ON STRONG SUMMABILITY OF FOURIER SERIES

BY

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1. A series $\sum_0^\infty A_\nu$, or the corresponding sequence of partial sums $s_n = \sum_0^n A_\nu$, $n=0, 1, 2, \cdots$, is said to be strongly summable $(C, 1)$ with index k to the sum s if $k>0$ and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_0^n | s_\nu - s |^k = 0 \text{ (1)}. \tag{1.1}$$

It follows from Hölder's inequality that the larger k the stronger is the assertion (1.1). Furthermore, for $k=1$, (1.1) evidently implies $(C, 1)$ summability to the sum s.

Suppose now that $f(t)$ is a periodic function of the class L. Let its Fourier series be

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_\nu \cos \nu t + b_\nu \sin \nu t) \equiv \sum_0^\infty A_\nu(t); \tag{1.2}$$

let

$$\phi(x, t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\}. \tag{1.3}$$

Hardy and Littlewood proved (1913):

THEOREM I. *The Fourier series of an integrable function $f(t)$ is strongly summable $(C, 1)$ with index 2 at a point x if $f(t)$ is of integrable square in a neighborhood of x and if for some s*

$$\int_0^t \{\phi(x, u)\}^2 du = o(t) \qquad \textit{as } t \downarrow 0. \tag{1.4}$$

In this paper we shall restrict ourselves to the index $k=2$, and speak simply of this case as "strong summability." For generalizations of Theorem I and for further references consult Hardy and Littlewood [2] and Zygmund [5, chap. 10].

For the special case in which $\phi(t) \to 0$ as $t \downarrow 0$, Fejér [1] recently gave two new proofs of the strong summability of the series (1.2) at $t=x$. We shall simplify his device and use it to give two new proofs of Theorem I. The essence of Fejér's method is to introduce double integrals with positive kernels while using the $(C, 3)$ and Abel summability methods. Replacing the partial sums s_n by $s_n - \frac{1}{2}A_n$ we get simpler (and also positive) kernels.

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(1) For generalizations to other summability methods cf. [3, §§7, 8 and 11]. Numbers in brackets refer to the bibliography at the end of the paper.

In the last section we prove still another theorem of Hardy and Littlewood [2]:

THEOREM II. *If*

$$\int_0^t |\phi(x, u)| \, du = o(t) \qquad \text{as } t \downarrow 0, \tag{1.5}$$

then

$$\sum_0^n (s_\nu(x) - s)^2 = o(n \log n) \qquad \text{as } n \to \infty. \tag{1.6}$$

Our proof is shorter and simpler, not involving complex function theory. Hardy and Littlewood also proved, by constructing examples, that (1.6) is the sharpest asymptotic estimate implied by the assumption (1.5).

2. We prove first the following lemma.

LEMMA. *Let* $s_0^* = 0$, $s_n^* = s_n - \frac{1}{2}A_n$, $n = 1, 2, \cdots$; *if* $\lim_{n\to\infty} A_n = 0$, *and if one of the sequences* s_n, s_n^* *is strongly summable, so is the other.*

This follows from the identities

$$\begin{aligned}
\sum_1^n (s_\nu - s)^2 - \sum_1^n (s_\nu^* - s)^2 &= \frac{1}{2}\sum_1^n A_\nu(s_\nu + s_\nu^* - 2s) \\
&= \sum_1^n A_\nu(s_\nu - s) - \frac{1}{4}\sum_1^n A_\nu^2 \\
&= \sum_1^n A_\nu(s_\nu^* - s) + \frac{1}{4}\sum_1^n A_\nu^2.
\end{aligned}$$

In view of this lemma, we may deal with $s_n^*(x)$ instead of $s_n(x)$ while discussing the series (1.2). Now

$$s_n^*(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + t) \cot \tfrac{1}{2}t \sin nt \, dt = \frac{1}{\pi}\int_0^{\pi} \psi(x, t) \cot \tfrac{1}{2}t \sin nt \, dt, \tag{2.1}$$

where $\psi(x, t) = \frac{1}{2}\{f(x+t) - f(x-t)\} = \psi(t)$. Hence

$$s_n^*(x)^2 = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \psi(t)\psi(u) \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \sin nt \sin nu \, dt du,$$

and

$$\sum_1^n (n + 1 - \nu)^2 s_\nu^*(x)^2 = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \psi(t)\psi(u) \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \, R_n(t, u) dt du, \tag{2.2}$$

where $R_n(t, u) = \sum_1^n (n+1-\nu)^2 \sin \nu t \sin \nu u$.

If $f(t)\equiv 1$, then $A_\nu(t)\equiv 1$, $\nu=1, 2, \cdots$, and $s_\nu^*(x)=1$, $\nu=1, 2, \cdots$; hence from (2.2)

$$\begin{aligned}(2.3)\qquad \sum_1^n (n+1-\nu)^2 &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u\, R_n(t, u)dtdu \\ &= \tfrac{1}{6}n(n+1)(2n+1).\end{aligned}$$

Now

$$(2.4)\qquad R_n(t, u) > 0 \qquad \text{for } 0 < t < \pi,\ 0 < u < \pi;$$

the proof is elementary (cf. [4, §2]).

As a first application of (2.2), (2.3) and (2.4) we get:

If $|f(t)| \leqq 1$ *in* $|t| \leqq \pi$, *then* $\sum_1^n (n+1-\nu)^2 s_\nu^*(x)^2 \leqq \sum_1^n \nu^2$.

Also from (2.1)

$$1 = \frac{1}{\pi}\int_0^\pi \cot \tfrac{1}{2}t \sin nt\, dt, \qquad n = 1, 2, \cdots;$$

hence

$$(2.5)\qquad s_n^* - s = \frac{1}{\pi}\int_0^\pi \phi(x, t) \cot \tfrac{1}{2}t \sin nt\, dt, \qquad n = 1, 2, \cdots,$$

and, writing $\psi(t)$ for $\psi(x, t)$,

$$\begin{aligned}\sum_1^n (n+1-\nu)^2(s_\nu^* - s)^2 &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \phi(t)\phi(u) \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u\, R_n(t, u)dtdu \\ &\equiv I_n(\phi).\end{aligned}$$

Now (2.4) yields

$$(2.6)\qquad | I_n(\phi) | \leqq I_n(\bar{\phi}),$$

whenever $|\phi(t)| \leqq \bar{\phi}(t)$ in $0<t<\pi$.

The proof of strong summability at a point where $\phi(x, t)\to 0$ as $t\downarrow 0$ now follows as in Fejér's method. We note first that $I_n(\phi)=o(n^3)$ as $n\to\infty$ is a necessary and sufficient condition for the strong summability of the series (1.2) at $t=x$, or, what is the same, of the cosine series of $\phi(t)$ at $t=0$. This follows from the following general inequalities for an arbitrary sequence of positive quantities $p_\nu \geqq 0$:

$$(2.7)\qquad (n+1)^{-2}\sum_0^n (n+1-\nu)^2 p_\nu \leqq \sum_0^n p_\nu \leqq n^{-2}\sum_0^{2n} (2n-\nu)^2 p_\nu.$$

Next, (2.6) yields the following theorem:

Whenever $|\phi(t)| \leqq \bar{\phi}(t)$ *in* $0<t<\pi$, *the strong summability of the cosine series of* $\bar{\phi}(t)$ *at* $t=0$ *implies that of the series of* $\phi(t)$.

If now $\phi(t)\rightarrow 0$ as $t\downarrow 0$, then there is an interval $0\leqq t\leqq \delta$ in which $\phi(t)$ is bounded. We choose the majorant function $\bar{\phi}$ to be

$$\bar{\phi}(t) = \begin{cases} \max\limits_{0\leqq\tau\leqq t} |\phi(\tau)| & \text{if} \quad 0 \leqq t \leqq \delta \\ |\phi(t)| & \text{if} \quad \delta < t \leqq \pi. \end{cases}$$

Now $\bar{\phi}$ is continuous at $t=0$ and monotonic in $0<t<\delta$; hence its cosine series converges at $t=0$, and it is, consequently, strongly summable. Thus the series (1.2) is strongly summable at $t=x$.

3. The symmetry of the integrand gives

$$I_n(\phi) = \frac{1}{\pi^2}\iint_{0\leqq t\leqq u\leqq \pi} \cdots + \frac{1}{\pi^2}\iint_{0\leqq u\leqq t\leqq \pi} \cdots = \frac{2}{\pi^2}\iint_{0\leqq t\leqq u\leqq \pi} \cdots .$$

Furthermore for the function

$$\phi_1(t) = \begin{cases} 0 & \text{for} \quad 0 < t < \delta \\ \phi(t) & \text{for} \quad \delta < t < \pi, \end{cases} \tag{3.1}$$

$$I_n(\phi_1) = \frac{2}{\pi^2}\iint_{\delta\leqq t\leqq u\leqq \pi} \phi(t)\phi(u) \cdots = o(n^3) \qquad \text{as } n \rightarrow \infty,$$

since the cosine series of $\phi_1(t)$ converges to 0 at $t=0$. This yields the following result:

A necessary and sufficient condition that (1.2) *be strongly summable at* $t=x$ *is that for a fixed* $\delta>0$

$$I_n^{(\delta)}(\phi) = \frac{1}{\pi^2}\iint_{0\leqq t\leqq u\leqq \delta} \phi(t)\phi(u) \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \; R_n(t, u)dtdu = o(n^3)$$

as $n\rightarrow\infty$.

We now use this criterion to prove Theorem I.

Schwarz's inequality yields

$$I_n^{(\delta)}(\phi)^2 \leqq \frac{1}{\pi^4}\iint_{0\leqq t\leqq u\leqq \delta} \phi(t)^2 \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \; R_n(t, u)dtdu$$

$$\cdot \iint_{0\leqq t\leqq u\leqq \delta} \phi(u)^2 \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \; R_n(t, u)dtdu.$$

Hence

$$I_n^{(\delta)}(\phi) \leqq \frac{1}{\pi^2}\int_0^\delta \int_0^\delta \phi(t)^2 \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \, R_n(t, u)dtdu$$

$$\leqq \frac{1}{\pi^2}\int_0^\delta \left\{\phi(t)^2 \cot \tfrac{1}{2}t \int_0^\pi \cot \tfrac{1}{2}u \, R_n(t, u)du\right\} dt.$$

But

$$\int_0^\pi \cot \tfrac{1}{2}u \, R_n(t, u)du = \sum_1^n (n + 1 - \nu)^2 \sin \nu t \int_0^\pi \cot \tfrac{1}{2}u \sin \nu u \, du$$

$$= \pi \sum_1^n (n + 1 - \nu)^2 \sin \nu t;$$

and the relation

$$\int_0^\delta \phi(t)^2 \cot \tfrac{1}{2}t \sum_1^n (n + 1 - \nu)^2 \sin \nu t \, dt = o(n^3)$$

follows from Lebesgue's theorem on $(C, 1)$ summability applied to $\phi(t)^2$, using (1.4) and (2.7). This proves Theorem I.

4. We shall now apply the Abel-Poisson summability method. From (2.1) for $0<r<1$

$$\sum_1^\infty (s_n^*(x))^2 r^n = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \psi(t)\psi(u) \cot \tfrac{1}{2}t \cot \tfrac{1}{2}u \left(\sum_0^\infty \sin nt \sin nu \, r^n\right) dtdu$$

$$= \frac{4r(1 - r^2)}{\pi^2}\int_0^\pi \int_0^\pi \psi(t)\psi(u) \cos^2 \tfrac{1}{2}t \cos^2 \tfrac{1}{2}u$$

$$\cdot [1 - 2r \cos (u - t) + r^2)]^{-1}[1 - 2r \cos (u + t) + r^2]^{-1} dtdu.$$

Putting $f(t) \equiv 1$, we get

$$\frac{r}{(1 - r)} = \frac{4r(1 - r^2)}{\pi^2}\int_0^\pi \int_0^\pi \cos^2 \tfrac{1}{2}t \cos^2 \tfrac{1}{2}u [1 - 2r \cos (u - t) + r^2]^{-1}$$

$$\cdot [1 - 2r \cos (u + t) + r^2]^{-1} dtdu.$$

The integrand will be denoted by $P(t, u; r)$. Evidently

$$P(t, u; r) > 0 \qquad \text{for } 0 < t < \pi,\ 0 < u < \pi,\ 0 < r < 1. \tag{4.1}$$

If $|f(t)| \leqq 1$, $0<t<2\pi$, this yields

$$\sum_1^\infty s_n^*(x)^2 r^n \leqq \sum_1^\infty r^n \qquad \text{for } 0 < r < 1. \tag{4.2}$$

Similarly from (2.5)

$$\sum_1^\infty (s_n^* - s)^2 r^n = \frac{4r(1-r^2)}{\pi^2}\int_0^\pi\int_0^\pi \phi(t)\phi(u)P(t, u; r)dtdu \equiv A(\phi; r),$$

and (4.1) gives $A(\phi; r) \leqq A(\bar{\phi}; r)$ whenever $|\phi(t)| \leqq \bar{\phi}(t)$. We first remark that

$$A(\phi; r) = o\left(\frac{1}{1-r}\right) \qquad \text{as } r \uparrow 1$$

is a necessary and sufficient condition for strong summability; i.e., for $\sum_0^n (s_\nu^* - s)^2$ to be $o(n)$ as $n \to \infty$.

The necessity is obvious; the sufficiency follows from the inequality (valid for any $p_\nu \geqq 0$):

$$\sum_0^n p_\nu \leqq \left(1 - \frac{1}{n}\right)^{-n} \sum_0^n p_\nu \left(1 - \frac{1}{n}\right)^\nu \leqq 4 \sum_0^\infty p_\nu \left(1 - \frac{1}{n}\right)^\nu, \qquad n \geqq 2.$$

If now $\phi(t) \to 0$ as $t \downarrow 0$, then, using the same majorant as in §2, we obtain still another proof of strong summability at points where $\phi(t) \to 0$. It is similar to Fejér's second proof except that we use a simpler kernel.

To prove Theorem I we observe that for the function $\phi_1(t)$ of (3.1) evidently

$$A(\phi_1; r) = \frac{4r(1-r^2)}{\pi^2}\int_\delta^\pi\int_\delta^\pi \phi(t)\phi(u)P(t, u; r)dtdu = o\left(\frac{1}{1-r}\right). \tag{4.3}$$

This together with the symmetry of the integrand gives (as in §3):

A necessary and sufficient condition for strong summability of (1.2) *at* $t = x$ *is that for a fixed* $\delta > 0$

$$A_\delta(\phi; r) \equiv \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\int_0^\delta \phi(t)\phi(u)P(t, u; r)dtdu = o\left(\frac{1}{1-r}\right).$$

Again using Schwarz's inequality, we obtain

$$\begin{aligned} |A_\delta(\phi; r)| &\leqq \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\int_0^\delta \phi(t)^2 P(t, u; r)dtdu \\ &\leqq \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\left[\phi(t)^2\int_0^\pi P(t, u; r)du\right]dt. \end{aligned} \tag{4.4}$$

But

$$\begin{aligned} 4r(1-r^2)\int_0^\pi P(t, u; r)du &= \cot \tfrac{1}{2}t \int_0^\pi \cot \tfrac{1}{2}u \left(\sum_1^\infty \sin nt \sin nu\, r^n\right) du \\ &= \pi \cot \tfrac{1}{2}t \sum_1^\infty \sin nt\, r^n. \end{aligned}$$

The right-hand side of (4.4) is $o(1-r)^{-1}$ since the cosine series of $\phi(t)^2$ is Poisson summable at $t=0$ under assumption (1.4). We have thus proved Theorem I once again.

5. In this section we assume

$$\int_0^t |\phi(x,u)|\,du \equiv \Phi(x,t) = o(t) \qquad \text{as } t\downarrow 0. \tag{5.1}$$

From (4.3)

$$\begin{aligned} A(\phi;r) \leqq A(|\phi|;r) = \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\int_0^\delta |\phi(t)\phi(u)|\,P(t,u;r)dtdu \\ + o\left(\frac{1}{1-r}\right). \end{aligned} \tag{5.2}$$

The first term on the right is

$$\begin{aligned} A_\delta(|\phi|;r) &= \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\int_0^\delta |\phi(t)\phi(u)| \\ &\quad\cdot\frac{\cos^2\frac{1}{2}t\cos^2\frac{1}{2}u\,dtdu}{[(1-r)^2+4r\sin^2\frac{1}{2}(u-t)][(1-r)^2+4r\sin^2\frac{1}{2}(u+t)]} \\ &\leqq \frac{4r(1-r^2)}{\pi^2}\int_0^\delta\int_0^\delta |\phi(t)\phi(u)| \\ &\quad\cdot\left[(1-r)^2+\frac{r}{\pi^2}(u-t)^2\right]^{-1}\left[(1-r)^2+\frac{r}{\pi^2}(u+t)^2\right]^{-1}dtdu \end{aligned} \tag{5.3}$$

assuming $0<\delta<\pi/2$. Let $1-r<\delta$, and decompose the range of integration into $0\leqq u+t\leqq 1-r$ and $1-r\leqq u+t\leqq 2\delta$. For the first part, using (5.1), we get

$$\iint_{0\leqq u+t\leqq 1-r}\cdots < (1-r)^{-4}\left(\int_0^{1-r}|\phi(t)|\,dt\right)^2 = o((1-r)^{-2}). \tag{5.4}$$

Hence, for $r\uparrow 1$,

$$\begin{aligned} A_\delta(|\phi|;r) < o\left(\frac{1}{1-r}\right) + 4(1-r^2)\iint_{1-r\leqq u+t\leqq 2\delta}|\phi(t)\phi(u)|\,(u+t)^{-2} \\ \cdot[(1-r)^2+r(u-t)^2]^{-1}dtdu. \end{aligned} \tag{5.5}$$

Now, the last integral is

$$\begin{aligned} 2\iint_{0\leqq t\leqq u}^{1-r\leqq u+t\leqq 2\delta}\cdots \leqq 2\iint_{0\leqq t\leqq u}|\phi(t)\phi(u)|\,u^{-2} \\ \cdot[(1-r)^2+r(u-t)^2]^{-1}dtdu \equiv 2B_\delta(r). \end{aligned} \tag{5.6}$$

Furthermore

$$B_\delta(r) \leqq \int_{(1-r)/2}^{2\delta} u^{-2}|\phi(u)|\left(\int_0^u |\phi(t)|\,[(1-r)^2 + r(u-t)^2]^{-1}dt\right)du$$

$$< (1-r)^{-2}\int_{(1-r)/2}^{2\delta} u^{-2}|\phi(u)|\,\Phi(u)du$$

$$= (1-r)^{-2}O\left(\int_{(1-r)/2}^{2\delta} u^{-1}|\phi(u)|\,du\right),$$

and

$$\int_{(1-r)/2}^{2\delta} u^{-1}|\phi(u)|\,du = u^{-1}\Phi(u)\Big]_{(1-r)/2}^{2\delta} + \int_{(1-r)/2}^{2\delta} u^{-2}\Phi(u)du$$

$$= O(1) + \int_{(1-r)/2}^{2\delta} u^{-2}\Phi(u)du.$$

Thus from (5.2), (5.5) and (5.6), as $r \uparrow 1$

$$A(\phi; r) < o\left(\frac{1}{1-r}\right) + o\left(\frac{1}{1-r}\right) + O\left(\frac{1}{1-r}\right)$$

$$+ \frac{1}{1-r}O\left(\int_{(1-r)/2}^{2\delta} u^{-2}\Phi(u)du\right).$$

Finally

$$\int_{(1-r)/2}^{2\delta} u^{-2}\Phi(u)du = \left(\int_{(1-r)/2}^{\epsilon(r)} + \int_{\epsilon(r)}^{2\delta}\right)u^{-2}\Phi(u)du \equiv C_1(r) + C_2(r),$$

where we may assume

$$\tfrac{1}{2}(1-r) < \epsilon(r) \equiv \exp\left[-(\log(1-r)^{-1})^{1/2}\right] < 2\delta.$$

Now

$$C_1(r) \leqq \max_{u \leqq \epsilon(r)} u^{-1}\Phi(u)\int_{(1-r)/2}^{\epsilon(r)} u^{-1}du = \max_{u \leqq \epsilon(r)} u^{-1}\Phi(u)\left[\log \epsilon(r) - \log \tfrac{1}{2}(1-r)\right]$$

$$= \max_{u \leqq \epsilon(r)} u^{-1}\Phi(u)\left[\log\frac{2}{1-r} - \left(\log\frac{1}{1-r}\right)^{1/2}\right]$$

$$= o\left(\log\frac{1}{1-r}\right) \qquad \text{as } r \uparrow 1,$$

and

$$C_2(r) = O\left(\int_{\epsilon(r)}^{2\delta} u^{-1}du\right) = O\left(\log 2\delta + \left(\log\frac{1}{1-r}\right)^{1/2}\right) = o\left(\log\frac{1}{1-r}\right)$$

as $r\uparrow 1$. Summarizing,

$$A(\phi; r) < O\left(\frac{1}{1-r}\right) + o\left(\frac{1}{1-r}\log\frac{1}{1-r}\right) + o\left(\frac{1}{1-r}\log\frac{1}{1-r}\right),$$

or

$$\sum_{1}^{\infty}(s_n^* - s)^2 r^n = o\left(\frac{1}{1-r}\log\frac{1}{1-r}\right).$$

Putting $r=1-1/n$ yields

$$\sum_{1}^{n}(s_n^* - s)^2 = o(n\log n), \qquad \text{as } n\to\infty,$$

which proves Theorem II.

Addendum (May 27, 1940): To complete the proof of the criterion in §3 we remark that

$$\int_0^\delta\left\{\phi(t)\int_\delta^\pi\phi(u)\cot\frac{u}{2}R_n(t,u)du\right\}\cot\frac{t}{2}\,dt = o(n^3).$$

This follows easily from the fact that strong summability at a point is a local property of the function. A similar remark holds for the criterion of §4. To prove Theorems I and II we could also confine ourselves to the case $\delta=\pi$.

I have learned from Mathematical Reviews, vol. 1 (1940), p. 139, that T. Kawata (Proceedings of the Imperial Academy, Tokyo, vol. 15 (1939), pp. 243–246) also gave a simpler proof of Theorem II.

Literature

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