TOPOLOGY IN LATTICES

BY

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1. **Introduction.** Many mathematical systems are at the same time lattices and topological spaces. It is natural to inquire whether the topology in such systems is definable in terms of the order relation alone. Sequential topologies of this kind are well known. For example, in the case of the Boolean algebra M/N of measurable sets modulo null sets, and of continuous geometries, the usual topology is that of a metric space, distance being defined in terms of a modular functional [5, pp. 70, 100]. In the theory of partially ordered linear spaces, the notions of sequential order convergence and star convergence, defined by Garrett Birkhoff and Kantorovitch [5, p. 49; 11] in terms of the order relation, are of importance.

However, the topology of many important systems cannot be defined in terms of the convergence of sequences. In this paper, various nonsequential topologies are studied which are definable in terms of the order relation in a lattice. One of these is the topology of Moore-Smith order convergence of directed sets, introduced by Garrett Birkhoff [5, 7]. Another is the *interval* topology, which results on taking the closed intervals of the lattice as a subbasis for the closed sets of the topology. It is shown that in a lattice which is the direct product of chains, these two topologies are equivalent. With respect to its interval topology, any complete lattice is bicompact, but this is not true with the Moore-Smith topology.

Modifications of these topologies are introduced for the complete lattice of all closed sets of a topological space, and the relation of these topologies to those of transformation spaces is considered.

2. **Definitions.** Lattice meet, join, and inclusion will be denoted by $x \cap y$, $x \cup y$, and $x \leqq y$. A system of elements $\{x_a\}$, not necessarily distinct, is called a *directed set* if the subscripts are partially ordered in such a way that given any two subscripts a and b, there exists a third subscript c such that $a \leqq c$ and $b \leqq c$. By a *residual* set of $\{x_a\}$ is meant the set of all x_a such that $a \geqq b$, for some b. A set A of elements of $\{x_a\}$ is said to be *cofinal* if, given any index b, there exists an element x_a of A such that $a \geqq b$. A property is said to hold *residually* or *cofinally* if it holds for a residual or cofinal set of $\{x_a\}$.

By a *closed interval* of a lattice is meant either the entire lattice or a set of elements of one of the three types: (i) all $x \geqq a$, (ii) all $x \leqq b$, (iii) all x such that $a \leqq x \leqq b$. A collection K of closed sets of a space is said to be a *basis* if every closed set of the space is an intersection of sets of K, and K is said to be a *subbasis* if finite unions of sets of K form a basis. The notions of a basis

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and a subbasis for the open sets of a space are defined dually. The *interval topology* of a lattice is that which results on taking the closed intervals of the lattice as a subbasis for the closed sets of the space.

THEOREM 1. *Any lattice is a T_1 space with respect to its interval topology.*

Proof. A set made up of a single element a is a closed set, since it is a closed interval consisting of all elements x such that $a \leqq x \leqq a$. The other conditions for a T_1 space hold automatically.

A directed set $\{x_a\}$ of elements of a lattice L is said to *converge* to an element x of L in the *Moore-Smith order topology* if there exist monotonic directed sets $\{u_a\}$ and $\{v_a\}$ such that (1) $u_a \leqq x_a \leqq v_a$, (2) $\sup u_a = x = \inf v_a$, and (3) if $a \leqq b$, then $u_a \leqq u_b$, and $v_a \geqq v_b$. This notion of convergence in a lattice, denoted by $x_a \rightarrow x$, was introduced by Garrett Birkhoff [5, 7]. If the suprema and infima involved exist, as they will in a complete lattice, we may also define

$$\limsup x_a = \inf_b \left(\sup_{a \geqq b} x_a \right), \qquad \liminf x_a = \sup_b \left(\inf_{a \geqq b} x_a \right).$$

It is easy to see that $\liminf x_a \leqq \limsup x_a$, and $x_a \rightarrow x$ in the Moore-Smith order topology if and only if the two are equal.

The *closure* $\overline{A}$ of a set A of elements of a lattice L in terms of Moore-Smith convergence is defined as follows. An element x of L is in $\overline{A}$ if and only if there exists a directed set of elements of A converging to x. It can be verified that this closure operation has the two properties: (1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, (2) a set made up of a single element is closed. On the other hand it is not always true in the Moore-Smith topology that closures are closed, that is, that $\overline{\overline{A}} = \overline{A}$. Hence a lattice with this topology is not necessarily a T_1 space.

Two methods of topologizing the same set of elements will be called *equivalent* if they lead to the same definition of closure. It will be shown later by an example that the Moore-Smith order topology of a lattice is not always equivalent to the interval topology.

THEOREM 2. *The lattice operations $x \cap y$ and $x \cup y$ of a distributive lattice are continuous in the Moore-Smith order topology.*

Proof. As Garrett Birkhoff has remarked, it is sufficient to show that the operations are continuous in the variables separately [5, p. 30]. Suppose that $x_a \rightarrow x$, where $u_a \leqq x_a \leqq v_a$, u_a and v_a are monotonic, and $\sup u_a = x = \inf v_a$. Then if y is any element of the lattice, $u_a \cap y \leqq x_a \cap y \leqq v_a \cap y$, and it must be shown that $\sup (u_a \cap y) = (\sup u_a) \cap y$, and $\inf (v_a \cap y) = (\inf v_a) \cap y$. The second of these equalities holds in any lattice, and the first is a form of the infinite distributive law. The other half of the theorem is proved dually.

3. **Topology of chains.** By a *chain* is meant a linearly ordered set (sometimes called a simply ordered set). It is natural to define the topology of a

chain by taking the open intervals, consisting of elements x such that $a<x$, or $x<b$, or $a<x<b$, as a basis for the open sets of the space. This gives what Garrett Birkhoff calls the *intrinsic* topology of the chain.

THEOREM 3. *Both the Moore-Smith order topology and the interval topology of a chain are equivalent to the intrinsic topology of the chain.*

Proof. That the interval topology and the intrinsic topology are equivalent follows from the fact that in a chain, the closed and open intervals with one end point are complementary. Suppose now that $x_a \to x$ in the Moore-Smith topology, and u is the lower end point of an open interval containing x. Then $u<x$, and since sup $u_a = x$, ultimately $u<u_a \leqq x_a$. Likewise, if v is the upper end point of an open interval containing x, ultimately $x_a \leqq v_a < v$. Hence x_a is ultimately in any open interval containing x. Conversely, if every open interval containing x ultimately contains x_a, it follows that lim inf $x_a = x$ = lim sup x_a, and consequently $x_a \to x$.

It should be noted that the *sequential* order topology of a chain is not always equivalent to the intrinsic topology. For example, in the chain of all ordinal numbers less than or equal to Ω, the first noncountable ordinal, no sequence of ordinals less than Ω converges to Ω, although some directed sets of such ordinals do.

4. **Cartesian products and direct products.** There are several equivalent methods of defining the cartesian product P of a collection of non-empty topological spaces (X_a). The points x of P consist of selections (x_a) of one point x_a, called the a-coordinate of x, from each space X_a. The topology of P is assigned by taking as a subbasis for the neighborhoods of a point x of P, the collection of all sets of points of P whose a-coordinate is in some neighborhood U_a of the a-coordinate x_a of x. A basis for the neighborhoods of a point x consists of all finite intersections of such subbasic neighborhoods.

If the topology of the spaces (X_a) is given by the convergence of directed sets, an equivalent topology of P results on defining convergence of directed sets in P to mean coordinatewise convergence of the coordinates in the spaces X_a [15, p. 71]. Still a third equivalent topology, if the spaces X_a are T_1 spaces, is obtained by taking as a subbasis for the closed sets of P the collection of all sets of points x of P such that for all a, the a-coordinate of x lies in a subbasic closed set of the space X_a.

By the *direct product* L of a collection of lattices (L_a) is meant the lattice whose elements x are selections (x_a) of one element x_a, called the a-coordinate of x, from each lattice L_a. The elements x and y of L are ordered by the rule that $x \leqq y$ if and only if $x_a \leqq y_a$ for each a, where x_a and y_a are the a-coordinates of x and y. Garrett Birkhoff [5, p. 29] has shown that the *sequential* order topology of the direct product of a *finite* number of lattices is equivalent topologically to the cartesian product of the factors, each with the sequential order topology. With the interval topology or the Moore-Smith order topology, this result can be extended to the product of an infinite number of factors.

THEOREM 4. *The cartesian product P of any collection of lattices* (L_a), *each with the interval topology* (*Moore-Smith order topology*), *is homeomorphic to the direct product L of these lattices, also topologized by the interval topology* (*Moore-Smith order topology*).

Proof. Since P and L have the same elements, it remains to show that their topologies are equivalent. For the interval topology this follows from the fact that the same system of closed sets can be taken as a subbasis for both P and L, namely the system of all sets of elements x of P or L whose a-coordinate x_a is in a fixed closed interval of L_a for each a.

In the case of the Moore-Smith order topology, the result is most easily seen by using the definition of cartesian product in terms of coordinatewise convergence of directed sets. The theorem then follows from the fact that betweenness, monotonicity, and suprema and infima are defined coordinatewise in the direct product of lattices.

5. **Some examples.** It follows from Theorems 3 and 4 that the interval topology and Moore-Smith order topology are equivalent, in lattices which are direct products of chains, both to each other and to the cartesian product topology based on the intrinsic topologies of the chains. Many important lattices are of this type.

Let R stand for the lattice of all real numbers ordered by magnitude, and K for the lattice of all complex numbers, ordered as follows: $a+bi \leqq c+di$ if and only if $a \leqq c$ and $b \leqq d$. Then R is a chain and K is lattice-isomorphic with R^2. The usual topology of R and K is clearly equivalent to the interval topology and the Moore-Smith order topology. If a is any cardinal number, then the generalized euclidean or cartesian spaces R^a and K^a are partially ordered linear topological spaces isomorphic with the space of all real- or complex-valued functions defined over a set A of cardinal number a [2, 16]. Considered as lattices, these spaces are isomorphic to the direct product of a lattices, each isomorphic to R or K.

THEOREM 5. *The topology of the generalized euclidean spaces* R^a *and* K^a, *considered as cartesian products of a spaces each homeomorphic to R or K, is equivalent to both the interval topology and the Moore-Smith order topology of these spaces, considered as lattices.*

This follows from Theorems 3 and 4. The topology in question is, of course, that of pointwise convergence of directed sets of functions. In many subspaces of R^a and K^a, the two lattice topologies are also of importance. If the elements of R^a are represented as real-valued functions $f(x)$ defined over a set A of cardinal number a, then the subspace consisting of all elements $f(x)$ such that $|f(x)| \leqq m$ is the so-called *Tychonoff cube* J^a, the cartesian product of a intervals J, each of the form $[-m, m]$.

THEOREM 6. *The interval topology and the Moore-Smith order topology of the Tychonoff cube J^a, considered as a complete distributive lattice, are equivalent to the ordinary cartesian product topology of J^a.*

This follows from Theorems 3 and 4. It also follows that the space of all elements $f(x)$ of R^a such that $g(x) \leqq f(x) \leqq h(x)$, for $g(x)$ and $h(x)$ fixed, is a bicompact space homeomorphic to J^a with respect to the lattice topologies. There are other subspaces of R^a and K^a in which the lattice topologies are significant. These include the spaces m and c of bounded and convergent sequences, respectively, and M and C of bounded and continuous functions. In certain cases the lattice topologies are equivalent to the weak neighborhood topology of these spaces [1, 10].

6. **Lattices of sets and Boolean algebras.** An important subspace of R^a is the space 2^a of all characteristic functions $f(x)$, that is, of functions taking only the values 0 and 1 over a set A of cardinal number a. The space 2^a is homeomorphic and lattice-isomorphic to the complete Boolean algebra of all subsets of the set A, and also to the direct product of a two-element Boolean algebras. The usual topology of the space 2^a is that defined by Stone [14], which is equivalent to its relative topology as a subspace of the Tychonoff cube J^a. With this topology 2^a is zero-dimensional and bicompact.

THEOREM 7. *The interval topology and the Moore-Smith order topology of the complete atomic Boolean algebra 2^a are both equivalent to the Stone topology.*

This follows from Theorem 4. For the case of the Moore-Smith topology it was first proved by Tukey [15, p. 77].

Since 2^a is a complete lattice, its Moore-Smith order topology may be defined in terms of lim sup x_α and lim inf x_α, where $\{x_\alpha\}$ is a directed set of elements of 2^a. Since lim sup x_α consists of all points cofinally in x_α, and lim inf x_α of all points ultimately (residually) in x_α, these notions are a direct generalization of the notions of complete and restricted limit of a sequence of sets, familiar in set theory.

Since any Boolean algebra may be imbedded in an algebra of the form 2^a [13], though only with preservation of finite sums and products, it might be expected that the lattice topologies of a Boolean algebra can be obtained by topological relativization from those of the enveloping algebra 2^a, and are consequently zero-dimensional. Examples show that this is not the case. The fact that the space 2^a is zero-dimensional (totally disconnected) is a consequence of the existence of atomic elements, which are not necessarily present in a sub-algebra. It would be interesting to study the lattice topologies of the complete Boolean algebra M/N of measurable sets modulo null sets, and of the complete Boolean algebra of the regular open sets of a topological space. In its metric topology, and in the equivalent sequential order topology, the algebra M/N is a complete metric space, but it is not bicompact. In its interval topology, however, M/N is bicompact, as will be shown.

7. **Bicompactness.** A space in which closures are defined is said to be *bicompact* if there is a point common to the closures of any collection of sets, if any finite number of the sets have a common point. In a T_1 space, an equivalent condition is that there be a point common to all sets of any collection of basic closed sets, any finite number of which have a common point.

THEOREM 8. *A T_1 space T is bicompact, if there exists a subbasis S for the closed sets of T, such that there is a point common to the members of any collection of sets of S, any finite number of which have a common point.*

This was proved by J. W. Alexander [3]. It can also be proved by means of a complicated argument due to H. Wallman [17, pp. 123–124]. The following proof is simpler.

If K is any collection of basic closed sets of the space T having the finite intersection property, extend K to be maximal by Zorn's lemma [15, p. 7] and call the extended collection M. Since S is a subbasis, any set m of the collection M is the union of a finite number of sets s_r of S. At least one of the sets s_r is also a member of M, since M is maximal. The elements s of S which are also in M have the finite intersection property, hence by hypothesis there is a point p common to them all. The point p is also in every set of K, since each such set is the union of subbasic sets s_r, at least one of which is in M. Hence the space T is bicompact.

THEOREM 9. *Every complete lattice is a bicompact space in its interval topology.*

Proof. A lattice L is said to be *complete* (or continuous) if all subsets of its elements have suprema and infima. Since the maximal and minimal elements I and O exist in a complete lattice, the closed intervals $[x, y]$ with two end points form a subbasis for the closed sets of the interval topology of L. Suppose a collection $\{J_a\}$ of closed intervals $[x_a, y_a]$ is given, every finite number of which have a common element. Then for every pair of indices a and b, we must have $x_a \leqq y_b$, since otherwise the intervals J_a and J_b would have no common element. Since L is complete, sup x_a and inf y_a exist, and sup $x_a \leqq$ inf y_a. Either of these elements is clearly common to all the intervals $\{J_a\}$. Hence, by Theorem 8, L is bicompact. This completes the proof.

8. **Unsolved problem number eleven.** The following is the eleventh of a list of seventeen unsolved problems given in Garrett Birkhoff's *Lattice Theory*, p. 146.

Is every complete lattice topologically bicompact? Is it true that if the intersection of any family of subsets of a complete lattice is void, and if the subsets are closed relative to Moore-Smith convergence, then there exists a finite subfamily having a void intersection?

As we have just seen (Theorem 9), for the interval topology of a complete lattice, the answer is affirmative. With the Moore-Smith order topology, however, the answer is negative. Consider the complete lattice F of all closed sets of a compact metric space. For simplicity, consider a space S consisting of a point x, and a sequence of points x_n converging to x. The lattice F of closed sets consists of finite sets and of infinite sets containing x. Consider the sets R_m consisting of all x_n such that $n \geqq m$. Each R_m may be thought of as consisting of elements of the lattice F, each made up of a single point.

There is an element common to any finite number of sets R_m, but no element common to them all. It remains to prove that these sets are closed relative to the Moore-Smith order topology of F. Suppose $\{x_a\}$ is any directed set of elements of a particular set R_m, which is not ultimately constant. It is easy to see that lim inf x_a is the empty set. However, lim sup $x_a = \inf v_a$, where v_a is a monotonic directed set of non-empty closed sets of a bicompact space. Hence lim sup x_a is not empty, and the directed set $\{x_a\}$ does not converge. Since the only directed sets of elements of R_m which converge are ultimately constant, R_m is closed in the Moore-Smith topology. Hence F is not bicom pact.

THEOREM 10. *In the complete lattice of all closed sets of a compact metric space, the interval topology is in general distinct from the Moore-Smith order topology.*

THEOREM 11. *There exist complete lattices which are not bicompact with re-respect to the Moore-Smith order topology.*

9. **Comparison of the two lattice topologies.** Of two methods of topologizing the same set of elements, that is, of assigning closures to subsets, one will be called *weaker* than the other if it assigns larger closures to the same sets. In the weaker topology there are fewer closed and open sets. Alexandroff and Hopf [4, p. 62] use the terms *weaker* and *stronger* in exactly the opposite sense. In the case of the lattice F of closed sets of a space, we have seen that the interval topology is sometimes weaker than the Moore-Smith topology. This is true in general.

THEOREM 12. *The interval topology of a lattice is weaker than or equivalent to the Moore-Smith order topology.*

Proof. Suppose the directed set $\{x_a\}$ of elements of a lattice L converges to an element x in the Moore-Smith order topology. In order to prove the theorem, it is sufficient to show that every neighborhood of x in some subbasic system of neighborhoods for the interval topology contains a residual set of $\{x_a\}$. Sets consisting of all elements x not $\leqq b$, or all x not $\geqq c$, being complements of closed intervals, form a subbasis for the neighborhoods of the interval topology. Since $x_a \rightarrow x$, there exist monotonic directed sets $\{u_a\}$ and $\{v_a\}$ such that $u_a \leqq x_a \leqq v_a$, and $\sup u_a = x = \inf v_a$. Suppose x is not $\leqq b$. Then it must be shown that ultimately x_a is not $\leqq b$. If this were not true, then $x_a \leqq b$ cofinally. Then since $u_a \leqq x_a$, we would have $u_a \leqq b$ cofinally, and since u_a is monotonic increasing, $u_a \leqq b$ for all a. It would follow that sup $u_a = x \leqq b$, which contradicts the assumption that x is not $\leqq b$. Likewise by duality, if x is not $\geqq c$, then ultimately x_a is not $\geqq c$. This proves the theorem.

10. **Tychonoff's theorem.** As J. W. Alexander has remarked [3], Tychonoff's theorem on the bicompactness of cartesian products is an easy consequence of Theorem 8.

THEOREM 13. *The cartesian product of bicompact T_1 spaces is a bicompact T_1 space.*

Proof. As a subbasis for the closed sets of the product space P we can take the system S of all closed sets of P of the form $F = \Pi_a F_a$, where F_a is an arbitrary closed set of the component space E_a. Since there is clearly a point common to the sets of any subcollection K of S which has the finite intersection property, the space P is bicompact by Theorem 8.

11. **The lattice F of closed sets.** We have seen that in the lattice F of all closed sets of a topological space, the Moore-Smith and interval topologies are not always equivalent. In this case, however, neither of these topologies is satisfactory. In the first place they do not specialize, in the case of metric spaces, to the metric and sequential topologies of Hausdorff [4, p. 111; 9, p. 145]. In the second place they do not specialize by topological relativization, for the subspace consisting of closed sets made up of single points, to the topology of the original space, as would be desirable. However, slight modifications of these two lattice topologies which are also definable in terms of the order relation alone, turn out to be much more useful for the special lattice F, and will now be considered. They will be called the *neighborhood topology* and the *convergence topology*.

12. **The neighborhood topology of F.** An analogue of the interval topology, called the *neighborhood* topology, is defined as follows. Two types of subbasic open sets (neighborhoods) of elements of the lattice F will be considered. Neighborhoods of the *first type* will consist of all elements x of F such that x is not $\leqq a$, where a is any element of F. These are also neighborhoods in the interval topology of F. Neighborhoods of the *second type* will consist of all elements of F not meeting a fixed element b of F, that is, of all x such that $x \cap b = O$, where O is the empty set, that is, the zero element of the lattice. The collection of all finite intersections of neighborhoods of the first or second type is taken as a basis for the open sets of F in the neighborhood topology.

13. **The convergence topology of F.** In the case of the lattice F of closed sets, a more useful limit topology than the Moore-Smith order topology is obtained by retaining the definition of lim sup x_a, while replacing lim inf x_a by a different kind of lower limit. We define LL x_a (which can be read the *lower*

limit of x_a) to be inf (sup A) for all *cofinal* sets A of elements of the directed set $\{x_a\}$. It is clear that LL $x_a \leqq \lim\sup x_a$, since lim sup x_a is by definition inf (sup B) for all *residual* sets B. We write $x_a \rightarrow x$, and say that the directed set $\{x_a\}$ converges to x in the *convergence* topology of F, if and only if LL $x_a = \lim\sup x_a = x$.

Neither the neighborhood topology nor the convergence topology is self-dual. The dual definitions, obtained by reversing the order relation and interchanging sup and inf, can be used to define topologies of the lattice dual to F, consisting of all *open* sets of the space. The definition of the convergence topology was given in the form above in order to show that it depends on the order relation alone. Actually, however, the convergence topology is a simple generalization to directed sets of the notion of topological limit (upper and lower closed limit) of a sequence of sets, due to Hausdorff [9, p. 147].

THEOREM 14. *If* $\{x_a\}$ *is a directed set of elements of the lattice* F *of all closed sets of a* T_1 *space* S, *then* lim sup x_a *is the set of all points* p *of* S *such that every neighborhood of* p *contains a point of all the elements of some cofinal set of elements of* $\{x_a\}$, *while* LL x_a *is the set of all points* p *of* S *such that every neighborhood of* p *has a point in common with all the elements of some residual set of elements of* $\{x_a\}$.

Proof. If every neighborhood u of a point p has points in common with every element of a cofinal set of $\{x_a\}$, then every closed set which contains every element of some residual set of $\{x_a\}$, contains p. For if a closed set f contained a residual set of $\{x_a\}$ without containing p, then the complement u of f would be a neighborhood of p not meeting all the elements of any cofinal set. It follows that p is in lim sup x_a.

Conversely, if p is in lim sup x_a, then every closed set which contains a residual set of $\{x_a\}$ contains p. It follows that every neighborhood of p contains a point of every element of some cofinal set of $\{x_a\}$. For if there were a neighborhood u of p which did not, then f, the complement of u, would contain a residual set of $\{x_a\}$ without containing p.

The second part of the theorem states that p is in LL x_a if and only if every neighborhood of p has points in common with all elements of a residual set of $\{x_a\}$. This is proved in the same way as the first part, interchanging the words *cofinal* and *residual*.

THEOREM 15. *In its neighborhood topology, the lattice* F *of all closed sets* (1) *of a* T_1 *space is a* T_1 *space*, (2) *of a regular* T_1 *space is a Hausdorff space*, (3) *of a bicompact* T_1 *space is a bicompact* T_1 *space*.

Proof. If a and b are elements of F and a is not $\leqq b$, then a is a member of the neighborhood of the first type consisting of all elements x of F such that x is not $\leqq b$, but b is not a member of this neighborhood. If p is a point of b but not of a, then b is a member of the neighborhood of the second type consisting of all elements x of F such that $x \cap p = O$, but a is not a member. This shows that F is a T_1 space in its neighborhood topology.

Now suppose S is a regular T_1 space, and a and b are distinct closed sets of S such that a is not $\leqq b$. If p is a point of a but not of b, there exist disjoint open sets u and v of S, containing p and b, respectively, since S is regular. If u' and v' denote the complements of u and v, respectively, then the neighborhood of the first type, consisting of all elements x of F such that x is not $\leqq u'$, is disjoint from the neighborhood of the second type, consisting of all elements of F such that $x \cap v' = O$. Since these neighborhoods contain a and b, respectively, the lattice F has the Hausdorff separation property.

Finally, suppose S is a bicompact T_1 space. If x and y are elements of the lattice F of closed sets of S, and $x \leqq y$, consider the set $K(x, y)$ consisting of all elements z of F such that $x \cap z \neq O$ and $z \leqq y$. These sets $K(x, y)$, which correspond to the closed intervals of the interval topology, clearly form a subbasis for the closed sets of F in the neighborhood topology. Suppose (K_a) is a collection of such sets $K(x_a, y_a)$, any finite number of which have a common element. It follows that any finite number of the sets (y_a) have a common point. Let y be the intersection of the sets (y_a). Since S is bicompact, y is not empty, and it is clear that we have $x_a \cap y \neq O$ for all a. Hence y is common to all the sets (K_a). It follows from Theorem 8 that F is bicompact.

THEOREM 16. *In the lattice F of all closed sets of a regular T_1 space S, the convergence topology is weaker than or equivalent to the neighborhood topology.*

Proof. Suppose every neighborhood of an element x of F of both the first and second types contains a residual set of the directed set $\{x_a\}$ of elements of F. It must be shown that $x_a \to x$ in the convergence topology. First it will be shown that $x \leqq \text{LL } x_a$. Since every neighborhood of x of the first type contains a residual set of $\{x_a\}$, it follows that if $x_a \leqq u$ cofinally, then $x \leqq u$, where u is an element of F. Hence $x \leqq \sup A$, if A is any cofinal set of elements of $\{x_a\}$. Consequently $x \leqq \text{LL } x_a = \inf (\sup A)$ for all A.

Next it will be shown that $x \geqq \limsup x_a$. Suppose on the contrary that p is a point of $\limsup x_a$, but not of x. Now S is regular; hence there exist disjoint open sets u and v of S containing x and p, respectively. Since every neighborhood of x of the second type contains a residual set of $\{x_a\}$, and $x \cap u' = O$, where u' is the complement of u, then ultimately $x_a \cap u' = O$. Since p is in $\limsup x_a$, then cofinally $x_a \cap v \neq O$, by Theorem 14. But this is a contradiction, since $v \leqq u'$.

This proves that $x \geqq \limsup x_a$, and it has been shown that $x \leqq \text{LL } x_a$. It follows that $x = \limsup x_a = \text{LL } x_a$, and consequently $x_a \to x$ in the convergence topology, which was to be proved.

THEOREM 17. *The neighborhood and convergence topologies are equivalent in the lattice F of all closed sets of a bicompact Hausdorff space S.*

Proof. Since a bicompact Hausdorff space is regular, the convergence topology is weaker than or equivalent to the neighborhood topology of F by Theorem 16. It remains to prove the reverse. Suppose then that the directed set $\{x_a\}$ of elements of F converges to x, that is, $x = \text{LL } x_a = \lim\sup x_a$. If $x_a \leqq u$ for a cofinal set A, then $\sup A \leqq u$, hence $\text{LL } x_a = \inf (\sup A) = x \leqq u$. Hence any neighborhood of x of the first type contains a residual set of $\{x_a\}$.

Now suppose some neighborhood of x of the second type does not contain a residual set of $\{x_a\}$, that is, that $x \cap f = O$, but x_b contains a point p_b of f for a cofinal set $\{x_b\}$ of $\{x_a\}$. Since the space S is bicompact, the directed set of points $\{p_b\}$ has a *cluster* point p, such that every neighborhood of p in S contains a cofinal set of $\{p_b\}$ [15, p. 36]. Since f is closed, p is in f. But by Theorem 14, p is in $\lim\sup x_a = x$. This is a contradiction, since $x \cap f = O$. This completes the proof.

THEOREM 18. *If D is any set of points everywhere dense in a T_1 space S, then the collection of all finite subsets of D is everywhere dense in the lattice F of all closed subsets of S, with respect to the neighborhood topology of F.*

Proof. Consider the collection K of all elements of F which are contained in an open set G of S, and have at least one point in common with each of a finite number of open sets G_r of S, each contained in G. It can be seen that the family (K) of all such collections K is a basis (not merely a subbasis) for the open sets of F in the neighborhood topology. But every such collection K contains at least one finite subset of the everywhere dense set D, namely a subset consisting of one point of D selected from each non-empty set G_r. Hence the finite subsets of D are everywhere dense in F, which was to be proved.

The following two theorems, which show the relation of the two topologies of F to other topologies, are easily established and will be stated without proof.

THEOREM 19. *The neighborhood topology and the convergence topology of the lattice F of all closed subsets of a compact metric space S are both equivalent to the Hausdorff metric topology of F.*

THEOREM 20. *The topologies of a Hausdorff space S obtained by topological relativization from the neighborhood topology and the interval topology of the lattice F of all closed sets of S, by considering the points x of S to be elements of F made up of single points, are both equivalent to the original topology of S.*

14. **Transformation spaces.** Since a transformation of a space X into a space Y is determined by its *graph*, which is a set of points in the product space $X \times Y$, any definition of convergence of sets leads to a definition of convergence of transformations.

THEOREM 21. *If $y(x)$ is a continuous transformation of a T_1 space X onto* *a Hausdorff space Y, then the graph G of $y(x)$ is a closed set of the cartesian product space $X\times Y$.*

Proof. By the *graph* G of $y(x)$ is meant the set of all points (x, y) of $P=X\times Y$ such that $y=y(x)$. If G is not closed, there is a point (a, b) of P which is in $\overline{G}$, but not in G. Then b is distinct from $y(a)$, and there exist neighborhoods U and V of b and $y(a)$ in Y which are disjoint. Since $y(x)$ is continuous, there is a neighborhood W of a in X such that $y(x)$ is in V if x is in W. The neighborhood $U\times W$ of (a, b) in P contains no points of G, since U and V are disjoint. Hence (a, b) is not in $\overline{G}$, contrary to assumption.

15. **Continuous convergence.** A directed set $\{y_a(x)\}$ of transformations of a space X into a space Y is said to *converge continuously* to the transformation $y(x)$ at a point c of X if for every neighborhood U of $y(c)$ there exists a neighborhood V of c and an index b such that if x is in V and $a\geqq b$, then $y_a(x)$ is in U.

The convergence is said to be *quasi-continuous* at c if for every neighborhood U of $y(c)$ there exists a neighborhood V of c such that if x is any point of V, then there exists an index b such that if $a\geqq b$, then $y_a(x)$ is in U.

THEOREM 22. *If $\{y_a(x)\}$ is a directed set of transformations of a T_1 space X into a regular T_1 space Y which converges pointwise to the transformation $y(x)$, then $y(x)$ is a continuous transformation if and only if the convergence is quasi-continuous at every point of X.*

Proof. If W is a neighborhood of $y(c)$, then there is a neighborhood U of $y(c)$ such that $\overline{U}\leqq W$. If the convergence is quasi-continuous at c, there is a neighborhood V of c such that if x is in V, then there is an index b such that $y_a(x)$ is in U if $a\geqq b$. This is the neighborhood V required by the definition of continuity of the limit transformation $y(x)$. For since the directed set converges pointwise to $y(x)$ at each point x of V, $y_a(x)$ is ultimately in any neighborhood of $y(x)$. But $y_a(x)$ is also ultimately in U, hence $y(x)$ is in $\overline{U}$ and consequently in W. This proves that $y(x)$ is continuous at $x=c$.

Conversely, if $y(x)$ is continuous at $x=c$ and U is any neighborhood of $y(c)$, then there is a neighborhood V of c such that $y(x)$ is in U if x is in V. Then U, being an open set, is also a neighborhood of $y(x)$, and there exists an index b such that $y_a(x)$ is in U if $a\geqq b$, from the convergence. Hence the convergence is quasi-continuous.

COROLLARY. *The limit of a continuously convergent directed set of transformations of a T_1 space into a regular T_1 space, is a continuous transformation.*

Note that it was not assumed that the transformations $y_a(x)$ were themselves continuous. Suppose now that $y_a(x)$ and $y(x)$ are *continuous* transformations of a T_1 space X into a Hausdorff space Y, and that G_a and G are the graphs of these transformations in the product space $P=X\times Y$. Then we have

THEOREM 23. *If the directed set $\{y_a(x)\}$ of transformations converges con-*

tinuously to $y(x)$, then the directed set $\{G_a\}$ of graphs converges to the graph G in the convergence topology of the closed sets of the product space P.

Proof. It is clear that $G \leqq \text{LL } G_a$, since every neighborhood of a point of G ultimately contains a point of G_a, from the convergence. Suppose now that the point (c, d) of P is in lim sup G_a, but not in G. Disjoint neighborhoods U and V of the distinct points d and $y(c)$ of Y exist. Since the convergence is continuous, there exists a neighborhood W of c and an index b such that $y_a(x)$ is in V, if x is in W and $a \geqq b$. Since the point (c, d) is in lim sup G_a, there is a point $(x, y_a(x))$ in the neighborhood $U \times W$ of P for some index $a \geqq b$. Since U and V are disjoint, this is a contradiction.

The converse of Theorem 23 is not always true. However, if the spaces X and Y are bicompact Hausdorff spaces, we have

THEOREM 24. *The directed set $\{y_a(x)\}$ of transformations from a bicompact Hausdorff space X to a bicompact Hausdorff space Y converges continuously to the transformation $y(x)$ if and only if the directed set of graphs $\{G_a\}$ converges to the graph G in the convergence topology (or in the equivalent neighborhood topology) of the closed sets of the product space.*

Proof. The necessity of the condition follows from Theorem 23. Suppose $\{y_a(x)\}$ does not converge continuously to $y(x)$ at $x = c$. Then there is a neighborhood U of $y(c)$ such that for every neighborhood V of c, $y_b(x_b)$ is not in U for a cofinal set of indices b, where x_b is in V. The directed set of points $\{x_b, y_b(x_b)\}$ has a cluster point (c, d), since the product space is bicompact. Since U is open, the point d is not in U. But if $G_a \rightarrow G$, the point (c, d), since it is not in G, lies in a neighborhood containing no points of a residual set of $\{G_a\}$, contrary to the assumption that (c, d) is a cluster point. This contradiction proves the sufficiency of the condition.

Theorems 23 and 24 suggest that the topology of continuous convergence might be a suitable one for the semi-group of all continuous transformations of a topological space into itself [8].

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