

ON THE PARTIAL SUMS OF HARMONIC DEVELOPMENTS AND OF POWER SERIES

BY
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1. **Introduction.** Consider the class E of power series $f(z) = \sum_0^\infty c_\nu z^\nu$, convergent for $|z| < 1$ and such that $|f(z)| \leq 1$. The following result is due to I. Schur and G. Szegő [5]⁽¹⁾.

For any series of the class E ,

$$|s_n(z)| \equiv \left| \sum_0^n c_\nu z^\nu \right| \leq 1$$

in $|z| \leq r_n$, but not always in $|z| < r_n + \epsilon$, $\epsilon > 0$, where r_n is the largest r for which

$$T_n(r, \theta) = \frac{1}{2} + \sum_1^n r^\nu \cos \nu\theta \geq 0 \quad \text{for all } \theta.$$

The r_n are non-decreasing,

$$r_n > 1 - \frac{\log 2n}{n}, \quad n = 1, 2, 3, \dots,$$

$$r_n = 1 - \frac{\log 2n - \log \log 2n + \epsilon_n}{n}, \quad \lim_{n \rightarrow \infty} \epsilon_n = 0.$$

We obtain the same constant r_n if we assume $Rf(z) \geq 0$ and require $Rs_n(z) \geq 0$. Here Ru means the real part of u ; Iu will denote the imaginary part.

In what follows, we consider harmonic sine developments

$$H(r, \theta) = \sum_1^\infty b_\nu r^\nu \sin \nu\theta,$$

convergent for $0 < r < 1$, and non-negative for $0 < \theta < \pi$. Evidently there exists an R_n with the following properties:

(a) Whenever

$$(1.1) \quad H(r, \theta) \geq 0, \quad 0 < r < 1; 0 < \theta < \pi,$$

then,

$$(1.2) \quad s_n(r, \theta) \equiv \sum_1^n b_\nu r^\nu \sin \nu\theta \geq 0, \quad 0 < r \leq R_n; 0 < \theta < \pi.$$

Presented to the Society, January 1, 1941; received by the editors March 10, 1941, and in revised form, May 7, 1941. The author is indebted to the referee for valuable suggestions.

⁽¹⁾ Numbers in brackets refer to the literature at the end of this paper.

(b) For any $\epsilon > 0$ we can find an H satisfying (1.1) and such that $s_n(r, \theta)$ becomes negative for some θ and some $r < R_n + \epsilon$.

We denote the class of harmonic functions satisfying (1.1) by T . On writing $f(z) = \sum_{\nu=0}^{\infty} b_{\nu} z^{\nu}$, the power series $f(z)$ is regular in $|z| < 1$, has all its coefficients real, and $\text{Im} f(z) \geq 0$ in $|z| < 1, \text{Im} z > 0$. The class T has been discussed by Rogosinski [4]; the function $f(z)$ is called typically real. (Cf. also S. Mandelbrojt [2].)

One of the results of the present paper is

$$R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n}{n} + O(1/n), \quad \text{as } n \rightarrow \infty.$$

M. S. Robertson [3] gave the erroneous estimate

$$R_n \geq 1 - 2 \log n/n \quad \text{for } n > 12.$$

His calculation yields however, as is seen easily, $R_n \geq 1 - 4 \log n/n$, for $n > n_0^{(2)}$. We then apply the properties of R_n to Fourier series of convex functions and to a certain class of power series.

Note that if $\phi(\theta) \sim \sum_{\nu=0}^{\infty} b_{\nu} \sin \nu \theta$, $\phi(\theta) \geq 0, 0 < \theta < \pi$, then

$$\begin{aligned} H(r, \theta) &= \frac{2}{\pi} \int_0^{\pi} \phi(x) \left(\sum_1^{\infty} r^{\nu} \sin \nu \theta \sin \nu x \right) dx \\ &= \frac{1}{\pi} \int_0^{\pi} \phi(x) \sum_1^{\infty} r^{\nu} [\cos \nu(\theta - x) - \cos \nu(\theta + x)] dx \\ &= \frac{1 - r^2}{2\pi} \int_0^{\pi} \phi(x) \left(\frac{1}{1 - 2r \cos(\theta - x) + r^2} - \frac{1}{1 - 2r \cos(\theta + x) + r^2} \right) dx \\ &= \frac{2r(1 - r^2)}{\pi} \sin \theta \int_0^{\pi} \frac{\phi(x) \sin x dx}{[1 - 2r \cos(\theta - x) + r^2][1 - 2r \cos(\theta + x) + r^2]}. \end{aligned}$$

Hence $H(r, \theta)$ belongs to the class T . (Cf. Zygmund [8, p. 57].)

2. Characterization of R_n . We quote the following lemma, due to Fejér (Turán [7]).

LEMMA 1. *In order that*

$$\sum_{\nu=1}^n \lambda_{\nu} \sin \nu x \sin \nu y \geq 0 \quad \text{for } 0 < x < \pi, 0 < y < \pi,$$

it is necessary and sufficient that

$$\sum_1^n \nu \lambda_{\nu} \sin \nu \theta \geq 0 \quad \text{for } 0 < \theta < \pi.$$

(²) Robertson, *Annals of Mathematics*, (2), vol. 42 (1941), pp. 829-838.

We now prove

THEOREM 1. *The quantity R_n as defined in §1 is the largest r for which*

$$(2.1) \quad S_n(r, \theta) \equiv \sum_1^n \nu r^\nu \sin \nu \theta \geq 0 \quad \text{for } 0 < \theta < \pi.$$

We have for $0 < \rho < 1$,

$$\rho^\nu b_\nu = \frac{2}{\pi} \int_0^\pi H(\rho, x) \sin \nu x \, dx, \quad \nu = 1, 2, 3, \dots;$$

hence

$$s_n(r, \theta) = \frac{2}{\pi} \int_0^\pi H(\rho, x) \left(\sum_1^n \left(\frac{r}{\rho} \right)^\nu \sin \nu \theta \sin \nu x \right) dx.$$

For any $r < R_n$ we can choose $\rho < 1$ so that $r/\rho < R_n$; we then obtain by Lemma 1 (for $\lambda_\nu = r^\nu$) $s_n(r, \theta) \geq 0$ for any $r < R_n$ and for $0 < \theta < \pi$; hence (a) holds for $r \leq R_n$. Conversely, for the function

$$H(r, \theta) = \sum_1^\infty \nu r^\nu \sin \nu \theta = r \sin \theta \frac{1 - r^2}{(1 - 2r \cos \theta + r^2)^2} > 0$$

the function

$$s_n(r, \theta) = \sum_1^n \nu r^\nu \sin \nu \theta$$

becomes negative for any $r > R_n$ and for some θ in $(0, \pi)$. This proves Theorem 1. To estimate R_n we first give another characterization for it. An easy calculation yields

$$\begin{aligned} & \frac{(1 - 2r \cos \theta + r^2)^2}{r \sin \theta} \sum_1^n \nu r^\nu \sin \nu \theta \\ &= 1 - r^2 - (n+1)r^{n+2} \frac{\sin(n-1)\theta}{\sin \theta} + r^{n+1}(2n+2+nr^2) \frac{\sin n\theta}{\sin \theta} \\ & \quad - r^n(n+1+2nr^2) \frac{\sin(n+1)\theta}{\sin \theta} + nr^{n+1} \frac{\sin(n+2)\theta}{\sin \theta} \\ & \equiv C_n(r, \theta). \end{aligned}$$

This furnishes

THEOREM 2. *R_n is the largest r for which*

$$C_n(r, \theta) \geq 0 \quad \text{for all } \theta.$$

Evidently

$$\begin{aligned}
C_n(r, \pi) &= 1 - r^2 + (n^2 - 1)r^{n+2}(-1)^{n-1} + nr^{n+1}(2n + 2 + nr^2)(-1)^{n-1} \\
&\quad + (n + 1)r^n(n + 1 + 2nr^2)(-1)^{n-1} + n(n + 2)r^{n+1}(-1)^{n-1} \\
&= 1 - r^2 + (-1)^{n-1}\{(n^2 - 1)r^{n+2} + nr^{n+1}(2n + 2 + nr^2) \\
&\quad + (n + 1)r^n(n + 1 + 2nr^2) + n(n + 2)r^{n+1}\}.
\end{aligned}$$

Thus

$$\begin{aligned}
C_n(r, \theta) &\geq 1 - r^2 - \{(n^2 - 1)r^{n+2} + nr^{n+1}(2n + 2 + nr^2) \\
&\quad + (n + 1)r^n(n + 1 + 2nr^2) + n(n + 2)r^{n+1}\}, \quad n \geq 1,
\end{aligned}$$

and equality holds if $n = 2k$, and $\theta = \pi$. This yields

THEOREM 3. Denote the unique positive root of the equation

$$p_n(r) \equiv 1 - r^2 - (n + 1)^2 r^n - n(3n + 4)r^{n+1} - (3n^2 + 2n - 1)r^{n+2} - n^2 r^{n+3} = 0$$

by ρ_n . Then $R_n \geq \rho_n$, and equality holds for $n = 2k$, $k \geq 1$.

Note that $p_n(0) = 1$, $p_n(1) < 0$, $p_n'(r) < 0$. Hence ρ_n is unique and

$$(2.2) \quad 0 < \rho_n < 1.$$

Evidently $p_n(-1) = 0$, hence $1 + r$ can be factored out, and we get

$$(2.3) \quad \frac{p_n(r)}{1 + r} = 1 - r - (n + 1)^2 r^n - (2n^2 + 2n - 1)r^{n+1} - n^2 r^{n+2} \equiv q_n(r),$$

so that $q_n(\rho_n) = 0$.

3. Estimation of ρ_n and R_n . Direct calculation gives

$$R_1 = 1; \rho_1 = 0.182 \dots$$

Also $\rho_2 = R_2$, and

$$S_2(r, \theta) = r \sin \theta + 2r^2 \sin 2\theta = r \sin \theta(1 + 4r \cos \theta),$$

which yields by Theorem 1: $R_2 = 1/4 = \rho_2$. A similar calculation yields $R_3 = 2^{1/2}/3$.

We shall prove

$$(3.1) \quad \rho_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \log 3/4 + \epsilon_n}{n}, \quad \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let c be a constant, and

$$(3.2) \quad r_n(c) = 1 - \frac{3 \log n}{n} + \frac{\log \log n + c}{n};$$

then from

$$\log(1 - x) = -x + O(x^2) \quad \text{as } x \rightarrow 0,$$

we conclude

$$(3.3) \quad \begin{aligned} \{r_n(c)\}^n &= \exp \{ -3 \log n + \log \log n + c + O(n^{-1} \log^2 n) \} \\ &= n^{-3} \log n \cdot e^c \{ 1 + O(n^{-1} \log^2 n) \} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Furthermore, from (2.3), (3.2), and (3.3)

$$q_n \{r_n(c)\} = \frac{3 \log n}{n} - \frac{\log \log n + c}{n} - \frac{4 \log n}{n} \cdot e^c \{ 1 + o(1) \},$$

hence

$$\frac{nq_n \{r_n(c)\}}{\log n} \rightarrow 3 - 4e^c \quad \text{as } n \rightarrow \infty.$$

Thus for

$$c = \log 3/4 + \epsilon,$$

ϵ a given small number, and for sufficiently large values of n

$$\operatorname{sgn} q_n \{r_n(c)\} = \operatorname{sgn} \epsilon,$$

from which follows (3.1).

We have thus proved

THEOREM 4. *If $\rho_n > 0$ and $p_n(\rho_n) = 0$, then*

$$\rho_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \log 3/4 + \epsilon_n}{n}, \quad \text{where } \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

4. Derivation of an asymptotic estimate for R_n . On writing

$$R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \delta_n}{n}, \quad n > 1,$$

it follows from Theorem 3 that

$$\delta_n \geq \log 3/4 + \epsilon_n,$$

and equality holds for $n = 2k$, $k \geq 1$; hence from Theorem 4

$$\liminf_{n \rightarrow \infty} \delta_n = \log 3/4, \quad \lim_{k \rightarrow \infty} \delta_{2k} = \log 3/4.$$

It remains to give an estimate for R_{2k-1} from above.

If for a particular value of θ and r , $C_{2k-1}(r, \theta) < 0$, then by Theorem 2, evidently $R_{2k-1} < r$. We now choose $\theta = \pi - (3\pi/4k)$; then

$$\begin{aligned}
C_{2k-1}(r, \theta) &= 1 - r^2 + \frac{1}{\sin(3\pi/4k)} \left\{ 2kr^{2k+1} \sin \frac{3(k-1)\pi}{2k} \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \sin \frac{3(2k-1)\pi}{4k} \\
&\quad + r^{2k-1}[2k + 2(2k-1)r^2] \sin \frac{3\pi}{2} \\
&\quad \left. + (2k-1)r^{2k} \sin \frac{3(2k+1)\pi}{4k} \right\} \\
&= 1 - r^2 - \frac{1}{\sin(3\pi/4k)} \left\{ 2kr^{2k+1} \cos \frac{3\pi}{2k} \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \left[\cos \frac{3\pi}{4k} \right] \\
&\quad \left. + 2kr^{2k-1} + (4k-2)r^{2k+1} + (2k-1)r^{2k} \cos \frac{3\pi}{4k} \right\} \\
&< 1 - r^2 - \frac{4k}{3\pi} \left\{ 2kr^{2k+1} \left(1 - \frac{9\pi^2}{8k^2} \right) \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \left(1 - \frac{9\pi^2}{32k^2} \right) \\
&\quad \left. + 2kr^{2k-1} + (4k-2)r^{2k+1} + (2k-1)r^{2k} \left(1 - \frac{9\pi^2}{32k^2} \right) \right\}, \quad k \geq 3
\end{aligned}$$

(since $\cos x > 1 - x^2/2$ for all x). Hence

$$\begin{aligned}
C_{2k-1}(r, \theta) &< 1 - r^2 - (2k/5) \{ kr^{2k+1} + 2kr^{2k} + (k-1/2)r^{2k+2} + 2kr^{2k-1} \\
&\quad + (4k-2)r^{2k+1} + (k-1/2)r^{2k} \}, \quad k \geq 5,
\end{aligned}$$

thus

$$C_{2k-1}(r, \theta) < 1 - r^2 - (2k/5)(11k-3)r^{2k+2} < 1 - r^2 - 4k^2r^{2k+2}.$$

Choosing r so that

$$(4.1) \quad 1 - r^2 - 4k^2r^{2k+2} \leq 0,$$

we get

$$C_{2k-1}(r, \theta) < 0, \quad R_{2k-1} < r.$$

To find an upper bound for r , we put

$$(4.2) \quad r = 1 - \frac{3 \log(2k-1)}{2k-1} + \frac{\log \log(2k-1) + c}{2k-1};$$

we obtain as in (3.3)

$$\begin{aligned} r^{2k-1} &= \exp \left\{ -3 \log (2k-1) + \log \log (2k-1) + c + O(k^{-1} \log^2 k) \right\} \\ &= (2k-1)^{-3} \log (2k-1) \cdot e^c \{ 1 + O(k^{-1} \log^2 k) \}. \end{aligned}$$

Thus, using (4.2),

$$\begin{aligned} \frac{4k^2 r^{2k+2}}{1-r^2} &= \left(\frac{2k}{2k-1} \right)^2 \cdot \frac{r^3}{1+r} \cdot \frac{r^{2k-1} (2k-1)^2}{1-r} \\ &= \frac{1+o(1)}{2+o(1)} \cdot \frac{1}{3} e^c \{ 1+o(1) \} \rightarrow \frac{1}{6} e^c \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Hence (4.1) is satisfied for all sufficiently large k provided $e^c/6 > 1$, that is, $c > \log 6$. It now follows that $\limsup_{k \rightarrow \infty} \delta_{2k-1} \leq 6$. Summarizing we have

THEOREM 5. *Let*

$$R_n = 1 - \frac{3 \log n}{n} + \frac{\log \log n + \delta_n}{n}, \quad n > 1;$$

then $\lim_{k \rightarrow \infty} \delta_{2k} = \log 3/4$, and

$$\log 3/4 \leq \liminf_{k \rightarrow \infty} \delta_{2k-1} \leq \limsup_{k \rightarrow \infty} \delta_{2k-1} \leq 6.$$

5. Application to Fourier series. Consider the roof-function

$$\frac{2b}{a(\pi-a)} \sum_1^\infty \frac{\sin \nu a \sin \nu \theta}{\nu^2} = \begin{cases} \frac{b\theta}{a} & \text{for } 0 \leq \theta \leq a, \\ b \frac{\pi-\theta}{\pi-a} & \text{for } a \leq \theta \leq \pi, \end{cases}$$

where $0 < a < \pi$, $0 < b$, and the corresponding harmonic function

$$\frac{2b}{a(\pi-a)} \sum_1^\infty r^\nu \frac{\sin \nu a \sin \nu \theta}{\nu^2} = H(r; a, b).$$

Denote its partial sums by

$$H_n(r, \theta) = \frac{2b}{a(\pi-a)} \sum_1^n r^\nu \frac{\sin \nu a \sin \nu \theta}{\nu^2};$$

then

$$\frac{\partial^2 H_n(r, \theta)}{\partial \theta^2} = - \frac{2b}{a(\pi-a)} \sum_1^n r^\nu \sin \nu a \sin \nu \theta \leq 0$$

for $0 < r \leq R_n$, $0 < \theta < \pi$, by Lemma 1 and Theorem 1. Hence $H_n(r, \theta)$ is con-

vex upwards for $0 < \theta < \pi$, $r \leq R_n$; but not convex for $r > R_n$. The same is true for the limiting cases $a \rightarrow 0$ and $a \rightarrow \pi$. In which cases

$$H(r; 0, b) = \frac{2b}{\pi} \sum_1^{\infty} r^{\nu} \frac{\sin \nu \theta}{\nu},$$

$$H(r; \pi, b) = \frac{2b}{\pi} \sum_1^{\infty} r^{\nu} \frac{\sin \nu(\pi - \theta)}{\nu}.$$

Moreover every polygon convex upwards and lying above the axis of abscissae is expressible as a finite sum with positive coefficients of roof-functions. Hence the partial sums of the corresponding harmonic development are convex upwards for $r \leq R_n$. Finally any function positive in $0 < \theta < \pi$, and convex upwards can be approximated uniformly by such polygons; hence we have

THEOREM 6. *If $f(\theta) > 0$ in $0 < \theta < \pi$, and is convex upwards, and if $f(\theta) \sim \sum_1^{\infty} b_{\nu} \sin \nu \theta$, then $\sum_1^n r^{\nu} b_{\nu} \sin \nu \theta$ is convex upwards in $0 < \theta < \pi$, $r \leq R_n$; but not always for $r < R_n + \epsilon$, $\epsilon > 0$.*

6. Cosine series. We now consider the cosine series of the step function

$$\frac{2b}{\pi} \left\{ \frac{\pi - a}{2} - \sum_1^{\infty} \frac{\sin \nu a \cos \nu \theta}{\nu} \right\} = \begin{cases} 0 & \text{for } 0 \leq \theta < a, \\ b & \text{for } a < \theta \leq \pi, \end{cases}$$

where $0 < a < \pi$, $b > 0$; and the corresponding harmonic development

$$K(r, \theta) = \frac{b}{\pi} (\pi - a) - \frac{2b}{\pi} \sum_1^{\infty} r^{\nu} \frac{\sin \nu a \cos \nu \theta}{\nu}.$$

For the partial sums $K_n(r, \theta)$ of this series we have

$$\frac{\partial K_n(r, \theta)}{\partial \theta} = \frac{2b}{\pi} \sum_1^n r^{\nu} \sin \nu a \sin \nu \theta \geq 0 \quad \text{for } 0 < r \leq R_n, 0 < \theta < \pi;$$

hence $K_n(r, \theta)$ is monotonic increasing in the same domain; R_n cannot be replaced by $R_n + \epsilon$, $\epsilon > 0$. The same statement for any monotonic increasing function follows now in an obvious way. Hence we have

THEOREM 7. *If $f(\theta)$ is monotonic in $0 < \theta < \pi$, and*

$$f(\theta) \sim a_0/2 + \sum_1^{\infty} a_{\nu} \cos \nu \theta,$$

then the n th partial sum of $a_0/2 + \sum_1^{\infty} a_{\nu} r^{\nu} \cos \nu \theta$ is monotonic in the same sense for $0 < r \leq R_n$, and here R_n cannot be replaced by $R_n + \epsilon$, $\epsilon > 0$.

7. Curves convex in direction of the v -axis. We say that a curve in the (u, v) -plane is convex in the direction of the v -axis if any parallel to the v -axis

has at most two points in common with the curve. This class of mappings was considered by L. Fejér [1] and the author [6]. We now prove

THEOREM 8. *Suppose the power series $\sum_0^\infty a_\nu z^\nu = f(z) = u + iv$ is regular in $|z| < 1$, and all a_ν are real. Suppose further that the images K_r of the circles $|z| = r$, $0 < r < 1$, are convex in the direction of the v -axis (thus $f(z)$ is univalent). Then the partial sum $\sum_0^n a_\nu z^\nu$ has the same property in $|z| \leq R_n$, but—in general—not in a larger circle.*

For the proof we may assume without loss of generality that the upper half of the circle $|z| < 1$ is mapped onto the upper half of the image in the w -plane. On writing $w(e^{i\theta}) = u(\theta) + iv(\theta) \sim \sum_0^\infty a_\nu \cos \nu\theta + i \sum_0^\infty a_\nu \sin \nu\theta$, we find that $v(\theta)$ is positive for $0 < \theta < \pi$, and (from the assumption) $u(\theta)$ is decreasing in the same interval. Our theorem follows now from Theorems 5 and 7.

8. Conclusion. Suppose $f(z) = \sum_1^\infty b_\nu z^\nu$ is a typically real function, that is,

$$\sum_1^\infty b_\nu r^\nu \sin \nu\theta \geq 0 \quad \text{for } 0 < r < 1, 0 < \theta < \pi.$$

Then the Riesz means of second order

$$P_n(z) = (n+1)^{-2} \sum_{\nu=1}^n (n-\nu+1)^2 b_\nu z^\nu, \quad n \geq 1,$$

are typically real in $|z| \leq 1$ (Szász [6]; cf. Theorem 1). Evidently $\lim_{n \rightarrow \infty} P_n(z) = f(z)$ in $|z| < 1$, uniformly in $|z| \leq r$, $r < 1$. Another such sequence of polynomials is

$$s_n(R_n z) = \sum_{\nu=1}^n b_\nu R_n^\nu z^\nu, \quad n \geq 1.$$

These polynomials are typically real in $|z| \leq 1$ by property (a) of §1. Furthermore for $|z| \leq r < 1$

$$\begin{aligned} |f(z) - s_n(R_n z)| &\leq \sum_1^n |b_\nu| r^\nu (1 - R_n^\nu) + \sum_{n+1}^\infty |b_\nu| r^\nu \\ &< (1 - R_n) \sum_1^n \nu |b_\nu| r^\nu + \sum_{n+1}^\infty |b_\nu| r^\nu \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} s_n(R_n z) = f(z)$$

uniformly in $|z| \leq r < 1$.

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