NETS AND GROUPS. II*

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In this note we continue the investigation of symmetry-properties of nets, in §8 of a previous paper.† We make free use of the results and definitions of the first five sections of this paper to which reference will be made by prefixing I.

11. In this section we analyze the following property of division-systems with unit.

(S) $x(yz)=1$ *if, and only if*, $(xy)z=1$.

This property may clearly be divided into the following two properties.

(S′) *If* $x(yz)=1$, *then* $(xy)z=1$.

(S″) *If* $(xy)z=1$, *then* $x(yz)=1$.

However the following proposition holds true.

THEOREM 11.1. *If D is a division-system with unit, then each of the properties* (S), (S′), (S″) *implies the others.*

Proof. Clearly it suffices to show that (S″) is a consequence of (S′). Thus let us assume that (S′) holds true, and that $(uv)w=1$. Since D is a division-system with unit, there exists one and only one element r so that $r[u(vw)]=1$. Applying (S′) twice we find that

$$1 = r[u(vw)] = (ru)(vw) = [(ru)v]w.$$

Since there exists one and only one solution of the equation $xw=1$ in D, it follows that $uv=(ru)v$, and that therefore $u=ru$, $r=1$, that is, $u(vw)=1$.

THEOREM 11.2. *Suppose that D is a division-system with unit, and that* $(S<G; r(X))$ *is a canonical representation of D. Then* (S) *is satisfied by D if, and only if,* $r(Z)^{-1}r(Y)^{-1}r(YZ)$ *is an element in S for any two elements Y and Z of D.*

REMARK. $r(YZ)r(Z)^{-1}r(Y)^{-1}$ is always contained in S. Thus our condition amounts to saying that D is not changed, if we consider left-cosets instead of right-cosets.

Proof. Suppose that X, Y, Z are three elements in D. Then $(XY)Z$ is

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† These Transactions, vol. 46 (1939), pp. 110–141.

represented by $r(X)r(Y)r(Z)$; and $X(YZ)$ is represented by $r(X)r(YZ) = r(X)r(Y)r(Z)r(Z)^{-1}r(Y)^{-1}r(YZ)$.

Suppose first that (S) is satisfied by D, and that Y, Z are any two elements in D. Since D is a division-system with unit, there exists one and only one element X in D so that $X(YZ)=1$; and this implies by (S) that both $r(X)r(Y)r(Z)$ and $r(X)r(YZ)$ are elements in S. As S is a subgroup of G, it follows that $r(Z)^{-1}r(Y)^{-1}r(YZ)$ is an element in S and this shows the necessity of our condition.

Suppose conversely that the condition of the theorem is satisfied. If $(XY)Z=1$, then $r(X)r(Y)r(Z)$ is an element in S; and it follows from our condition that $r(X)r(YZ)$ is an element in S, that is, that $X(YZ)=1$. Thus (S″) is satisfied by D; and it follows from Theorem 11.1 that (S) is satisfied by D. This completes the proof.

Remark. That condition (S) is satisfied by the so-called quasigroups has been proved by Bol.* That the converse does not hold true may be seen from the following example† of a *division-system with unit which satisfies condition* (S), *but which does not satisfy the rule*:

(+) *if* x^{-1} *signifies the solution of* $xy=1$, *then* $(uv)^{-1}=v^{-1}u^{-1}$.

That this last rule is satisfied by all the quasi-groups has been shown by Bol.‡

Denote by H the abelian group of order 4 which is not cyclic; and denote by Z a (cyclic) group of order 2. Then H contains apart from 1 three elements u, v, w of order 2; and Z contains 1 and z. We put $1=f(1, x)=f(x, 1)$ for every x in H; $1=f(u, u)=f(v, v)=f(u, v)=f(v, u)$ and $z=f(w, w)=f(u, w)=f(w, u)=f(v, w)=f(w, v)$.

The system D consists of all the (ordered) pairs (h, k) for h in H and k in Z; and multiplication in D is defined by the rule

$$(x, x')(y, y') = (xy, x'y'f(x, y)).$$

It is easily verified that D is a division-system with unit $(1, 1)$, and that D is commutative. We have

$$(1, 1) = (u, 1)^2 = (v, 1)^2 = (w, 1)(w, z),$$

though $(u, 1)(v, 1)=(uv, 1)=(w, 1)\neq(w, 1)^{-1}$, that is, the rule (+) is not satisfied in D.

In order to prove that (S) is satisfied in D, it suffices to consider prod-

* G. Bol, Mathematische Annalen, vol. 114 (1937), pp. 414–431; cf. p. 425.

† This example belongs to a class of systems discovered by Zassenhaus; cf. Bol, op. cit., p. 530.

‡ Cf. Bol, op. cit., p. 425.

ucts of the form $(r, r')[(s, s')(t, t')]$ for $rst=1$. This product is equal to $(1, r's't'f(s, t)f(r, r))$ and we find furthermore that $[(r, r')(s, s')](t, t')$ $=(1, r's't'f(r, s)f(t, t))$. But one verifies that $f(r, r)f(s, t)=f(t, t)f(r, s)$ whenever $rst=1$, since all these products are equal to z, whenever none of the elements r, s, t is equal to 1. Thus (S) is clearly satisfied by D.

If D is a division-system with unit, and if x is an element in D, then we denote by x^{-1} the right inverse of x, that is, x^{-1} is uniquely determined as a solution of the equation $xx^{-1}=1$.

THEOREM 11.3. *If D is a division-system with unit, and if D satisfies condition* (S), *then each of the following properties implies all the others.*

(i) $(xy)(y^{-1}x^{-1})=1$ *for every pair x, y in D.*

(ii) $y(y^{-1}x)=x$ *for every pair x, y in D.*

(iii) $(xy)y^{-1}=x$ *for every pair x, y in D.*

REMARK. The example, discussed just now, shows that these conditions are *not* consequences of (S).

Proof. Suppose first that (i) is satisfied by D. Then it follows from (S) that

$$1 = (xy)(y^{-1}x^{-1}) = x[y(y^{-1}x^{-1})] = [(xy)y^{-1}]x^{-1};$$

and this implies, since D is a division-system, that $x^{-1}=y(y^{-1}x^{-1})$ and $x=(xy)y^{-1}$ so that (ii) and (iii) are both consequences of (i).

If (ii) holds true, then $y(y^{-1}x^{-1})=x^{-1}$ and consequently

$$1 = xx^{-1} = x[y(y^{-1}x^{-1})] = (xy)(y^{-1}x^{-1})$$

by (S) so that (i) is a consequence of (ii); and if (iii) is true, then

$$1 = xx^{-1} = [(xy)y^{-1}]x^{-1} = (xy)(y^{-1}x^{-1})$$

by (S) so that (i) is a consequence of (iii).

The following remark may be of some interest. Condition (iii) is obviously a special case of the following condition:

(iv) $wx=[w(xy)]y^{-1}$ *for any three elements w, x, y, in D.*

Substituting $w=1$, $x=y^{-1}$ we find $y^{-1}=(y^{-1}y)y^{-1}$, that is, $y^{-1}y=1$; and $w=1$ alone gives (iii). Thus we find

$$(wx)y = \{[w(xy)]y^{-1}\}y = \{[w(xy)]y^{-1}\}(y^{-1})^{-1} = w(xy),$$

so that (iv) implies the associative law; that is, *a division-system with unit, satisfying* (iv), *is a group.*

12. In this section we shall need the following special case of the property *R-S*.

PROPERTY *R-S-e*. *If b_1, b_2 are points on $T(e)$, c_1, d_1 on $R(e)$ and a_2, d_2 on $S(e)$, and if $R(a_i)=R(b_i)$, $R(c_i)=R(d_i)$, $S(a_i)=S(d_i)$, $S(b_i)=S(c_i)$, $T(a_1)=T(a_2)$, $T(c_1)=T(c_2)$, then d_1 and d_2 are points on the same T-line.*

This property is illustrated by Fig. 6, I, p. 134.

THEOREM 12.1. *Each of the following properties of a point e in a net N implies the other properties.*

(a) *The system $M(R/S; e)$ satisfies condition* (S) *of* §11.

(b) *The net has Property R-S-e.*

(c) *An anti-isomorphism of $M(R/S; e)$ upon $M(S/R; e)$ is defined in mapping $G(R/S; e)r(R/S; T(e)-X)$ upon $G(S/R; e)r(S/R; T(e)-X)$.*

REMARK. The author is indebted to Dr. Max Zorn for pointing out that the above property (c) is satisfied whenever the system $M(R/S; e)$ is a quasigroup—as has been remarked in §11, condition (S) is satisfied by every quasigroup, though the converse does not hold true. The statement of I, Theorem 8.1, that (2) is equivalent to each of the other conditions is thus erroneous. Regarding its proof the following may be said. The paragraphs preceding I, Fig. 7, p. 135, contain a nearly complete proof of Theorem 12.1, the paragraph containing this figure does not prove anything; and the remaining paragraphs contain a proof of the facts that (3) implies (4), (4) implies (5) and that (5) implies (1). As has been remarked before, it may be verified by a simple computation that (1) implies (3).

The following proof of Theorem 12.1 is slightly different from the one suggested by the discussion in I.

Proof. If Z is any T-line whatsoever, then we shall write for brevity $r(Z)$ instead of $r(R/S; T(e)-Z)$—if X and Y are any two T-lines, then there exists one and only one T-line W so that $e^{r(W)}=e^{r(X)r(Y)}$ and there exists one and only one T-line V so that $e=e^{r(Y)^{-1}r(X)^{-1}r(V)}$, as follows from I, Theorem 4.4. Using the notations of I, Fig. 6, we have $W=T(d_1)$ and $V=T(d_2)$. Thus the net has Property R-S-e if, and only if, $W=V$, that is, if, and only if, $e=e^{r(Y)^{-1}r(X)^{-1}r(W)}$, that is, if, and only if, $r(Y)^{-1}r(X)^{-1}r(W)$ is an element in $G(R/S; e)$ for any two T-lines X and Y, that is, by Theorem 11.2, if, and only if, $M(R/S; e)$ satisfies condition (S) of §11; and thus we have proved the equivalence of the properties (a) and (b).

Using again the notations of I, Fig. 6, and applying I, Theorem 4.3, it follows that $r(R/S; T(e)-X)r(R/S; T(e)-Y)$ maps e upon the same point as $r(R/S; T(e)-T(d_1))$ and that $r(S/R; T(e)-Y)r(S/R; T(e)-X)$ maps e upon the same point as $r(S/R; T(e)-T(d_2))$. Thus condition (c) is satisfied if, and only if, $T(d_1)=T(d_2)$, that is, if, and only if, the net has Property R-S-e; and thus we have proved that (b) and (c) are equivalent properties.

If D is a division-system with unit, then a *net* $K(D)$ *may be derived from* D in the following fashion. The points of this net $K(D)$ are all the triples (r, s, t) of elements r, s, t in D which satisfy $r(ts)=1$. All the points with the same first coordinate r lie on the same R-line which shall be denoted by r; and the S- and T-lines are defined accordingly. That $K(D)$ is really a net under these definitions is readily derived from the fact that D is a division-system with unit.

THEOREM 12.2.* *Suppose that D is a division-system with unit, that N is the net $K(D)$, just defined, and that $e=(1, 1, 1)$. Then an isomorphism of D upon $M(R/S; e)$ is defined in mapping the element x in D upon the element $G(R/S; e)r(R/S; T(e)-x)$ in $M(R/S; e)$ if, and only if, condition* (S) *of* §11 *is satisfied by D.*

Proof. It is a consequence of I, Theorem 4.3, that $r(R/S; T(e)-x)$ maps the point $p=(r, s, t)$ upon the point $R(p)S\{xR[T(e)S(p)]\}$. As $T(e)=1$, $S(p)=s$, we have $T(e)S(p)=(s', s, 1)$ where s' is the uniquely determined left inverse of s, that is, $s's=1$. Hence $s'=R[T(e)S(p)]$ and $xR[T(e)S(p)]=(s', y, x)$ where y is uniquely determined as the solution of $s'(xy)=1$—the y is the uniquely determined solution of $xy=s$. Thus $S\{xR[T(e)S(p)\}=y$ and the image of p under $r(R/S; T(e)-x)$ is exactly (r, y, v) where v is the uniquely determined solution of $r(vy)=1$ or $vy=r^{-1}$ where r^{-1} is the right inverse to r.

Thus in particular $e=(1, 1, 1)$ is mapped by $r(R/S; T(e)-x)$ upon the point $(1, x^{-1}, x)$.

If u is some other element in D, then it follows furthermore that $(1, x^{-1}, x)$ is mapped by $r(R/S; T(e)-u)$ upon the point $(1, w, w')$ where w' is the left inverse of w and where w is uniquely determined as the solution of the equation $uw=x^{-1}$. The product $r(R/S; T(e)-x)r(R/S; T(e)-u)$ consequently maps the point $e=(1, 1, 1)$ upon the point $(1, w, w')$ where w is determined as the solution of $x(uw)=1$ and w' as the solution of $w'w=1$.

Note now that $r(R/S; T(e)-xu)$ maps the point $e=(1, 1, 1)$ upon the point $(1, (xu)^{-1}, xu)$ so that $r(R/S; T(e)-xu)$ and $r(R/S; T(e)-x)r(R/S; T(e)-u)$ have the same effect upon the point e if, and only if, $w'=xu$. An isomorphism of D upon $M(R/S; e)$ is consequently defined in mapping the element x in D upon the element $G(R/S; e)r(R/S; T(e)-x)$ if, and only if, $x(uw)=1$ implies $(xu)w=1$, that is, if, and only if, D satisfies condition (S′) of §11; and now our theorem is a consequence of Theorem 11.1.

* This theorem shows that just the nets whose coordinates are taken from a division-system with unit, satisfying (S), admit properly of a Kneserian representation; cf. Bol, op. cit., p. 424.

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