only if the sum of the supplementary curvatures κ_s and κ_s' of T and T_1 is zero. That is*

$$\kappa_s + \kappa_s' = 0. \tag{65}$$

Any two tangent turbine directions at an element E of an umbilical field F are reciprocal.

In general, given a one-parameter family of series $\phi(u, v) = \text{const.}$ of a general or special ruled field F, we can find another one-parameter family of series $\psi(u, v) = \text{const.}$ of F such that the two tangent turbine directions of the two series of the two families passing through any element E of F are reciprocal at E.

The self-reciprocal directions of a field F are called *asymptotic directions*. Any series S of a field F whose tangent turbine direction at any element E of S is an asymptotic direction is termed an *asymptotic series*. The differential equation of all asymptotic series of a field F is

$$w_{vv}v'^2 + (1 + w_v^2 + 2w_{uv})v' + (w_{uu} + w_u w_v) = 0. \tag{66}$$

This means that a series S is an asymptotic series if and only if the supplementary curvature κ_s of the supplementary section tangent to S at any element E of S is zero at E.

In a general field F, there are $2\infty^1$ asymptotic series, all general series. A special ruled field F possesses $2\infty^1$ asymptotic series, namely, (1) the ∞^1 general series which satisfy (66), and (2) the ∞^1 equiparallel series of F. Every series of an umbilical field F is an asymptotic series.

From Theorem 30 we pass to the following conclusion.

THEOREM 32. *Let T_0 be the principal tangent turbine (the principal direction), and let T be any other tangent turbine at an element E of a general field F. Let δ be the distance between the centers of T_0 and T. The tangent turbine direction T is an asymptotic direction if and only if*

$$\delta = (-\kappa_0/\lambda)^{1/2}. \tag{67}$$

The osculating flat field of a general series S of a general or special ruled field F at any element E of S will coincide with the tangent flat field of F at E if and only if the normal angles of the central elements of these two flat fields are identical. This means that the angle of equations (52) must be zero. Hence the supplementary curvature κ_s of the supplementary section tangent to S at E is zero at E. Therefore S is an asymptotic series. Thus we obtain

THEOREM 33. *A general series S of a general or special ruled field F is an* *asymptotic series of F if and only if its osculating flat fields coincide with the tangent flat fields of F at the elements of S.*

By this result and by (4) and (24), it follows that the torsion τ of an asymptotic series S at any element E of S is the same as that of the geodesic series which is tangent to S at E. Hence τ is given by (57). Upon squaring this value of τ and noting that S satisfies (66), we obtain the following analogue of the Beltrami-Enneper theorem.

THEOREM 34. *The torsion τ of any asymptotic general series S of a general or special ruled field F at any element E of S is equal to the square root of the gaussian curvature K of F at E. That is,*

$$\tau = K^{1/2}. \tag{68}$$

Thus we have discussed in the geometry of the whirl-motion group G_6 the analogues of some of the classic theorems in the differential geometry of curves and surfaces embedded in a euclidean three-dimensional space. Of significant interest is the fact that our geometric configurations and invariants may be constructed by ordinary geometric means. Some of our results which seem to be completely analogous in content are nevertheless entirely distinct when we think of the meanings of the terms used.

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BROOKLYN COLLEGE,
BROOKLYN, N. Y.