HARMONIC MINIMAL SURFACES*

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1. **Introduction.** A minimal surface in a Euclidean space of three dimensions is harmonic if it is representable in terms of Cartesian coordinates (x_1, x_2, x_3) by an equation of the form $U(x_1, x_2, x_3)=\text{const.}$, where U is a harmonic function. The problem of the determination of all harmonic minimal surfaces is equivalent to a problem in hydrodynamics which will be described in §2. Hamel† has recently solved this problem in two ways. His first method demands in itself that the functions under consideration be real. On the other hand, the second method places no restriction on the functions involved and, since it appears to lead to the same results as the first, one is tempted to infer that all solutions $U(x_1, x_2, x_3)$ of the problem are real. Actually, there exist imaginary solutions and they are geometrically far more intriguing than the real solutions.

It is shown in this paper that all solutions, real and imaginary, except those with isotropic gradients, are reducible, by means of a change of coordinate axes and an integral linear transformation on the function itself, to one of the following forms:

(I) $$U = \tan^{-1}(x_2/x_3) + ax_1,$$

(IIa) $$(2zU + ix_1)^2 = z\tan(4zU^2 + 4ix_1U + \bar{z}),$$

(IIb) $$z^4U^6 + 3(x_1^2 + x_2^2 + x_3^2) = 0,$$

(IIIa) $$U = -\int\frac{dy}{(1-y^3)^{1/2}} = \frac{2}{3}\int\frac{du}{(1+u^2)^{5/6}},$$
$$(z^2 - 2ix_1)^3u^2 + (z^3 - 3ix_1z + \tfrac{3}{2}\bar{z})^2 = 0,$$

(IV) $$U = f(z)x_1 + \phi(z),$$

where $z=x_2+ix_3$ and $\bar{z}=x_2-ix_3$ throughout. It is to be noted that only in the last case does the solution involve arbitrary functions.

The families of minimal surfaces $U=\text{const.}$ in the five cases are: (I) a family of helicoids or a pencil of planes with Euclidean axis; (IIa) a family of imaginary transcendental surfaces; (IIb) a family of imaginary quartic surfaces; (IIIa) a family of imaginary sextic surfaces; and (IV) families of

* Presented to the Society, December 29, 1939; received by the editors December 2, 1939.

† *Potentialströmungen mit konstanter Geschwindigkeit*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1937, pp. 5-20.

imaginary cylinders with isotropic rulings, or a pencil of planes with isotropic axis, or a pencil of parallel planes.

The helicoids of a family (I) are all congruent and each of them admits a one-parameter group of screw motions about a Euclidean axis into itself. The transcendental surfaces (IIa) are all congruent under a one-parameter group of "rotations" about an isotropic line. Each of the sextic surfaces (IIIa) admits a "screw motion" about a line at infinity into itself. Finally, the quartic surfaces of the family (IIb) are all congruent and each admits a one-parameter group of "rotations" about an isotropic line into itself. Furthermore, this family of surfaces belongs to a triply orthogonal system of surfaces which admits a two-parameter group of rigid motions into itself.

It is of interest to note that in every case, not merely those just cited, a one-parameter group of rigid motions plays an important role, and it is perhaps still more striking that these groups exhaust all the one-parameter groups of complex rigid motions.

Analytically, the problem calls for the simultaneous solutions of two partial differential equations of the second order in three independent variables. A frontal attack on it from this point of view seems hopeless. In fact, no matter how it is approached, the analytic complications are severe. The method here adopted turned out to be the same as the second method employed by Hamel. It makes use of the intrinsic geometry of surfaces and congruences of curves. In particular, it introduces three mutually orthogonal congruences of curves, with unit tangent vector fields α, β, γ, which are closely associated with the required family of surfaces $U=$const., and expresses the desired properties of these surfaces by a suitable choice of the coefficients in the equations of variations of α, β, γ with respect to the arcs of the curves of the three congruences. These equations of variation constitute the differential system finally to be integrated. Their conditions of integrability yield a second differential system of ten partial differential equations of the first order in five dependent and three independent variables. The analytic difficulties lie primarily in the solution of this second system and they are not rendered any easier by the fact that the independent variables are the nonholonomic arcs of the curves of the three congruences.

The paper falls into five parts. In Part A, the general case of the problem is formulated after the manner just outlined, and the solutions of the secondary or scalar differential system are listed. In Part B, the primary or vector differential system is integrated in the various cases and the functions U found, and in Part C, the properties of the corresponding minimal surfaces $U=$const. are discussed. The deductions of the solutions of the scalar differential system and the proof that there are no other solutions is given in Part

D, and Part E is devoted to a special case, previously excluded, which gives rise to the solution (IV).

It is assumed that all functions are analytic in the complex variables x_1, x_2, x_3.

A. Formulation of problem and method of solution

2. **The physical problem.** Let there be given a stationary irrotational flow of a frictionless incompressible liquid with the special property that the velocity along an arbitrary line of flow is constant along this line.

A flow of the general type described is characterized by the existence of a harmonic function $U(x_1, x_2, x_3)$ whose gradient is the flow-vector. It will have the desired special property provided the gradient of the velocity of flow, or of any variable function of the velocity, is orthogonal always to the flow-vector. Thus, the problem of determining all flows of the kind required is identical with the problem of finding the simultaneous solutions of the two partial differential equations

$$(1) \qquad \Delta_2 U = 0, \qquad \Delta_1(U, V) = 0,$$

where

$$(2) \qquad V = \log (\Delta_1 U)^{1/2}.$$

Equivalent to these equations are the relations $\Delta_2 U = 0, \Delta_2 U - \Delta_1(U, V) = 0$ which characterize the function U as harmonic and the surfaces $U = \text{const.}$ as minimal.

In the general case in which $\Delta_1 V \neq 0$, we prefer to employ the following equations:

$$(3) \qquad \Delta_2 U - \Delta_1(U, V) = 0, \qquad \Delta_1(U, V) = 0.$$

Thus, our problem becomes that of determining the families of minimal surfaces $U = \text{const.}$ which are cut orthogonally by the corresponding families of surfaces $V = \text{const.}$, where V and U are related by conditions (2). It is in this form that we shall solve the general problem in the complex domain.

The special case in which $\Delta_1 V = 0$ will be treated in §17.

3. **Geometric formulation of the analytic problem.** If U is a solution of (3) for which $\Delta_1 V \neq 0$, the families of surfaces $U = \text{const.}$ and $V = \text{const.}$ are mutually orthogonal and determine three mutually orthogonal congruences of curves: the orthogonal trajectories C_1 of the surfaces $U = \text{const.}$, the orthogonal trajectories C_2 of the surfaces $V = \text{const.}$, and the curves of intersection C_3 of the two families of surfaces. The curves of these congruences, properly directed, have respectively the unit tangent vectors

$$(4)\qquad \alpha = \frac{\nabla U}{(\Delta_1 U)^{1/2}}, \qquad \beta = \frac{\nabla V}{(\Delta_1 V)^{1/2}}, \qquad \gamma = \widetilde{\alpha\beta},$$

where ∇U, for example, is the gradient of U and γ is the vector product of α and β.

The curves C_2 and C_3 lying on a generic surface S of the family $U=\text{const.}$ form an orthogonal system. If $1/r_2$, $1/\tau_2$, $1/\rho_2$ are the normal curvature, geodesic torsion, and geodesic curvature of the directed curves C_2, with respect to $+\alpha$ as the unit vector normal to S, and $1/r_3$, $1/\tau_3$, $1/\rho_3$ have the same meanings for the directed curves C_3, then $1/\tau_2+1/\tau_3=0$ and, since S is minimal, $1/r_2+1/r_3=0$.

From (4) and (2) it follows that the differential of arc ds_1 of the curves C_1 has the value $e^{-V}dU$. Hence, the curves C_1 are geodesics on the surfaces S' of the family $V=\text{const.}$, and β and γ play for them the roles of principal normal vector and binomial vector, respectively. Furthermore, the torsion $1/T_1$ of the curves C_1 is equal to the geodesic torsion of these curves, as curves on the surfaces S'. This geodesic torsion is the negative of the geodesic torsion of the curves C_3, as curves on the surfaces S' or, since the surfaces S and S' intersect under a constant angle, as curves on the surfaces S, and hence it is equal to $1/\tau_2$. Thus, $1/T_1=1/\tau_2$.

By means of these results we obtain from the Frenet-Serret formulas for the curves C_1, and from the formulas for the variation of the surface trihedrals* of the curves C_2 and C_3, as curves on the surfaces S, the following equations:

$$(5)\qquad \begin{aligned}
&\frac{\partial\alpha}{\partial s_1} = A\beta, &&\frac{\partial\alpha}{\partial s_2} = B\beta + C\gamma, &&\frac{\partial\alpha}{\partial s_3} = C\beta - B\gamma,\\
&\frac{\partial\beta}{\partial s_1} = -A\alpha - C\gamma, &&\frac{\partial\beta}{\partial s_2} = -B\alpha + E\gamma, &&\frac{\partial\beta}{\partial s_3} = -C\alpha + F\gamma,\\
&\frac{\partial\gamma}{\partial s_1} = C\beta, &&\frac{\partial\gamma}{\partial s_2} = -C\alpha - E\beta, &&\frac{\partial\gamma}{\partial s_3} = B\alpha - F\beta,
\end{aligned}$$

where $\partial/\partial s_1$, $\partial/\partial s_2$, $\partial/\partial s_3$ represent directional differentiation in the positive directions of the curves C_1, C_2, C_3 respectively, and

$$(6)\quad A = \frac{1}{R_1},\quad B = \frac{1}{r_3} = -\frac{1}{r_2},\quad C = \frac{1}{T_1} = \frac{1}{\tau_2} = -\frac{1}{\tau_3},\quad E = \frac{1}{\rho_2},\quad F = \frac{1}{\rho_3},$$

$1/R_1$ being, of course, the curvature of the curves C_1.

Inasmuch as, for an arbitrary function $f(x_1, x_2, x_3)$,

* See Graustein, *Differential Geometry*, p. 165.

$$\frac{\partial f}{\partial s_1} = \sum \alpha_i \frac{\partial f}{\partial x_i}, \qquad \frac{\partial f}{\partial s_2} = \sum \beta_i \frac{\partial f}{\partial x_i}, \qquad \frac{\partial f}{\partial s_3} = \sum \gamma_i \frac{\partial}{\partial x_i},$$

it follows, by (5), that

$$\begin{aligned} \frac{\partial}{\partial s_2}\frac{\partial f}{\partial s_1} - \frac{\partial}{\partial s_1}\frac{\partial f}{\partial s_2} &= A\frac{\partial f}{\partial s_1} + B\frac{\partial f}{\partial s_2} + 2C\frac{\partial f}{\partial s_3}, \\ \frac{\partial}{\partial s_3}\frac{\partial f}{\partial s_2} - \frac{\partial}{\partial s_2}\frac{\partial f}{\partial s_3} &= E\frac{\partial f}{\partial s_2} + F\frac{\partial f}{\partial s_3}, \\ \frac{\partial}{\partial s_1}\frac{\partial f}{\partial s_3} - \frac{\partial}{\partial s_3}\frac{\partial f}{\partial s_1} &= B\frac{\partial f}{\partial s_3}. \end{aligned} \tag{7}$$

These relations we shall refer to as the conditions of integrability $(f;\ s_1,\ s_2)$, $(f;\ s_2,\ s_3)$, $(f;\ s_3,\ s_1)$, respectively.

From (4) we have, in view of (2),

$$\frac{\partial U}{\partial s_1} = e^V, \qquad \frac{\partial U}{\partial s_2} = 0, \qquad \frac{\partial U}{\partial s_3} = 0, \tag{8}$$

$$\frac{\partial V}{\partial s_1} = 0, \qquad \frac{\partial V}{\partial s_2} = A, \qquad \frac{\partial V}{\partial s_3} = 0, \tag{9}$$

where

$$A = (\Delta_1 V)^{1/2} \neq 0. \tag{10}$$

The conditions of integrability of equations (8) simply require that the quantity A in (9) and (10) be identical with the quantity A in equations (5) to (7).

The conditions of integrability of equations (9) are

$$\frac{\partial A}{\partial s_1} = -AB, \qquad \frac{\partial A}{\partial s_3} = AE. \tag{11}$$

Equations (5) and (11), together with the inequality (10), constitute necessary conditions. Suppose, conversely, that the scalar functions A $(\neq 0)$, B, C, E, F and the vector functions α, β, γ, representing three mutually orthogonal unit vector fields with the same disposition as the coordinate axes, are known solutions of equations (5) and (11). Equations (9), for the given α, β, γ, and A, are then integrable and the function V is determined to within an additive constant; and equations (8), for the given α, β, γ and the V just found, are integrable and U is determined to within a multiplicative and an additive constant.

Since (8) and (9) are equivalent to the relations $\nabla U = e^V \alpha$, $\nabla V = A\beta$, it follows that $V = \log (\Delta_1 U)^{1/2}$, $\Delta_1 V = A^2$, and $\Delta_1(U, V) = 0$. Hence, the function V is related to U as prescribed by (2), $\Delta_1 V \neq 0$, and the families of surfaces $U = \text{const.}$ and $V = \text{const.}$ cut orthogonally. Finally, since α is a unit vector normal to the surfaces $U = \text{const.}$ and the curves C_2 and the curves C_3 determined respectively by the unit vector fields β and γ lie on these surfaces, it follows from (5) that the surfaces are minimal.

Thus, we have established the following existence theorem.

THEOREM 1. *A necessary and sufficient condition that there exist a harmonic function* $U(x_1, x_2, x_3)$ *such that the surfaces* $U = \text{const.}$ *are minimal is that there exist three mutually orthogonal unit vector fields* α, β, γ, *with the same disposition as the axes, and five scalar functions* A $(\neq 0)$, B, C, E, F *which satisfy equations* (5) *and* (11). *The function* U *is then determined to within a multiplicative and an additive constant and can be found by quadratures.*

COROLLARY. *If two solutions* α, β, γ, A $(\neq 0)$, B, C, E, F *of equations* (5) *and* (11) *are related to one another by a rigid motion, the two resulting families of surfaces* $U = \text{const.}$ *are congruent.*

The corollary is an obvious consequence of the fact that the equations with which we are dealing are invariant with respect to the group of rigid motions.

4. **Outline of the solution.** The nine conditions of integrability of equations (5), combined with the two equations in (11), yield the following ten independent equations in A, B, C, E, F and their directional derivatives:

$$\text{(12a)} \qquad \frac{\partial A}{\partial s_1} = -AB, \qquad \frac{\partial A}{\partial s_2} = A^2 + 2B^2 + 2C^2 - AF, \qquad \frac{\partial A}{\partial s_3} = AE,$$

$$\text{(12b)} \qquad \begin{aligned} &\frac{\partial B}{\partial s_1} = B^2 - C^2 - AF, \qquad &&\frac{\partial B}{\partial s_3} - \frac{\partial C}{\partial s_2} = 2CF + 2BE, \\ &\frac{\partial C}{\partial s_1} = AE + 2BC, &&\frac{\partial B}{\partial s_2} + \frac{\partial C}{\partial s_3} = 2CE - 2BF, \end{aligned}$$

$$\text{(12c)} \qquad \begin{aligned} &\frac{\partial E}{\partial s_1} + \frac{\partial C}{\partial s_2} = -BE - 2CF, \\ &\frac{\partial F}{\partial s_1} + \frac{\partial C}{\partial s_3} = AB + BF, \\ &\frac{\partial E}{\partial s_3} - \frac{\partial F}{\partial s_2} = -B^2 - C^2 + E^2 + F^2. \end{aligned}$$

The process of solving our problem now becomes clearer. First, the system of equations (12) is to be solved for the unknown functions A, B, C, E, F. For the values found for these functions, equations (5) will be completely integrable and will determine, to within a rigid motion, three mutually perpendicular unit vector fields α, β, γ, and equations (9) and (8) will, then, yield the desired functions V and U. This, at least, would be the general procedure if, instead of the directional derivatives, we had ordinary partial derivatives with which to deal. Actually, equations (12) involve the unknown vector functions α, β, γ (through the directional derivatives), as well as the unknown scalars A, B, C, E, F, and equations (5) involve α, β, γ both in the derivatives and as the unknown functions. Nevertheless, the general procedure described remains valid, as we shall proceed to show.

The essential requirement for the employment of this procedure is that the three mutually orthogonal unit vector fields α, β, γ and the directional derivatives in the directions of α, β, γ which are employed when equations (12) are solved for A, B, C, E, F should later be found to satisfy equations (5) for the values of A, B, C, E, F obtained. This requirement is actually fulfilled by the inherent demand that the directional derivatives in question enjoy the conditions of integrability (7). For, it is readily proved that, if (7) are satisfied, the vector fields α, β, γ and the derivatives in their directions satisfy (5).

Equations (12), subject to the integrability conditions (7), have the following solutions:

$$\text{(I)}\qquad \begin{array}{lll} B = 0, & E = 0, & AF + C^2 = 0, \\ \dfrac{\partial A}{\partial s_1} = 0, & \dfrac{\partial A}{\partial s_2} = A^2 + 3C^2, & \dfrac{\partial A}{\partial s_3} = 0, \\ \dfrac{\partial C}{\partial s_1} = 0, & \dfrac{\partial C}{\partial s_2} = -2CF, & \dfrac{\partial C}{\partial s_3} = 0, \\ \dfrac{\partial F}{\partial s_1} = 0, & \dfrac{\partial F}{\partial s_2} = C^2 - F^2, & \dfrac{\partial F}{\partial s_3} = 0, \end{array}$$

$$\text{(II)}\qquad \begin{array}{lll} AE = -2BC, & AF = 2B^2, & A^2 + 4(B^2 + C^2) = 0, \\ \dfrac{\partial A}{\partial s_1} = -AB, & \dfrac{\partial A}{\partial s_2} = \tfrac{1}{2}A^2 - 2B^2, & \dfrac{\partial A}{\partial s_3} = -2BC, \\ \dfrac{\partial B}{\partial s_1} = \tfrac{1}{4}A^2, & \dfrac{\partial B}{\partial s_2} = AB, & \dfrac{\partial B}{\partial s_3} = \tfrac{1}{2}AC, \\ \dfrac{\partial C}{\partial s_1} = 0, & \dfrac{\partial C}{\partial s_2} = \tfrac{1}{2}AC, & \dfrac{\partial C}{\partial s_3} = 0, \end{array}$$

$$\text{(III)}\quad \begin{aligned} &E = 0, && F = -\tfrac{1}{2}A, && A^2 + 4(B^2 + C)^2 = 0,\\ &\frac{\partial A}{\partial s_1} = -AB, && \frac{\partial A}{\partial s_2} = A^2, && \frac{\partial A}{\partial s_3} = 0,\\ &\frac{\partial B}{\partial s_1} = \tfrac{1}{2}A^2 + B^2 - C^2, && \frac{\partial B}{\partial s_2} = AB, && \frac{\partial B}{\partial s_3} = 0,\\ &\frac{\partial C}{\partial s_1} = 2BC, && \frac{\partial C}{\partial s_2} = AC, && \frac{\partial C}{\partial s_3} = 0. \end{aligned}$$

In all three cases, $A \neq 0$. Henceforth, this condition will always be tactily understood.

Solutions (II) and (III) are imaginary, whereas (I) exists in the real domain. It may be readily verified that all three satisfy equations (12) and the integrability conditions (7).

The derivation of the three solutions from (12) and the proof that (12) has no other solutions present analytic problems of unusual complexity. In order not to interrupt the present development, we shall postpone the consideration of them to Part D.

Since, for the values of A, B, C, E, F furnished by a solution of (12), equations (5), (9), and (8) are completely integrable, it is theoretically possible to solve equations (5) for the three mutually orthogonal unit vector functions α, β, γ and hence (9) and (8) for the scalar functions V and U. Practically, however, this procedure is complicated by the presence of directional, rather than partial, derivatives, and we find ourselves forced to adopt a different method.

We remark, first, that equations (8) and (9) are of the same type as the differential equations in one of the solutions of (12), and hence that U and V are just as much known as the quantities involved in these differential equations. Consequently, we have at our disposal the seven functions U, V, A, B, C, E, F. It is evident from (I), (II), (III) that at most two of the last five are functionally independent.

By integration of equations (5) it is possible to find α, β, γ in terms of certain of the seven functions. For these values of α, β, γ, the equations

$$\frac{\partial x}{\partial s_1} = \alpha, \qquad \frac{\partial x}{\partial s_2} = \beta, \qquad \frac{\partial x}{\partial s_3} = \gamma \tag{13}$$

are completely integrable, by virtue of (5), and x_1, x_2, x_3 may be found in terms of three independent functions, or parameters, one of which is U. Elimination of the other two parameters from the three equations results in the desired value of U as a function of x_1, x_2, x_3.

Geometrically, each of the three solutions (I), (II), (III) has two cases according as $C \neq 0$ or $C=0$, that is, by (6), according as the curves C are twisted or plane. The two cases we shall distinguish by attaching the letters a and b to the Roman numerical. It is readily verified that the solutions (IIb) and (IIIb) are identical.

B. The harmonic functions

5. **The real solutions.** *Case* Ia. Inasmuch as $B=E=0$, we conclude from (6) that the curves C_2 are straight lines. Since $C \neq 0$, the Gaussian curvature* of the surfaces S is not zero. Hence, the surfaces S are right helicoids and a normal form for the function U is

$$U = \tan^{-1}(x_2/x_3) + ax_1 = 0, \qquad a \neq 0. \tag{14}$$

The lines of flow in the physical problem are the circular helices cutting the helicoids orthogonally.

Case Ib. It may be shown geometrically that the surfaces S form a pencil of intersecting planes and that a normal form for U is (14), where $a=0$. The lines of flow are circles.

It will be advantageous to illustrate the analytic method described at the end of the preceding section in this simple case. Since $B=C=E=F=0$, equations (I) reduce to

$$\frac{\partial A}{\partial s_1} = 0, \qquad \frac{\partial A}{\partial s_2} = A^2, \qquad \frac{\partial A}{\partial s_3} = 0. \tag{15}$$

Comparison of these equations with (9) shows that we may take $V=\log A$. Then, (8) becomes

$$\frac{\partial U}{\partial s_1} = A, \qquad \frac{\partial U}{\partial s_2} = 0, \qquad \frac{\partial U}{\partial s_3} = 0. \tag{16}$$

Equations (5) reduce to $\partial\alpha/\partial s_1 = A\beta$, $\partial\beta/\partial s_1 = -A\alpha$, with the remaining derivatives of α and β, and all of those of γ, zero. Consequently, in view of (16), if a, b, c are three fixed mutually perpendicular unit vectors with the disposition of the axes, we have

$$\alpha = b\cos U - c\sin U, \qquad \beta = -b\sin U - c\cos U, \qquad \gamma = -a.$$

But then equations (13), since a function W exists satisfying the equations

$$\frac{\partial W}{\partial s_1} = 0, \qquad \frac{\partial W}{\partial s_2} = 0, \qquad \frac{\partial W}{\partial s_3} = -1, \tag{17}$$

* For the formula employed, see Graustein, *Invariant methods in classical differential geometry*, Bulletin of American Mathematical Society, vol. 36 (1930), p. 508.

have the integral

$$x = Wa + e^{-V}(b \sin U + c \cos U) + k,$$

where k represents a triple of constants.

By means of a rigid motion this representation is reducible to the normal form

$$x_1 = W, \qquad x_2 = e^{-V} \sin U, \qquad x_3 = e^{-V} \cos U. \tag{18}$$

Eliminating V, we obtain for U the normal form $\tan^{-1}(x_2/x_3)$.

Equations (18) represent a change from Cartesian coordinates to curvilinear coordinates (U, V, W). In view of equations (15), (16), (17) and the fact that $A = e^V$, the congruences of parametric curves consist of the curves C_1, the curves C_2, and the curves C_3, respectively, and the parametric surfaces form a triply orthogonal system, that of cylindrical coordinates.

6. **The imaginary solution** IIa. The point of departure here is solution (II) of equations (12) for $C \neq 0$, namely,

$$\begin{gathered} AE = -2BC, \qquad AF = 2B^2, \qquad A^2 + 4(B^2 + C^2) = 0, \qquad C \neq 0, \\ \frac{\partial A}{\partial s_1} = -AB, \qquad \frac{\partial A}{\partial s_2} = \tfrac{1}{2}A^2 - 2B^2, \qquad \frac{\partial A}{\partial s_3} = -2BC, \\ \frac{\partial B}{\partial s_1} = \tfrac{1}{4}A^2, \qquad \frac{\partial B}{\partial s_2} = AB, \qquad \frac{\partial B}{\partial s_3} = \tfrac{1}{2}AC, \\ \frac{\partial C}{\partial s_1} = 0, \qquad \frac{\partial C}{\partial s_2} = \tfrac{1}{2}AC, \qquad \frac{\partial C}{\partial s_3} = 0. \end{gathered} \tag{19}$$

Equations (5) become

$$\begin{gathered} \frac{\partial \alpha}{\partial s_1} = A\beta, \qquad \frac{\partial \alpha}{\partial s_2} = B\beta + C\gamma, \qquad \frac{\partial \alpha}{\partial s_3} = C\beta - B\gamma, \\ \frac{\partial \beta}{\partial s_1} = -A\alpha - C\gamma, \qquad \frac{\partial \beta}{\partial s_2} = -B\alpha - \frac{2BC}{A}\gamma, \qquad \frac{\partial \beta}{\partial s_3} = -C\alpha + \frac{2B^2}{A}\gamma, \\ \frac{\partial \gamma}{\partial s_1} = C\beta, \qquad \frac{\partial \gamma}{\partial s_2} = -C\alpha + \frac{2BC}{A}\beta, \qquad \frac{\partial \gamma}{\partial s_3} = B\alpha - \frac{2B^2}{A}\beta. \end{gathered} \tag{20}$$

It follows from (19) and (20) that the determinant of β and its first two derivatives with respect to s_2 vanishes. Hence, the curves C_2 are plane curves. The planes in which they lie are determined by the vector fields β and $B\alpha - E\gamma$.

Similarly, it can be shown that the curves C_3 are plane curves, lying in the planes determined by the vector fields γ and $B\alpha - F\beta$.

A vector field common to the planes of C_2 and C_3 is $B\alpha-F\beta-E\gamma$ or, since $B\neq 0$ by (19), $A\alpha-2B\beta+2C\gamma$. The vectors of this field are isotropic. Moreover, they are fixed in direction. In fact, it is readily shown, by (19) and (20), that the derivatives of the vector field

$$a=\frac{A^2}{C^3}\alpha-\frac{2AB}{C^3}\beta+\frac{2AC}{C^3}\gamma \tag{21}$$

all vanish.

A vector field normal to the planes of the curves C_3 is $2B\alpha+A\beta$. Setting

$$\eta=\frac{2B}{C}\alpha+\frac{A}{C}\beta, \tag{22}$$

we find that

$$\frac{\partial\eta}{\partial s_1}=-\tfrac{1}{2}C^2a,\qquad \frac{\partial\eta}{\partial s_2}=0,\qquad \frac{\partial\eta}{\partial s_3}=0.$$

Comparison of (9) with the derivatives of C in (19) shows that $e^V=kC^2$. Since k becomes the multiplicative constant in the value of U, we may without loss of generality specialize it. Taking $k=-1/2$, and noting that (8) then becomes

$$\frac{\partial U}{\partial s_1}=-\tfrac{1}{2}C^2,\qquad \frac{\partial U}{\partial s_2}=0,\qquad \frac{\partial U}{\partial s_3}=0, \tag{23}$$

we conclude that

$$\eta=Ua+b, \tag{24}$$

where b is a triple of constants.

Equation (21) and the equation obtained by equating the values of η in (22) and (24) can be solved for β and γ as linear combinations of α, a, b. Substituting these values of β and γ in the equations for the derivatives of α in (20), we find that the resulting equations can be written in the forms:

$$\frac{\partial}{\partial s_1}\left(\frac{C^3}{A^2}\alpha\right)=\frac{C^4}{A^2}Ua+\frac{C^4}{A^2}b,$$

$$\frac{\partial}{\partial s_2}\left(\frac{C^3}{A^2}\alpha\right)=\left(\frac{C^6}{2A^3}+\frac{2BC^4}{A^3}U\right)a+\frac{2BC^4}{A^3}b,$$

$$\frac{\partial}{\partial s_3}\left(\frac{C^3}{A^2}\alpha\right)=\left(-\frac{BC^5}{2A^3}+\frac{C^5-B^2C^3}{A^3}U\right)a+\frac{C^5-B^2C^3}{A^3}b.$$

It may be shown, by means of (19) and (23), that the coefficients of b in these three equations are respectively the directional derivatives of the function $U/4-BC^2/2A^2$ and that $-BC^4/2A^2$, $C^6/2A^3$, $-BC^5/2A^3$ are the derivatives of the function $-C^4/8A^2$. Hence, the equations are integrable and yield the relation

$$\frac{C^3}{A^2}\alpha = \left(\frac{1}{8}U^2 - \frac{BC^2}{2A^2}U - \frac{C^4}{8A^2}\right)a + \left(\frac{1}{4}U - \frac{BC^2}{2A^2}\right)b + c, \tag{25}$$

where c is a triple of constants.

Having solved equations (20) for α, β, γ, we could proceed by the method outlined at the end of §4 to find x_1, x_2, x_3. We adopt a different, but equivalent, method. Instead of solving equations (21), (22), (24), (25) for α, β, γ in terms of a, b, c, we solve for a, b, c in terms of α, β, γ. Making use of the function $H=4B/C^3$, whose directional derivatives are the coefficients of α, β, γ in (21), we find the expressions

$$\begin{aligned} a &= \frac{\partial H}{\partial s_1}\alpha + \frac{\partial H}{\partial s_2}\beta + \frac{\partial H}{\partial s_3}\gamma, \\ b &= \left(-\frac{\partial H}{\partial s_1}U + \frac{2B}{C}\right)\alpha + \left(-\frac{\partial H}{\partial s_2}U + \frac{A}{C}\right)\beta - \frac{\partial H}{\partial s_3}U\gamma, \\ c &= \left(\frac{1}{8}\frac{\partial H}{\partial s_1}U^2 - \frac{B}{2C}U - \frac{C}{8}\right)\alpha + \left(\frac{1}{8}\frac{\partial H}{\partial s_2}U^2 - \frac{A}{4C}U + \frac{BC}{4A}\right)\beta \\ &\quad + \left(\frac{1}{8}\frac{\partial H}{\partial s_3}U^2 + \frac{C^2}{4A}\right)\gamma. \end{aligned} \tag{26}$$

The conditions on a, b, c guaranteeing that α, β, γ are mutually perpendicular unit vector fields with the same disposition as the axes are now readily found to be*

$$\begin{aligned} &(a\,|\,a) = 0, \quad (b\,|\,b) = -4, \quad (c\,|\,c) = 0, \\ &(b\,|\,c) = 0, \quad (c\,|\,a) = 1, \quad (a\,|\,b) = 0, \end{aligned} \qquad (a\;b\;c) = -2. \tag{27}$$

It is to be noted that the vector c, as well as the vector a, is isotropic.

The integrals of equations (26) are

$$\begin{aligned} &(a\,|\,x) + a_0 = H, \qquad H = 4B/C^3, \\ &(b\,|\,x) + b_0 = -HU - 2/C, \\ &(c\,|\,x) + c_0 = \tfrac{1}{8}HU^2 + \tfrac{1}{2}(1/C)\,U + \tfrac{1}{8}\tan^{-1}(C/B), \end{aligned} \tag{28}$$

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* By $(a\,|\,b)$ is meant the scalar product of the vectors a and b, and by $(a\;b\;c)$, the determinant of the components of three vectors a, b, c.

for equations (26) simply state that a, b, c are respectively the gradients of the functions which appear on the right-hand sides of (28).

Equations (28) define x_1, x_2, x_3 as functions of U, B, C—three parameters which by (19) and (23) are independent. Denoting the linear functions of x_1, x_2, x_3 on the left-hand sides by $a(x)$, $b(x)$, $c(x)$ respectively, and eliminating B and C, we obtain the equation

$$(29)\qquad [a(x)U + b(x)]^2 = a(x) \tan [a(x)U^2 + 2b(x)U + 8c(x)],$$

which with the attendant conditions (27) serves to define, within a multiplicative and an additive constant, the general solution U in this case.

It follows from (27) that the equations

$$a(x) = 4(x_2' + ix_3'), \qquad b(x) = 2ix_1', \qquad 8c(x) = x_2' - ix_3'$$

define a rigid motion. By means of this rigid motion or, what is the same thing, by setting a_0, b_0, c_0 equal to zero and taking as a, b, c respectively the triples

$$(30)\qquad 0,\ 4,\ 4i, \qquad 2i,\ 0,\ 0, \qquad 0,\ \tfrac{1}{8},\ -\tfrac{1}{8}i,$$

(29) reduces to the normal form

$$(29a)\qquad (2zU + ix_1)^2 = z \tan (4zU^2 + 4ix_1U + \bar{z}),$$

where $z = x_2 + ix_3$ and $\bar{z} = x_2 - ix_3$. Hence, there is essentially only one solution U of our problem in this case.

7. **The imaginary solution** IIb. When $C=0$, equations (II) become

$$E = 0, \qquad F = -\tfrac{1}{2}A, \qquad A^2 + 4B^2 = 0, \qquad C = 0, \qquad B \neq 0,$$

$$(31)\qquad \begin{aligned} &\frac{\partial A}{\partial s_1} = -AB, \qquad &\frac{\partial A}{\partial s_2} = A^2, \qquad &\frac{\partial A}{\partial s_3} = 0,\\ &\frac{\partial B}{\partial s_1} = -B^2, \qquad &\frac{\partial B}{\partial s_2} = AB, \qquad &\frac{\partial B}{\partial s_3} = 0. \end{aligned}$$

Hence, equations (5) reduce to

$$(32)\qquad \begin{aligned} &\frac{\partial \alpha}{\partial s_1} = A\beta, \qquad &\frac{\partial \alpha}{\partial s_2} = B\beta, \qquad &\frac{\partial \alpha}{\partial s_3} = -B\gamma,\\ &\frac{\partial \beta}{\partial s_1} = -A\alpha, \qquad &\frac{\partial \beta}{\partial s_2} = -B\alpha, \qquad &\frac{\partial \beta}{\partial s_3} = -\tfrac{1}{2}A\gamma,\\ &\frac{\partial \gamma}{\partial s_1} = 0, \qquad &\frac{\partial \gamma}{\partial s_2} = 0, \qquad &\frac{\partial \gamma}{\partial s_3} = B\alpha + \tfrac{1}{2}A\beta. \end{aligned}$$

The curves C_3 turn out to be plane curves, lying in the isotropic planes determined by the vector field γ and the isotropic vector field $B\alpha+\frac{1}{2}A\beta$. The vectors of the latter field are fixed in direction. In fact, we find that the vector

$$a = BK\alpha + \tfrac{1}{2}AK\beta, \tag{33}$$

where K ($\neq 0$) is a function defined by the compatible equations

$$\frac{\partial K}{\partial s_1} = -BK, \qquad \frac{\partial K}{\partial s_2} = -\tfrac{1}{2}AK, \qquad \frac{\partial K}{\partial s_3} = 0, \tag{34}$$

is a fixed vector.

It follows from (32) that

$$\gamma = -Wa + b, \tag{35}$$

where W is defined by the integrable equations

$$\frac{\partial W}{\partial s_1} = 0, \qquad \frac{\partial W}{\partial s_2} = 0, \qquad \frac{\partial W}{\partial s_3} = -\frac{1}{K}, \tag{36}$$

and b is a triple of constants.

When the value of β in terms of α and a from (33) and the value of γ from (35) are substituted in the differential equations for α in (32), these equations are readily integrated and have the solution

$$\frac{1}{AK}\alpha = \left(\frac{1}{2ABK^2} - \frac{B}{2A}W^2\right)a + \frac{B}{A}Wb + c, \tag{37}$$

where c is a fixed vector.

From equations (33), (35), (37), we find a, b, c as linear combinations of α, β, γ:

$$\begin{aligned} a &= -\frac{\partial K}{\partial s_1}\alpha - \frac{\partial K}{\partial s_2}\beta - \frac{\partial K}{\partial s_3}\gamma, \\ b &= -W\frac{\partial K}{\partial s_1}\alpha - W\frac{\partial K}{\partial s_2}\beta + \gamma, \\ c &= \left(\frac{B}{2A}W^2\frac{\partial K}{\partial s_1} + \frac{1}{2AK}\right)\alpha + \left(\frac{B}{2A}W^2\frac{\partial K}{\partial s_1} - \frac{1}{4BK}\right)\beta - \frac{B}{A}W\gamma, \end{aligned} \tag{38}$$

and hence obtain

$$\begin{matrix} (a\,|\,a) = 0, & (b\,|\,b) = 1, & (c\,|\,c) = 0, \\ (b\,|\,c) = 0, & (c\,|\,a) = B/A, & (a\,|\,b) = 0, \end{matrix} \qquad (a\,b\,c) = 1/2, \tag{39}$$

as the conditions on a, b, c equivalent to the initial conditions on α, β, γ. It should be noted, from (31), that B/A is a constant.

It is readily shown that equations (38) have the solutions

$$(40)\qquad \begin{aligned} (a \mid x) + a_0 &= -K, \\ (b \mid x) + b_0 &= -KW, \\ (c \mid x) + c_0 &= \frac{B}{2A}\left(KW^2 + \frac{1}{3B^2K}\right), \end{aligned}$$

defining x_1, x_2, x_3 in terms of B, K, W.

We proceed to find values of B, K in terms of U, V. For this purpose, it will be convenient to replace e^{-V} by $-\bar{V}^2$ in equations (8) and (9). These equations then become, after dropping the bars,

$$(41)\qquad \frac{\partial U}{\partial s_1} = -\frac{1}{V^2}, \qquad \frac{\partial U}{\partial s_2} = 0, \qquad \frac{\partial U}{\partial s_3} = 0,$$

$$(42)\qquad \frac{\partial V}{\partial s_1} = 0, \qquad \frac{\partial V}{\partial s_2} = -\tfrac{1}{2}AV, \qquad \frac{\partial V}{\partial s_3} = 0.$$

From (31), (34), (41), (42), it is readily verified that we may write

$$(43)\qquad B = -\frac{1}{UV^2}, \qquad K = \frac{V}{U}, \qquad UV \neq 0.$$

Denoting the left-hand sides of (40) by $a(x)$, $b(x)$, and $c(x)$, substituting the values of B and K from (43), and eliminating V, W, we obtain the equation

$$(44)\qquad [a(x)]^4U^6 + 3[b(x)]^2 + \frac{6A}{B}a(x)c(x) = 0,$$

which with (39) serves to define the required function U to within an additive and a multiplicative constant.

By means of the equations

$$a(x) = x_2' + ix_3', \qquad b(x) = \frac{2B}{A}ix_1', \qquad c(x) = \frac{B}{2A}(x_2' - ix_3'),$$

which by virtue of (39) represent a rigid motion, or, what is the same thing, by setting $a_0 = b_0 = c_0 = 0$, and taking as a, b, c the triples

$$(45)\qquad 0,\ 1,\ i, \qquad \frac{2B}{A}i,\ 0,\ 0, \qquad 0,\ \frac{B}{2A},\ -\frac{B}{2A}i,$$

(44) reduces to the following normal form:

$$(x_2 + ix_3)^4 U^6 + 3(x_1^2 + x_2^2 + x_3^2) = 0. \tag{44a}$$

Thus here, too, the solution of the problem is essentially unique.

Equations (40), when B and K are replaced by their values from (43), represent the transformation from (x_1, x_2, x_3) to the curvilinear coordinates (U, V, W). From (41), (42), (36), it follows that the parametric surfaces for these coordinates form a triply orthogonal system and that the congruences of parametric curves consist precisely of the curves C_1, the curves C_2, and the curves C_3.

8. **The imaginary solution** IIIa. In this case we have

$$E = 0, \qquad F = -\tfrac{1}{2}A, \qquad A^2 + 4(B^2 + C^2) = 0, \qquad C \neq 0,$$

$$\begin{aligned} &\frac{\partial A}{\partial s_1} = -AB, && \frac{\partial A}{\partial s_2} = A^2, && \frac{\partial A}{\partial s_3} = 0, \\ &\frac{\partial B}{\partial s_1} = \tfrac{1}{2}A^2 + B^2 - C^2, && \frac{\partial B}{\partial s_2} = AB, && \frac{\partial B}{\partial s_3} = 0, \\ &\frac{\partial C}{\partial s_1} = 2BC, && \frac{\partial C}{\partial s_2} = AC, && \frac{\partial C}{\partial s_3} = 0, \end{aligned} \tag{46}$$

and equations (5) become

$$\begin{aligned} &\frac{\partial \alpha}{\partial s_1} = A\beta, && \frac{\partial \alpha}{\partial s_2} = B\beta + C\gamma, && \frac{\partial \alpha}{\partial s_3} = C\beta - B\gamma, \\ &\frac{\partial \beta}{\partial s_1} = -A\alpha - C\gamma, && \frac{\partial \beta}{\partial s_2} = -B\alpha, && \frac{\partial \beta}{\partial s_3} = -C\alpha - \tfrac{1}{2}A\gamma, \\ &\frac{\partial \gamma}{\partial s_1} = C\beta, && \frac{\partial \gamma}{\partial s_2} = -C\alpha, && \frac{\partial \gamma}{\partial s_3} = B\alpha + \tfrac{1}{2}A\beta. \end{aligned} \tag{47}$$

The principal normals of the curves C_3 are all parallel to one and the same isotropic plane, whose aspect is given by the fixed isotropic vector field

$$a = -\frac{A}{C^{1/2}}\alpha + \frac{2B}{C^{1/2}}\beta + 2C^{1/2}\gamma. \tag{48}$$

For the field of vectors

$$\xi = \frac{B}{C}\alpha + \frac{A}{2C}\beta$$

in the directions of the principal normals of the curves C_3 we find, in terms of the function L defined by the compatible equations

$$\frac{\partial L}{\partial s_1} = -\frac{A}{4C^{1/2}}, \qquad \frac{\partial L}{\partial s_2} = \frac{B}{2C^{1/2}}, \qquad \frac{\partial L}{\partial s_3} = \frac{1}{2}C^{1/2},$$

the relation

$$\xi = La + b,$$

where b is a fixed vector.

Using (48) and the equation which results from equating the two values of ξ, we may eliminate β, γ from the differential equations for α in (47). The resulting equations are found to have the integral

$$\frac{C^{1/2}}{A}\alpha = \left(-\frac{1}{2}L^2 - \frac{B}{AC^{1/2}}L + \frac{1}{8C}\right)a - \left(L + \frac{B}{AC^{1/2}}\right)b + c,$$

where c is a triple of constants.

We may now compute a, b, c in terms of α, β, γ:

$$(49) \qquad \begin{aligned} a &= 4\left(\frac{\partial L}{\partial s_1}\alpha + \frac{\partial L}{\partial s_2}\beta + \frac{\partial L}{\partial s_3}\gamma\right), \\ b &= \left(-4L\frac{\partial L}{\partial s_1} + \frac{B}{C}\right)\alpha + \left(-4L\frac{\partial L}{\partial s_2} + \frac{A}{2C}\right)\beta - 4L\frac{\partial L}{\partial s_3}\gamma, \\ c &= \left(-2L^2\frac{\partial L}{\partial s_1} + \frac{B}{C}L - \frac{A}{8C^{3/2}}\right)\alpha \\ &\quad + \left(-2L^2\frac{\partial L}{\partial s_2} + \frac{A}{2C}L + \frac{B}{4C^{3/2}}\right)\beta + \left(-2L^2\frac{\partial L}{\partial s_3} - \frac{1}{4C^{1/2}}\right)\gamma, \end{aligned}$$

and hence obtain the conditions

$$(50) \qquad \begin{matrix} (a\,|\,a) = 0, & (b\,|\,b) = -1, & (c\,|\,c) = 0, \\ (b\,|\,c) = 0, & (c\,|\,a) = -1, & (a\,|\,b) = 0, \end{matrix} \qquad (a\;b\;c) = -1,$$

which insure that α, β, γ are mutually perpendicular unit vector fields with the disposition of the axes.

Integration of equations (49) yields the relations

$$(51) \qquad \begin{aligned} (a\,|\,x) + a_0 &= 4L, \\ (b\,|\,x) + b_0 &= -2L^2 - \frac{1}{2C}, \\ (c\,|\,x) + c_0 &= -\frac{2}{3}L^3 - \frac{1}{2C}L - \frac{B}{3AC^{3/2}}, \end{aligned}$$

where a_0, b_0, c_0 are arbitrary constants.

The directional derivatives of the function u defined by the equations

$$(52) \qquad \frac{2B}{A} = u, \qquad \frac{2C}{A} = \epsilon(1 + u^2)^{1/2}, \qquad (\epsilon = \pm i),$$

are $-6C^2/A, 0, 0$. Furthermore, it follows from (9) and (46) that $e^V = k(A^2C)^{1/3}$, where k is a constant, not zero. Comparison of the derivatives of U in (8) with those of u shows that U is a function of u. In particular, if we take $k = (2/\epsilon)^{1/3}$,

$$(53) \qquad U = \frac{2}{3}\int \frac{du}{(1 + u^2)^{5/6}}.$$

Denoting the left-hand sides of (51) by $a(x)$, $b(x)$, and $c(x)$, setting $B/A = u/2$, and eliminating L and C, we obtain the equation

$$(54) \qquad \{[a(x)]^2 + 8b(x)\}^3 u^2 + \{[a(x)]^3 + 12a(x)b(x) - 48c(x)\}^2 = 0,$$

which with (53) and the conditions (50) define the general solution U to within a multiplicative constant.

Without loss of generality we may take $a_0 = b_0 = c_0 = 0$ and as a, b, c the triples

$$(55) \qquad 0,\ 2,\ 2i, \qquad -i,\ 0,\ 0, \qquad 0,\ -\tfrac{1}{4},\ \tfrac{1}{4}i.$$

Equation (54) then assumes the normal form

$$(54a) \qquad (z^2 - 2ix_1)^3 u^2 + (z^3 - 3ix_1 z + \tfrac{3}{2}\bar{z})^2 = 0,$$

where $z = x_2 + ix_3$ and $\bar{z} = x_2 - ix_3$.

If in (53) we set $1 + u^2 = 1/y^3$, we find that

$$U = -\int \frac{dy}{(1 - y^3)^{1/2}}.$$

Thus, the function U involves an elliptic integral. The surfaces $U = \text{const.}$ are, however, obviously algebraic.

C. The minimal surfaces

9. **The imaginary surfaces** IIa. We have to do here with equations (28) where $a_0 = b_0 = c_0 = 0$ and a, b, c have the values (30). When we set $U = u$, $1/C = v$, $B/C^2 = w$, these equations become*

$$(56) \qquad x_1 = i(2uvw + v), \quad z = vw, \quad \bar{z} = 4u^2vw + 4uv + \cot^{-1}(w/v), \quad vw \neq 0,$$

and represent a transformation from the coordinates (x_1, x_2, x_3), or $(x, z, \bar{z})$, to curvilinear coordinates (u, v, w). Corresponding to a generic point $(x, z, \bar{z})$

* Throughout the paper, $z = x_2 + ix_3$ and $\bar{z} = x_2 - ix_3$.

there are two sets of values of the curvilinear coordinates, namely, (u, v, w) and $(u+1/w, -v, -w)$. In other words, the transformation $u'=u+1/w$, $v'=-v$, $w'=-w$ has the effect of the identity.

From (56) we find, as the equations of the three families of parametric surfaces,

$$(57)\qquad \begin{aligned} S:&\quad (ix_1+2uz)^2 = z\tan(4iux_1+4u^2z+\bar{z}),\\ S':&\quad z = v^2\cot((x\mid x)/z+v^2/z),\\ S'':&\quad z = w^2\tan((x\mid x)/z+z/w^2). \end{aligned}$$

Since $u=U$, the surfaces S are actually the minimal surfaces $U=$const., and inasmuch as $v=1/C$ and C is, according to §6, a function of V, the surfaces S' are actually the surfaces $V=$const. Moreover, since

$$(58)\qquad (x_u\mid x_v) = 0,\qquad (x_u\mid x_w) = 2v^2,\qquad (x_v\mid x_w) = 0,$$

the surfaces S' cut the surfaces S'', as well as the surfaces S, orthogonally. The curves $u=$const., $v=$const. are, then, the curves C_3, and the curves $u=$const., $w=$const. are the curves C_2. The curves $v=$const., $w=$const., which, according to (57), are the parabolic circles in which the isotropic planes $z=$const. intersect the spheres $(x\mid x)=$const., are not, however, the curves C_1.

THEOREM 2. *The system of surfaces S, S', S'' admits the one-parameter group of rigid motions into itself for which the parabolic circles* $v=$const., $w=$const. *are the path curves. Each surface S', and each surface S'', admits this group of motions into itself, and the minimal surfaces S, which are permuted by it, are all congruent to the surface S_0: $u=0$, namely,* $x_1^2+z\tan\bar{z}=0$.

The surfaces S all contain the isotropic line L: $x_1=z=0$ and are tangent all along L to the isotropic plane Π: $z=0$. The group of motions in question consists of the "rotations" about L and may be represented by the equations

$$x_1' = x_1+2icz,\qquad z'=z,\qquad \bar{z}' = -4icx_1+4c^2z+\bar{z},$$

where c is the parameter. Since $(x'\mid x')=(x\mid x)$ and $z'=z$, the path curves of the group are actually the parabolic circles described.

The equations of the group in terms of the curvilinear coordinates, as found from (56), are, to within the identical transformation $u'=u+1/w$, $v'=-v$, $w'=-w$,

$$u' = u+c,\qquad v'=v,\qquad w'=w.$$

Hence, the theorem is proved.

From (56) follows the relation

$$ix_1+2uz+v = 0.$$

Hence, the curves C_3 are plane curves, and those on a specific surface S lie in parallel planes. In particular, the curves C_3 on the surface S_0 lie in the planes $x_1 = iv$ and have in these planes the equation $z = v^2 \cot \bar{z}$, and those on an arbitrary surface S are congruent to them.

Inasmuch as

$$iwx_1 + (1 + 2uw)z = 0,$$

the curves C_2 are plane curves lying in the Euclidean planes which pass through L. The general curve C_2 on S_0 is the intersection of the plane $iwx_1 + z = 0$ with the cylinder $z = w^2 \tan \bar{z}$ and the curves C_2 on the general surface S are congruent to those on S_0.

Using (58) and the formulas

$$(59) \qquad (x_u \mid x_u) = -4v^2w^2, \qquad (x_v \mid x_v) = (x_w \mid x_w) = -v^2/(v^2 + w^2),$$

we find that the curves C_1 are the curves $v = \text{const.}$, $2uv + \cot^{-1}(w/v) = \text{const.}$ Since, by (6), C is the torsion of these curves, it follows from (19) that they are twisted curves of constant torsion, and that this torsion is the same for all of them which lie on a given surface S'.

According to (59), the curves C_2 and C_3 on a surface S form an isometric system and v, w are isometric parameters.

10. **The imaginary surfaces** IIb. Consider the equations (40), where B and K are given by (43) in terms of U and V, a, b, c have the values (45), and $a_0 = b_0 = c_0 = 0$. According to (31), $B/2A = \pm i$. We may take $B/2A = i$, since, if $B/2A = -i$, changing the sign of W would yield the same results. If, then, U, V, W are replaced by $1/u$, v, w, the equations become

$$(60) \qquad x_1 = uvw, \qquad z = -uv, \qquad \bar{z} = uvw^2 + v^3/3u^3, \qquad uv \neq 0.$$

According to the last paragraph of §7, the parametric surfaces for the curvilinear coordinates (u, v, w) form a triply orthogonal system whose curves of intersection are the curves C_1, C_2, C_3. Corresponding to a generic point $(x, z, \bar{z})$ of space there are six sets of values of the curvilinear coordinates, of the form $(ku, v/k, w)$, where k takes on in turn the sixth roots of unity. Thus, the transformations $u' = ku$, $v' = v/k$, $w' = w$, $k^6 = 1$, have the same effect as the identity.

We find from (60), as the equations of the three families of parametric surfaces,

$$(61) \qquad \begin{aligned} S&: \quad z^4 + 3u^6(x \mid x) = 0, \\ S'&: \quad 3z^2(x \mid x) + v^6 = 0, \\ S''&: \quad x_1 + wz = 0. \end{aligned}$$

The minimal surfaces S are algebraic surfaces of the fourth order. Each surface contains the isotropic line L: $z=x_1=0$ and is tangent all along L to the isotropic plane Π: $z=0$. The line at infinity in Π is a cuspidal edge of the surface with the plane at infinity as the cuspidal tangent plane; in particular, the point at infinity on L is a triple point at which the plane at infinity counts twice, and the plane Π once, as tangent planes. The origin O (on L) is a conical point with the isotropic cone at O as the tangent cone.

The surfaces S' are also surfaces of the fourth order. For each of them the line at infinity in Π is a cuspidal edge with the plane Π as the cuspidal tangent plane; in particular, the point at infinity on L is a triple point at which Π counts three times as tangent plane. There are no other singular points.

The surfaces S'' consist of the Euclidean planes through the line L.

THEOREM 3. *The triply orthogonal system of surfaces admits a two-parameter group of rigid motions into itself, consisting of the* ∞^1 *one-parameter groups of rotations about the lines of the pencil of lines which lies in the plane* Π *and has its vertex at* O. *Each surface of the system admits at least a one-parameter group of rigid motions into itself and each two surfaces of the same family are congruent.*

The one-parameter group of rotations about L has the equations

$$x_1' = x_1 - cz, \qquad z' = z, \qquad \bar{z}' = 2cx_1 - c^2 z + \bar{z},$$

where c is the parameter. The corresponding equations in the curvilinear coordinates, to within one of the identical transformations, are

$$u' = u, \qquad v' = v, \qquad w' = w + c. \tag{62}$$

Hence each surface S, and each surface S', is carried into itself by the group of rotations about L. The path curves are the curves C_3 in which the surfaces S and S' intersect, namely, the parabolic circles in which the planes $z=$const. cut the sphere $(x|x)=$const.

The parabolic circles C_3 are the orthogonal trajectories of the planes S''. Since $w=-x_1/z$ is a harmonic function, these planes, too, constitute a family of harmonic minimal surfaces. The lines of flow are the curves C_3 and hence are the path curves of a one-parameter group of rigid motions.

One of the Euclidean lines through O in the plane Π is the x_1 axis. The one-parameter group of rotations about this axis has the equations $x_1'=x_1$, $z'=e^{\theta i}z$, $\bar{z}'=e^{-\theta i}\bar{z}$, or, in terms of the curvilinear coordinates, to within an identical transformation,

$$u' = e^{(2/3)\theta i}u, \qquad v' = e^{(1/3)\theta i}v, \qquad w' = e^{-\theta i}w. \tag{63}$$

The product of the general transformation (63) and the general transformation (62) is the *two-parameter* group of rigid motions,

$$u' = e^{(2/3)\theta i}u, \qquad v' = e^{(1/3)\theta i}v, \qquad w' = e^{-\theta i}w + c, \tag{64}$$

which we shall think of as consisting of the rotations (62) about L and the transformations

$$u' = e^{(2/3)\theta i}u, \qquad v' = e^{(1/3)\theta i}v, \qquad w' - a = e^{-\theta i}(w - a). \tag{65}$$

For a fixed value of a these transformations are simply the rotations about the line through O with direction components 1, a, ai. For, they leave O and the plane $w=a$, or $x_1+az=0$, fixed and hence leave fixed every point of the line in question.

Thus the two-parameter group (64) actually consists of the ∞^1 one-parameter groups of rotations about the ∞^1 lines passing through O and lying in the plane Π. Obviously, every transformation of the group carries each family of surfaces of the triply orthogonal system into itself. Moreover, it is clear from (65) that, if two surfaces S, or two surfaces S', are given, there exists one rotation about each Euclidean axis which carries the one surface into the other. Thus, the theorem is completely established.

THEOREM 4. *The curves C_1 are plane quartic curves which lie in the planes S'' and are all congruent to the curve $z^3\bar{z}=1$ in the plane $x_1=0$. The curves C_2 are plane cubics which lie in the planes S'' and are all congruent to the curve $z^3=\bar{z}$ in the plane $x_1=0$. The curves C_3 are the parabolic circles in which the isotropic planes parallel to Π cut the spheres with center at O.*

The curves C_1, which are the lines of flow in the physical problem, are the intersections of the planes S'' with the surfaces S' and hence are plane quartics. Since they are given by $v=$const., $w=$const., and since $v\neq 0$, there exists a unique rigid motion (64) which carries a given one of them into a second. They are, then, all congruent to the particular one $v=(-3)^{1/6}$, $w=0$, which lies in the plane $x_1=0$ and has the equation $z^3\bar{z}=1$.

Since each of the surfaces S contains the line L, the curves C_2 in which they are met by the planes S'' are plane cubics. These curves are given by $u=$const., $w=$const., where $u\neq 0$. It follows from (64) that they are all congruent to the particular one $u=(-1/3)^{1/6}$, $w=0$, which lies in the plane $x_1=0$ and has the equation $z^3=\bar{z}$.

The facts concerning the parabolic circles C_3 have already been established. Each two of them which lie on the same sphere are congruent, while two lying on different spheres are not.

It should perhaps be remarked that the plane cubics and the parabolic circles are the lines of curvature on the minimal surfaces S.

11. **The imaginary surfaces** IIIa. We are concerned in this case with equations (51), where $a_0=b_0=c_0=0$ and a, b, c have the values (55). When we set $2B/A=u$, $1/C=v^2$, $L=w/2$, the equations become

$$(66)\quad x_1 = -\tfrac{1}{2}i(v^2+w^2), \quad z = w, \quad \bar{z} = \tfrac{2}{3}uv^3 + v^2w + \tfrac{1}{3}w^3, \quad 1+u^2 \neq 0.$$

For a generic point of space there are two sets of values of the curvilinear coordinates thus introduced, namely, (u, v, w) and $(-u, -v, w)$. Thus, the transformation $u'=-u$, $v'=-v$, $w'=w$ is in effect the identity.

The equations of the three families of parametric surfaces are found to be

$$(67)\quad \begin{aligned} S:&\quad u^2(2ix_1 - z^2)^3 = (z^3 - 3ix_1z + \tfrac{3}{2}\bar{z})^2,\\ S':&\quad 2ix_1 - z^2 = v^2,\\ S'':&\quad z = w. \end{aligned}$$

According to §8, u is a function of U. Hence, the surfaces S are actually the minimal surfaces $U=\text{const}$. The surfaces S' are *not*, in this case, the survaces $V=\text{const}$.

The surfaces S' are parabolic cylinders which are tangent to the plane at infinity along the ideal line in the isotropic plane Π: $z=0$ and whose rulings are parallel to the isotropic line L: $z=x_1=0$. The surfaces S'' are the planes parallel to Π.

The w-curves are parabolic helices* lying on the parabolic cylinders S', except for the curve $v=0$:

$$(68)\quad x_1 = -\tfrac{1}{2}iw^2, \quad z = w, \quad \bar{z} = \tfrac{1}{3}w^3,$$

which is an isotropic cubic. The v-curves are plane cubics lying in the isotropic planes parallel to Π except for those for which $u=0$, which are Euclidean straight lines. Finally, the u-curves are the isotropic lines parallel to L.

The surfaces S for which $u\neq 0$ are algebraic surfaces of the sixth order. For each of them, the isotropic cubic K defined by (68) is a cuspidal edge, with the isotropic osculating planes of K as the cuspidal tangent planes, and the points of the line at infinity in the plane Π are all singular points, with the plane at infinity counting at least three times as tangent plane. The parametric curves on these surfaces are the parabolic circles and plane cubics just mentioned.

The surface $u=0$ is the cubic surface

$$z^3 - 3ix_1z + \tfrac{3}{2}\bar{z} = 0,$$

counted twice. The cubic surface has the line at infinity in Π as a double

* A parabolic helix is a curve whose curvature and torsion are constant and in the ratio $\pm i$. The tangent indicatrix is a parabolic circle, lying in this case in a plane parallel to Π.

line, the two tangent planes at the ideal point in the direction a, 1, i being $iaz=1$ and the plane at infinity. Furthermore, it contains the isotropic cubic K as an asymptotic line. The remaining asymptotic lines are parabolic circles (w-curves) and straight lines (the special v-curves described above). Thus, this surface S is a *ruled* minimal surface, a parabolic helicoid.

The surfaces S for which $1+u^2=0$ have been excluded. They are the same surface, namely, the isotropic developable which is the tangent surface of the isotropic cubic K.

The differential coefficients of the general surface S, referred to v, w as parameters, where $D^2=EF-G^2$, are found to be

$$(69)\quad E = -v^2, \quad De = uv^2, \quad F = uv^2, \quad Df = v^2, \quad G = v^2, \quad Dg = -uv^2.$$

Since these coefficients are independent of w, the surface S admits the one-parameter group of rigid motions

$$(70)\qquad u' = u, \qquad v' = v, \qquad w' = w + c$$

into itself. In terms of the coordinates $(x, z, \bar{z})$, the equations of this group are

$$(71)\qquad x_1' = x_1 - icz - \tfrac{1}{2}ic^2, \qquad z' = z + c, \qquad \bar{z}' = 2icx_1 + c^2z + \bar{z} + \tfrac{1}{3}c^3.$$

Since each of these motions leaves fixed only the point at infinity in the direction of the isotropic line L and the tangent to the absolute at this point, the group may properly be described as the group of screw motions about the tangent to the absolute in question. Since $2ix_1-z^2$ is an absolute invariant, the path curves lie on the parabolic cylinders $v=\text{const}$. The path curves on the cylinders for which $v\neq 0$ are the parabolic helices (the w-curves), whereas those on the cylinder $v=0$ are isotropic cubics similar to the curve K.

THEOREM 5. *Each surface of the family of minimal surfaces is carried into itself by the group of screw motions about the tangent to the absolute at the ideal point of the isotropic line L. The path curves are the parabolic helices on the surface and the isotropic cubic K.*

From §8 and the relation $1/C=v^2$, it follows that u, v are functions of U, V. Hence, the w-curves (the parabolic helices and K) are actually the curves C_3.

The curves C_2 are the orthogonal trajectories of the w-curves on the surfaces S. As such, they are found to have the equation $uv+w=\text{const}$. Those on a specific surface S are all congruent, inasmuch as they cut the path curves of the rigid deformation of S into itself orthogonally. Hence, it suffices to consider the family of curves, one on each of the surfaces S, which is defined by the equation $w=-uv$. It is found that these curves are helices which lie on cubic cylinders all congruent to the cylinder $\bar{z}=z^3$ and have on these cylinders varying pitches. When the curves are moved along the surfaces S by the group of screw motions, the point at infinity in the direction of the rulings of the cylinders on which they lie traces the axis of the group (the ideal line in the plane Π), since in its original position it is the ideal point in the direction 1, 0, 0, and hence a point on this axis. Finally, it may be shown that the curves C_2 are all tangent to the isotropic cubic K.*

On each surface S there is a single family of lines of curvature which covers the surface twice, and all of its members are tangent to the isotropic cubic K. For, the lines of curvature on an arbitrary surface S are the curves $r=\text{const.}$ and $s=\text{const.}$, where $w+iv=r$, $w-iv=s$, and since the identical transformation $u'=-u$, $v'=-v$, $w'=w$ interchanges r and s, the two families coincide. Furthermore, the locus $r=s$ is the curve $v=0$ or K, so that the tangency of the lines of curvature with K is indicated and readily verified.

Inasmuch as the lines of curvature on a specific surface S are all congruent, it suffices to consider those lines of curvature, one on each surface S, which are given by $r=0$ or $s=0$. It turns out that these lines of curvature are plane cubics all lying in the plane $x_1=0$ and all congruent to the cubic $\bar{z}=z^3$ in this plane. When they are moved along the surfaces S by the group of screw motions, their common plane envelopes the invariant parabolic cylinder $2ix_1-z^2=0$, since in its original position it is a tangent plane to this cylinder. All the lines of curvature on the surfaces S are, of course, congruent.

Employing the method of §9, we find as the parametric representation of the curves C_3, in terms of the parameter y of §8,

$$u=\frac{(1-y^3)^{1/2}}{y^{3/2}},\qquad v=ky^{1/2},\qquad w=\frac{k}{2}\int\frac{y\,dy}{(1-y^3)^{1/2}}+l,$$

where k and l are arbitrary constants. Thus, though the equipotential surfaces are algebraic, the lines of flow of the physical problem are transcendental.

D. Solution of the differential system

12. **First special case. Change of variables.** The crux of our problem, namely, the deduction of the solutions (I), (II), (III) of equations (12) and the proof that these are the only solutions, must finally be met. The three conditions of integrability on the derivatives of A given by (12a) yield two new relations, namely the finite relation

$$4ACE-4ABF+4B(B^2+C^2)=A^2B, \tag{72}$$

* In this discussion we have tacitly excluded the curves C_2 on the surface $u=0$. As we have already seen, these curves are straight lines.

and the differential equation

$$2\frac{\partial}{\partial s_3}(B^2+C^2) - A\left(\frac{\partial E}{\partial s_2}+\frac{\partial F}{\partial s_3}\right) = 4E(B^2+C^2). \tag{73}$$

First special case. Suppose that $B^2+C^2=0$. Differentiating with respect to s_1 and using (12b), we find that $ACE-ABF=0$, and therefore conclude, from (72), that $B=0$ and hence $C=0$. It follows from (12b) that $E=0$ and $F=0$. Thus, solution (Ib) is obtained.

Change of variables. We may assume henceforth that $B^2+C^2\neq 0$. Equations (12b) and (73) suggest the substitutions

$$2B = N^{1/2}\sin\phi, \quad 2C = N^{1/2}\cos\phi, \quad 4(B^2+C^2) = N. \tag{74}$$

Relation (72) becomes

$$4AE\cos\phi - 4AF\sin\phi + N\sin\phi = A^2\sin\phi,$$

and, when we adjoin the equation

$$4AE\sin\phi + 4AF\cos\phi - N\cos\phi = M,$$

the two relations yield values for E and F:

$$4AE = M\sin\phi + A^2\sin\phi\cos\phi, \quad 4AF = M\cos\phi - A^2\sin^2\phi + N. \tag{75}$$

Thus, we have introduced the new set of unknowns A, M, N, ϕ, in terms of which B, C, E, F are given by (74) and (75).

Equations (12a) become

$$\begin{aligned} 4\frac{\partial A}{\partial s_1} &= -2AN^{1/2}\sin\phi, \\ 4\frac{\partial A}{\partial s_2} &= A^2\sin^2\phi - M\cos\phi + 4A^2 + N, \\ 4\frac{\partial A}{\partial s_3} &= (M + A^2\cos\phi)\sin\phi, \end{aligned} \tag{76a}$$

and the equations resulting from (12b) are

$$\frac{\partial N}{\partial s_1} = A^2N^{1/2}\sin\phi, \quad 2N^{1/2}\frac{\partial\phi}{\partial s_1} = -M - 2N\cos\phi, \tag{77a}$$

$$2N\frac{\partial\phi}{\partial s_2} + \frac{\partial N}{\partial s_3} = 4NE, \quad 2N\frac{\partial\phi}{\partial s_3} - \frac{\partial N}{\partial s_2} = 4NF, \tag{77b}$$

where E and F are given by (75).

From the relations obtained by making the foregoing substitutions in the *first two* equations in (12c), we obtain the equations

$$2A\frac{\partial N}{\partial s_2} = M^2 + A^2M\cos\phi + 2N(A^2 - N), \tag{78}$$

$$2N^{1/2}\frac{\partial M}{\partial s_1} + 2A\frac{\partial N}{\partial s_3} = -A^2M\sin\phi. \tag{79}$$

It is clear from (77) and (78) that, in order to find finite values for all the derivatives of N and ϕ, we need the value of $\partial N/\partial s_3$. This derivative enters into the condition of integrability (N; s_1, s_2), which is found to reduce to

$$(M - N\cos\phi)\frac{\partial N}{\partial s_3} + (A^2 - N)\frac{\partial N}{\partial s_2}\sin\phi + 4AN^2\sin\phi = 0. \tag{80}$$

Evidently, two cases arise, according as $M-N\cos\phi$ vanishes or not.

13. **Second special case.** If

$$M - N\cos\phi = 0, \tag{81}$$

it follows immediately from (78) and (80), since $N\neq 0$, that

$$\sin\phi(A^2 + N)[(A^2 - N)\cos^2\phi + 2(A^2 + N)] = 0.$$

(i) If $A^2+N=0$, we have from (75) and (81) that $E=0$ and $F=-A/2$. Substitution to these values in equations (12) leads immediately to solution (III).

(ii) If $\sin\phi=0$, the second equation in (77a), in conjunction with (81), yields $N=0$, a contradiction.

(iii) Suppose, finally, that

$$(A^2 - N)\cos^2\phi + 2(A^2 + N) = 0. \tag{82}$$

Differentiating this relation with respect to s_1, we find, since $N\sin\phi\neq 0$, that

$$(A^2 - 3N)\cos^2\phi = 0.$$

If $\cos\phi=0$, it follows from (82) that $A^2+N=0$ and we are thus led to solution (IIIb).

If $A^2-3N=0$, differentiation with respect to s_1 gives rise to an immediate contradiction.

14. **The derivatives in the general case.** We assume henceforth that $AN(M-N\cos\phi)\neq 0$, and introduce the quantity

$$P = \frac{S\sin\phi}{M - N\cos\phi}, \tag{83}$$

where

$$S = M^2 - 2Y^2 - A^4 \cos^2 \phi, \tag{84}$$

and

$$Y = A^2 + N = A^2 + 4(B^2 + C^2). \tag{85}$$

Since $M - N \cos \phi \neq 0$, equations (77), (78), and (80) may be solved for the derivatives of N and ϕ. From (79) follows then a finite value for $\partial M/\partial s_1$. Finally, when the substitutions (74) and (75) are carried out in (73) and the last equation in (12c), values for $\partial M/\partial s_2$ and $\partial M/\partial s_3$ result. Thus, the values of all twelve derivatives of the four unknowns A, M, N, ϕ are obtained, and from them may be computed the derivatives of Y and S.

The derivatives of A are already listed in (76a). Those of the other quantities are

$$\begin{aligned} \frac{\partial N}{\partial s_1} &= A^2 N^{1/2} \sin \phi, \\ 2A \frac{\partial N}{\partial s_2} &= M^2 + A^2 M \cos \phi + 2N(A^2 - N), \\ 2A \frac{\partial N}{\partial s_3} &= - A^2(M + A^2 \cos \phi) \sin \phi + P, \end{aligned} \tag{76b}$$

$$\begin{aligned} 2N^{1/2} \frac{\partial \phi}{\partial s_1} &= - M - 2N \cos \phi, \\ 4AN \frac{\partial \phi}{\partial s_2} &= (A^2 + 2N)(M + A^2 \cos \phi) \sin \phi - P, \\ 4AN \frac{\partial \phi}{\partial s_3} &= M^2 + (A^2 + 2N)M \cos \phi + 2A^2 N \cos^2 \phi, \end{aligned} \tag{76c}$$

$$\begin{aligned} 2N^{1/2} \frac{\partial M}{\partial s_1} &= A^4 \sin \phi \cos \phi - P, \\ 4AN \frac{\partial M}{\partial s_2} &= (3A^2 + 10N + A^2 \cos^2 \phi)A^2 M \\ &\quad + (5A^2 + 4N + A^2 \cos^2 \phi)A^4 \cos \phi \\ &\quad + [M + 2(A^2 - N) \cos \phi]S + \sin \phi (2N - A^2)P, \\ 4AN \frac{\partial M}{\partial s_3} &= - A^4(M + A^2 \cos \phi) \sin \phi \cos \phi + (3M + A^2 \cos \phi)P, \end{aligned} \tag{76d}$$

(76e) $$\frac{\partial Y}{\partial s_1} = 0, \qquad 2A\frac{\partial Y}{\partial s_2} = S + 7A^2Y, \qquad 2A\frac{\partial Y}{\partial s_3} = P,$$

$$N^{1/2}\frac{\partial S}{\partial s_1} = -MP,$$

(76f) $$2AN\frac{\partial S}{\partial s_2} = [6A^2(M + A^2\cos\phi)\cos\phi + (6A^2 - 8N)Y]A^2Y + kS + lP,$$

$$2AN\frac{\partial S}{\partial s_3} = (3M^2 + A^2M\cos\phi - 4NY)P,$$

where k and l are functions of A, N, M, ϕ whose explicit values we shall not need.

We distinguish two cases, according as P is, or is not, zero.

15. **Third special case.** If $P=0$, it follows from (83) that $S\sin\phi=0$.

(i) If $\sin\phi=0$, then $B=0$ and since $N\neq 0$, $C\neq 0$. From (75), $E=0$, and from (12b), $AF+C^2=0$. We are thus led to solution (Ia).

(ii) If $\sin\phi\neq 0$ and $S=0$, it follows from the equation $\partial S/\partial s_2=0$ that $YZ=0$, where Z may be read off from (76f).

Suppose that $Y=0$. From $S=0$, it follows, since $M-N\cos\phi\neq 0$, that $M=A^2\cos\phi$. Hence, it is found from (75) and (74) that $AE=-2BC$ and $AF=2B^2$, and we arrive at solution (II).

If $Y\neq 0$, then

$$Z \equiv 3A^2(M + A^2\cos\phi)\cos\phi + (3A^2 - 4N)Y = 0.$$

Eliminating M from this equation and the equation $S=0$, we obtain the relation

$$T \equiv (3A^2 - 4N)^2Y - 42A^4N\cos^2\phi = 0.$$

By means of $S=0$, $Z=0$, $T=0$, it follows from the equation $\partial T/\partial s_2=0$ that $3A^2-4N=0$ or $3A^2+10N=0$. But differentiation of either of these equations with respect to s_1 leads to the contradiction $\sin\phi=0$. Thus, the discussion of this special case is complete.

16. **General case. Conclusion.** When $P\neq 0$, then $Y\neq 0$, for if Y were zero, it would follow from (76e) that $P=0$. Consequently we have in the present case $ANSY(M-N\cos\phi)\sin\phi\neq 0$.

The integrability conditions on A we know to be satisfied. Those on N may be replaced by the conditions on Y. The conditions $(Y;\, s_1, s_2)$ and $(\phi;\, s_1, s_3)$ are found to be identities, and the conditions $(Y;\, s_1, s_3)$ and $(\phi;\, s_1, s_2)$ both reduce to the equation

$$\frac{\partial P}{\partial s_1} = 0, \tag{86a}$$

which, when P is replaced by its value from (83), can be written in the form

$$P \sin \phi = M^2 \cos \phi + (N - 2A^2 \sin^2 \phi)M + A^4 \sin^2 \phi \cos \phi - 2N^2 \cos \phi. \tag{86b}$$

The condition of integrability $(\phi; s_2, s_3)$ reduces, by (86b), to

$$AN \frac{\partial \log P}{\partial s_3} \sin \phi = M^2 \cos \phi + (N - A^2 \sin^2 \phi)M - 2N^2 \cos \phi.$$

On the other hand, we find that the value of $\partial \log P/\partial s_3$, computed from (83), becomes, by virtue of (86b),

$$A(M - N \cos \phi) \frac{\partial \log P}{\partial s_3} = (M^2 - 2NY) \sin \phi.$$

Eliminating $\partial \log P/\partial s_3$, we obtain the relation

$$\begin{aligned} M^3 \cos \phi - A^2 \sin^2 \phi\, M^2 + (A^2 N \sin^2 \phi - 3N^2)M \cos \phi \\ + 2A^2 N^2 \sin^2 \phi + 2N^3 = 0. \end{aligned} \tag{87}$$

If we differentiate (86b) with respect to s_1, remembering that $\partial P/\partial s_1 = 0$, and eliminate P from the resulting equation by means of (86b), we get

$$\begin{aligned} M^3 \cos 2\phi - 8A^2 \sin^2 \phi \cos \phi\, M^2 - (3N^2 \cos 2\phi + A^2 r)M \\ + (2N^3 + A^2 s) \cos \phi = 0, \end{aligned} \tag{88}$$

where r and s are polynomials in A, N, $\cos^2 \phi$.

When the value of P from (83) is substituted in (86b) and the resulting equation is cleared of fractions, there is obtained an equation of the form $H = 0$, where H is a cubic polynomial in M whose leading term is $M^3 \cos \phi$. Subtraction of this equation from (87) yields the simple relation

$$M^2 - YM \cos \phi - 2NY = 0. \tag{89}$$

LEMMA 1. *The condition on A and N that the three equations* (87), (88), (89), *considered as equations in M and ϕ, be compatible is not an identity in A and N.*

LEMMA 2. *There exists no functional relation between A and N.*

Once these contradictory lemmas are proved it will follow that no further solution of the differential system exists.

To prove Lemma 2, we assume that A and N are functionally related: $\Phi(A, N) = 0$. Inasmuch as $\partial\Phi/\partial s_1 = 0$ and $\partial Y/\partial s_1 = 0$, it follows, since $\partial A/\partial s_1 \neq 0$, that $\Phi \equiv f(Y)$. But, if we differentiate the relation $f(Y) = 0$ with respect to s_3, we get $P = 0$, a contradiction.

To prove that the condition on A and N described in Lemma 1 is not an identity in A and N, it suffices to show that it is not an identity in N when $A=0$. In this case, the expressions on the left-hand sides of (87), (88), and (89) become

$$a \equiv M^3 \cos\phi - 3N^2M\cos\phi + 2N^3,$$
$$b \equiv M^3 \cos 2\phi - 3N^2M\cos 2\phi + 2N^3\cos\phi,$$
$$c \equiv M^2 - NM\cos\phi - 2N^2,$$

and satisfy the relations

$$a\cos 2\phi - b\cos\phi \equiv -2N^3\sin^2\phi, \tag{90a}$$

$$a \equiv (M\cos\phi + N\cos^2\phi)c + N^2 d, \tag{90b}$$

where

$$d \equiv M(\cos^2\phi - 1)\cos\phi + 2N(\cos^2\phi + 1).$$

From (90a) it follows that, if the equations $a=0$, $b=0$ have a common solution in M and ϕ, the solution must be $M=M$, $\sin\phi=0$. If this solution is also to satisfy the equation $c=0$, it must, by (90b), satisfy $d=0$. But, when $\sin\phi=0$, then $d\equiv 4N$, and the condition $N=0$ is not an identity in N. Thus, Lemma 1 is established.

E. Special cases

17. **The special case $\Delta_1 V=0$.** When $\Delta_1 V=0$ and V is not a constant, ∇V is an isotropic vector field which, since $\Delta_1(U, V)=0$, is "orthogonal" to α. There then exists a second isotropic vector function η such that the three vector fields

$$\alpha = \frac{\nabla U}{(\Delta_1 U)^{1/2}}, \qquad \xi = \nabla V, \qquad \eta \tag{91}$$

enjoy the relations

$$\begin{matrix} (\alpha \mid \alpha) = 1, & (\xi \mid \xi) = 0, & (\eta \mid \eta) = 0, \\ (\alpha \mid \xi) = 0, & (\alpha \mid \eta) = 0, & (\xi \mid \eta) = -i, \end{matrix} \qquad (\alpha\ \xi\ \eta) = 1, \tag{92}$$

provided merely that, to insure $+1$ as the value of $(\alpha\ \xi\ \eta)$, we are permitted, if necessary, to change the sign of U.

The nonholonomic derivatives corresponding to the three vector fields α, η, ξ are

$$\frac{\partial f}{\partial t_1} = \sum \alpha_i \frac{\partial f}{\partial x_i}, \qquad \frac{\partial f}{\partial t_2} = \sum \eta_i \frac{\partial f}{\partial x_i}, \qquad \frac{\partial f}{\partial t_3} = \sum \xi_i \frac{\partial f}{\partial x_i}. \tag{93}$$

In terms of them, the gradient of the function f has the form

$$(94) \qquad \nabla f = \alpha \frac{\partial f}{\partial t_1} + i\xi \frac{\partial f}{\partial t_2} + i\eta \frac{\partial f}{\partial t_3},$$

as is readily shown by means of the relations

$$\widetilde{\xi\eta} = \alpha, \qquad \widetilde{\alpha\xi} = i\xi, \qquad \widetilde{\eta\alpha} = i\eta.$$

It follows from (91) and (93) that

$$(95) \qquad \begin{aligned} &\frac{\partial U}{\partial t_1} = e^V, \qquad \frac{\partial U}{\partial t_2} = 0, \qquad \frac{\partial U}{\partial t_3} = 0, \\ &\frac{\partial V}{\partial t_1} = 0, \qquad \frac{\partial V}{\partial t_2} = -i, \qquad \frac{\partial V}{\partial t_3} = 0. \end{aligned}$$

The equations of variations of α, ξ, η, in terms of the nonholonomic derivatives, have, in view of relations (92), the general form

$$(96) \qquad \frac{\partial \alpha}{\partial t_j} = A_j\xi + B_j\eta, \qquad \frac{\partial \xi}{\partial t_j} = iB_j\alpha + C_j\xi, \qquad \frac{\partial \eta}{\partial t_j} = iA_j\alpha - C_j\eta,$$

where $j=1, 2, 3$.

From these equations are obtained the conditions of integrability for the nonholonomic derivatives. When these conditions are applied to the derivatives of U and V given in (95), the following relations result:

$$\begin{aligned} B_3 = 0, \qquad C_3 = 0, \qquad B_2 + C_1 = 0, \\ A_1 = 1, \qquad B_1 = 0, \qquad A_3 - B_2 = 0. \end{aligned}$$

To these is to be added, in accordance with (1), the condition $A_3+B_2=0$ which guarantees that $\Delta_2 U=0$. Consequently, all of the functions A_j, B_j, C_j vanish except A_2 and C_2, which we shall henceforth denote by A and C.

Equations (96) now become

$$(97) \qquad \begin{aligned} &\frac{\partial \alpha}{\partial t_1} = \xi, \qquad \frac{\partial \alpha}{\partial t_2} = A\xi, \qquad \frac{\partial \alpha}{\partial t_3} = 0, \\ &\frac{\partial \xi}{\partial t_1} = 0, \qquad \frac{\partial \xi}{\partial t_2} = C\xi, \qquad \frac{\partial \xi}{\partial t_3} = 0, \\ &\frac{\partial \eta}{\partial t_1} = i\alpha, \qquad \frac{\partial \eta}{\partial t_2} = iA\alpha - C\eta, \qquad \frac{\partial \eta}{\partial t_3} = 0, \end{aligned}$$

and the conditions of integrability reduce to

$$
(98)\qquad
\begin{aligned}
\frac{\partial}{\partial t_2}\frac{\partial f}{\partial t_1}-\frac{\partial}{\partial t_1}\frac{\partial f}{\partial t_2} &= -i\frac{\partial f}{\partial t_1}+A\frac{\partial f}{\partial t_3},\\
\frac{\partial}{\partial t_3}\frac{\partial f}{\partial t_2}-\frac{\partial}{\partial t_2}\frac{\partial f}{\partial t_3} &= \qquad\qquad -C\frac{\partial f}{\partial t_3},\\
\frac{\partial}{\partial t_1}\frac{\partial f}{\partial t_3}-\frac{\partial}{\partial t_3}\frac{\partial f}{\partial t_1} &= 0.
\end{aligned}
$$

Equations (97) are completely integrable if A and C satisfy the differential equations

$$
(99)\qquad \frac{\partial C}{\partial t_1}=0,\qquad \frac{\partial C}{\partial t_3}=0,\qquad \frac{\partial A}{\partial t_1}=C+i,\qquad \frac{\partial A}{\partial t_3}=0,
$$

and this system of equations, unlike the corresponding system in the general case, is readily integrated. In fact, C is an arbitrary function of V which we choose to write in the form $C=iF'(V)$, where the prime denotes differentiation, and A has, then, the value $A=i(1+F')e^{-V}U+e^{F}\Phi'$, where $\Phi(V)$ is a second arbitrary function of V.

It follows from (97) that

$$
(100)\qquad \alpha-(e^{-V}U+ie^{F}\Phi)\xi=a,\qquad e^{F}\xi=b,
$$

where a and b are constant vector fields, which, on account of (92), must satisfy the relations

$$
(101)\qquad (a\,|\,a)=1,\qquad (b\,|\,b)=0,\qquad (a\,|\,b)=0.
$$

The nonholonomic derivatives of the functions whose gradients are the expressions on the left-hand sides of equations (100) are obtainable by comparison of these expressions with (94). Thus, the integrals of the equations are found to be

$$
e^{-V}U-i\int e^{F}\Phi dV=(a\,|\,x)+a_0,\qquad \int e^{F}dV=(b\,|\,x)+b_0.
$$

Denoting $(a|x)+a_0$, $(b|x)+b_0$ by $a(x)$, $b(x)$, and eliminating V, we obtain the equation

$$
(102)\qquad U=f(b(x))a(x)+\phi(b(x)),
$$

where, since $F(V)$ and $\Phi(V)$ are arbitrary functions, $f(y)$ and $\phi(y)$ are arbitrary functions, the first of which is not a constant. This equation, subject to the attendant conditions (101), defines the general solution U in this special case. For, it is readily found that

(103) $\nabla U = fa + [f'a(x) + \phi']b, \quad \Delta_2 U = 0, \quad V = \log f, \quad \nabla V = (f'/f)b,$

whence the conclusion follows.

It is clear from (101) that there exists a rigid motion which transforms $-a(x)$ and $2b(x)$ respectively into x_1 and x_2+ix_3. By means of this rigid motion or, what is the same thing, by setting a_0 and b_0 equal to zero and taking a and b as the triples $-1, 0, 0$ and $0, 1/2^{1/2}, i/2^{1/2}$, we reduce equation (102) to the normal form

$$U = f_1(z)x_1 + \phi_1(z), \tag{102a}$$

where f_1 and ϕ_1 are arbitrary functions, the first of which is not a constant.

The surfaces $U=$const. defined by (102a) are cylinders with isotropic rulings parallel to the line L: $x_1=0$, $z=0$, except in the case $f_1(z)=1/z$, $\phi_1(z)=0$, when they consist of the Euclidean planes through L. The surfaces $V=$const. are always the isotropic planes $z=$const. parallel to L, and the lines of flow—the orthogonal trajectories of the surfaces $U=$const.—are parabolic circles lying in these planes and cut by them from spheres whose centers are in the plane $\bar{z}=0$. The group of rotations about L, or that about any line parallel to L, carries the totality of surfaces of all the families $U=$const. into itself.

We have thus far excluded the case in which V is a constant, and consequently have demanded that $f_1(z)$ be not a constant. As a matter of fact, the function

$$U = x_1 + g(z), \tag{104}$$

where $g(z)$ is arbitrary, is a solution of our problem, as is clear from (103). Moreover, it gives a normal form for all solutions in the case $V\equiv$const. For, since in this case $\Delta_1 U$ is a constant, not zero, the surfaces $U=$const. are geodesically parallel and, inasmuch as they are minimal, they must be Euclidean planes or non-isotropic minimal developables. A minimal developable has isotropic rulings and hence, if it has Euclidean tangent planes, must be a cylinder with isotropic rulings. Therefore, in any case, the parallel surfaces are given by $U=$const., where U is of the form (102a) subject to the condition $f_1(z)\neq 0$. But, since $\Delta_1 U$ must be constant, it follows from (103) that (102a) reduces to (104) and the proof is complete.

Hence, in case $\Delta_1 U$ is constant, not zero, the minimal surfaces $U=$const. consist of parallel planes or special parallel cylinders with isotropic rulings. There are no surfaces $V=$const., and the lines of flow in both cases are straight lines.

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