ERGODIC THEOREMS FOR ABELIAN SEMI-GROUPS[1]

BY

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This paper adapts the method of F. Riesz to the proof of certain general ergodic theorems for Abelian semi-groups of operators on a Banach space to itself. The main features of the method are that no measurability conditions are imposed on the semi-group under consideration and that consistent use of the second conjugate space and its compactness properties make it possible to replace the compactness conditions often imposed by a more natural restriction on the transforms of points. Theorem 1 and the various supplementary results include as special cases theorems of Lorch [10], Dunford [7], Yosida [15], F. Riesz [12], and Cohen [6]. It overlaps the work of Alaoglu and Birkhoff [3] at those points where they consider Abelian cases; for example, Corollary 8 is a great generalization of their Theorem 5.

Section 1 contains some introductory material on conjugate spaces and adjoint operations. Section 2 introduces bounded Abelian semi-groups of operators and near invariance of a system of set functions on such a semi-group; this section also contains the principal theorem (Theorem 1) of the paper. The form of this theorem raises three questions ((A) to (C) at the beginning of §3). The answer to (A) shows, among other things, that every Abelian semi-group has a property much like "ergodicity" in the sense of Alaoglu and Birkhoff; Theorem 3 is the main result here. The answer to (B) again indicates the importance of reflexivity in theorems of this type; Corollary 8 is one example. Two special cases of (C) give a generalization of Dunford's theorem (Theorem 5) and a theorem on bounded Abelian semi-groups of projections (Theorem 6) which has not, so far as I know, been considered before.

1. **Some properties of Banach spaces.** If B is a Banach space[2], let B^* be the set of all linear—that is, additive and continuous—real-valued functions on B. If, for β in B^*, $\|\beta\| = \sup_{\|b\| \leq 1} |\beta(b)|$, then B^* is also a Banach space. As is usual, the weak neighborhood topology in B is defined as follows: For each b_0 in B the weak neighborhoods of b_0 are the sets of the form[3] $\{b \mid |\beta_i(b) - \beta_i(b_0)| < \epsilon \text{ for } i = 1, \cdots, k\}$ for every choice of $\epsilon > 0$, k a positive integer, and $\beta_1, \cdots, \beta_k$ in B^*. With this topology B is a linear topological

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(1) Part of the work on this paper was done at the Institute for Advanced Study while the author held the Corinna Borden Keen Research Fellowship of Brown University.

(2) See Banach [4]. A Banach space is a complete norm vector space.

(3) $\{b \mid \cdots\}$ is the set of all points b satisfying the condition after the vertical bar.

space (see Wehausen [12]) and hence is a regular Hausdorff space in which addition of elements and multiplication by real numbers are continuous operations.

Since B^* is a Banach space, it has a weak topology defined just as in B; however, there is another topology in a conjugate space which cannot be defined in every space. This is the weak* topology in which the neighborhoods of a point β_0 in B^* are the sets of the form $\{\beta \mid |\beta(b_i)-\beta_0(b_i)|<\epsilon$ for $i=1, \cdots, k\}$ for all choices of $\epsilon>0$, k a positive integer, and $b_1, \cdots, b_k$ in B. Each weak* neighborhood of β_0 is a weak neighborhood of β_0 but the converse is not true unless B is reflexive. The importance of the weak* topology in this paper arises from

LEMMA 1. *The unit sphere in B^* is always compact*(4) *in the weak* neighborhood topology.*

This has been proved by Alaoglu [2], Kakutani [9], and Šmulian [13].

If B is a Banach space, let B^{**} be the space $(B^*)^*$. Then there is a natural imbedding of B in B^{**} which associates to each b in B the point $\mathfrak{b}_b$ in B^{**} such that $\mathfrak{b}_b(\beta)=\beta(b)$ for every β in B^*. B is *reflexive* if B fills up B^{**} under this imbedding. For the rest of this paper B will be considered to be imbedded in this way in B^{**} whenever it seems convenient.

If T is a linear operator defined on B with values in B, let T^*, the adjoint of T, be the operator on B^* to B^* such that $\beta(Tb)=T^*\beta(b)$ for every β in B^* and b in B. Then:

(1) $\|T^*\|=\|T\|$.

(2) $(T_1T_2)^*=T_2^*T_1^*$ so T_1^* and T_2^* commute if T_1 and T_2 do.

(3) If $T^{**}=(T^*)^*$, then T^{**} agrees with T in B; that is, $T^{**}\mathfrak{b}_b=\mathfrak{b}_{Tb}$, since $T^{**}\mathfrak{b}_b(\beta)=\mathfrak{b}_b(T^*\beta)=T^*\beta(b)=\beta(Tb)=\mathfrak{b}_{Tb}(\beta)$ for every β in B^*.

For brevity T will sometimes be used for T^{**}.

If Y is any set of elements y, M_Y is the Banach space of all real-valued bounded functions ϕ on Y with(5) $\|\phi\|_{M_Y}=\sup_{y\in Y}|\phi(y)|$. If B is any Banach space, $M_Y(B)$ is the Banach space of all bounded functions f on Y with values in B where $\|f\|_{M_Y(B)}=\sup_{y\in Y}\|f(y)\|_B$. If Υ is any element of M_Y^*, it is possible to define U on $M_Y(B)$ to B^{**} by letting $U(f)$ be that point $\mathfrak{b}$ of B^{**} such that $\mathfrak{b}(\beta)=\Upsilon(\beta f)$ for every β in B^*, where βf is the element of M_Y defined by $\beta f(y)=\beta(f(y))$. For each Υ in M_Y^* there is defined a unique, bounded, additive(6) set function Ψ by the relation $\Psi(E)=\Upsilon(\phi_E)$, where ϕ_E is the charac-

(4) E is compact (bicompact in the sense of Alexandroff and Hopf, *Topologie*, I) if every covering of E by open sets contains a finite subcovering; that is, if $E\subset\sum_\alpha O_\alpha$, where the O_α are open, there exist $\alpha_1, \cdots, \alpha_k$ such that $E\subset\sum_{i\leq k}O_{\alpha_i}$; this is equivalent to the following condition on closed sets: If the closed sets $C_\alpha\subset E$ are such that every finite set of the C_α have a point in common, then $\prod_\alpha C_\alpha$ is not empty.

(5) The subscript on the norm symbol indicates the space in question; it will be omitted when there is no danger of confusion.

(6) Ψ is bounded if $\Psi(E)\leq K$ if $E\subset Y$, Ψ is additive if $\Psi(E_1+E_2)=\Psi(E_1)+\Psi(E_2)$ whenever

teristic function of E. Conversely, for each bounded additive Ψ, setting $\Upsilon(\phi)=\int\phi d\Psi$ defines an Υ in M_Y^*, where the integral is, say, that of Radon-Stieltjes(7); moreover $\|\Upsilon\|=V\Psi(Y)$. Because of this relation between Ψ and Υ it is possible to define $\int fd\Psi$ for f in $M_Y(B)$ to be the element $\mathfrak{b}$ of B^{**} for which $\mathfrak{b}(\beta)=\int\beta fd\Psi$ for every β in B^*. All integrals used hereafter will be of this nature(8).

A set X is *directed* if there is a relation $>$ ("follows") among some pairs of its points such that $x>x'$ and $x'>x''$ implies $x>x''$ and such that each pair, x' and x'', of points in X has a common successor, x in X; that is, $x>x'$ and $x>x''$. If for each x in X, s_x is a point of the topological space S, then $s=\lim_x s_x$ if and only if for each neighborhood N of s there is an x_N in X such that $s_x\in N$ if $x>x_N$.

LEMMA 2. *If X is a directed set, if for each x in X $\mathfrak{b}_x$ is a point in B^{**} (B any Banach space), and if $\|\mathfrak{b}_x\|$ is ultimately bounded—that is, if there exist $K>0$ and $x_0\in X$ such that $\|\mathfrak{b}_x\|\leqq K$ if $x>x_0$—then there is a $\mathfrak{b}_0$ in B^{**} such that $\mathfrak{b}_0$ is in the weak* closure of $\{\mathfrak{b}_{x'}\mid x'>x\}$ for every x in X; that is, for every $\epsilon>0, \beta_1, \cdots, \beta_k$ in B^* and x in X there is an x' in X such that $x'>x$ and $|b_0(\beta_i)-b_{x'}(\beta_i)|<\epsilon$ for $i=1,\cdots,k$.*

For each x in X let $E_x=\{\mathfrak{b}_{x'}\mid x'>x\}$; since X is directed, any finite number of the E_x have a point in common, so the sets F_x which are obtained by taking the weak* closure of E_x are weak* closed sets with non-empty finite intersections. Since the sphere $\|\mathfrak{b}_x\|\leqq K$ is weak* compact, by the condition for this in terms of closed sets some $\mathfrak{b}_0$ exists in all these F_x; the last clause in the lemma is merely a full statement of the fact that $\mathfrak{b}_0$ is in the weak* closure of every E_x.

2. **The principal theorem.** The terms next defined are the ones used in the statement of the theorem and not merely in its proof.

A set Y is called an Abelian semi-group if there is defined for each pair of elements y, y' in Y a sum $y+y'$ in Y such that $y+y'=y'+y$, and such that $y+(y'+y'')=(y+y')+y''$; that is, addition is commutative and associative. If E is a subset of Y, then $E\dot{-}y$ is the set $\{y'\mid y'+y\in E\}$. It is clear that if $\{E\}$ is any partition of Y into any number of disjoint sets, then $\{E\dot{-}y\}$ is also such a partition.

Let X be a directed set and Y an Abelian semi-group; for each x in X let Ψ_x be a bounded, additive set function over Y. For each y in Y and

E_1 and E_2 are disjoint subsets of Y. If Ψ is bounded $V\Psi(Y)=\sup\sum_{i\leqq k}|\Psi(E_i)|\leqq 2K$, where the "sup" is taken over all partitions of Y into a finite number of disjoint subsets E_i.

(7) The pertinent properties are these: (1) If $\phi=\sum_{i\leqq k}\alpha_i\phi_{E_i}$, $\int\phi d\Psi=\sum_{i\leqq k}\alpha_i\Psi(E_i)$. (2) If $\|\phi_n-\phi\|_{M_Y}\to 0$, then $\int\phi_n d\Psi\to\int\phi d\Psi$. See Fichtenholz and Kantorovich [7].

(8) Integrals whose values lie in B^{**} instead of in B were introduced by N. Dunford, *Uniformity in linear spaces*, these Transactions, vol. 44 (1938), pp. 305–356, and I. Gelfand, *Abstrakte Funktionen und lineare Operatoren*, Recueil Mathématique, (n.s.), vol. 4 (1938), pp. 235–284.

x in X let Ψ_{xy} be the bounded additive set function over Y defined by $\Psi_{xy}(E)=\Psi_x(E\dot{-}y)-\Psi_x(E)$ for every $E\subset Y$; the system of set functions $\{\Psi_x\}$ is called nearly invariant over Y if $\lim_x V\Psi_{xy}(Y)=0$ for each y in Y, if $V\Psi_x(Y)$ is bounded, and if $\lim_x \Psi_x(Y)=1$. Ψ is called invariant if $\Psi(E\dot{-}y)=\Psi(E)$ for every $E\subset Y$ and $\Psi(Y)=1$.

If Y is an Abelian semi-group and B is a Banach space and if for each y in Y, T^y is an element of $\mathfrak{T}$, the space of linear operators on B to B, the semi-group $\{T^y\}$ is called a bounded representation of Y if $T^yT^{y'}=T^{y+y'}$ for every y and y' in Y, and if there is a $K\geqq 0$ such that $\|T^y\|\leqq K$ for every y. If $\{T^y\}$ is any bounded representation of Y, let B' and B'' be the subsets of B defined by $B''=\{b \mid T^yb=b \text{ for every } y \text{ in } Y\}$ and B' is the smallest closed linear subset of B containing all the points $b-T^yb$ for every b in B and y in Y; M is the smallest linear subset of B containing B' and B''. Clearly, $b\in M$ if and only if one (or all) $T^yb\in M$, since $b=T^yb+(b-T^yb)$ and $b-T^yb$ is in B'.

THEOREM 1. *Let Y be any Abelian semi-group, B a Banach space and $\{T^y\}$ a bounded representation of Y in $\mathfrak{T}$. Let X be a directed set and $\{\Psi_x\}$ a nearly invariant system of set functions over Y, and for each b in B and x in X let $\tau_xb=\int f^bd\Psi_x$ where f^b in $M_Y(B)$ is defined by $f^b(y)=T^yb$ for each y in Y; for each b in B let τb be one of the points which Lemma* 1 *associates with the points τ_xb. Then:*

(1) *τb can be taken in B instead of merely in B^{**} if and only if b is in M.*

(2) *M is a closed linear subset of B and is the direct sum of B' and B'', that is, each b in M is the sum of a b' in B' and a b'' in B'', where b' and b'' are uniquely determined; in fact $b''=\tau b$.*

(3) *τb is uniquely determined if $b\in M$ and, in M, τ is a linear operator with values in B''.*

(4) *If $b\in M$, $\tau\tau b=\tau b$ and $T^y\tau b=\tau b=\tau T^yb$ for every y in Y.*

(5) *$\tau b=b$ if and only if $b\in B''$; $\tau b=\theta$, the zero element in B, if and only if $b\in B'$.*

(6) *If $b\in M$, $\|\tau_xb-\tau b\|_{B^{**}}\to 0$.*

The main body of the proof will be divided into a number of simple steps.

(a) $T^y\tau_xb=\tau_xT^yb$ for every x in X, y in Y and b in B.

For each β in B^*,

$$T^y\tau_xb(\beta)=\tau_xb(T^{y*}\beta)=\int T^{y*}\beta f^bd\Psi_x=\int \beta f^{T^yb}d\Psi_x=\tau_xT^yb(\beta)$$

since

$$T^{y*}\beta f^b(t)=T^{y*}\beta(T^tb)=\beta(T^yT^tb)=\beta(T^tT^yb)=\beta f^{T^yb}(t)$$

for each t in Y.

(b) $\|T^y\tau_xb-\tau_xb\|\to 0$ for each b in B and y in Y[9].

[9] It should be noted that the condition that $V\Psi_{xy}(Y)\to 0$ for each y which is used in the

$$\|T^y\tau_x b - \tau_x b\| = \|\tau_x T^y b - \tau_x b\| = \sup_{\|\beta\|\leq 1} |\tau_x T^y b(\beta) - \tau_x b(\beta)|$$

$$= \sup_{\|\beta\|\leq 1} \left| \int \beta f^{T^y b} d\Psi_x - \int \beta f^b d\Psi_x \right|$$

$$= \sup_{\|\beta\|\leq 1} \left| \int \beta f^b d\Psi_{xy} \right| \leqq \|f^b\| V\Psi_{xy}(Y)$$

which tends to zero for each y. The only difficulty is in justifying the last equality which can be done as follows: since $\beta f^b \in M_Y$ for each β in B^* and b in B, it suffices to prove that $\int\phi d\Psi_{xy} = \int\phi_y d\Psi_x - \int\phi d\Psi_x$ where ϕ_y is defined by $\phi_y(y') = \phi(y+y')$ and ϕ is any function in M_Y. If ϕ is a simple function; that is, $\phi = \sum_{i\leq k}\alpha_i\phi_{E_i}$ where the E_i are disjoint subsets of Y and the α_i are real numbers; then $\int\phi d\Psi_{xy} = \sum_{i\leq k}\alpha_i\Psi_{xy}(E_i) = \sum_{i\leq k}\alpha_i\Psi_x(E_i \dot{-} y) - \sum_{i\leq k}\alpha_i\Psi_x(E_i)$ $= \int\phi_y d\Psi_x - \int\phi d\Psi_x$ since $\phi_y = \sum_{i\leq k}\alpha_i\phi_{E_i \dot{-} y}$; since the simple functions are dense in M_Y the same is true for any ϕ in M_Y.

(c) $T^y\tau b = \tau b$ for any y in Y and b in B.

It suffices to show for any $\epsilon > 0$ and β in B^* that $|T^y\tau b(\beta) - \tau b(\beta)| < \epsilon$. But

$$|T^y\tau b(\beta) - \tau b(\beta)| \leqq |T^y\tau b(\beta) - T^y\tau_x b(\beta)| + |T^y\tau_x b(\beta) - \tau_x b(\beta)| + |\tau_x b(\beta) - \tau b(\beta)|;$$

by (b) the middle term is less than $\epsilon/3$ if $x > x_0$. The first term is equal to $|\tau b(T^{y*}\beta) - \tau_x b(T^{y*}\beta)|$; by the definition of τb and Lemma 1, τb is in the weak* closure of $\{\tau_x b \mid x > x_0\}$ so there is an $x > x_0$ such that $|\tau_x b(\beta) - \tau b(\beta)| < \epsilon/3$ and $|\tau_x b(T^{y*}\beta) - \tau b(T^{y*}\beta)| < \epsilon/3$. For this x all three terms are less than $\epsilon/3$ and (c) follows.

(d) If $d \in B'$, then $\|\tau_x d\| \to 0$.

If $d = b - T^y b$ for some b in B and y in Y then $\|\tau_x d\| = \|T^y\tau_x b - \tau_x b\| \to 0$ by (b). Since all the operations in question are additive and homogeneous, $\|\tau_x d\| \to 0$ if $d = \sum_{i\leq k}\alpha_i(b_i - T^{y_i}b_i)$ for any choice of y_i in Y, b_i in B, and α_i real. If $d \in B'$ then there is a d_ϵ of this last form such that $\|d - d_\epsilon\| < \epsilon$; then $\|\tau_x d\| < \|\tau_x(d - d_\epsilon)\| + \|\tau_x d_\epsilon\| < (K+1)\epsilon$, where K is the upper bound of $V\Psi_x(Y)$, if x is large enough.

(e) $\|\tau_x\tau b - \tau b\| \to 0$ for each b in M', where M' is the set of those b in B such that τb can be chosen in B, not only in B^{**}.

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definition of near invariance is stronger than the corresponding condition used by Alaoglu and Birkhoff [3]. Since this property is used only in this step of the proof of the theorem, it is easily seen that the property that must be required of the system $\{\Psi_x\}$ is that $\|\int f^b d\Psi_{xy}\| \to 0$ for each b and y. This shows that the hypotheses on $\{\Psi_x\}$ can be weakened if some restriction is placed on the semi-group $\{T^y\}$; for example, it is sufficient that $\Psi_{xy}(E) \to 0$ for each $E \subset Y$ provided that the semi-group $\{T^y\}$ is so restricted that each f^b is integrable in the sense used in [3]. It seems to me that the extra restrictions are more properly placed on the system $\{\Psi_x\}$ than on the semi-group $\{T^y\}$ since the results are to a large extent independent of all but the existence of $\{\Psi_x\}$ and since (see Theorem 3) a nearly invariant system always exists.

$\tau_x\tau b(\beta)=\int\beta f^{\tau b}d\Psi_x$, but by (c) $f^{\tau b}(y)=T^y\tau b=\tau b$ for each y so $\tau_x\tau b(\beta)=\int\beta(\tau b)\phi_Y d\Psi_x=\Psi_x(Y)\beta(\tau b)=\beta(\Psi_x(Y)\tau b)$, and $\|\tau_x\tau b-\tau b\|=\|\tau b\|\,|\Psi_x(Y)-1|$ which tends to 0 by the hypotheses on $\{\Psi_x\}$.

(f) $b-\tau b\in B'$ for each b in M' so $\|\tau_x b-\tau_x\tau b\|\to 0$ for each b in M'.

If $b_1=b-\tau b$ is in $B-B'$, there exists a β_0 in B^* for which $\beta_0(d)=0$ if $d\in B'$ while $\beta_0(b_1)=1$. Then $\tau_x b(\beta_0)=\int\beta_0 f^b d\Psi_x$, but $\beta_0 f^b(y)=\beta_0(T^y b)=\beta_0(b)$ for each y so $\tau_x b(\beta_0)=\Psi_x(Y)\beta_0(b)$ and $\lim_x \tau_x b(\beta_0)=\lim_x \Psi_x(Y)\beta_0(b)=\beta_0(b)$. On the other hand, for each x in X and $\epsilon>0$ there is an x' in X such that $x'>x$ and $|\tau_{x'}b(\beta_0)-\beta_0(\tau b)|<\epsilon$, so $\beta_0(\tau b)=\beta_0(b)$ and $\beta_0(b_1)=0$ contrary to the choice of β_0.

The conclusions mentioned can now be drawn. (1) is true if $M'=M$. $B'\subset M'$ by (d); if $b\in B''$, $\tau_x b=\Psi_x(Y)b$ so $\|b-\tau_x b\|\to 0$ and $B''\subset M'$; hence $M'\supset M$. If $b\in M'$, by (c) $\tau b\in B''$ and $b-\tau b\in B'$ by (f); hence $b=b+(b-\tau b)\in M$ if $b\in M'$ and $M'\subset M$. (6) follows from (e), (f), and (1). For (3) note that (6) implies that τb is uniquely determined if $b\in M$; τ is clearly additive on M and for each b in M $\|\tau b\|\leqq\sup_x\|\tau_x b\|=\sup_x\|\int f^b d\Psi_x\|\leqq\|f^b\|\sup_x V\Psi_x(Y)\leqq\|b\|\sup_y\|T^y\|\sup_x V\Psi_x(Y)$, so τ is linear on M. All of (2) is proved except that M is closed; let $N=\{b\mid \tau_0 b=\lim_x\tau_x b$ exists in the norm topology in $B^{**}\}$. Then N, as the set of points of convergence of $\{\tau_x\}$, is a closed linear manifold and M is the inverse image by τ_0 of B (considered as imbedded in B^{**}). Hence M is closed in N and therefore in B. For (5) the remarks above showed that $\tau b=b$ if $b\in B''$; if $\tau b=b$, $T^y b=T^y\tau b=\tau b=b$ by (c), so $b\in B''$. (d) shows that $\tau b=\theta$ if $b\in B'$; if $\tau b=\theta$, $b=b-\tau b\in B'$ by (f) since $b\in M$ in this case. For (4) $\tau b\in B''$ if $b\in M$ so $\tau\tau b=\tau b$ and τ is idempotent; $T^y\tau b=\tau b$ by (c); if $b\in M$, $T^y b=(T^y b-b)+b\in M$ too, since $T^y b-b\in B'\subset M$; then $\tau T^y b=\lim_x\tau_x T^y b=\lim_x T^y\tau_x b=T^y(\lim_x\tau_x b)=T^y\tau b=\tau b$.

3. **Related and supplementary theorems.** One point stands out strongly in this set of conclusions: most of them do not directly involve X or $\{\Psi_x\}$ except in so far as the existence of the nearly invariant system $\{\Psi_x\}$ was required to prove existence here. M is already defined in terms of the bounded Abelian semi-group $\{T^y\}$ and (5) defines τ in M in terms of $\{T^y\}$ alone, so that M and τ are the same no matter what X and $\{\Psi_x\}$ are used so long as the system $\{\Psi_x\}$ is nearly invariant over Y. This raises three questions:

(A) If Y is a given Abelian semi-group, is there a nearly invariant system $\{\Psi_x\}$ for some X?

(B) Under what conditions on Y and $\{T^y\}$ does $M=B$?

(C) If some natural choice of X and $\{\Psi_x\}$ is suggested by the nature of Y, is this system nearly invariant?

(A) can be completely answered (yes, for any X); (B) partially; (C) depends on the case in question and obviously has no general answer. This section contains the discussion of (A) and (B).

THEOREM 2. *If a nearly invariant system $\{\Psi_x\}$ of set functions over Y exists* *then there is an invariant function* Ψ; *that is, a* Ψ *such that* $\Psi(E \dot{-} y) = \Psi(E)$ *for every* $E \subset Y$ *and* $y \in Y$ *and* $\Psi(Y) = 1$.

Proof. If $\{\Psi_x\}$ is the nearly invariant system, let Υ_x be the element of M_Y^* defined by $\Upsilon_x(\phi) = \int \phi d\Psi_x$ for every $\phi \in M_Y$. Applying Lemma 1 with $B^* = M_Y$, $\mathfrak{b}_x = \Upsilon_x$ gives an Υ in M_Y^* such that Υ is in the weak* closure of every set $\{\Upsilon_x \mid x > x_0\}$ for every x_0 in X. Let $\Psi(E) = \Upsilon(\phi_E)$ for each $E \subset Y$, then Ψ is additive and bounded, of course, since $V\Psi(Y) \leqq \lim \sup_x V\Psi_x(Y)$ and, since $\lim_x \Psi_x(Y) = 1$, $\Psi(Y) = 1$.

To show Ψ invariant it suffices to show that for each $\epsilon > 0$ $|\Psi(E \dot{-} y) - \Psi(E)| < \epsilon$. Now

$$\begin{aligned} |\Psi(E \dot{-} y) - \Psi(E)| &\leqq |\Psi(E \dot{-} y) - \Psi_x(E \dot{-} y)| + |\Psi_x(E \dot{-} y) - \Psi_x(E)| \\ &\quad + |\Psi_x(E) - \Psi(E)| \\ &= |\Upsilon(\phi_{E \dot{-} y}) - \Upsilon_x(\phi_{E \dot{-} y})| + |\Upsilon_x(\phi_{E \dot{-} y}) - \Upsilon_x(\phi_E)| \\ &\quad + |\Upsilon_x(\phi_E) - \Upsilon(\phi_E)|. \end{aligned}$$

By the near invariance of $\{\Psi_x\}$, there is an x_0 such that the middle term is less than $\epsilon/3$ whenever $x > x_0$; then by the fact that Υ is in the weak* closure of $\{\Upsilon_x | x > x_0\}$ the other two terms can be made less than $\epsilon/3$ by proper choice of $x > x_0$, so $\Psi(E \dot{-} y) = \Psi(E)$ for all $E \subset Y$ and $y \in Y$.

Note that the full strength of near invariance is not used in this proof but only that $\Psi_{xy}(E) \to 0$ for every $E \subset Y$. Naturally if an invariant function Ψ exists, the system $\{\Psi_x\}$ such that every $\Psi_x = \Psi$ for each x in X, is nearly invariant over Y no matter what X is.

COROLLARY 1. *If* Y *is a family of subsets* y *of a given set* A, *with addition in* Y *ordinary addition as subsets of* A, *then there exists an invariant function* Ψ *over* Y.

This can be proved directly by proper application of the Hahn-Banach theorem. To derive it from Theorem 3, let $X = Y$, order Y by letting $y > y'$ mean $y \supset y'$, and let Ψ_y for each y in Y be defined on the subsets E of Y by $\Psi_y(E) = 1$ if $y \in E$, $\Psi_y(E) = 0$ if not. To show that the system $\{\Psi_y\}$ is nearly invariant, for any y_0 in Y let $y \supset y_0$; then $\Psi_y(E \dot{-} y_0) = 1$ if and only if y is in $E \dot{-} y_0$, that is, if and only if $y + y_0$ is in E, but $y + y_0 = y$ if $y \supset y_0$ so $\Psi_y(E \dot{-} y_0) = 1$ if and only if $y \in E$, i.e., if and only if $\Psi_y(E) = 1$. Therefore $\Psi_y(E) = \Psi_y(E \dot{-} y_0)$ for every E or $V\Psi_{yy_0}(Y) = 0$ if $y > y_0$.

This particular nearly invariant system will be used later.

COROLLARY 2. *To the conclusions of Theorem* 1 *can be added*:

(7) *There exists an invariant function* Ψ *defined over* Y *such that* $\tau b = \int f^b d\Psi$ *if* $b \in M$, *while* $\int f^b d\Psi \in B^{**}$ *if* $b \notin M$. *Moreover* $\int f^b d\Psi$ *is in the weak* closure of* $\{\tau_x b \mid x > x_0\}$ *for every* x_0 *in* X.

Letting $\Psi_x' = \Psi$ for any X gives a nearly invariant system; $\tau' b = \tau_x' b = \int f^b d\Psi$

shows that $\tau' b \in B$ if and only if $b \in M$ (by Theorem 1) and in $M\ \tau' b = \tau b$. The definition of Υ, and hence of Ψ, in Theorem 2 shows that $\int f^b d\Psi$ lies in the given sets.

The important result is

THEOREM 3. *If Y is an Abelian semi-group, there is an invariant set function Ψ over Y such that $\Psi(Y) = V\Psi(Y) = 1$ and $\Psi(E) \geqq 0$ if $E \subset Y$.*

The proof is essentially that of a result of Morse and Agnew [1, Lemma 2.01]. Let Υ_0 be defined on the multiples of ϕ_Y by $\Upsilon_0(\alpha\phi_Y) = \alpha$; then by the Hahn-Banach theorem there is an Υ_1 in M_Y^* such that $\|\Upsilon_1\| = 1$ and $\Upsilon_1(\alpha\phi_Y) = \Upsilon_0(\alpha\phi_Y)$. For each y in Y let T^y be the function on M_Y to M_Y defined by $T^y\phi(y') = \phi(y+y')$ for each ϕ in M_Y and y' in Y. $\|T^y\| = 1$ for each y since $\sup_{y' \in Y} |\phi(y+y')| \leqq \sup_{y' \in Y} |\phi(y')|$ for every y in Y and $T^y\phi_Y = \phi_Y$; Υ_0 is invariant under all T^y. Following [1] let

$$p(\phi) = \inf \left[\sup_{y_0 \in y} \frac{1}{n} \sum_{i=1}^{n} \Upsilon_1(T^{y_0} T^{y_i} \phi)\right]$$

where the "inf" is taken over all choices of the integer n and the points $y_1, \cdots, y_n$ in Y. As in [1] it can be shown that $p(\alpha\phi) = \alpha p(\phi)$ if $\alpha \geqq 0$, that $p(\phi + \phi') \leqq p(\phi) + p(\phi')$, and that $p(\phi) \leqq \|\phi\|$ for every ϕ in M_Y. Since $\Upsilon_0(\alpha\phi_Y) \leqq p(\alpha\phi_Y)$, applying the Hahn-Banach theorem again gives an Υ in M_Y^* such that $\Upsilon(\alpha\phi_Y) = \alpha$ and $\Upsilon(\phi) \leqq p(\phi) \leqq \|\phi\|$ for every ϕ; the proof that Υ is invariant under the T^y is as in [1].

Let $\Psi(E) = \Upsilon(\phi_E)$ if $E \subset Y$. Then $\Psi(E \dot{-} y) = \Upsilon(\phi_{E \dot{-} y}) = \Upsilon(T^y\phi_E) = \Upsilon(\phi_E) = \Psi(E)$, so Ψ is invariant; $\|\Upsilon\| = \Upsilon(\phi_Y) = 1$ so $V\Psi(Y) = \Psi(Y) = 1$. Ψ is non-negative, for if E exists with $\Psi(E) < 0$, then $V\Psi(Y) \geqq |\Psi(E)| + |\Psi(Y-E)| = -\Psi(E) + 1 - \Psi(E) > 1$.

COROLLARY 3. *$\|\tau b\|_B \leqq \|f^b\|_{M_Y(B)} = \sup_{y \in Y} \|T^y b\|_B \leqq \|b\| \sup_{y \in Y} \|T^y\|$ if $b \in M$.*

If $b \in M$, $\tau b = \int f^b d\Psi$ where Ψ is defined by Theorem 3; hence $\|\tau b\| = \|\int f^b d\Psi\| \leqq \|f^b\|\ V\Psi(Y) = \|f^b\|$.

COROLLARY 4. *If $b \in M$, then τb is in the closed convex hull $K(b)$ of the set $\{T^y b \mid y \in Y\}$.*

For each β in B^*, $\beta(\tau b) = \int \beta f^b d\Psi \leqq V\Psi(Y) \sup_{y \in Y} \beta f^b(y) = \sup_{y \in Y} \beta(T^y b)$. If b_0 is not in $K(b)$ there is a β_0 such that $\beta_0(b_0) > \sup_y \beta_0(T^y b)$ (by a theorem of Mazur [9]) so τb is in $K(b)$.

Alaoglu and Birkhoff call a point b in B ergodic relative to the bounded semi-group $\{T^y\}$ if there is just one point of B'' in $K(b)$ and all $K(T^y b)$(10).

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(10) This is not their definition but one of the properties shown to be necessary and sufficient.

COROLLARY 5. *b is ergodic if and only if $b\in M$.*

If $b\in M, \tau b\in B''$ and to $K(b)$ by Theorem 2 and Corollary 4; all $T^y b$ are in M too and $\tau T^y b=\tau b$, so τb is all $K(T^y b)$ also and b is ergodic if these sets contain no other points in B''; this is asserted by Lemmas 1 and 2 of [2]. If b is ergodic and b_0 is the unique point in $B''\cdot K(b)$, then θ is the unique point in $B''\cdot K(b-b_0)$; let $b_1=b-b_0$. Since θ is in the closed convex hull of the set $\{T^y b_1 \mid y\in Y\}$, for each $\epsilon>0$ there is a point $b_2=\sum_{i\leq k}\alpha_i T^{y_i} b_1$ with $\|b_2\|<\epsilon/K$, where $K=\sup_{y\in Y}\|T^y\|$. If $\tau b'=\int f^{b'}d\Psi$, with Ψ as in Theorem 3, then $\tau b_2=\sum_{i\leq k}\alpha_i\tau T^{y_i}b_1=\tau b_1$ and $\|\tau b_1\|=\|\tau b_2\|<\epsilon$. Hence $\|\tau b_1\|=0$ or $b_1\in B'$; so $b=b_1+b_0\in M$.

To complete the relation of these results to those of Alaoglu and Birkhoff requires some study of Problem (B).

LEMMA 3. *If $f\in M_Y(B)$ and Ψ is a bounded additive set function over Y, $\int fd\Psi$ is in Y if P, the smallest closed linear subspace of B containing all the points $\{f(y) \mid y\in Y\}$, is reflexive.*

Each β in B^* defines a π_β in P^* by $\pi_\beta(p)=\beta(p)$ for each p in P; then $\int fd\Psi(\beta)=\int\beta fd\Psi=\int\pi_\beta fd\Psi$. Since the π_β cover P^* this defines a unique p in P^{**} such that $p(\pi)=\int\pi fd\Psi$ if $\pi\in P^*$. Since P is reflexive there is a p in P such that $\pi(p)=\int\pi fd\Psi$ if $\pi\in P^*$. Hence $\beta(p)=\pi_\beta(p)=\int\pi_\beta fd\Psi=\int\beta fd\Psi$; so $\int fd\Psi=p\in P\subset B$.

COROLLARY 6. *If the set $\{T^y b \mid y\in Y\}$ lies in some reflexive subspace of B, then $b\in M$.*

COROLLARY 7. *If B is reflexive, $M=B$.*

COROLLARY 8. *If B is reflexive and $\{T^y\}$ is any bounded Abelian semi-group of operators on B to B, then each b in B is ergodic; that is, for each b in B $K(b)$ contains just one fixed point τb of all the T^y.*

It is to be noticed that this is a great strengthening of Theorem 5 of [2] since, as is known, every uniformly convex space is reflexive, and since $\|T^y\|\leqq 1$ can be replaced by $\|T^y\|\leqq K$. On the other hand, that result can be proved with far less machinery.

Corollaries 3 and 7 together imply that if B is reflexive the set B'' of common fixed points of any bounded Abelian semi-group $\{T\}$ of operators on B to B is the range of a projection operator τ defined on all of B and $\|\tau\|\leqq\sup\|T\|$.

A trivial result is this:

If $\{T^y\}$ is any bounded Abelian semi-group of operators on B to B and if there is a y_0 such that $\|T^{y_0}\|<1$, then $B'=B$; in fact $\lim_y\|T^y\|=0$ if y is directed by letting $y>y'$ if there is a y'' such that $y=y''+y'$.

For each $\epsilon>0$, there is an n such that $\|(T^{y_0})^n\|<\epsilon/K$, where K is the bound of the norms of all T^y, then $\|T^y(T^{y_0})^n\|<\epsilon$ for every y or $\|T^y\|<\epsilon$ if $y>ny_0$.

In some cases a certain X and a nearly invariant system $\{\Psi_x\}$ over Y arise naturally. Under certain conditions on X the reflexivity condition of Corollary 7 can be weakened (at least formally). A directed set X has a *countable cofinal subset* if there is a countable subset X' of X such that each x in X is followed by some x' in X'.

THEOREM 4. *If $\{T^y\}$ is a bounded representation of the Abelian semi-group Y, if X is a directed set with a countable cofinal subset, and if the system $\{\Psi_x\}$ is nearly invariant over Y, then $b\in M$ if and only if there is a countable cofinal subset $\{x_n\}$ of X and a b_0 in B such that $\lim_n \tau_{x_n} b(\beta)=\beta(b_0)$ for every β in B^*. In this case $b_0=\tau b$.*

For such an X this sequential compactness condition assures that τb is in B; that is, that b is in M. Since norm convergence implies weak convergence this condition is satisfied if $b\in M$.

COROLLARY 9. *If B is a Banach space with sequentially weakly compact*(11) *unit sphere, if $\{T^y\}$ is a bounded Abelian semi-group of operators on B to B, if X has a countable cofinal subset and if $\{\Psi_x\}$ is a nearly invariant system such that $\int f^b d\Psi_x\in B$ for each b in B, then $M=B$.*

Since every reflexive space has a sequentially weakly compact unit sphere, this result is related to Corollary 7; since it is not known whether or not sequential weak compactness implies reflexivity, it is not known whether the hypotheses on X and $\{\Psi_x\}$ are needed.

4. **Special semi-groups and systems of set functions.** A theorem of Dunford [6] uses E_n, euclidean n-space with coordinates $y_1, \cdots, y_n$, for Y and the class of n-dimensional cubes $x=\{y \mid \alpha_j<y_j<\alpha_j+r, j=1, \cdots, n\}$, where $r>0$ and the α_j are arbitrary real numbers, for X, defining $\Psi_x(E)=m(Ex)/m(x)$, where m is Lebesgue measure, for every Lebesgue measurable set $E\subset Y$. (He has then to assume measurability for each f^b in order to integrate.) $\tau_x b$, then, is the arithmetic mean of f^b over the cube x; that is, $\tau_x b=r^{-n}\int_x f^b\, dm$, where r is the length of edge of the cube x. X is ordered by the size of the cubes, that is, $x>x'$ if the edges of x are longer than those of x'.

A more general result follows from a simple property of convex bodies with interior points in E_n. In what follows let $S_\alpha(y)$ be the closed sphere about y of radius α: as in any linear space if $E, E'\subset E_n$ let $E+E'=\{e+e' \mid e \text{ in } E \text{ and } e' \text{ in } E'\}$ and for any real α and $E\subset E_n$ let $\alpha E=\{\alpha e \mid e \text{ in } E\}$.

LEMMA 4. *If E is a convex subset of E_n and if E contains a sphere $S_r=S_r(0)$, and if $S_\alpha=S_\alpha(0)$, then $S_\alpha+E\subset[(r+\alpha)/r]E$*(12).

(11) A set $B_0\subset B$ is sequentially weakly compact if for each sequence $\{b_n\}\subset B_0$ there is a subsequence $\{n_i\}$ and a b_0 in B_0 such that $\lim_i \beta(b_{n_i})=\beta(b_0)$ for every β in B^*.

(12) This proof is due to the referee who notes that it holds in any normed vector space.

$y \in S_\alpha$ if and only if $y = (\alpha/r)y'$ for some y' in S_r. For each y in S_α and y'' in E

$$y + y'' = \frac{\alpha}{r} y' + y'' = \frac{r+\alpha}{r}\left[\frac{\alpha}{r+\alpha} y' + \frac{r}{r+\alpha} y''\right] = \frac{r+\alpha}{r} y_0$$

where $y_0 \in E$ since $y', y'' \in E$, $\alpha/(r+\alpha) + r/(r+\alpha) = 1$, and $\alpha/(r+\alpha) > 0$' $r/(r+\alpha) > 0$. Hence $E + S_\alpha \subset [(r+\alpha)/r]E$.

For each bounded convex set E with interior points contained in E_n let $r(E)$ be the least upper bound of the radii of the spheres contained in E.Then there will be at least one sphere of radius $r(E)$ contained in $\overline{E}$, the closure of E, since any bounded closed set in E_n is compact.

LEMMA 5. *If X is the set of all bounded convex sets with interior points in E_n and if X is directed by the relation $x > x'$ if and only if $r(x) \geqq r(x')$, then for each y in E_n,* $\lim_x m[x(x \dot{-} y)]/m(x) = 1$.

Since $x \dot{-} y = x(x \dot{-} y) + [(x \dot{-} y) - x]$, it suffices to show that $m[(x \dot{-} y) - x]/m(x) \to 0$. If α is the distance from y to 0, then $x \dot{-} y \subset x \dot{+} S_\alpha$; by Lemma 4, $x \dot{+} S_\alpha$ is contained in a dilation of $\bar{x}$ in the ratio $(r(x)+\alpha)/r(x)$ about the center of any sphere of radius $r(x)$ contained in x. The ratio of the measures of the dilated set and x is precisely $[(r(x)+\alpha)/r(x)]^n$ so $m[(x \dot{-} y) - x]/m(x) \leqq m[(x \dot{+} S_\alpha) - x]/m(x) \leqq [(r(x)+\alpha)/r(x)]^n - 1$ which tends to 0 as $r(x) \to \infty$.

LEMMA 6. *There is a non-negative additive set function μ defined on all subsets of E_n of finite outer measure, such that $\mu(\alpha A \dot{-} y) = |\alpha| \mu(A)$ and such that $\mu(A) = m(A)$ if A is Lebesgue measurable and of finite measure.*

This follows from the work of Morse and Agnew [1]; they gave the construction for the case $n = 1$.

THEOREM 5. *Let A be any convex set with interior points in E_n and let $Y = \{\alpha y \mid y \text{ in } A \text{ and } \alpha \geqq 1\}$; then Y is an Abelian semi-group under vector addition in E_n; for any Banach space B let $\{T^y\}$ be a bounded representation of Y. Let X be the set of all convex sets of finite nonzero measure contained in Y and for each x in X define Ψ_x by $\Psi_x(E) = \mu(Ex)/\mu(x)$. Then the system $\{\Psi_x\}$ is nearly invariant and the conclusions of Theorem 1 hold.*

Near invariance of $\{\Psi_x\}$ is all that needs be verified. $V\Psi_x(Y) = \Psi_x(Y) = \mu(x)/\mu(x) = 1$;

$$\begin{aligned}
\Psi_{xy}(E) &= \Psi_x(E \dot{-} y) - \Psi_x(E) = [\mu(x)]^{-1}\{\mu[(E \dot{-} y)x] - \mu(Ex)\} \\
&= [\mu(x)]^{-1}\{\mu[(E \dot{-} y)(x \dot{-} y)] - \mu[(E \dot{-} y)((x \dot{-} y) - x)] \\
&\quad + \mu[(E \dot{-} y)(x - (x \dot{-} y))] - \mu(Ex)\} \\
&= [\mu(x)]^{-1}\{\mu(Ex \dot{-} y) - \mu(Ex) - \mu[(E \dot{-} y)((x \dot{-} y) - x)] \\
&\quad + \mu[(E \dot{-} y)(x - (x \dot{-} y))]\} \\
&= [\mu(x)]^{-1}\{-\mu[(E \dot{-} y)((x \dot{-} y) - x)] + \mu[(E \dot{-} y)(x - (x \dot{-} y))]\}.
\end{aligned}$$

If $Y=\sum_{i\leqq k}E_i$ with the E_i disjoint, then the sets $E_i\dot{-}y$ are disjoint and have sum Y. Hence

$$\begin{aligned}\sum_{i\leqq k}|\Psi_{xy}(E_i)| &\leqq [\mu(x)]^{-1}\sum_{i\leqq k}\{\mu[(E_i\dot{-}y)((x\dot{-}y)-x)] \\ &\qquad + \mu[(E_i\dot{-}y)(x-(x\dot{-}y))]\} \\ &\leqq [\mu(x)]^{-1}\left\{\mu\left[\sum_{i\leqq k}(E_i\dot{-}y)((x\dot{-}y)-x)\right]\right. \\ &\qquad \left.+\mu\left[\sum_{i\leqq k}(E_i\dot{-}y)(x-(x\dot{-}y))\right]\right\} \\ &\leqq [\mu(x)]^{-1}\{\mu[(x\dot{-}y)-x]+\mu[x-(x\dot{-}y)]\}.\end{aligned}$$

By Lemma 3, this is small for $r(x)$ large, independent of the decomposition of Y into the sets E_i, so $V\Psi_{xy}(Y)\to 0$ and this system $\{\Psi_x\}$ is nearly invariant.

The set X' of cubes used by Dunford if ordered by edge length has the same ordering as if ordered by the radius of the largest sphere inside; so X' is a cofinal subset of this family X of convex sets of finite, nonzero measure; so $\lim_{x'}\tau_{x'}b$ exists for every b in M since $\lim_x \tau_x b$ exists for such b; moreover if the functions f^b are all measurable, the $\tau_{x'}$ reduce to Dunford's transformations and Theorem 5 offers a simple proof, without differentiation theorems, of Dunford's result.

Note that this X has a countable cofinal subset, in fact any sequence $\{x_n\}\subset X$ such that $r(x_n)\to\infty$ will do. Hence if the assumption is made that each $T^y b$ is a measurable function, each $\tau_x b\in B$ and Corollary 9 can be applied with proper choice of B.

For a second application (not considered anywhere in the literature so far as I know) take Y to be the *stack* Δ whose elements are the finite subsets of some given set D of elements d where addition is, as in Corollary 1, ordinary point set addition. If B is a Banach space and $\{T^\delta \mid \delta\in\Delta\}$ is a bounded representation of Δ in the space of linear operators on B to B, $T^\delta T^\delta=T^{\delta+\delta}=T^\delta$ so every T^δ must be a projection; moreover $T^\delta=\prod_{d\in\delta}T^d$ for each $\delta\in\Delta$. Δ is also a directed set if $\delta>\delta'$ means $\delta\supset\delta'$; for each $\delta\in\Delta$ let Ψ_δ be defined over the subsets of Δ by $\Psi_\delta(E)=1$ if $\delta\in E$, $\Psi_\delta(E)=0$ if $\delta\notin E$. Then, as in Corollary 1, the system $\{\Psi_\delta\}$ is nearly invariant over Δ.

THEOREM 6. *If B is a Banach space and the T^d are commuting projections on B to B such that $\|T^\delta\|\leqq K$ for all δ, where $T^\delta=\prod_{d\in\delta}T^d$, then* $\lim_\delta T^\delta b$ *exists (in the norm topology) if and only if $b\in M$ (where M is defined as in Theorem* 1) *and for such b,* $\lim_\delta\|T^\delta b-\tau b\|=0$.

By Theorem 1, $\|\tau_\delta b-\tau b\|\to 0$ with τb in B if and only if $b\in M$. But $\tau_\delta b=\int f^b d\Psi_\delta=T^\delta b$, and if the $T^\delta b$ converge at all their limit must be in B.

It is to be noted that this theorem has a much stronger conclusion than can be obtained in general; this is true because the bounded representations of a stack are so greatly restricted.

COROLLARY 10. *If B is reflexive and if T^d are commuting projections with $\|T^\delta\|$ uniformly bounded, then* $\lim_\delta T^\delta b = \tau b$ *for every b in B, where τ is a projection defined on all of B which has all the properties ascribed to τ in Theorem* 1.

One more consequence of Theorem 1 is this: If the collection $\{T^y\}$ of operators on B to B is a bounded representation of the semi-group Y, then the collection $\{T^{y*}\}$ is such a representation in the set of operators on B^* to B^*. In general, defining $B^{(n*)}$ by induction from $B^{(n*)} = (B^{((n-1)*)})^*$, and $T^{(n*)} = (T^{((n-1)*)})^*$, the same is true of the collection $\{T^{y(n*)}\}$. Hence Theorem 1 defines in $B^{(n*)}$ a set $M^{(n*)}$ consisting of the direct sum of $B''^{(n*)}$, the set of fixed points of the $T^{y(n*)}$, and $B'^{(n*)}$, the smallest closed linear set in $B^{(n*)}$ containing all $T^{y(n*)}b^{(n*)} - b^{(n*)}$ for all choices of y in Y and $b^{(n*)}$ in $B^{(n*)}$. Since $B^{((n+2)*)} \supset B^{(n*)}$ for every n, and the $T^{y((n+2)*)}$ agree with the $T^{y(n*)}$ in $B^{(n*)}$, $B''^{((n+2)*)} \cdot B^{(n*)} = B''^{(n*)}$ and similar relations hold for $B'^{(n*)}$ and $M^{(n*)}$. (c) of the proof of Theorem 1 shows that for any b in B and any possible choice of τb, τb is in $B''^{(2*)}$.

An example of the results obtained from this point of view is

THEOREM 7. *If B is reflexive and if $\{T^y \mid y \in Y\}$ is a bounded Abelian semi-group of operators on B to B, then there exists a projection τ defined over all B such that for every nearly invariant system of set functions $\{\Psi_x\}$ over Y, $\lim_x \|\tau_x b - \tau b\| = 0$ for each b in B and $\lim_x \|\tau_x^* \beta - \tau^* \beta\| = 0$ for each β in B^*. τ and τ^* have in their respective spaces the properties specified by Theorem* 1.

All that need be verified is that τ_0^*, the projection in B^* that exists by direct application of Theorem 1 to that space, is equal to τ^*, the adjoint of τ. Since $\|\tau_x^*\beta - \tau_0^*\beta\| \to 0$ for each β in B^*, $\tau_x^*\beta(b) - \tau_0^*\beta(b) = \beta(\tau_x b) - \tau_0^*\beta(b) \to 0$ for each b in B but $\beta(\tau_x b) \to \beta(\tau b)$ so $\tau^*\beta(b) = \beta(\tau b) = \tau_0^*\beta(b)$ for each b in B and β in B^*, or $\tau^* = \tau_0^*$.

I see no way of proving anything quite of this nature if $M \neq B$.

It may be noted that neither the methods nor the results of this paper carry over to noncommutative semi-groups; in fact, an example [2, §13, Example 1] shows that for non-abelian semi-groups the set M and the set of ergodic points of B need not be the same.

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