# A THEOREM ON QUADRATIC FORMS AND ITS APPLICATION IN THE CALCULUS OF VARIATIONS*

BY

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1. **Principal result.** Let $P(z), Q_1(z), \cdots, Q_r(z)$ be quadratic forms in the real variables $z_1, \cdots, z_m$ with real coefficients having the following properties: the form $P(z)$ is positive at each point $(z)\neq(0)$ at which the forms $Q_1(z), \cdots, Q_r(z)$ vanish simultaneously; for every set of constants $S_1, \cdots, S_r$, not all zero, the quadratic form $S_iQ_i(z)$† is indefinite; for every linear subspace $L$ on which the quadratic forms $Q_1, \cdots, Q_r$ do not vanish simultaneously, save at $(z)=(0)$, there is a linear combination $S_iQ_i(z)$ of these forms which is positive definite on $L$. The principal result given in the present note is given in the following:

THEOREM 1. *Under the above hypotheses there exists a set of constants $S_1, \cdots, S_r$ such that the quadratic form*

$$P(z, S) = P(z) + S_iQ_i(z) \qquad (i = 1, \cdots, r)$$

*is positive definite in the variables $z_1, \cdots, z_m$. The last two hypotheses may be dropped when $r=1$.*

This theorem was proposed in a somewhat different form by Bliss in 1937 in a seminar on the calculus of variations. It is useful in sufficiency proofs for multiple integral problems. Proofs of the theorem for the case $r=1$ were given in the seminar by Albert,‡ Reid,§ McShane and Hestenes, each using a different method. The last two of these proofs have not been published. The proof of Theorem 1 here given is due to McShane and is an immediate extension of the one given by McShane for the case $r=1$. In the next section Theorem 1 will be applied to the case in which $(z)=(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and the forms $Q_1, \cdots, Q_r$ are the two-rowed minors of the $n\times 2$-dimensional matrix $(x_k\ y_k)$. If $n=2$ then $r=1$ and Theorem 2 below is an immediate consequence of Theorem 1. If $n>2$ the further result described in Theorem 3 below is

* Presented separately to the Society, by McShane on April 8, 1939, and by M. R. Hestenes on December 29, 1939; received by the editors November 29, 1939.

† A repeated index denotes summation.

‡ A. A. Albert, *A quadratic form problem in the calculus of variations*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 250–252.

§ W. T. Reid, *A theorem on quadratic forms*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 437–440.

needed. This further result for the case $n=3$ has been established by McShane.* The proofs of Theorems 3, 4 and 5 here given are due to Hestenes.†

The proof of Theorem 1 is based on two lemmas, the first of which is the following:

LEMMA A. *Let $m(S)$ be the minimum of the quadratic form $P(z, S)$ on the unit $(m-1)$-sphere $z_k z_k = 1$. There exists a set of constants $(S_0)$ which maximizes the function $m(S)$.*

For a set of constants $S_1, \cdots, S_r$ let $M(S)$ be the maximum of the quadratic form $-S_i Q_i$ on the unit sphere $z_k z_k = 1$. We have $M(S) > 0$ if $(S) \neq (0)$ since the quadratic form $-S_i Q_i$ is indefinite by hypothesis. Moreover $M(hS) = hM(S)$ for every positive number $h$. Let $M_0$ be the minimum of $M(S)$ on the set $S_i S_i = 1$. Since this minimum is attained, we have $M_0 > 0$. Finally let $N$ be the maximum of $P(z)$ on the set $z_k z_k = 1$. Then for every set of constants $(S)$ there is a point $(z)$ such that $z_k z_k = 1$, $-S_i Q_i(z) = M(S) \geqq hM_0$, $S_i S_i = h^2$ and

$$m(S) \leqq P(z, S) \leqq N - M(S) \leqq N - hM_0.$$

It follows that there is a positive constant $h_0$ such that when $h > h_0$ we have $m(S) < m(0)$ for every set $(S)$ with $S_i S_i = h^2$. There is accordingly a set $(S_0)$ such that the relation $m(S) \leqq m(S_0)$ holds for every set $(S)$, as was to be proved.

LEMMA B. *Let $(S_0)$ be a set of constants which maximizes the function $m(S)$ and set $m_0 = m(S_0)$. The set of points $(z)$ satisfying the equation $P(z, S_0) = m_0 z_k z_k$ forms a linear space $L$. There is no quadratic form $S_i Q_i$ which is positive definite on $L$.*

The set $L$ consists of all points $z$ at which the function $P(z, S_0) - m_0 z_k z_k$ attains its minimum value 0. Hence the partial derivatives of that function vanish on $L$, and therefore the equations

$$P_{z_k}(z, S_0) = 2m_0 z_k \tag{1}$$

hold for all $z$ in $L$. Conversely, if we multiply equations (1) by $z_k$ and sum on $k$ we see that every point $z$ which satisfies (1) lies in $L$. Thus $L$ is the set of solutions of the linear equations (1), and is therefore linear. Suppose now that there exists a quadratic form $Q = S_i Q_i$ that is positive definite on $L$. Let $K$ be the unit sphere $z_k z_k = 1$ and $L_1$ the set of points in $L$ on $K$. Choose $b > 0$ such

* E. J. McShane, *The condition of Legendre for double integral problems of the calculus of variations*, Bulletin of the American Mathematical Society, abstract 45-5-209.

† M. R. Hestenes, *A theorem on quadratic forms and its application in the calculus of variations*, Bulletin of the American Mathematical Society, abstract 46-1-83.

that $Q(z)>b$ on $L_1$, and let $N$ be a neighborhood of $L_1$ relative to $K$ on which $Q(z)>b$. Let $m_1$ be the minimum of $P(z, S_0)$ on the closed set $K-N$. Then $m_1>m_0$. It follows that for a sufficiently small positive constant $h$ one will have

$$P(z, S_0 + hS) = P(z, S_0) + hQ > m_0$$

on $K-N$. But $P(z, S_0+hS)>m_0+hb$ on $N$, and hence $m(S_0+hS)>m(S_0)$, contrary to our choice of the set $(S_0)$. This proves Lemma B.

In order to prove Theorem 1 we note that there is a point $(z)\neq(0)$ on the set $L$ described above at which the forms $Q_1, \cdots, Q_r$ vanish simultaneously. Otherwise by virtue of the last hypothesis made in Theorem 1 there would exist a quadratic form $S_iQ_i(z)$ which is positive definite on $L$, contrary to Lemma B. At this point $(z)$ we have accordingly $m_0z_kz_k=P(z, S_0)=P(z)>0$ in view of the first hypothesis of the theorem. It follows that $m_0>0$ and hence that $P(z, S_0)$ is positive definite. This proves the first statement in the theorem. The second statement is readily verified. It should be observed that the hypotheses of Theorem 1 imply that there is a point $(z)\neq(0)$ at which the forms $Q_1, \cdots, Q_r$ vanish simultaneously. Otherwise the last hypothesis would imply the existence of a positive definite form $S_iQ_i$, contrary to the second hypothesis.

The last sentence in Theorem 1 remains to be established. It is easy to see that if $r=1$ the last hypothesis in §1 is automatically satisfied. Suppose then that $r=1$ and that $Q$ is not indefinite. Let $K$ be the sphere $z_kz_k=1$. If $Q_1$ is positive definite, the sum $P(z, S)$ is positive on $K$, hence is a positive definite form, provided that $S_1$ is large. If $Q_1$ is negative definite, $P(z, S)$ is positive definite provided that $-S_1$ is large. If $Q_1$ is semi-definite, say positive, let $L$ be the set on which $Q_1$ vanishes. As in Lemma B, this is linear. On $LK$ the form $P(z)$ is positive; it then remains positive on a neighborhood $N$ of $LK$ relative to $K$. On $K-N$ the form $Q_1$ is positive. Choose $S_1$ large enough so that $P(z, S)$ is positive on $K-N$. On $N$ we have $P(z, S)\geqq P(z)>0$, so that $P(z, S)$ is positive on $K$, and is therefore a positive definite form.

In the proof of the first part of Theorem 1 made above we have established essentially the following more general result:

COROLLARY. *Suppose the last two of the hypotheses made in Theorem* 1 *hold, and let m be the minimum of the form $P(z)$ on the set of points $(z)$ satisfying the conditions $Q_1(z)=\cdots=Q_r(z)=0$, $z_kz_k=1$. There exists a set of constants $S_i$ such that the inequality $P(z, S)\geqq mz_kz_k$ holds for all points $(z)$.*

2. **A further result.** Consider now the case in which the space of points $(z)$ described in the last section is of dimension $2n$. For the purposes of this sec-

tion it will be convenient to denote the $i$th ($i \leqq n$) and the $(n+i)$th coordinates of $(z)$ by $x_i$ and $y_i$ respectively. Thus the points of our space will be denoted by the symbol $(x, y)$. To each point $(x, y)$ there is associated a $n \times 2$-dimensional matrix $(x_i\, y_i)$ whose $i$th row is composed of the coordinates $x_i$, $y_i$ of the point $(x, y)$. This matrix will be used below to classify the points $(x, y)$ of our space. By a quadratic form in the variables $(x, y)$ will be meant an expression of the form

$$R_{ik}x_ix_k + S_{ik}x_iy_k + T_{ik}y_iy_k.$$

In particular the expression $S_{ik}x_iy_k$ is a quadratic form in the variables $(x, y)$. Finally by a linear space $L$ of points $(x, y)$ is meant a subspace such that if $(x, y)$ and $(x', y')$ belong to $L$ so also does the point $(ax+bx', ay+by')$, where $a$ and $b$ are arbitrary real constants.

The results described in the last section will be used to prove the following:

THEOREM 2. *Let $P(x, y)$ be a quadratic form in the $2n$ real variables $x_1, \cdots, x_n, y_1, \cdots, y_n$ with real coefficients. Suppose that the inequality $P(x, y) > 0$ holds whenever the $n \times 2$-dimensional matrix $(x_i\, y_i)$ has rank 1. Then there exists an $n$-rowed skew-symmetric matrix $S = (S_{ik})$ such that the quadratic form $P(x, y) + S_{ik}x_iy_k$ $(i, k = 1, \cdots, n)$ is positive definite.*

To prove this result let $Q_{ik}$ $(i<k)$ be the quadratic form $x_iy_k - x_ky_i$. A linear combination $S_{ik}Q_{ik}$ (summed with $i<k$) is easily seen to be equal to $S_{ik}x_iy_k$ (summed for all $i$, $k$) if we set $S_{ii}=0$, $S_{ki} = -S_{ik}$ $(i<k)$. Thus we see that the theorem will be established if we show that the hypotheses of Theorem 1 with $Q_i$ replaced by $Q_{ik}$ are satisfied. The first hypothesis holds since the matrix $(x_i\, y_i) \neq (0\ 0)$ has rank 1 if and only if the forms $Q_{ik}$ vanish simultaneously. Moreover a linear combination $S_{ik}Q_{ik}$ with $S \neq 0$ is indefinite. Finally the last hypothesis of Theorem 1 holds by virtue of the following:

THEOREM 3. *Let $L$ be a linear set of points $(x, y)$ such that the $n \times 2$-dimensional matrix $(x_i\, y_i)$ has rank 2 at each point $(x, y) \neq (0, 0)$ on $L$. There exists a skew-symmetric matrix $S = (S_{ik})$ such that the quadratic form $S_{ik}x_iy_k$ is positive definite on $L$.*

Let $m$ be the dimension of $L$ and let $(X_{1\alpha}, \cdots, X_{n\alpha}, Y_{1\alpha}, \cdots, Y_{n\alpha})$ $(\alpha = 1, \cdots, m)$ be a basis for $L$, that is, a set of $m$ points $(X_\alpha, Y_\alpha)$ in $L$ such that the coordinates of each point $(x, y)$ in $L$ are expressible uniquely in the form

$$x_i = X_{i\alpha}u_\alpha, \qquad y_i = Y_{i\alpha}u_\alpha \qquad (\alpha = 1, \cdots, m). \tag{2}$$

Since the matrix $(x_i\, y_i)$ has rank 2 at each point $(x, y) \neq (0, 0)$ on $L$, the matrix

$\lambda X+\mu Y$, where $X=(X_{i\alpha})$ and $Y=(Y_{i\alpha})$, has rank $m$ for every pair of real numbers $\lambda$, $\mu$, not both zero. Conversely for every pair of matrices $X$, $Y$, having $\lambda X+\mu Y$ of rank $m$ when $(\lambda,\mu)\neq(0,0)$, the corresponding linear space $L$ defined by equations (2) is such that the matrix $(x_i\ y_i)$ has rank 2 at each point $(x, y)\neq(0,0)$ on $L$. Moreover by the use of equations (2) it is seen that $S_{ik}x_iy_k=R_{\alpha\beta}u_\alpha u_\beta$, where $R=X'SY$ and $X'$ is the transpose of $X$. Here and elsewhere it is understood that the symbol for an element of a matrix is obtained by adding a pair of subscripts to the symbol for the matrix. Theorem 3 is accordingly equivalent to the following theorem on matrices:

THEOREM 4. *Let $X$, $Y$ be two $n\times m$-dimensional matrices such that the matrix $\lambda X+\mu Y$ has rank $m$ for every pair of real numbers $\lambda$, $\mu$, not both zero. There exists an $n$-rowed skew-symmetric matrix $S$ such that the matrix $R=X'SY$ is positive definite, that is, the quadratic form $R_{\alpha\beta}u_\alpha u_\beta$ $(\alpha,\beta=1,\cdots,m)$ is positive definite. Here $X'$ is the transpose of $X$. The matrix $R$ in general will not be symmetric.*

In order to prove Theorem 4 we first observe that in the proof of the theorem we may replace the matrices $X$, $Y$ by $X_1=AXB$, $Y_1=AYB$, where $A$ and $B$ are arbitrary nonsingular matrices of dimensions $n$ and $m$ respectively. For, the matrix $\lambda X_1+\mu Y_1$ has rank $m$ when $(\lambda,\mu)\neq(0,0)$. Moreover, suppose there exists a skew-symmetric matrix $S_1$ such that the matrix $X_1'S_1Y_1$ is positive definite. Let $S$ be the skew-symmetric matrix $A'S_1A$, and set $R=X'SY$. Then the matrix $B'RB=X_1'S_1Y_1$ is positive definite. It follows readily that $R$ is positive definite and hence that the matrices $X$, $Y$ can be replaced by $X_1$, $Y_1$ respectively. In fact $X$, $Y$ can be replaced by matrices of the form

$$X=\begin{pmatrix} I \\ O \end{pmatrix},\qquad Y=\begin{pmatrix} C \\ D \end{pmatrix},\tag{3}$$

where $I$ is the $m$-rowed identity matrix, $C$ is an $m$-rowed matrix and $O$ is the zero matrix. Suppose that $X$, $Y$ are of this form. Then the condition that the matrix $\lambda X+\mu Y$ be of rank $m$ for all real numbers $\lambda$, $\mu$, not both zero, is equivalent to the condition that the matrix

$$\begin{pmatrix} \lambda I - C \\ -D \end{pmatrix}\tag{4}$$

be of rank $m$ for all real values of $\lambda$. Moreover the equation $AXB=X$ holds for two nonsingular matrices $A$ and $B$ if and only if $A$ is of the form

$$A=\begin{pmatrix} B^{-1} & B^{-1}E \\ O & F \end{pmatrix}.\tag{5}$$

When $A$ is of this form we have

$$AYB = \begin{pmatrix} B^{-1}(C+ED)B \\ FDB \end{pmatrix}. \tag{6}$$

In the sequel we shall assume that the matrices $X$, $Y$, $A$ are of the forms (3) and (5). It will be understood that if $n=m$ the matrices $O$, $D$, $E$, $F$ do not appear in these matrices. Theorem 4 will be established by replacing $Y$ by a matrix $AYB$ having special properties. This will be done with the help of three lemmas, the first of which is the following:

LEMMA 1. *Let $\phi_1(\lambda), \cdots, \phi_k(\lambda)$ be the elementary divisors of the matrix $\lambda I - C$ and let $\epsilon$ be an arbitrary constant different from zero. With an elementary divisor of the form $\phi_i(\lambda) = [(\lambda-\alpha_i)^2+\beta_i^2]^{m_i}$ $(\beta_i \neq 0)$ associate the $2m_i$-rowed matrix*

$$M_i = \begin{Vmatrix} N_i & O & O & \cdots & O \\ \epsilon I & N_i & O & \cdots & O \\ O & \epsilon I & N_i & \cdots & O \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ O & O & O & \cdots \epsilon I & N_i \end{Vmatrix}, \quad N_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

*With an elementary divisor of the form $\phi_j(\lambda) = (\lambda-\alpha_j)^{m_j}$ associate the matrix*

$$M_j = \begin{Vmatrix} \alpha_j & 0 & 0 & \cdots & 0 \\ 1 & \alpha_j & 0 & \cdots & 0 \\ 0 & 1 & \alpha_j & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots 1 & \alpha_j \end{Vmatrix}. \tag{8}$$

*There exists a nonsingular $m$-rowed matrix $B$ such that $B^{-1}CB = C_1$, where*

$$C_1 = \begin{Vmatrix} M_1 & O & O & \cdots & O \\ O & M_2 & O & \cdots & O \\ O & O & M_3 & \cdots & O \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ O & O & O & \cdots & M_k \end{Vmatrix}. \tag{9}$$

For it is clear from the special forms (7) and (8) that $\phi_i(\lambda) = |\lambda I - M_i|$. If $M_i$ is of the form (8), the $(m_i-1)$-rowed minor obtained by deleting the first row and last column has determinant unity. Hence $\phi_i(\lambda)$ is the only nontrivial invariant factor of $\lambda I - M_i$ in this case. Suppose next that $M_i$ is of the form (7) and let $\phi(\lambda)$ be the determinant of the minor of $\lambda I - M_i$ obtained by deleting the first row and the last column. The values $\phi(\alpha_i + i\beta_i), \phi(\alpha_i - i\beta_i)$ are different from zero since $\epsilon \neq 0$, as one readily verifies. It follows that $\phi(\lambda)$ and $\phi_i(\lambda)$ have no common factors and hence that $\phi_i(\lambda)$ is the only non-trivial invariant factor of $\lambda I - M_i$ when $M_i$ is of the form (7). The matrices $\lambda I - C_1$ and $\lambda I - C$ accordingly have the same elementary divisors and hence the same invariant factors. The matrix $C_1$ is therefore similar to $C$, that is, $C_1$ is of the form $B^{-1}CB$, where $B$ is nonsingular.†

LEMMA 2. *If the equation* $|\lambda I - C| = 0$ *has no real roots, there exists a matrix* $C^*$ *similar to* $C$ *and a skew-symmetric matrix* $S$ *such that the matrix* $SC^*$ *is positive definite. The matrix* $SC^*$ *is not in general symmetric.*

For in this case the diagonal blocks in (9) are of the form (7). Consider $C_1$ as a function $C_1(\alpha, \beta, \epsilon)$ of the values $\alpha_i$, $\beta_i$, $\epsilon$ described in Lemma 1. Let $R(\epsilon) = C_1(0, \beta, \epsilon)$, $S = -R(0)$ and $T = C_1(\alpha, 0, 0)$. Then $S$ and $ST$ are skew-symmetric, $S$ is nonsingular and $C_1 = R(\epsilon) + T$. Since $ST$ is skew-symmetric one has $u_\alpha S_{\alpha\beta} T_{\beta\gamma} u_\gamma = 0$ $(\alpha, \beta, \gamma = 1, \cdots, m)$ and

$$u_\alpha S_{\alpha\beta}(R_{\beta\gamma} + T_{\beta\gamma})u_\gamma = u_\alpha S_{\alpha\beta} R_{\beta\gamma} u_\gamma$$

for arbitrary values of $(u)$. Since $R(0) = -S$ and $S$ is nonsingular, the last quadratic form is positive definite when $\epsilon = 0$ and hence for a value $\epsilon' \neq 0$. The matrices $S$ and $C^* = R(\epsilon') + T$ have the properties described in the lemma.

LEMMA 3. *Let* $C$, $D$ *be the matrices appearing in the matrix* (4) *and suppose the matrix* (4) *has rank* $m$ *for all real values of* $\lambda$. *There exists an* $m \times (n-m)$*-dimensional matrix* $E$ *such that the equation*

$$|\lambda I - C - ED| = 0 \tag{10}$$

*has no real roots if the dimension* $m$ *of* $C$ *is even and a single real root if* $m$ *is odd.*

Let us begin by disposing of the case $m = n$. If $m = n$, the conclusion of Lemma 3 is that $|\lambda I - C| \neq 0$ for all real $\lambda$. This is an immediate consequence of the hypothesis that matrix (4) has rank $m = n$ for all real $\lambda$. Incidentally, if $m = n$, both $m$ and $n$ must be even; otherwise the equation $|\lambda I - C| = 0$ would be of odd degree, and would therefore have a real root.

Suppose then that $m < n$. We shall consider first the case in which $m = 2$ and $C$, $D$ are of the form

$$C = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad D = (d \;\; e). \tag{11}$$

† A. A. Albert, *Modern Higher Algebra*, University of Chicago Press, 1937, pp. 84–85.

If $D$ is the two-rowed identity matrix, let

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad E = G - C.$$

Then the equation (10) takes the form $\lambda^2+1=0$ and has no real roots. Suppose next that $D=(d\ e)$. We may suppose that $b=0$ since this can be brought about by replacing $\lambda$ by $\lambda+b$. Since the matrix (4) has rank 2 for $\lambda=0$ and $\lambda=a$, we have $e\neq 0$ and $ad+ec\neq 0$. Choose numbers $\alpha$, $\beta$ such that

$$\alpha ce - \beta ae = -1, \qquad \alpha d + \beta e + a = 0.$$

Then equation (10) with $E'=(\alpha\ \beta)$ as the transpose of $E$ reduces to $\lambda^2+1=0$. The lemma is accordingly true for the matrices (11).

To prove the lemma as stated, let $h$ be the number of real roots of the equation $|\lambda I - C| = 0$, each root counted a number of times equal to its multiplicity. We may suppose that $h \geqq 2$. It is sufficient to show that the matrix $E$ can be chosen so that the equation (10) has exactly $h-2$ real roots. In view of equation (6) it is sufficient to prove this result when $C$, $D$ are replaced respectively by matrices of the form $C_1 = B^{-1}CB$, $D_1 = FDB$, where $B$ and $F$ are nonsingular. By virtue of Lemma 1 we may select $B$ so that $C_1$ is given by (9). In fact $B$ may be chosen so that the matrices (8) corresponding to real roots of $|\lambda I - C| = 0$ have higher subscripts than the matrices (7) corresponding to roots that are not real. It follows that, after a suitable choice of the matrix $F$ in $D_1$, one has

$$C_1 = \begin{pmatrix} M & O \\ N & C_2 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} U & O \\ V & D_2 \end{pmatrix},$$

where the matrices $C_2$, $D_2$ are the matrices $C$, $D$ in (11). The matrix (4) with $C=C_2$, $D=D_2$ has rank 2, since otherwise the corresponding matrix with $C=C_1$, $D=D_1$ could not have rank $m$. It follows that there exists a matrix $E_2$ such that the equation (10) with $C=C_2$, $D=D_2$, $E=E_2$ has no real roots. Choosing

$$E = \begin{pmatrix} O & O \\ O & E_2 \end{pmatrix},$$

the equation (10) with $C=C_1$, $D=D_1$ reduces to the product

$$|\lambda I_{m-2} - M| \cdot |\lambda I_2 - C_2 - E_2 D_2| = 0,$$

where the subscript on $I$ denotes its dimension. This equation has $h-2$ real roots and the proof of Lemma 3 is complete.

We are now in position to prove Theorem 4. As was seen above we can assume that the matrices $X$, $Y$ are of the form (3). Since we can replace $Y$

by the matrix (6) we can suppose, by virtue of Lemma 3, that the equation $|\lambda I - C| = 0$ has no real roots if the dimension $m$ of $C$ is even and one real root if $m$ is odd. Consider first the case in which $m$ is even. Then by Lemma 2 there is a nonsingular matrix $B$ and a skew-symmetric matrix $S$ such that the matrix $SC^*$ with $C^* = B^{-1}CB$ is positive definite. Replace $Y$ by $AYB$, where $A$ is defined by equation (5) with $E = 0$ and $F = I$. Then $C$ is replaced by $C^*$ so that the matrix $SC$ is now positive definite. Let $S_1$ be the $n$-rowed skew-symmetric matrix

$$\begin{pmatrix} S & O \\ O & O \end{pmatrix}.$$

The product $X'S_1Y$ is then equal to $SC$ and is accordingly positive definite. This proves Theorem 4 for the case in which $m$ is even.

The case in which $m$ is odd will be reduced to the case in which $m$ is even. As was seen above we can assume that the equation $|\beta I - C| = 0$ has only one real root $\lambda = a$. Choose nonsingular matrices $B$, $F$ such that the matrices $C_1 = B'CB$, $D_1 = FDB$ are of the form

$$C_1 = \begin{pmatrix} M & O \\ O & a \end{pmatrix}, \qquad D_1 = \begin{pmatrix} U & b \\ V & O \end{pmatrix} \qquad (b \neq 0).$$

This choice is possible by virtue of Lemma 1 with $M_k = (a)$ and the fact that the matrix (4) has rank $m$ when $\lambda = a$. We can suppose that $C = C_1$, $D = D_1$, since this result can be brought about by replacing $Y$ by $AYB$, where $A$ is given by (5) with $E = O$. Clearly $m < n$ since $D$ has at least one row. Let $X_1, Y_1$ be the matrices

$$X_1 = \begin{pmatrix} I & O & O \\ O & 1 & 0 \\ O & 0 & 1 \\ O & O & O \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} M & O & O \\ O & a & -b \\ U & b & a \\ V & O & O \end{pmatrix}$$

obtained by adding a suitable column to each of the matrices $X$ and $Y$. Since the equations $|\lambda I - M| = 0$, $(\lambda - a)^2 + b^2 = 0$ have no real roots, the matrix $\lambda X_1 - Y_1$ corresponding to (4) has rank $m+1$ for all real values of $\lambda$. As was seen in the last paragraph, there exists, since $m+1$ is even, a skew-symmetric matrix $S$ such that the matrix $X_1'SY_1$ is positive definite. The matrix $X'SY$, being a principal minor of $X_1'SY_1$, is also positive definite. This completes the proof of Theorem 4 and hence also of Theorem 3.

Incidentally, by an argument similar to that just made, it can be shown that if $L$ is a maximal linear space having the properties described in Theorem 3 then its dimension is $n$ if $n$ is even and $n-1$ if $n$ is odd.

3. **Application to the calculus of variations.** Consider the problem of minimizing the double integral

$$I = \iint_A f(x, y, z_1, \cdots, z_n, p_1, \cdots, p_n, q_1, \cdots, q_n)dxdy,$$

where $p_i = \partial z_i/\partial x$, $q_i = \partial z_i/\partial y$, in a class of subspaces

$$z_i(x, y) \qquad ((x, y) \text{ on } A + B; i = 1, \cdots, n) \tag{12}$$

having a common boundary, where $B$ is the boundary of $A$. The integrand $f(x, y, z, p, q)$ is assumed to be continuous and to have continuous first and second partial derivatives on a region $\mathfrak{R}$ of points $(x, y, z, p, q)$. The subspaces (12) are assumed to be continuous, to have continuous second partial derivatives and to have their elements $(x, y, z, p, q)$ in $\mathfrak{R}$. The boundary $B$ of $A$ is supposed to be a simply closed continuous curve having a piecewise continuously turning tangent. Weaker differentiability assumptions could be made.

LEMMA 4. *Let $S_{ik}(x, y) = -S_{ki}(x, y)$ $(i, k = 1, \cdots, n)$ be arbitrary continuous functions having continuous first and second derivatives in a neighborhood of the set $A + B$. Then the integral*

$$\begin{aligned} J &= \iint_A g(x, y, z, p, q)dxdy \\ &\equiv \iint_A (1/2)\left\{p_i \frac{\partial}{\partial y}(s_{ik}z_k) - q_i \frac{\partial}{\partial x}(s_{ik}z_k)\right\}dxdy \end{aligned} \tag{13}$$

*has the same value for all subspaces* (12) *here considered.*

For by virtue of Green's theorem the value of $2J$ is given by the formula

$$2J = \iint_A \left\{\frac{\partial}{\partial y}(S_{ik}p_iz_k) - \frac{\partial}{\partial x}(S_{ik}q_iz_k)\right\}dxdy = \int_B z_kS_{ki}dz_i,$$

and hence is completely determined by the common boundary of the subspaces (12).

Let $E$ be a minimizing subspace. Then $E$ must satisfy the *condition of Legendre*,* that is, at each element $(x, y, z, p, q)$ on $E$ the inequality

$$P(\xi, \eta) = f_{p_ip_k}\xi_i\xi_k + 2f_{p_iq_k}\xi_i\eta_k + f_{q_iq_k}\eta_i\eta_k \geqq 0 \tag{14}$$

must hold for every set $(\xi, \eta)$ whose $n \times 2$-dimensional matrix has rank 1. If

* Graves, *The Weierstrass condition for multiple integral variation problems*, Duke Mathematical Journal, vol. 5 (1939), pp. 656–660. Graves proves only the Weierstrass condition. The condition of Legendre is a well known consequence of that of Weierstrass.

the condition (14) with the equality excluded holds on $E$ for every set $(\xi, \eta)$ whose matrix $(\xi_i\ \eta_i)$ has rank 1, then $E$ will be said to satisfy the *strengthened condition of Legendre*. It should be emphasized that in this strengthened condition of Legendre we require that the inequality $P(\xi, \eta)>0$ hold only for sets $(\xi, \eta)$ whose $n\times 2$-dimensional matrix $(\xi_i\ \eta_i)$ has rank 1 and not for all sets $(\xi, \eta)\neq(0, 0)$. We have the following:

THEOREM 5. *Let $E$ be a subspace* (12) *satisfying the strengthened condition of Legendre. There exists an invariant integral of the form* (13) *such that for the function $F=f+g$ the inequality*

$$F_{p_ip_k}\xi_i\xi_k + 2F_{p_iq_k}\xi_i\eta_k + F_{q_iq_k}\eta_i\eta_k > 0 \tag{15}$$

*holds at each element $(x, y, z, p, q)$ on $E$ for every set $(\xi, \eta)\neq(0, 0)$.*

For by Theorem 2 there exists for each point on $E$ a skew-symmetric matrix $(S_{ik})$ such that the quadratic form $P(\xi, \eta)+S_{ik}\xi_i\eta_k$ is positive definite at this point and hence in a neighborhood of this point. It follows readily that the set $A+B$ can be covered by a finite number of circles $C_1, \cdots, C_m$ having a common radius $r$ such that to the circle $C_\alpha'$ of radius $2r$ concentric with the circle $C_\alpha$ there is associated a skew-symmetric matrix $(S^\alpha_{ik})$ of constant elements such that the form $P_\alpha(\xi, \eta)=P(\xi, \eta)+S^\alpha_{ik}\xi_i\eta_k$ is positive definite at each point on $E$ whose projection in the $xy$-plane is in $C_\alpha'$. We propose now to construct a set of functions $S_{ik}(x, y)=-S_{ki}(x, y)$ having continuous second derivatives on a neighborhood of $A+B$ and such that the form $P(\xi, \eta)+S_{ik}(x, y)\xi_i\eta_k$ is positive definite on $E$. To this end let $h(t)$ be a function having continuous second derivatives and such that $h(t)=0$ when $t\leqq r$, $h(t)=1$ when $t\geqq 2r$, and $0<h(t)<1$ when $r<t<2r$. Let $(x_\alpha, y_\alpha)$ be the center of the circle $C_\alpha$, and let $d_\alpha(x, y)$ be the distance between the points $(x, y)$ and $(x_\alpha, y_\alpha)$. Set $h_\alpha(x, y)=h[d_\alpha(x, y)]$. It should be observed that the product $h_1h_2\cdots h_m$ is identically zero on the set $A+B$. This follows because $h_\alpha\equiv 0$ on $C_\alpha$ and the circles $C_1, \cdots, C_m$ cover $A+B$. Denote by $S_{ik}(x, y)$ the function

$$(1-h_1)\overset{1}{S}_{ik} + h_1(1-h_2)\overset{2}{S}_{ik} + \cdots + h_1h_2\cdots h_{m-1}(1-h_m)\overset{m}{S}_{ik},$$

where $S^\alpha_{ik}$ $(\alpha=1, \cdots, m)$ are the constants described above. The quadratic form $P(\xi, \eta)+S_{ik}(x, y)\xi_i\eta_k$ determined by these functions is easily seen to be identical with the sum

$$(1-h_1)P_1(\xi,\eta) + h_1(1-h_2)P_2(\xi,\eta) + \cdots + h_1h_2\cdots h_{m-1}(1-h_m)P_m(\xi,\eta),$$

where $P_\alpha(\xi, \eta)$ is the quadratic form associated with the circle $C'$. Each term in this sum is positive or zero, since for each integer $\alpha$ the product

$(1-h_\alpha)P_\alpha(\xi, \eta)$ is positive definite at points on $E$ corresponding to points in $C_2'$ and identically zero elsewhere. Moreover the terms do not vanish simultaneously when $(\xi, \eta) \neq (0, 0)$. The quadratic form $P(\xi, \eta)+S_{ik}\xi_i\eta_k$ is accordingly positive definite on $E$.

We now use these functions $S_{ik}$ to define the function $g$ of equation (13). From the definition of $g$ we at once obtain

$$g(x, y, z, p, q) = (1/2)\left[(\partial S_{ik}/\partial y)p_i z_k + (\partial S_{ki}/\partial x)q_i z_k\right] + S_{ik}p_i q_k.$$

This shows that the Legendre quadratic form (15) for $g$ is $S_{ik}\xi_i\eta_k$. Hence if $F=f+g$ the Legendre form (15) for $F$ is exactly the positive definite form $P(\xi, \eta)+S_{ik}\xi_i\eta_k$. This establishes Theorem 5.

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