ARC- AND TREE-PRESERVING TRANSFORMATIONS

BY

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1. Introduction. In an earlier paper[2] by one of us, referred to hereafter as
A.P.T., arc-preserving transformations were defined and studied in connec-
tion with an irreducibility condition on the transformation. It was shown,
for example, that if A and B are compact locally connected metric continua
which are cyclic (that is, without cut points) any single valued continuous
arc-preserving and irreducible transformation $T(A) = B$ of A onto B is neces-
sarily a homeomorphism. ("Arc-preserving" means that the image of every
simple arc in A is either a simple arc or a single point of B; irreducibility of T
means that no proper subcontinuum of A maps onto all of B.) It was shown,
furthermore, that in case A is hereditarily locally connected the same con-
clusion holds without the assumption of irreducibility; and the prediction was
made that this is true in the general case.

Now as pointed out in A.P.T., if A is a compact continuum and $T(A) = B$
is continuous, then, since the property of being a subcontinuum of A mapping
onto all of B under T is inducible, there always exists a subcontinuum A_1
of A such that $T(A_1) = B$ and T is irreducible on A_1. However, since local
connectedness of A would certainly not in general insure local connectedness
of A_1, it is not possible always to reduce the set A so as to make the trans-
formation irreducible without sacrificing essential properties of A.

In the present paper we propose not only to completely justify the earlier
prediction referred to above, but also to obtain theorems concerning a much
more general type of transformation than "arc-preserving" which will give
all the theorems of the first three sections of A.P.T. as immediate corollaries.

R. G. Simond[3] has studied tree-preserving transformations on locally
connected compact and metric continua (that is, transformations $T(A) = B$
satisfying the condition that the image of every tree (or dendrite) in A is a
tree in B). Miss Simond has proved with considerable difficulty that every
arc-preserving transformation is tree-preserving. We show that this result
is an immediate consequence of one of our theorems, as it is also of a theorem
of A.P.T. In fact our first principal result, the proof of which is very much
simpler than that given by Simond, shows that in order that $T(A) = B$ be
tree-preserving it is necessary and sufficient that the image of every simple

arc in A shall be a tree in B. We also give an independent proof of the Simond theorem.

If A is a tree, every simple arc in A is a cyclic chain([4]) in A. This shows at once that if A and B are both trees and $T(A)=B$ is continuous, then in order that T be arc-preserving it is necessary and sufficient that T be cyclic chain-preserving. This immediately suggests another of our main results: If A is a compact locally connected continuum and $T(A)=B$ is continuous, then in order that T be arc-preserving it is necessary and sufficient that T be both tree-preserving and cyclic chain-preserving. We give this theorem added meaning by obtaining a somewhat unexpected characterization of tree-preserving transformations in terms of the action of these transformations and their inverses on the sets A and B.

In §5 a characterization of A-set reversing transformations([5]) is given which supplements the treatment of this type of transformation initiated in A.P.T.

In conclusion we might mention that if B is cyclic, the following types of transformations are equivalent: (a) arc-preserving, (b) tree-preserving, (c) A-set reversing, (d) monotone retracting.

Throughout the paper all transformations are assumed to be single valued and continuous and all continua compact and metric.

2. **Principal results.** We assume throughout this section that A and B are locally connected continua and that $T(A)=B$ maps every arc of A onto a dendrite in B.

(2.0)   *The image of every dendritic graph in A is a dendrite in B.*

Proof. (By induction on the number of end points.) If the number of end points in the graph is 2, then the hypothesis gives the conclusion since the graph is an arc. Suppose that any dendrite in A having k or less end points ($k>0$) maps onto a dendrite in B. Let D be a dendrite in A having $k+1$ end points. Let p be a branch point in D giving a decomposition $D=D_1+D_2+D_3$, where D_1, D_2, and D_3 are dendrites intersecting by pairs in p. Since D_1, D_2, D_3, D_1+D_2, D_1+D_3, D_2+D_3 are dendrites having at most k end points, each of their transforms is a tree. Hence $T(D_1) \cdot T(D_3)$ and $T(D_2) \cdot T(D_3)$ are con-

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([4]) If M is a locally connected compact and metric continuum and A is a closed subset of M containing every simple arc axb of M such that a and b are points of A, then A is called an A-set. By the cyclic chain in M determined by two points a and b of M and designated by $C(a, b)$ is meant the product of all A-sets in M containing both a and b. It is the minimal A-set in M containing both these points. The cyclic chains in M are closely related to the decomposition of M into its cyclic elements, for which see Kuratowski and Whyburn, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331. See also G. T. Whyburn, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, and W. L. Ayres, these Transactions, vol. 30 (1928), pp. 567–578, and vol. 31 (1929), pp. 595–695.

([5]) For definition, see §5.

nected; and since both of these sets contain $T(p)$, their sum $T(D_3) \cdot T(D_1+D_2)$ is connected. Thus $T(D) = T(D_1+D_2+D_3)$ is a dendrite.

(2.1) *If B is cyclic, T is A-set reversing.*

Proof. Otherwise there exists a simple arc b_1xb_2 in A such that $T(b_1) = T(b_2) = b$, but $T^{-1}(b) \cdot b_1xb_2 = b_1+b_2$. Let $T(b_1xb_2) = X$ and $K = T^{-1}(X)$. From (2.0) it follows that X is a dendrite; hence $X \neq B$. Thus $A - K \neq 0$.

(i) *For any component R of $A - K$, $T(F(R))$ is a single point*[6].

Otherwise there exists a simple arc cyd in $R+c+d$ such that c and d lie in $F(R)$ and $T(c) \neq T(d)$. But now if both c and d are on $b'xb''$ (in the order b', c, d, b'') we let $t = b'c+cyd+db''$; if not let t be a dendritic graph in A which contains both b_1xb_2 and cyd. This is impossible by (2.0) since in either case $T(t)$ must contain a simple closed curve.

(ii) *There exist two components R and S of $A - K$ such that $a = T(F(R))$ $\neq T(F(S)) = c$ and $T(R) \cdot T(S) \neq 0$.*

For let R be any component of $A - K$ and let Q be the sum of all those components U of $A - K$ such that $T(F(U)) = T(F(R)) = a$. Then since $Q+T^{-1}(a)$ is closed, it follows that $T(Q)+a$ is closed. Since a is not a cut point of B, it follows that some point x of $T(Q)$ must be a limit point of $B - T(Q)$. (We know that $T(Q)+a \neq B$ since X is not a single point.) But Q is open in A; consequently $T^{-1}(x)$ must intersect some component S of $A - K$ which does not belong to Q. Thus if we set $c = T(F(S))$, (ii) is satisfied.

Now to prove (2.1), let y' and y'' be points of R and S, respectively, such that $T(y') = T(y'') = y$. Then there exists a dendritic graph t' in A containing y' and b_1xb_2. If t' contains y'', let $t = t'$. Otherwise there exists a dendritic graph t in A containing both y'' and t'. It is immediate that $T(t)$ contains a simple closed curve, contradicting (2.0).

We have at once:

(2.11) *If A and B are both cyclic, T is a homeomorphism.*

(2.2) *In order that a single valued continuous transformation $T(A) = B$ shall be tree-preserving it is necessary and sufficient that the image of every arc in A shall be a tree in B.*

Proof. The necessity is trivial. To prove the sufficiency suppose A is a tree and that B has a true cyclic element E_b. Let $W(B) = E_b$ be monotone and retracting. Then $WT(A) = E_b$ is a transformation which maps arcs into trees. Thus, by (2.1), WT is A-set reversing, hence monotone. But this makes E_b a tree, which is absurd.

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[6] For any open set G, $F(G)$ denotes the boundary of G, that is, the set $\overline{G} - G$.

(2.3) *If B is cyclic and no A-set in A other than A itself maps onto all of B, then A is cyclic and T is a homeomorphism.*

Proof. For if A had a cut point p, we could write $A = A_1 + A_2$, where A_1 and A_2 are A-sets with $A_1 \cdot A_2 = p$. Then since, by (2.1), T is monotone, either $A_1 - p$ or $A_2 - p$, say $A_1 - p$, contains the set $T^{-1}(B - T(p))$, as this latter set is connected. But this gives $T(A_1) = B$.

(2.4) *If B is cyclic, T is equivalent to a monotone transformation retracting A onto some true cyclic element of A. Thus, in this situation, "arc-preserving," "tree-preserving," "A-set reversing," and "monotone retracting" are all equivalent.*

Proof. For let E_a be a minimal A-set in A mapping onto all of B under T. Then since $T(E_a) = B$ is a transformation mapping arcs into trees, it follows from (2.3) that E_a is a cyclic element of A and E_a maps onto all of B topologically under T. Thus if for each y in B we set $h(y) = E_a \cdot T^{-1}(y)$, then h is topological and the transformation $hT(A) = E_a$ is retracting. Furthermore, hT is monotone since both h and T are monotone. Obviously hT is equivalent to T, since $h^{-1}(hT) = T$.

DEFINITION. *For any true cyclic element E_a of A we define E_a^0 as the set of all internal points of E_a, that is, all points x of E_a which are non-cut points of A. It is well known that the set of all non-internal points of any such E_a is countable.*

(2.5) *For each true cyclic element E_b of B there exists a unique true cyclic element E_a of A such that $T(E_a) = E_b$. The transformation T is a homeomorphism on E_a and T^{-1} is single valued on $T(E_a^0)$.*

Proof. For let $W(B) = E_b$ be monotone and retracting. Since $WT(A) = E_b$ is a transformation which maps arcs into trees, it follows from (2.4) that there exists a true cyclic element E_a of A and a homeomorphism $h(E_b) = E_a$ such that $hWT(A) = E_a$ is monotone and retracting. Since E_a maps topologically under hWT, it must therefore map topologically under T. Let $y = T(x)$ be a point of $T(E_a^0)$, where x lies in E_a^0. Since x is an internal point of E_a and hWT is monotone and retracting, we see that $x = (hWT)^{-1}(x) = T^{-1}W^{-1}h^{-1}(x) = T^{-1}W^{-1}(y) \supset T^{-1}(y)$, and hence $x = T^{-1}(y)$. The uniqueness of E_a follows at once from this single-valuedness of T^{-1}.

(2.6) *Let A be a compact locally connected continuum and $T(A) = B$ be continuous. Then in order that T be tree-preserving it is necessary and sufficient that for each true cyclic element E_b of B there exist a true cyclic element E_a of A mapping onto E_b topologically under T and such that T^{-1} is single valued on the set $T(E_a^0)$.*

Proof. The necessity follows from (2.5). To establish the sufficiency we need only, in virtue of (2.2), show that the image of every simple arc t in A

is a tree in B. Assuming the contrary, $T(t)$ must contain a simple closed curve J'. Let E_b be the true cyclic element of B containing J' and E_a the true cyclic element of A which satisfies the conditions of the theorem. It follows at once that $t \cdot E_a$ is a simple arc axb which maps into an arc $a'x'b'$ of J'. Let $a'y'b'$ be the other arc of J', and suppose that y' is the image of an internal point of E_a. Then $T^{-1}(y')$ contains a point of E_a and a point of $t - E_a$, which is impossible.

(2.7) *Let A be a compact locally connected continuum and let $T(A) = B$ be continuous. Then in order that T be arc-preserving it is necessary and sufficient that it be tree-preserving and that the image of each cyclic chain([7]) in A be a cyclic chain in B.*

Proof. *Necessity*: The first condition is necessary by (2.2). That the second condition is necessary results essentially from the fact that, for arc-preserving transformations, (1) A-sets map onto A-sets, and (2) the property of having any three points on an arc is invariant.

We first show that A-sets map onto A-sets. Let A' be an A-set in A and $T(A') = B'$. For any cyclic element E_b of B intersecting B' in at least two points, let E_a be the corresponding cyclic element of A given by (2.6). Since $E_b \cdot B'$ is a nondegenerate continuum and $T(E_a) = E_b$ is topological, $E_b \cdot B'$ must contain the image y of at least one internal point x of E_a. Then since $x = T^{-1}(y)$ we see that x must lie in A'. Thus E_a is contained in A', consequently E_b is contained in B'. Therefore, B' is an A-set in B.

Now to prove the necessity of the second condition of the theorem, let $C(a, b)$ be a cyclic chain in A. Then $T(C(a, b)) = K$ is an A-set in B. Let x, y, z be points of K and x', y', z' be points of $C(a, b) \cdot T^{-1}(x)$, $C(a, b) \cdot T^{-1}(y)$, $C(a, b) \cdot T^{-1}(z)$, respectively. There exists an arc cd in $C(a, b)$ containing x', y', and z'. Hence $T(cd)$ is an arc in K containing x, y, and z. Therefore, K is a cyclic chain (since for A-sets the property of being a cyclic chain is equivalent to the property of containing an arc through any three points).

Sufficiency: Let ab be any simple arc in A. We first show that if E_b is any true cyclic element of B such that $E_b \cdot T(ab)$ is nondegenerate and E_a is the corresponding cyclic element of A given by (2.6) and xy is the arc $E_a \cdot ab$, then $T(ax + yb) \cdot E_b = x + y$. (We may suppose the order a, x, y, b.) If this is not so, then $T(ax) \cdot E_b$ or $T(yb) \cdot E_b$, say $T(ax) \cdot E_b$, is a nondegenerate continuum; hence there must exist a point z distinct from $T(x)$ of $T(E_a^0)$ which belongs to $E_b \cdot T(ax)$. This contradicts (2.6), since $T^{-1}(z)$ intersects both E_a and $ax - x$. Thus $E_b \cdot T(ab)$ is a simple arc $x'y' = T(xy)$. Furthermore, no interior point of $x'y'$ is a limit point of $T(ab) - x'y'$.

Now if $T(ab)$ is not a simple arc it cannot be a simple closed curve. This follows either from (2.2) or from what was just shown. Hence $T(ab)$ must con-

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([7]) See footnote 4.

tain a triod $oc+od+oe=t$. But, since T is cyclic chain-preserving, $T(C(a, b))$ must contain a simple arc through the three points c, d, e, say cde. Then either cd or de does not contain o, say de does not contain o. Thus $od+oe+de$ contains a simple closed curve J_b containing nondegenerate subarcs od' and oe' of od and oe; and if E_b is the cyclic element of B containing J_b, $E_b \cdot T(ab)$ $\supset E_b \cdot t \supset d'e' = od' + oe'$ and o is a limit point of $T(ab) - d'e'$ contrary to what was shown above.

3. **Supplementary results.** We give here some additional results, throwing light on the action of arc-preserving transformations. We assume throughout this section that A is a compact locally connected continuum and that $T(A) = B$ is arc-preserving.

(3.1) *The image of each A-set in A is an A-set in B; the image of each cyclic chain in A is a cyclic chain in B.*

(3.2) *For each true cyclic element E_b of B there exists a unique true cyclic element E_a of A which maps onto E_b topologically under T.*

These are direct consequences of (2.6) and (2.7).

(3.3) *The image of every true cyclic element E_a of A is either a single point, a true cyclic element E_b of B, or a free arc of B which is also a cyclic chain of B. If $T(E_a) = E_b$, then T is topological on E_a.*

Proof. We have at once that $T(E_a)$ is a cyclic chain $C(x, y)$ in B, if we assume that it is not a single point. If $C(x, y)$ is a single true cyclic element E_b of B, then our conclusion follows at once. Hence we may assume that x and y are distinct and that there exists at least one point z which separates x and y in B.

Let x' and y' be points of E_a mapping into x and y, respectively, and let J be a simple closed curve in E_a containing x' and y', and define $J' = T(J)$. We first show that J' is a free arc of B. Regarding J as our space, we see that T is arc-preserving on J. If J' contains a true cyclic element F of itself, then by (3.2), $T(J) = F$ is a homeomorphism. This is impossible since x and y are separated in B by the point z. Thus J' is a dendrite. But J' contains no triod, since if it did we could easily find an arc of J having a triod in its image. Therefore, J' is a simple arc.

Assume that $J' = a'd'b'$ is not a free arc of B. Then there exists a triod t in B having d' as center and a', b' as two of its end points, where d' is some interior point of J'. Let c' be the other end point of t. Then the three arcs $a'd'$, $b'd'$, $c'd'$ of t are disjoint except for d' and we may let $\{c_i'\}$ be a sequence of distinct points on $c'd'$ converging to d' as a limit. It follows at once that there exists a point d in $T^{-1}(d')$ and a sequence of points $\{c_i\}$ in $\{T^{-1}(c_i')\}$ converging to d.

Since A is a locally connected continuum, there exists a region R in A

containing b but disjoint from both $T^{-1}(a')$ and $T^{-1}(b')$. This region is arc-wise connected and hence contains a simple arc cd, where c is one of the points c_i.

If d lies on J we let z be the first intersection of cd with J, and consider an arc G defined as the sum of cz and an arc of J intersecting both $T^{-1}(a')$ and $T^{-1}(b')$. It is immediate that $T(G)$ contains a triod, which is impossible since T is arc-preserving. Thus d does not lie on J.

Let z be the first intersection of the arc cd with the closed set $T^{-1}(J')$, and define cxe as a simple arc in A having the unique point e in common with J, and on cxe let y be the last intersection with cz. Define an arc H as follows: (a') if y is not z then H is the sum of the subarc zy of zc and the subarc ye of cxe; (b') if y is z then H is the sum of cz and the subarc ze of cxe. Let G be a simple arc of A composed of H and a subarc of J intersecting both $T^{-1}(a')$ and $T^{-1}(b')$. Then $T(G)$ contains a triod, which is impossible.

Thus J' is a free arc $a'b'$ of B. Thus every point of the open subarc xy of $a'b'$ must separate x and y in B. Accordingly, $C(x, y) = xy$ and $C(x, y)$ is a free arc of B.

(3.4)[8] *If a and b are two points of A having the same image point under T, then no true cyclic element of the chain $C(a, b)$ can map topologically under T. Thus each true cyclic element in $C(a, b)$ maps into either a single point or a free arc of B.*

Proof. Let $T(a) = T(b)$ and suppose there is a cyclic element E_α of $C(a, b)$ which maps onto a cyclic element E_β of B topologically. Let $aqrb$ be a simple arc in A where $aq \cdot E_\alpha = q$, $rb \cdot E_\alpha = r$. Then either aq or rb is nondegenerate, and we may suppose aq is nondegenerate (rb may or may not be nondegenerate). Then $T(aq+rb) = K$ is a continuum. Furthermore, $K \cdot E_\beta$ contains the two distinct points $T(q)$ and $T(r)$. Hence $K \cdot E_\beta$ contains a point x distinct from both $T(q)$ and $T(r)$ which is the image of an internal point x_0 of E_α. This is impossible, since $T^{-1}(x)$ also intersects $aq+rb-(q+r)$, whereas by (2.7) and (2.6), $T^{-1}(x)$ must consist of a single point.

4. **Dendrite-preserving property of arc-preserving transformations.** If A is a dendrite (or tree) and $T(A) = B$ is arc-preserving, then it follows at once from (2.2) that T is dendrite-preserving. This fact can also be seen from (2.5), since if B had a true cyclic element E_b then A would also have one. This dendrite-preserving property of arc-preserving transformations was first noted and proven by R. G. Simond[9] as mentioned in the introduction of this paper. However, it is interesting to note that it follows directly from (2.4) of A.P.T. by the reasoning just given above, since the irreducibility of T

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(8) This is closely related to (2.3) of A.P.T. and, indeed, yields (2.3) of A.P.T. as a special case.

(9) See footnote 3.

assumed in (2.4) of A.P.T. does not limit the generality because we can take a sub-dendrite of A on which T is irreducible.

In view of the considerable length and difficulty of Miss Simond's proof, the following one which is self-contained and independent of all other results on arc-preserving transformations may be of interest.

THEOREM (Simond). *Arc-preserving transformations are dendrite-preserving.*

Proof. Let $T(A) = B$ be arc-preserving, where A is a dendrite. We first show:

I. (*Whether A is a dendrite or not.*) *If $t = oa + ob + oc$ is a triod such that $T(o) = o'$, $T(a) = a'$, $T(b) = b'$, $T(c) = c'$, and if $t \cdot T^{-1}(a' + b' + c') = a + b + c$, then $T(oa) \cdot T(ob) = T(oa) \cdot T(oc) = T(ob) \cdot T(oc) = T(o) = o'$.*

For if, say, $T(oa) \cdot T(ob)$ contains a point q' distinct from o', we may suppose the order a', o', q', b' on the arc $a'o'b' = T(aob)$. Then $T(ao)$ is a subarc $a'o'q'$ of $a'o'b'$. Hence the arc $T(aoc)$ consists of $a'o'q'$ plus an arc $q'c'$ from q' to c' which contains neither o' nor b'. But then the arc $T(boc) = b'o'c'$ would contain both the arc $o'q'b'$ of $a'o'b'$ and the arc $q'c'$, which is impossible since clearly $o'q'b' + q'c'$ contains a triod. This proves I.

Now suppose, contrary to the theorem, that B has a true cyclic element B'. Let A' be a minimal A-set in A such that $T(A')$ contains B'. Then since A' is a dendrite but not an arc, there exists a point o in A' and three continua X, Y, Z such that $A' = X + Y + Z$ and $X \cdot Y = Y \cdot Z = Z \cdot X = o$. Let $T(o) = o'$. Since B' is cyclic and $T(Y+Z)$ does not contain B', there exists a point q' in $B' - o'$ and points x in X, y in $(Y+Z)$, such that $T(x) = T(y) = q'$. Clearly we may suppose y in Y. Take the arcs xo and yo in X and Y respectively. Then since both $T(xo)$ and $Y(yo)$ contain arcs from o' to q' whereas $T(xo + oy)$ must be a simple arc, clearly $T(xo) \cdot T(oy)$ contains an arc from o' to q'. Hence there is no loss of generality in assuming (as we shall do) that both x and y are cut points of A'. Let R_x and R_y be components of $A' - x$ and $A' - y$ lying in $X - x$ and $Y - y$ respectively. Then since no one of the sets $T(A' - R_x)$, $T(A' - R_y)$, $T(X + Y)$ contains B', there exist points a', b', and c' in B' such that $T^{-1}(a') \cdot A' \subset R_x$, $T^{-1}(b') \cdot A' \subset R_y$, $T^{-1}(c') \cdot A' \subset Z - o$. Let $a \, \varepsilon \, T^{-1}(a')$, $b \, \varepsilon \, T^{-1}(b')$, and $c \, \varepsilon \, T^{-1}(c')$ be so chosen that for the arcs ax, yb, and oc in A' we have $ax \cdot T^{-1}(a') = a$, $yb \cdot T^{-1}(b') = b$, and $T^{-1}(c') \cdot oc = c$. Let $oa = ox + xa$, $ob = oy + yb$. Then $t = oa + ob + oc$ is a triod satisfying the conditions in I. However, since each of the sets $T(oa)$ and $T(ob)$ contains both o' and q' we have a contradiction to I. Thus B can have no true cyclic element and hence must be a dendrite.

5. A-set reversing transformations. In conclusion, we give a characterization of A-set reversing transformations which is made possible by (2.6) and (2.7). We recall that $T(A) = B$ is A-set reversing provided that for each b

in B, $T^{-1}(b)$ is either a single point or an A-set in A, it being assumed that A is a compact locally connected continuum. We make use here of certain results concerning this type of transformation which were established in §4 of A.P.T.

THEOREM. *If A is a compact locally connected continuum and $T(A) = B$ is arc-preserving, then in order that T be A-set reversing it is necessary and sufficient that the following conditions hold*: (a) *there exists no true cyclic element E in A such that $T(E)$ is a free arc of B*; (b) *if K is the set of all cut points and end points of A, then T is monotone on $T(K)$.*

Proof. The necessity follows at once from the definition and the fact that every A-set reversing transformation is monotone (A.P.T., (4.12)).

Sufficiency: By (A.P.T., (4.1)) we must show that T is monotone on each simple arc in A. If this is not the case, there exists a simple arc axb in A such that $T(a) = T(b) \neq T(x)$ for any point x interior to axb. If a and b are conjugate points, they lie in the same true cyclic element E of A and it is immediate from (a) and (3.3) that T is monotone on axb. Thus there exists a point q interior to axb which separates a and b in A. It follows at once from (b) that a is an internal point of a true cyclic element E of A which maps topologically onto a true cyclic element F of B. But T^{-1} is single valued on $T(E^0)$, where E^0 represents the set of all internal points of E.

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