NÖRLUND SUMMABILITY OF DOUBLE FOURIER SERIES

BY

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1. **Introduction.** Throughout this paper the function $f(t, u)$ is assumed to to be Lebesgue integrable over the square Q $(-\pi, \pi; -\pi, \pi)$ and to have period 2π in each variable. The double Fourier series of f is denoted by $\sigma(f)$ and the rectangular partial sums of $\sigma(f)$ are denoted by $s_{mn}(x, y; f)$. To say that a method of summability S possesses the localization property means that if f vanishes in a neighborhood of (x, y) then S sums $\sigma(f)$ at (x, y) to 0. It is well known that the Cesàro method $(C, 1, 1)$, for example, does not possess the localization property. G. Grünwald [2](1) has shown that at any point (x, y) of continuity of f the square partial sums $s_{nn}(x, y; f)$ are summable $(C, 1)$ to $f(x, y)$. Thus $(C, 1)$ applied to the square partial sums possesses the localization property. We show in §5 that this is the best possible result.

In this paper we shall apply Nörlund means to $\sigma(f)$. To define the Nörlund mean of $\{s_{nn}(x, y; f)\}$ let $\{p_n\}$ be any sequence of constants. Let $P_n = \sum_{k=0}^{n} p_k \neq 0$. The Nörlund mean is

$$(1.01)\qquad t_n(x, y; f) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_{kk}(x, y; f).$$

If $t_n(x, y; f)$ tends to a limit as $n \to \infty$ the sequence $\{s_{nn}(x, y; f)\}$ is said to be summable N_p to this limit. We shall consider only regular Nörlund methods of summability. The conditions of regularity for N_p are(2)

$$(1.02)\qquad \sum_{k=0}^{n} | p_k | = O(| P_n |), \qquad p_n/P_n \to 0 \quad \text{as} \quad n \to \infty.$$

Cesàro (C, α), $\alpha > 0$, is clearly a regular Nörlund method.

We shall also consider a double Nörlund transform of $\{s_{mn}(x, y; f)\}$. Let $\{p_n^{(k)}\}$ $(k = 1, 2)$ be two sequences of constants. Let $P_n^{(k)} = \sum_{j=0}^{n} p_j^{(k)} \neq 0$. Then the double Nörlund transform is

$$(1.03)\qquad t_{mn}(x, y; f) = \frac{1}{P_m^{(1)} P_n^{(2)}} \sum_{j,k=0}^{m,n} p_{m-j}^{(1)} p_{n-k}^{(2)} s_{jk}(x, y; f).$$

We shall restrict the manner in which $m, n \to \infty$. If, for any $\lambda \geq 1$, $t_{mn}(x, y; f)$ tends to a limit when $m, n \to \infty$ in such a manner that $m/n \leq \lambda$, $n/m \leq \lambda$, this

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(1) The numbers in square brackets refer to the bibliography at the end of the paper.

(2) See, for example, Hille and Tamarkin [4, p. 758].

limit being independent of λ, then $\sigma(f)$ is said to be restrictedly summable N_p at (x, y) to this limit. (C, α, β) is clearly a double Nörlund method.

In §§5 and 6 of this paper local conditions are imposed on the function whose double Fourier series is under consideration in order to discover which of these methods of summability possess the localization property and which do not. In §§7 to 11 methods of summability which sum $\sigma(f)$ almost everywhere to f are studied. Theorem 5 is a generalization of and includes the result of Marcinkiewicz and Zygmund [6]. When the present paper had been prepared for publication the author received a copy of a paper just published by Grünwald [3] in which it was shown that the sequence $\{s_{nn}(x, y; f)\}$ is summable $(C, 1)$ almost everywhere to $f(x, y)$. However, by Corollary 6.1 of the present paper, this result is true also for (C, α), $\alpha > 0$. Both Corollary 6.1 and Theorem 6 from which it follows were established several months before the appearance of Grünwald's paper. Indeed the result of Corollary 6.1 was known much earlier, for, on reading the proofs of a paper of Marcinkiewicz [5] in which it was shown that the sequence $\{s_{nn}(x, y; f)\}$ is summable $(C, 2)$ almost everywhere to $f(x, y)$, Zygmund pointed out that the result could be extended to (C, α), $\alpha > 0$. But Marcinkiewicz did not wish to change his paper and so the result was not published.

2. **Basic formulas.** The following notation will be employed throughout this paper. Let

$$\begin{aligned}\phi_{xy}(t, u) &= f(x + t, y + u) + f(x + t, y - u) + f(x - t, y + u) \\ &\quad + f(x - t, y - u) - 4f(x, y). \end{aligned} \tag{2.01}$$

It is well known that

$$s_{mn}(x, y; f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x + t, y + u)D_m(t)D_n(u)dtdu \tag{2.02}$$

where $D_m(t)$ denotes the Dirichlet kernel. Then

$$t_n(x, y; f) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x + t, y + u)K_n(t, u)dtdu \tag{2.03}$$

where

$$\begin{aligned} K_n(t, u) &= \frac{1}{\pi^2 P_n}\sum_{k=0}^{n} p_{n-k}D_k(t)D_k(u) \\ &= \frac{1}{\pi^2 P_n}\sum_{k=0}^{n} \frac{p_{n-k}\sin (k + \frac{1}{2})t \sin (k + \frac{1}{2})u}{4 \sin \frac{1}{2}t \sin \frac{1}{2}u}. \end{aligned} \tag{2.04}$$

Clearly $K_n(t, u)$ is an even-even function of t and u and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_n(t, u)dtdu = 1.$$

It follows that

$$(2.05)\qquad t_n(x, y; f) - f(x, y) = \int_0^\pi \int_0^\pi \phi_{xy}(t, u) K_n(t, u)\,dt\,du.$$

In order to obtain alternative forms for $K_n(t, u)$ we set

$$(2.06)\qquad \mathfrak{P}_n(t) = \sum_{k=0}^{n} p_k e^{ikt} = \sum_{k=0}^{n} p_k \cos kt + i \sum_{k=0}^{n} p_k \sin kt = \mathfrak{C}_n(t) + i\mathfrak{S}_n(t).$$

Now $\sin (k+\frac{1}{2})t \sin (k+\frac{1}{2})u = -\frac{1}{2}[\cos (k+\frac{1}{2})(t+u) - \cos (k+\frac{1}{2})(t-u)]$ and

$$\begin{aligned}\sum_{k=0}^{n} p_{n-k} \cos (k + \tfrac{1}{2})(t \pm u) &= \sum_{k=0}^{n} p_k \cos (n - k + \tfrac{1}{2})(t \pm u) \\ &= \mathfrak{C}_n(t \pm u) \cos (n + \tfrac{1}{2})(t \pm u) \\ &\quad + \mathfrak{S}_n(t \pm u) \sin (n + \tfrac{1}{2})(t \pm u).\end{aligned}$$

Substituting in (2.04) we have

$$(2.07)\qquad \begin{aligned} K_n(t, u) = &- (8\pi^2 P_n \sin \tfrac{1}{2}t \sin \tfrac{1}{2}u)^{-1}\{\mathfrak{C}_n(t + u) \cos (n + \tfrac{1}{2})(t + u) \\ &+ \mathfrak{S}_n(t + u) \sin (n + \tfrac{1}{2})(t + u) - \mathfrak{C}_n(t - u) \cos (n + \tfrac{1}{2})(t - u) \\ &- \mathfrak{S}_n(t - u) \sin (n + \tfrac{1}{2})(t - u)\}.\end{aligned}$$

If we apply the mean value theorem to this we obtain

$$(2.08)\qquad \begin{aligned} K_n(t, u) = &- u(4\pi^2 P_n \sin \tfrac{1}{2}t \sin \tfrac{1}{2}u)^{-1}\{- \mathfrak{C}_n(\xi_1)\cdot(n + \tfrac{1}{2}) \sin (n + \tfrac{1}{2})\xi_1 \\ &+ \mathfrak{C}_n'(\xi_1) \cos (n + \tfrac{1}{2})\xi_1 + \mathfrak{S}_n(\xi_2)\cdot(n + \tfrac{1}{2}) \cos (n + \tfrac{1}{2})\xi_2 \\ &+ \mathfrak{S}_n'(\xi_2) \sin (n + \tfrac{1}{2})\xi_2\}, \qquad t - u \leqq \xi_1, \xi_2 \leqq t + u.\end{aligned}$$

Forming the double Nörlund transform of $s_{mn}(x, y; f)$ we have

$$(2.09)\qquad t_{mn}(x, y; f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + t, y + u) N_m^{(1)}(t) N_n^{(2)}(u)\,dt\,du$$

where

$$(2.10)\quad N_n^{(k)}(t) = \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n} p_{n-j}^{(k)} D_j(t) = \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n} \frac{p_{n-j}^{(k)} \sin (j + \frac{1}{2})t}{2 \sin \frac{1}{2}t}, \qquad k = 1, 2.$$

Thus $N_n^{(k)}(t)$ is an even function of t and

$$\int_{-\pi}^{\pi} N_n^{(k)}(t)\,dt = 1, \qquad k = 1, 2.$$

We easily deduce that

$$(2.11)\qquad t_{mn}(x, y; f) - f(x, y) = \int_0^\pi \int_0^\pi \phi_{xy}(t, u) N_m^{(1)}(t) N_n^{(2)}(u)\,dt\,du.$$

Defining $\mathfrak{P}_n^{(k)}(t)$, $\mathfrak{C}_n^{(k)}(t)$, $\mathfrak{S}_n^{(k)}(t)$ analogously to (2.06) and proceeding as in the deduction of (2.07) we obtain

$$(2.12)\quad N_n^{(k)}(t) = (2\pi P_n^{(k)} \sin \tfrac{1}{2}t)^{-1}\{\mathfrak{C}_n^{(k)}(t) \sin (n+\tfrac{1}{2})t - \mathfrak{S}_n^{(k)}(t) \cos (n+\tfrac{1}{2})t\}, \qquad k = 1, 2.$$

3. **Estimates of the kernels.** We require estimates for $K_n(t, u)$ and $N_n^{(k)}(t)$. We shall assume throughout this section that the sequences $\{p_n\}$ and $\{p_n^{(k)}\}$ $(k=1, 2)$ satisfy (1.02) and that $n|p_n| = O(|P_n|)$, $n|p_n^{(k)}| = O(|P_n^{(k)}|)$. All $\{p_n\}$ and $\{p_n^{(k)}\}$ used in our theorems satisfy these conditions.

Since $|D_k(t)| \leqq k+1/2$, it follows from (2.04) that[3]

$$(3.01)\qquad |K_n(t, u)| \leqq An^2, \qquad n \geqq 1, \text{ all } t, u.$$

Also from (2.04) we have

$$(3.02)\qquad |K_n(t, u)| \leqq A/tu, \qquad 0 < t, u \leqq \pi.$$

In the same way from (2.10) we obtain

$$(3.03)\qquad |N_n^{(k)}(t)| \leqq An, \qquad n \geqq 1, \text{ all } t,\ k = 1, 2.$$

In order to obtain further estimates for the kernels we need to estimate $\mathfrak{P}_n(t)$ and $\mathfrak{P}_n'(t)$. We put

$$(3.04)\qquad |p_n| = r_n, \qquad R_n = \sum_{k=0}^{n} r_k, \qquad V_0 = 0, \qquad V_n = \sum_{k=1}^{n} |p_k - p_{k-1}|,$$

and introduce the step functions

$$(3.05)\qquad r(u) = r_{[u]}, \qquad R(u) = R_{[u]}, \qquad V(u) = V_{[u]},$$

where $[u]$ as usual denotes the largest integer less than or equal to u. Let us note that by (1.02)

$$(3.06)\qquad |P_n| = \left|\sum_{k=0}^{n} p_k\right| \leqq \sum_{k=0}^{n} |p_k| = R_n = \sum_{k=0}^{n} |p_k| \leqq A|P_n|.$$

Proceeding as on p. 768 of the paper of Hille and Tamarkin [4] and noting that $t^{-1}r(1/t) \leqq AR(1/t)$ we have

$$(3.07)\qquad |\mathfrak{P}_n(t)| \leqq A\left\{R\left(\frac{1}{t}\right) + \frac{1}{t}\left[r_n + V_n - V\left(\frac{1}{t}\right)\right]\right\}, \qquad \frac{1}{n} \leqq t \leqq \frac{3\pi}{2}.$$

If we set

$$(3.08)\qquad U_k' = i\sum_{j=0}^{k} je^{ijt} = \frac{d}{dt}\left\{\sum_{j=0}^{k} e^{ijt}\right\},$$

(3) Here and in the sequel the letter A denotes an absolute constant. The constant need not be the same at every occurrence.

then for $1/n \leqq t \leqq 3\pi/2$ we have $|tU_k'| \leqq An$ ($k=0, 1, 2, \cdots, n$). Using this fact and proceeding as in proving (3.07) we get

$$(3.09)\quad |\mathfrak{P}_n'(t)| \leqq An\left\{R\left(\frac{1}{t}\right)+\frac{1}{t}\left[r_n+V_n-V\left(\frac{1}{t}\right)\right]\right\},\qquad \frac{1}{n}\leqq t\leqq\frac{3\pi}{2}.$$

Then for $t, u>0$, $1/n\leqq t+u\leqq 3\pi/2$, $|t-u|\geqq 1/n$ we have

$$(3.10)\quad \begin{aligned}|K_n(t,u)| \leqq \frac{A}{R_n tu}\Bigg\{&R\left(\frac{1}{t+u}\right)+\frac{1}{t+u}\left[r_n+V_n-V\left(\frac{1}{t+u}\right)\right]\\ &+R\left(\frac{1}{|t-u|}\right)+\frac{1}{|t-u|}\left[r_n+V_n-V\left(\frac{1}{|t-u|}\right)\right]\Bigg\},\end{aligned}$$

$$(3.11)\quad \begin{aligned}|K_n(t,u)| \leqq \frac{An}{R_n(t+u)}\Bigg\{&R\left(\frac{1}{|t-u|}\right)\\ &+\frac{1}{|t-u|}\left[r_n+V_n-V\left(\frac{1}{t+u}\right)\right]\Bigg\}.\end{aligned}$$

Relation (3.10) follows from (2.07) and (3.07); (3.11) follows from (2.08), (3.07) and (3.09) if we note that $K_n(t, u)=K_n(u, t)$ and that $t+u\leqq 2t$ in case $t-u\geqq 1/n$.

Analogously to (3.04) and (3.05) we can define $r_n^{(k)}$, $R_n^{(k)}$, $V_n^{(k)}$, $r^{(k)}(u)$, $R^{(k)}(u)$, $V^{(k)}(u)$ and obtain an estimate for $|\mathfrak{P}_n^{(k)}(t)|$ similar to (3.07). Then from (2.12) we have

$$(3.12)\quad |N_n^{(k)}(t)| \leqq A\{M_{n1}^{(k)}(t)+M_{n2}^{(k)}(t)+M_{n3}^{(k)}(t)\},\quad 1/n\leqq t\leqq\pi,\ k=1,2,$$

where

$$(3.13)\quad \begin{aligned}M_{n1}^{(k)}(t)&=\frac{1}{tR_n^{(k)}}R^{(k)}\left(\frac{1}{t}\right),\qquad M_{n2}^{(k)}(t)=\frac{1}{t^2R_n^{(k)}}r_n^{(k)},\\ M_{n3}^{(k)}(t)&=\frac{1}{t^2R_n^{(k)}}\left\{V_n^{(k)}-V^{(k)}\left(\frac{1}{t}\right)\right\},\qquad k=1,2.\end{aligned}$$

Estimating $\mathfrak{P}_n(t)$ and $\mathfrak{P}_n^{(k)}(t)$ as on p. 767 of the paper of Hille and Tamarkin [4] we obtain from (2.07) and (2.12), respectively,

$$(3.14)\quad \begin{aligned}&|K_n(t,u)| \leqq \{A(\delta)/R_n\}\{V_n+r_n\},\\ &\qquad 0<\delta\leqq t,\ u\leqq\pi,\ t+u\leqq 2\pi-\delta,\ |t-u|\geqq\delta,\end{aligned}$$

$$(3.15)\quad |N_n^{(k)}(t)| \leqq \{A(\delta)/R_n^{(k)}\}\{V_n^{(k)}+r_n^{(k)}\},\qquad 0<\delta\leqq t\leqq\pi,\ k=1,2,$$

where $A(\delta)$ depends only on δ.

Finally we consider the $(C, 1)$ kernel $K_n^1(t, u)$ which is a special case of $K_n(t, u)$ when $p_n\equiv 1$. In case $n\geqq 1$, $0\leqq t\leqq\pi$, $0\leqq u\leqq\pi/2$ or $0\leqq t\leqq\pi/2$,

$\pi/2 \leqq u \leqq \pi$ we shall show that

$$|\overset{1}{K}_n(t, u)| \leqq \frac{An^2}{(1 + n^{3/2}t^{3/2})(1 + n^{3/2}u^{3/2})} + \frac{An^2}{[1 + n^{3/2}|t - u|^{3/2}][1 + n^{3/2}(t + u)^{3/2}]}, \tag{3.16}$$

the positive square root being taken in all cases. Since $p_n \equiv 1$, we have $r_n = 1$, $P_n = R_n = n+1$, $V_n = 0$ $(n = 0, 1, 2, \cdots)$. Then, from (3.01), (3.02), (3.10) and (3.11) we have, respectively,

$$|\overset{1}{K}_n(t, u)| \leqq An^2, \qquad n \geqq 1, \text{ all } t, u, \tag{3.17}$$

$$|\overset{1}{K}_n(t, u)| \leqq A/tu, \qquad 0 < t, u \leqq \pi, \tag{3.18}$$

$$|\overset{1}{K}_n(t, u)| \leqq \frac{A}{ntu(t + u)} + \frac{A}{ntu|t - u|} \qquad \frac{1}{n} \leqq t + u \leqq \frac{3\pi}{2}, \tag{3.19}$$

$$|\overset{1}{K}_n(t, u)| \leqq \frac{A}{|t - u|(t + u)} \qquad |t - u| \geqq \frac{1}{n}, \; t, u > 0. \tag{3.20}$$

Let D_1 be the part of the domain under consideration in which $t \leqq 2/n$, $u \leqq 2/n$, D_2 that part in which $t > 2/n$, $u \leqq 1/n$, D_4 the part in which $0 \leqq t - u \leqq 1/n$, $t > 2/n$, $u > 1/n$, D_6 the part in which $t > 2/n$, $1/n < u \leqq t/2$, D_8 the part in which $t > 2/n$, $t/2 < u < t - 1/n$, and D_3, D_5, D_7, D_9 the domains symmetric to D_2, D_4, D_6, D_8, respectively. Then (3.16) follows from (3.17) in D_1, from (3.20) in D_2, from (3.18) in D_4, and from (3.19) in D_6 and D_8. It follows in D_3, D_5 D_7, D_9 by symmetry. Thus (3.16) is completely established.

4. **Preliminary lemmas.** The following lemmas concerning the Nörlund coefficients p_n and P_n will be useful.

LEMMA 1. *If* $\sum_{k=1}^{n} k|p_k - p_{k-1}| = O(|P_n|)$, *then* $n|p_n| = O(|P_n|)$ *and* $\sum_{k=0}^{n}|p_k| = O(|P_n|)$.

LEMMA 2. *If* $n\sum_{k=1}^{n}|p_k - p_{k-1}| = O(|P_n|)$, *then* $n = O(|P_n|)$ *and* $\sum_{k=1}^{n}|P_k|/k = O(|P_n|)$.

It is clear that the hypothesis of Lemma 2 implies that of Lemma 1. These lemmas follow easily from the relations

$$P_k = (k + 1)p_k + \sum_{j=1}^{k} j(p_{j-1} - p_j),$$

$$p_k = p_n + \sum_{j=k+1}^{n} (p_{j-1} - p_j), \qquad k = 0, 1, 2, \cdots, n - 1.$$

We may also easily establish the following analogue of Abel's partial sum formula.

LEMMA 3. *Let* $\{a_{jk}\}$, $\{b_{jk}\}$ *be two sequences. Let*

$$\Delta_{10}a_{jk} = a_{jk} - a_{j+1,k}, \qquad \Delta_{01}a_{jk} = a_{jk} - a_{j,k+1}, \qquad \Delta_{11}a_{jk} = \Delta_{01}\Delta_{10}a_{jk}.$$

Similarly define $\Delta_{10}b_{jk}$, $\Delta_{01}b_{jk}$, $\Delta_{11}b_{jk}$. *Then*

$$\begin{aligned} \sum_{j=c,k=d}^{m,n} a_{jk}\Delta_{11}b_{jk} = {} & \sum_{j=c,k=d}^{m,n} b_{jk}\Delta_{11}a_{j-1,k-1} - \sum_{j=c}^{m} b_{jd}\Delta_{10}a_{j-1,d-1} \\ & + \sum_{j=c}^{m} b_{j,n+1}\Delta_{10}a_{j-1,n} - \sum_{k=d}^{n} b_{ck}\Delta_{01}a_{c-1,k-1} \\ & + \sum_{k=d}^{n} b_{m+1,k}\Delta_{01}a_{m,k-1} + a_{c-1,d-1}b_{cd} \\ & - a_{m,d-1}b_{m+1,d} - a_{c-1,n}b_{c,n+1} + a_{mn}b_{m+1,n+1}. \end{aligned} \tag{4.01}$$

5. **Local results making use of square partial sums.** Our first theorem extends the result of Grünwald [2] in two directions and also includes his result.

THEOREM 1. *Let* N_p *be a regular Nörlund method of summability satisfying the condition*

$$\sum_{k=1}^{n-1} (n-k) \mid p_k - p_{k-1} \mid = O(\mid P_n \mid). \tag{5.01}$$

Then at any point (x, y) *such that*

$$\begin{aligned} \Phi(h, k) &= \int_0^h dt \int_0^k \mid \phi_{xy}(t, u) \mid du = o(hk), \\ \Phi^*(h, k) &= \int_0^h dt \int_0^k \left| \phi_{xy}\left(\frac{t-u}{2^{1/2}}, \frac{t+u}{2^{1/2}}\right) \right| du = o(hk) \end{aligned} \tag{5.02}$$

as $h, k \to 0$ *simultaneously but independently, the sequence* $\{s_{nn}(x, y; f)\}$ *is summable* N_p *to* $f(x, y)$.

It should be noted that the second condition on the function at (x, y) is similar to the first. The first is applied to rectangles along the axes, the second to rectangles along the bisectors of the angles between the axes. The factors $2^{-1/2}$ are not essential, but are introduced for convenience.

Proof. A regular Nörlund method N_p includes $(C, 1)$ if(4)

$$n \mid p_0 \mid + \sum_{k=1}^{n-1} (n-k) \mid p_k - p_{k-1} \mid < A \mid P_n \mid. \tag{5.03}$$

Hence if N_p satisfies (5.01) and is regular, then it includes $(C, 1)$. Thus it

(4) See, for example, Hille and Tamarkin [4, p. 782].

suffices to prove the theorem for $(C, 1)$. Let $t_n^1(x, y; f)$ denote the $(C, 1)$ transform of the sequence $\{s_{nn}(x, y; f)\}$. From (2.05) we have

$$(5.04) \qquad t_n^1(x, y; f) - f(x, y) = \int_0^\pi \int_0^\pi \phi_{xy}(t, u) K_n^1(t, u) dt du$$

where $K_n^1(t, u)$ is given by (2.04) with $p_n \equiv 1$. Fix (x, y).

Given $\epsilon > 0$ we can choose δ such that $0 < \delta < \pi/4$ and such that

$$(5.05) \quad |\Phi(h, k)| < \epsilon |hk|, \qquad |\Phi^*(h, k)| < \epsilon |hk|, \text{ for } 0 < |h|, \ |k| \leqq 2\delta.$$

Suppose $n > 2/\delta$. Let $B_\delta = [0, \pi; 0, \pi] - [0, \delta; 0, \delta]$. Then

$$(5.06) \quad \begin{aligned} |t_n^1(x, y; f) - f(x, y)| &\leqq \left(\int_0^\delta \int_0^\delta + \iint_{B_\delta} \right) \cdot |\phi_{xy}(t, u) K_n^1(t, u)| dt du \\ &= J_1 + J_2. \end{aligned}$$

Then by (3.16)

$$\begin{aligned} J_1 \leqq An^2 \int_0^\delta \int_0^\delta & \frac{|\phi_{xy}(t, u)| dt du}{(1 + n^{3/2} t^{3/2})(1 + n^{3/2} u^{3/2})} \\ &+ An^2 \int_0^\delta \int_0^\delta \frac{|\phi_{xy}(t, u)| dt du}{[1 + n^{3/2} |t - u|^{3/2}][1 + n^{3/2}(t + u)^{3/2}]} = J_{11} + J_{12}. \end{aligned}$$

Integrating J_{11} twice by parts and applying (5.05) we get

$$\begin{aligned} J_{11} \leqq \frac{An^2 \epsilon \delta^2}{n^3 \delta^3} &+ \frac{An^{7/2} \delta \epsilon}{n^{3/2} \delta^{3/2}} \int_0^\delta \frac{u^{3/2} du}{(1 + n^{3/2} u^{3/2})^2} \\ &+ An^5 \epsilon \int_0^\delta \int_0^\delta \frac{t^{3/2} u^{3/2} dt du}{(1 + n^{3/2} t^{3/2})^2 (1 + n^{3/2} u^{3/2})^2}. \end{aligned}$$

But

$$\int_0^\delta \frac{u^{3/2} du}{(1 + n^{3/2} u^{3/2})^2} = \int_0^{1/n} + \int_{1/n}^\delta \leqq \frac{1}{n} \cdot \frac{1}{n^{3/2}} + \frac{1}{n^3} \int_{1/n}^\delta \frac{du}{u^{3/2}} \leqq \frac{3}{n^{5/2}}.$$

Hence, since $n\delta > 2$, we easily obtain $J_{11} \leqq A\epsilon$. Applying the transformation $t = 2^{-1/2}(t' - u')$, $u = 2^{-1/2}(t' + u')$ to J_{12} and proceeding as above we get $J_{12} \leqq A\epsilon$. Thus $J_1 \leqq A\epsilon$.

Next let B_1 be that part of B_δ in which $t \geqq \delta$, $u \leqq \delta' < \delta/4$, B_2 the domain symmetric to B_1, B_3 that part of B_δ in which $|t - u| \leqq \delta'$, B_4 the rest of B_δ. Clearly $B_1 \cdot B_3 = B_2 \cdot B_3 = 0$ since $\delta' < \delta/4$. In $B_1 + B_2 + B_3$ we have $|K_n^1(t, u)| \leqq A/(3\delta/4)^2$. This follows from (3.20) in $B_1 + B_2$ and from (3.18) in B_3. Since $\phi_{xy}(t, u)$ is integrable we can choose δ' depending only on δ (and hence on ϵ), $0 < \delta' < \delta/4$, such that

$$(5.07) \qquad \iint_{B_1 + B_2 + B_3} |\phi_{xy}(t, u) K_n^1(t, u)| dt du < \epsilon.$$

Fixing δ', we see that, on account of (3.14), $K_n^1(t, u) \to 0$ uniformly in B_4. Thus for all sufficiently large n we have $J_2 < 2\epsilon$ and consequently $|t_n^1(x, y; f) - f(x, y)| \leqq A\epsilon$. That is, $t_n^1(x, y; f) \to f(x, y)$ as $n \to \infty$. This completes the proof of the theorem.

COROLLARY 1.1. *Let N_p be a regular Nörlund method of summability satisfying* (5.01). *Then N_p applied to the square partial sums of the double Fourier series possesses the localization property.*

For if f vanishes in a neighborhood of (x, y), $\phi_{xy}(t, u)$ satisfies (5.02).

Before showing that (5.01) is also partly necessary in order that N_p applied to the square partial sums should possess the localization property we prove the following lemma.

LEMMA 4. *Let N_p be a regular Nörlund method of summability with $p_n \geqq 0$, p_n non-increasing, $p_1 < p_0$, $n/P_n \to \infty$ as $n \to \infty$. Suppose $0 < \delta < \pi$. Let $E = [-\pi, \pi; -\pi, \pi] - (-\delta, \delta; -\delta, \delta)$. Then there exists $N > 0$ such that*

$$\operatorname*{ess\,sup}_{(t,u) \in E} |K_n(t, u)| > An/P_n, \qquad all\ n > N, A > 0. \tag{5.08}$$

Proof. From (2.04) we have

$$\begin{aligned} K_n(\pi, 0) &= \frac{1}{\pi^2 P_n} \sum_{k=0}^{n} p_{n-k} \cdot \tfrac{1}{2}(-1)^k (k + \tfrac{1}{2}) \\ &= \frac{(-1)^n}{2\pi^2 P_n} (n + \tfrac{1}{2}) \sum_{k=0}^{n} (-1)^k p_k - \frac{(-1)^n}{2\pi^2 P_n} \sum_{k=0}^{n} (-1)^k k p_k \\ &= J_1 - J_2. \end{aligned}$$

Since p_n is non-increasing we have immediately $|J_1| \geqq n(p_0 - p_1)/2\pi^2 P_n$. If we set $W_k = \sum_{j=0}^{k} (-1)^j j$, then $|W_k| \leqq k$ and we easily get

$$\begin{aligned} \left| \sum_{k=0}^{n} (-1)^k k p_k \right| &= \left| \sum_{k=0}^{n-1} W_k (p_k - p_{k+1}) + W_n p_n \right| \leqq \sum_{k=0}^{n-1} k (p_k - p_{k+1}) + n p_n \\ &= \sum_{k=1}^{n} p_k \leqq P_n. \end{aligned}$$

Hence $|J_2| \leqq 1/2\pi^2$. But we can choose $N > 0$ such that $n/P_n > 2/(p_0 - p_1)$ for all $n > N$. Then for $n > N$ we find $|K_n(\pi, 0)| \geqq |J_1| - |J_2| \geqq n(p_0 - p_1)/4\pi^2 P_n = An/P_n$, $A > 0$. But E is closed and for each n, $K_n(t, u)$ is continuous. Thus (5.08) follows.

THEOREM 2. *Let N_p be a regular Nörlund method of summability with $p_n \geqq 0$, p_n non-increasing, $p_1 < p_0$, $n/P_n \to \infty$ as $n \to \infty$. Then there exists f vanishing in a neighborhood of* $(0, 0)$ *such that* $\limsup_{n \to \infty} |t_n(0, 0; f)| = +\infty$.

Proof. Let $E=[-\pi, \pi; -\pi, \pi]-(-\delta, \delta; -\delta, \delta)$, $0<\delta<\pi$. Consider the class of functions $f\in L[-\pi, \pi; -\pi, \pi]$ which vanish in $(-\delta, \delta; -\delta, \delta)$, that is, the class of functions $f\in L(E)$. Then

$$t_n(0, 0; f) = \iint_E f(t, u)K_n(t, u)dtdu = T_n(f) \tag{5.09}$$

defines a linear functional on the space $L(E)$ with norm

$$\|T_n\| = \operatorname*{ess\,sup}_{(t,u)\in E} |K_n(t, u)|. \tag{5.10}$$

Now suppose that the conclusion of our theorem does not hold. Then for every $f\in L(E)$, $\limsup_{n\to\infty} |T_n(f)| < \infty$. By a well known theorem of Banach and Steinhaus([5]) it follows that $\|T_n\| \leqq M<\infty$ for all n. Thus by (5.10) and (5.08) we have $An/P_n \leqq M$, $A>0$, for all $n>N$. This contradicts the hypothesis.

Corollary 2.1. *Let N_p be a regular Nörlund method of summability with $p_n\geqq 0$, p_n non-increasing, $p_1<p_0$. Then* (5.01) *is necessary as well as sufficient for N_p applied to the square partial sums of the double Fourier series to possess the localization property.*

Proof. To prove the necessity we first note that n/P_n is non-decreasing since p_n is non-increasing. Then in order that N_p applied to the square partial sums should possess the localization property we must have n/P_n bounded by Theorem 2. The condition (5.01) is an immediate consequence of this.

The case in which $p_n\geqq 0$, p_n non-increasing, is especially important as it includes Cesàro (C, α), $0<\alpha\leqq 1$. Because of the simplicity of the result in this case we state it separately.

Corollary 2.2. *Under the hypotheses of Corollary* 2.1, *a necessary and sufficient condition that N_p applied to the square partial sums of the double Fourier series should possess the localization property is $n=O(P_n)$.*

From this it follows that (C, α) applied to the square partial sums possesses the localization property if and only if $\alpha\geqq 1$. Thus Grünwald's [2] result is the best possible in the sense that it cannot be extended to (C, α), $\alpha<1$.

6. **Local properties of restricted summability.** We turn now to restricted double Nörlund summability of the rectangular partial sums of the double Fourier series. The results are similar to those in §5.

Theorem 3. *Let N_p be a double Nörlund method of summability satisfying the conditions*

$$n\cdot\sum_{j=1}^{n} |p_j^{(k)} - p_{j-1}^{(k)}| = O(|P_n^{(k)}|), \qquad k = 1, 2. \tag{6.01}$$

([5]) See, for example, Banach [1, p. 80].

Then at any point (x, y) *such that*

$$\Phi(h, k) = \int_0^h dt \int_0^k |\phi_{xy}(t, u)| du = o(hk) \tag{6.02}$$

as $h, k \to 0$ *simultaneously but independently,* $\sigma(f)$ *is restrictedly summable* N_p *to* $f(x, y)$.

Proof. Let $\lambda \geqq 1$ be any fixed number. It suffices to show that $t_{mn}(x, y; f) \to f(x, y)$ as $m, n \to \infty$ in such a manner that $m/n \leqq \lambda$, $n/m \leqq \lambda$. Fix (x, y). Given $\epsilon > 0$, we can choose $\delta > 0$ such that $1/\delta$ is an integer greater than 2 and such that

$$\Phi(h, k) < \epsilon hk, \tag{6.03}$$

whenever $0 < h, k \leqq \delta$. Then from (2.11) we obtain

$$\begin{aligned} |t_{mn}(x, y; f) - f(x, y)| &\leqq \left(\int_\delta^\pi \int_\delta^\pi + \int_0^\delta \int_\delta^\pi + \int_\delta^\pi \int_0^\delta \right. \\ &\quad \left. + \int_0^\delta \int_0^\delta \right) | \phi_{xy}(t, u) N_m^{(1)}(t) N_n^{(2)}(u) | dudt \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{6.04}$$

On account of Lemma 1, the estimates of §3 may be applied here. From (3.15), (6.01) and Lemma 1 we have that $N_m^{(1)}(t) N_n^{(2)}(u) \to 0$ uniformly in $[\delta, \pi; \delta, \pi]$. Thus J_1 is small for all sufficiently large m and n. If $m/n \leqq \lambda$, $n/m \leqq \lambda$ we see from (3.03), (3.15), (6.01) and Lemma 1 that $N_m^{(1)}(t) N_n^{(2)}(u)$ is bounded in the domains of integration of J_2 and J_3. Thus we can find δ' such that $0 < \delta' < \delta$ and so small that

$$\left(\int_0^{\delta'} \int_\delta^\pi + \int_\delta^\pi \int_0^{\delta'} \right) | \phi_{xy}(t, u) N_m^{(1)}(t) N_n^{(2)}(u) | dudt$$

is uniformly small. In the remainder of the domains of integration of J_2 and J_3, $N_m^{(1)}(t) N_n^{(2)}(u) \to 0$ uniformly and thus $J_2 + J_3$ is small for all sufficiently large m and n such that $m/n \leqq \lambda$, $n/m \leqq \lambda$. Thus we can find $N_0 > 1/\delta$ such that $J_1 + J_2 + J_3 < A\epsilon$ if $m, n > N_0$, $m/n \leqq \lambda$, $n/m \leqq \lambda$. In the following we suppose $m, n > N_0$. Then

$$\begin{aligned} J_4 &= \left(\int_0^{1/m} \int_0^{1/n} + \int_{1/m}^\delta \int_0^{1/n} + \int_0^{1/m} \int_{1/n}^\delta \right. \\ &\quad \left. + \int_{1/m}^\delta \int_{1/n}^\delta \right) | \phi_{xy}(t, u) N_m^{(1)}(t) N_n^{(2)}(u) | dudt \\ &= J_{41} + J_{42} + J_{43} + J_{44}. \end{aligned} \tag{6.05}$$

Then from (3.03) and (6.03) we have at once $J_{41} \leqq A\epsilon$. Also by (3.03) and (3.12) we have

$$J_{42} \leqq An\int_0^{1/n} du \int_{1/m}^{\delta} |\phi_{xy}(t, u)| \{M_{m1}^{(1)}(t) + M_{m2}^{(1)}(t) + M_{m3}^{(1)}(t)\} dt \tag{6.06}$$
$$= \overset{1}{J}_{42} + \overset{2}{J}_{42} + \overset{3}{J}_{42}.$$

Then from (3.13)

$$\overset{1}{J}_{42} = An\sum_{j=1/\delta}^{m-1} \int_0^{1/n} du \int_{1/(j+1)}^{1/j} |\phi_{xy}(t, u)| \frac{1}{tR_m^{(1)}} R^{(1)}\left(\frac{1}{t}\right) dt$$
$$\leqq \frac{An}{R_m^{(1)}} \sum_{j=1/\delta}^{m-1} (j+1)R_{j+1}^{(1)}\Delta_{10}\Phi\left(\frac{1}{j}, \frac{1}{n}\right)$$
$$= \frac{An}{R_m^{(1)}}\left\{ \sum_{j=1/\delta}^{m-1} \Phi\left(\frac{1}{j}, \frac{1}{n}\right)[(j+1)R_{j+1}^{(1)} - jR_j^{(1)}] + \frac{1}{\delta} R_{1/\delta}^{(1)}\Phi\left(\delta, \frac{1}{n}\right) - mR_m^{(1)}\Phi\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$$
$$\leqq \frac{A\epsilon}{R_m^{(1)}}\left\{ \sum_{j=1/\delta}^{m-1} \frac{R_{j+1}^{(1)} + jr_{j+1}^{(1)}}{j} + R_{1/\delta}^{(1)}\right\}.$$

But by Lemma 2 and (3.06) we get

$$\sum_{j=1/\delta}^{m-1} \frac{R_{j+1}^{(k)} + jr_{j+1}^{(k)}}{j} \leqq \sum_{j=1/\delta}^{m-1} r_{j+1}^{(k)} + 2\sum_{j=1/\delta}^{m-1} \frac{R_{j+1}^{(k)}}{j+1} \leqq AR_m^{(k)}, \qquad k = 1, 2. \tag{6.07}$$

Thus $\overset{1}{J}_{42} \leqq A\epsilon$. Substituting from (3.13) in $\overset{2}{J}_{42}$, integrating twice by parts and using Lemma 1 we have $\overset{2}{J}_{42} \leqq A\epsilon$. Again from (3.13) we have

$$\overset{3}{J}_{42} = An\sum_{j=1/\delta}^{m-1} \int_0^{1/n} du \int_{1/(j+1)}^{1/j} |\phi_{xy}(t, u)| \frac{1}{t^2R_m^{(1)}}\left[V_m^{(1)} - V^{(1)}\left(\frac{1}{t}\right)\right] dt$$
$$\leqq \frac{An}{R_m^{(1)}} \sum_{j=1/\delta}^{m-1} (j+1)^2(V_m^{(1)} - V_j^{(1)})\Delta_{10}\Phi\left(\frac{1}{j}, \frac{1}{n}\right)$$
$$= \frac{An}{R_m^{(1)}}\left\{ \sum_{j=1/\delta}^{m-1} \Phi\left(\frac{1}{j}, \frac{1}{n}\right)[(2j+1)(V_m^{(1)} - V_j^{(1)}) - j^2|p_j^{(1)} - p_{j-1}^{(1)}|] + \frac{1}{\delta^2}(V_m^{(1)} - V_{(1/\delta)-1}^{(1)})\Phi\left(\delta, \frac{1}{n}\right) - m^2(V_m^{(1)} - V_{m-1}^{(1)})\Phi\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$$
$$\leqq \frac{A\epsilon}{R_m^{(1)}}\left\{ \sum_{j=1/\delta}^{m-1} \frac{2j+1}{j}(V_m^{(1)} - V_j^{(1)}) + \frac{1}{\delta}(V_m^{(1)} - V_{(1/\delta)-1}^{(1)})\right\}.$$

But by (3.06) and (6.01)

$$\begin{aligned}\sum_{j=1/\delta}^{m-1} \frac{2j+1}{j} (V_m^{(k)} - V_j^{(k)}) &\leqq 3 \sum_{j=1/\delta}^{m-1} \sum_{s=j+1}^{m} | p_s^{(k)} - p_{s-1}^{(k)} | \\ &= 3 \sum_{s=(1/\delta)+1}^{m} \sum_{j=1/\delta}^{s-1} | p_s^{(k)} - p_{s-1}^{(k)} | \\ &\leqq 3 \sum_{s=(1/\delta)+1}^{m} s | p_s^{(k)} - p_{s-1}^{(k)} | \leqq A R_m^{(k)}, \quad k = 1, 2. \end{aligned} \tag{6.08}$$

Likewise

$$\frac{1}{\delta} (V_m^{(k)} - V_{(1/\delta)-1}^{(k)}) \leqq m \sum_{j=1}^{m} | p_j^{(k)} - p_{j-1}^{(k)} | \leqq A R_m^{(k)}, \qquad k = 1, 2. \tag{6.09}$$

Thus $J_{42}^3 \leqq A\epsilon$. Substituting in (6.06) we have $J_{42} \leqq A\epsilon$. In the same way $J_{43} \leqq A\epsilon$. Turning now to J_{44} we have by (3.12)

$$\begin{aligned} J_{44} &\leqq A \int_{1/m}^{\delta} dt \int_{1/n}^{\delta} | \phi_{xy}(t, u) | \{ M_{m1}^{(1)}(t) + M_{m2}^{(1)}(t) + M_{m3}^{(1)}(t) \} \\ &\qquad \cdot \{ M_{n1}^{(2)}(u) + M_{n2}^{(2)}(u) + M_{n3}^{(2)}(u) \} du \\ &= \sum_{j=1}^{9} J_{44}^{j}. \end{aligned} \tag{6.10}$$

We now show that $J_{44}^j \leqq A\epsilon$ $(j = 1, 2, 3, \cdots, 9)$ as was done with the J_{42}^j. For example, let us take J_{44}^6. Then from (3.13)

$$\begin{aligned} J_{44}^6 &= A \sum_{j,k=1/\delta}^{m-1,n-1} \int_{1/(j+1)}^{1/j} dt \int_{1/(k+1)}^{1/k} | \phi_{xy}(t, u) | \frac{r_m^{(1)}}{t^2 u^2 R_m^{(1)} R_n^{(2)}} \left[V_n^{(2)} - V^{(2)}\left(\frac{1}{u}\right) \right] du \\ &\leqq \frac{A r_m^{(1)}}{R_m^{(1)} R_n^{(2)}} \sum_{j,k=1/\delta}^{m-1,n-1} (j+1)^2 (k+1)^2 (V_n^{(2)} - V_k^{(2)}) \Delta_{11} \Phi\left(\frac{1}{j}, \frac{1}{k}\right). \end{aligned}$$

Applying Lemma 3 and dropping clearly negative terms we obtain

$$\begin{aligned} J_{44}^6 \leqq \frac{A r_m^{(1)}}{R_m^{(1)} R_n^{(2)}} \Bigg\{ & \sum_{j,k=1/\delta}^{m-1,n-1} \Phi\left(\frac{1}{j}, \frac{1}{k}\right) \\ & \cdot (2j+1) [(2k+1)(V_n^{(2)} - V_k^{(2)}) - k^2 | p_k^{(2)} - p_{k-1}^{(2)} |] \\ & + \sum_{j=1/\delta}^{m-1} \Phi\left(\frac{1}{j}, \delta\right) \cdot (2j+1) \frac{1}{\delta^2} (V_n^{(2)} - V_{(1/\delta)-1}^{(2)}) \\ & + \sum_{k=1/\delta}^{n-1} \left[\frac{1}{\delta^2} \Phi\left(\delta, \frac{1}{k}\right) - m^2 \Phi\left(\frac{1}{m}, \frac{1}{k}\right) \right] \\ & \cdot [(2k+1)(V_n^{(2)} - V_k^{(2)}) - k^2 | p_k^{(2)} - p_{k-1}^{(2)} |] \\ & + \frac{1}{\delta^4} (V_n^{(2)} - V_{(1/\delta)-1}^{(2)}) \Phi(\delta, \delta) + m^2 n^2 (V_n^{(2)} - V_{n-1}^{(2)}) \Phi\left(\frac{1}{m}, \frac{1}{n}\right) \Bigg\}. \end{aligned}$$

Again dropping those terms which are negative and applying (6.03), (6.01), (6.08), (6.09) and Lemma 1 we obtain $J_{44}^6 \leqq A\epsilon$. Altogether, then, $J_{44} \leqq A\epsilon$. Combining our estimates in (6.05) and (6.04) we have $|t_{mn}(x, y; f) - f(x, y)| \leqq A\epsilon$ if $m, n > N_0$, $m/n \leqq \lambda$, $n/m \leqq \lambda$. This completes the proof of the theorem.

COROLLARY 3.1. *Let N_p be a double Nörlund method of summability satisfying* (6.01). *Then restricted N_p summability possesses the localization property.*

Before showing (6.01) is also partly necessary in order that restricted N_p possess the localization property we prove the following lemma.

LEMMA 5. *Let N_p be a double Nörlund method of summability with $p_n^{(k)} \geqq 0$, $p_n^{(k)}$ non-increasing* ($k = 1, 2$). *Then*

$$(6.11) \qquad |N_n^{(k)}(\pi)| \geqq (p_0^{(k)} - p_1^{(k)})/2\pi P_n^{(k)}, \quad |N_n^{(k)}(0)| \geqq n/2\pi, \quad k = 1, 2.$$

Proof. From (2.10) we have

$$N_n^{(k)}(\pi) = \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n} p_{n-j}^{(k)} \cdot \tfrac{1}{2}(-1)^j = \frac{(-1)^n}{2\pi P_n^{(k)}} \sum_{j=0}^{n} (-1)^j p_j^{(k)}.$$

The first inequality of (6.11) follows immediately. Also from (2.10)

$$\begin{aligned} N_n^{(k)}(0) &= \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n} p_{n-j}^{(k)}(j + \tfrac{1}{2}) = \frac{1}{2\pi} + \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n} (n - j) p_j^{(k)} \\ &= \frac{1}{2\pi} + \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n-1} P_j^{(k)}. \end{aligned}$$

Thus

$$(6.12) \qquad |N_n^{(k)}(0)| \geqq \frac{1}{\pi P_n^{(k)}} \sum_{j=0}^{n-1} P_j^{(k)}.$$

But since $P_n^{(k)}$ is non-decreasing we have $(P_n^{(k)})^{-1} \sum_{j=0}^{n-1} P_j^{(k)} \leqq n$ and thus

$$\begin{aligned} 0 \leqq n - \frac{1}{P_n^{(k)}} \sum_{j=0}^{n-1} P_j^{(k)} &= \frac{1}{P_n^{(k)}} \sum_{j=0}^{n-1} (P_n^{(k)} - P_j^{(k)}) = \frac{1}{P_n^{(k)}} \sum_{j=0}^{n-1} \sum_{s=j+1}^{n} p_s^{(k)} \\ &= \frac{1}{P_n^{(k)}} \sum_{s=1}^{n} s p_s^{(k)} \leqq \frac{1}{P_n^{(k)}} \sum_{s=1}^{n} P_{s-1}^{(k)} = \frac{1}{P_n^{(k)}} \sum_{j=0}^{n-1} P_j^{(k)}, \end{aligned}$$

or

$$n \leqq \frac{2}{P_n^{(k)}} \sum_{j=0}^{n-1} P_j^{(k)}.$$

Substituting this in (6.12) we obtain the second inequality of (6.11).

THEOREM 4. *Let N_p be a double Nörlund method of summability with $p_n^{(k)} \geqq 0$,*

$p_n^{(k)}$ *non-increasing* $(k=1, 2)$. *Suppose* $p_1^{(k)} < p_0^{(k)}$, $n/P_n^{(k)} \to \infty$ *as* $n \to \infty$ *for* $k=1$ *or* 2 *or both. Then there exists f vanishing in a neighborhood of* (0, 0) *such that* $\limsup_{n\to\infty} |t_{nn}(0, 0; f)| = +\infty$.

The proof is analogous to that of Theorem 2, using Lemma 5 instead of Lemma 4.

As in §5 we may prove the following corollaries:

COROLLARY 4.1. *Let* N_p *be a double Nörlund method of summability with* $p_n^{(k)} \geqq 0$, $p_n^{(k)}$ *non-increasing*, $p_1^{(k)} < p_0^{(k)}$ $(k=1, 2)$. *Then* (6.01) *is necessary as well as sufficient in order that restricted* N_p *possess the localization property.*

COROLLARY 4.2. *Under the hypotheses of Corollary* 4.1, *necessary and sufficient conditions that restricted* N_p *summability should possess the localization property are* $n = O(P_n^{(k)})$ $(k=1, 2)$.

It follows that restricted (C, α, β) possesses the localization property if and only if $\alpha \geqq 1, \beta \geqq 1$.

7. **Preliminary lemmas for almost everywhere results.** We turn now to the study of methods of summability which sum the double Fourier series almost everywhere. The results are generalizations and extensions of those due to Marcinkiewicz and Zygmund [5, 6] and Grünwald [3]. The proofs are based on those given by Marcinkiewicz and Zygmund. We shall require the following lemmas.

LEMMA 6. *Let* α *be any fixed positive number. For* (x, y) *belonging to the square* Q $[-\pi, \pi; -\pi, \pi]$, *we write*

$$(7.01) \qquad f_\alpha^*(x, y) = \sup_h \frac{1}{4\alpha h^2} \int_{-\alpha h}^{\alpha h} du \int_{-h}^{h} | f(x+t, y+u) | \, dt,$$

$$(7.02) \qquad f_\alpha^{**}(x, y) = \sup_h \frac{1}{4\alpha h^2} \int_{-\alpha h}^{\alpha h} du \int_{-h}^{h} \left| f\left(x + \frac{t-u}{2^{1/2}}, \, y + \frac{t+u}{2^{1/2}}\right) \right| dt$$

where the number h is so small that the rectangles over which the integrals are taken are contained in Q' $[-2\pi, 2\pi; -2\pi, 2\pi]$. *Let*

$$\mathcal{E}_\alpha^*(\xi) = \underset{(x,y)}{E} \, [f_\alpha^*(x, y) > \xi], \qquad \mathcal{E}_\alpha^{**}(\xi) = \underset{(x,y)}{E} \, [f_\alpha^{**}(x, y) > \xi]$$

for any $\xi > 0$. *Then*

$$(7.03) \qquad \begin{matrix} | \mathcal{E}_\alpha^*(\xi) | \\ | \mathcal{E}_\alpha^{**}(\xi) | \end{matrix} \leqq \frac{A}{\xi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | f(x, y) | \, dx dy.$$

In the case of $f_\alpha^*(x, y)$ the proof was given by Marcinkiewicz and Zygmund [6]. For the case of $f_\alpha^{**}(x, y)$ the proof can be carried through in the same way.

LEMMA 7. *Let a be any fixed positive number. For (x, y) belonging to the square $Q\,[-\pi, \pi; -\pi, \pi]$, we write*

$$(7.04)\qquad f^{*a}(x, y) = \sup_s \left(f^*_{2^s}(x, y)\cdot 2^{-a|s|}\right), \qquad \text{for } s = 0, \pm 1, \pm 2, \cdots,$$

$$(7.05)\qquad f^{**a}(x, y) = \sup_s \left(f^{**}_{2^s}(x, y)\cdot 2^{-a|s|}\right), \qquad \text{for } s = 0, \pm 1, \pm 2, \cdots.$$

We write

$$\mathcal{E}^{*a}(\xi) = \underset{(x,y)}{E}\,[f^{*a}(x, y) > \xi], \qquad \mathcal{E}^{**a}(\xi) = \underset{(x,y)}{E}\,[f^{**a}(x, y) > \xi]$$

for any $\xi > 0$. Then

$$(7.06)\qquad \begin{matrix} |\mathcal{E}^{*a}(\xi)| \\ |\mathcal{E}^{**a}(\xi)| \end{matrix} \leqq \frac{A(a)}{\xi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)|\, dxdy,$$

where $A(a)$ depends only on a.

The proof is similar to that given by Marcinkiewicz and Zygmund for their Lemma 3 [6].

LEMMA 8. *Suppose $P_n \geqq 0$, P_n non-decreasing, $a \geqq 0$. Then the condition*

$$(7.07)\qquad \sum_{k=1}^{n} \frac{P_k}{k}\left(\frac{n}{k}\right)^a = O(P_n)$$

is equivalent to the condition

$$(7.08)\qquad \sum_{k=0}^{n-1} P_{2^k}\cdot 2^{a(n-k)} = O(P_{2^n}).$$

Proof. Suppose (7.08) is satisfied. Let j be an integer such that $2^j \leqq n < 2^{j+1}$. Then

$$\begin{aligned} \sum_{k=1}^{n} \frac{P_k}{k}\left(\frac{n}{k}\right)^a &= \sum_{k=0}^{j-1}\left\{\frac{P_{2^k}}{2^k}\left(\frac{n}{2^k}\right)^a + \frac{P_{2^k+1}}{2^k+1}\left(\frac{n}{2^k+1}\right)^a + \cdots \right. \\ &\quad \left. + \frac{P_{2^{k+1}-1}}{2^{k+1}-1}\left(\frac{n}{2^{k+1}-1}\right)^a\right\} + \sum_{k=2^j}^{n} \frac{P_k}{k}\left(\frac{n}{k}\right)^a \\ &\leqq \sum_{k=0}^{j-1} P_{2^{k+1}}\left(\frac{2^{j+1}}{2^k}\right)^a + P_n\cdot 2^{-j}\cdot 2^j\cdot 2^a \\ &\leqq 2^{2a}\sum_{k=0}^{j-1} P_{2^k}\cdot 2^{a(j-k)} + 2^{2a}P_{2^j} + O(P_n) \\ &= O(P_{2^j}) + O(P_n) = O(P_n) \end{aligned}$$

showing (7.07) to be satisfied. The proof of the converse is similar.

8. **Lemma for restricted Nörlund summability.** Before proving our first result on almost everywhere summability we need a lemma.

LEMMA 9. *Let N_p be a double Nörlund method of summability. Suppose there exists a constant $a>0$ such that*

$$\sum_{j=1}^{n} j\,|\,p_j^{(k)} - p_{j-1}^{(k)}\,|\left(\frac{n}{j}\right)^a = O(|\,P_n^{(k)}\,|), \qquad k = 1, 2, \tag{8.01}$$

and

$$\sum_{j=1}^{n} \frac{|\,P_j^{(k)}\,|}{j}\left(\frac{n}{j}\right)^a = O(|\,P_n^{(k)}\,|), \qquad k = 1, 2. \tag{8.02}$$

Let $\lambda \geqq 1$ be any fixed number. Let

$$h_\lambda(x, y; f) = \sup_{m,n} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\,f(x+t, y+u)N_m^{(1)}(t)N_n^{(2)}(u)\,|\,dtdu \tag{8.03}$$

where $m, n \geqq 1$, $m/n \leqq \lambda$, $n/m \leqq \lambda$. Then for any $\xi>0$

$$\left|\, \underset{(x,y)}{E}\,\{[(x, y) \in Q][h_\lambda(x, y; f) > \xi]\}\,\right| \leqq \frac{A(a)\lambda^a}{\xi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\,f(x, y)\,|\,dxdy \tag{8.04}$$

where $A(a)$ depends only on a.

Proof. If (8.01) and (8.02) hold for any $a=a_0>0$, then they also hold for all a such that $0\leqq a\leqq a_0$. Hence we may suppose that $0<a<1$.

Let j, k be integers such that $2^j \leqq m < 2^{j+1}$, $2^k \leqq n < 2^{k+1}$. Also let $m/n \leqq \lambda$, $n/m \leqq \lambda$. Then $2^{|j-k|} \leqq 2\lambda$. Let

$$\begin{aligned} \overset{*}{t}_{mn}(x, y) &= \int_0^\pi\int_0^\pi |\,f(x+t, y+u)N_m^{(1)}(t)N_n^{(2)}(u)\,|\,dtdu \\ &= \left(\int_0^{\pi 2^{-j}}\int_{\pi 2^{-k}}^{\pi} + \int_{\pi 2^{-j}}^{\pi}\int_0^{\pi 2^{-k}} + \int_{\pi 2^{-j}}^{\pi}\int_{\pi 2^{-k}}^{\pi} \right. \\ &\qquad \left. + \int_0^{\pi 2^{-j}}\int_0^{\pi 2^{-k}}\right) |\,f(x+t, y+u)N_m^{(1)}(t)N_n^{(2)}(u)\,|\,dudt \\ &= P_{jk}(x, y) + Q_{jk}(x, y) + R_{jk}(x, y) + S_{jk}(x, y). \end{aligned} \tag{8.05}$$

On account of Lemma 1, the estimates of §3 may be applied. Then from (3.03) and (3.12) we have

$$\begin{aligned} P_{jk}(x, y) \leqq \frac{Am}{R_n^{(2)}} \sum_{r=0}^{k-1} \int_0^{\pi 2^{-j}} dt \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} \frac{|\,f(x+t, y+u)\,|}{u} \\ \cdot \left\{R^{(2)}\left(\frac{1}{u}\right) + \frac{1}{u}\left[r_n^{(2)} + V_n^{(2)} - V^{(2)}\left(\frac{1}{u}\right)\right]\right\} du. \end{aligned}$$

By (7.01) and (7.04) the rth term of the sum on the right does not exceed

$$\frac{Am}{R_n^{(2)}}2^r\left\{R^{(2)}\left(\frac{2^{r+1}}{\pi}\right)+2^r\left[r_n^{(2)}+V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)\right]\right\}$$

$$(8.06)\qquad \cdot\int_{-\pi 2^{-j}}^{\pi 2^{-j}}dt\int_{-\pi 2^{-r}}^{\pi 2^{-r}}|f(x+t,y+u)|\,du$$

$$\leqq \frac{Am2^{-i}}{R_n^{(2)}}\left\{R_{2^r}^{(2)}\cdot 2^{a|j-r|}\right.$$

$$\left.+2^{r+a|j-r|}\left[r_n^{(2)}+V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)\right]\right\}f^{*a}(x,y).$$

In order to sum these terms we shall need the following:

$$(8.07)\qquad \sum_{r=0}^{k-1}R_{2^r}^{(i)}\cdot 2^{a(k-r)}\leqq AR_{2^k}^{(i)},\qquad i=1,2,$$

$$(8.08)\qquad \sum_{r=0}^{k-1}2^{r+a(k-r)}\leqq A2^k,$$

$$(8.09)\qquad \sum_{r=0}^{k-1}2^{r+a(k-r)}\left[V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)\right]\leqq AR_n^{(2)}.$$

To prove (8.07) we apply (3.06) to (8.02) and make use of Lemma 8. (8.08) is immediate. For (8.09) we first note that

$$V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)=V_n^{(2)}=\sum_{s=1}^{n}|p_s^{(2)}-p_{s-1}^{(2)}|,\qquad r=0,1,$$

$$V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)\leqq V_n^{(2)}-V^{(2)}(2^{r-2})=\sum_{s=2^{r-2}+1}^{n}|p_s^{(2)}-p_{s-1}^{(2)}|,$$

$$r=2,3,4,\cdots,k-1.$$

Substituting in the left side of (8.09), reversing the order of the summations and denoting the greatest integer less than or equal to $2+\log_2(s-1)$ by $g(s)$, we have

$$\sum_{r=0}^{k-1}2^{r+a(k-r)}\left[V_n^{(2)}-V^{(2)}\left(\frac{2^r}{\pi}\right)\right]$$

$$\leqq\sum_{s=2}^{n}|p_s^{(2)}-p_{s-1}^{(2)}|\sum_{r=2}^{g(s)}2^{r+a(k-r)}+\sum_{s=1}^{n}(2^{ak}+2^{1+a(k-1)})|p_s^{(2)}-p_{s-1}^{(2)}|$$

$$=2^{ak}\sum_{s=2}^{n}|p_s^{(2)}-p_{s-1}^{(2)}|\frac{2^{(1-a)(g(s)+1)}-1}{2^{1-a}-1}+2^{ak}(1+2^{1-a})|p_1^{(2)}-p_0^{(2)}|$$

$$\leqq An^a\sum_{s=1}^{n}s^{1-a}|p_s^{(2)}-p_{s-1}^{(2)}|.$$

Then (8.09) follows from (8.01).

Summing (8.06) from $r=0$ to $k-1$, considering separately the cases $j\geqq k$ and $j<k$, and using (8.07)–(8.09) we get

$$P_{jk}(x, y) \leqq A\lambda^a f^{*a}(x, y). \tag{8.10}$$

In the same way we obtain

$$Q_{jk}(x, y) \leqq A\lambda^a f^{*a}(x, y). \tag{8.11}$$

Next from (3.12) we have

$$R_{jk}(x, y) \leqq A \sum_{r,s=0}^{j-1,k-1} Z_{rs}$$

where

$$Z_{rs} = \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} dt \int_{\pi 2^{-s-1}}^{\pi 2^{-s}} \left| f(x+t, y+u) \right| \left\{ M_{m1}^{(1)}(t) + M_{m2}^{(1)}(t) + M_{m3}^{(1)}(t) \right\}$$
$$\cdot \left\{ M_{n1}^{(2)}(u) + M_{n2}^{(2)}(u) + M_{n3}^{(2)}(u) \right\} du.$$

Each term of this sum consists of 9 parts each of which may be summed by making use of (8.07)–(8.09) and the analogue of (8.09). For example, let us consider that part arising from $M_{m1}^{(1)}(t) M_{n3}^{(2)}(u)$. The general term in this sum does not exceed

$$\frac{A}{R_m^{(1)} R_n^{(2)}} 2^{s+a|r-s|} R_{2^r}^{(1)} \left[V_n^{(2)} - V^{(2)}\left(\frac{2^s}{\pi}\right) \right] f^{*a}(x, y).$$

Considering the case $j \geqq k$ we have

$$\sum_{r,s=0}^{j-1,k-1} 2^{s+a|r-s|} R_{2^r}^{(1)} \left[V_n^{(2)} - V^{(2)}\left(\frac{2^s}{\pi}\right) \right]$$
$$\leqq \sum_{s=0}^{k-1} 2^s \left[V_n^{(2)} - V^{(2)}\left(\frac{2^s}{\pi}\right) \right] \left\{ \sum_{r=0}^{s-1} R_{2^r}^{(1)} \cdot 2^{a(j-r)} + \sum_{r=s}^{j-1} R_{2^r}^{(1)} \cdot 2^{a(j-s)} \right\}$$
$$\leqq \sum_{s=0}^{k-1} 2^s \left[V_n^{(2)} - V^{(2)}\left(\frac{2^s}{\pi}\right) \right] \sum_{r=0}^{j-1} R_{2^r}^{(1)} \cdot 2^{a(j-r)}$$
$$+ 2^{a(j-k)} \sum_{s=0}^{k-1} 2^{s+a(k-s)} \left[V_n^{(2)} - V^{(2)}\left(\frac{2^s}{\pi}\right) \right] \sum_{r=0}^{j-1} R_{2^r}^{(1)}$$
$$\leqq A\lambda^a R_m^{(1)} R_n^{(2)}.$$

The case $k>j$ can be treated similarly. Thus it follows that

$$R_{jk}(x, y) \leqq A\lambda^a f^{*a}(x, y). \tag{8.12}$$

Finally by (3.03)

$$(8.13)\qquad \begin{aligned} S_{jk}(x, y) &\leqq Amn \int_{-\pi 2^{-j}}^{\pi 2^{-j}} dt \int_{-\pi 2^{-k}}^{\pi 2^{-k}} | f(x+t, y+u) | \, du \\ &\leqq Amn \cdot 2^{-j-k+a|j-k|} f^{*a}(x, y) \leqq A\lambda^a f^{*a}(x, y). \end{aligned}$$

Combining (8.10)–(8.13) we see that $t^*_{mn}(x, y) \leqq A\lambda^a f^{*a}(x, y)$. But the integral on the right in (8.03) is the sum of four integrals, all analogous to $t^*_{mn}(x, y)$. Thus $h_\lambda(x, y; f) \leqq A\lambda^a f^{*a}(x, y)$. (8.04) now follows directly from Lemma 7.

9. **Restricted Nörlund summability almost everywhere.** We are now ready to prove our first theorem on almost everywhere summability.

THEOREM 5. *Let N_p be a double Nörlund method of summability. Suppose there exists a constant $a > 0$ such that* (8.01) *and* (8.02) *are satisfied. Then $\sigma(f)$ is restrictly summable N_p almost everywhere to f.*

Proof. This theorem follows immediately from Lemma 9. It suffices to make a decomposition $f = f_1 + f_2$ where f_1 is a trigonometrical polynomial and f_2 is such that

$$\left| \underset{(x,y)}{E} \left\{ | f_2(x, y) | > \delta \right\} \right| < \delta,$$

and

$$\left| \underset{(x,y)}{E} \left\{ \limsup \left| t_{mn}(x, y; f_2) \right| > \delta \right\} \right| < \delta$$

($m/n \leqq \lambda$, $n/m \leqq \lambda$, $\lambda \geqq 1$ any fixed number), where δ is a fixed positive number as small as we please. Since $t_{mn}(x, y; f_1) \to f_1(x, y)$ it follows that $\limsup | t_{mn}(x, y; f) - f(x, y) |$ where $m, n \to \infty$ in such a manner that $m/n \leqq \lambda$, $n/m \leqq \lambda$ does not exceed 2δ except on a set of measure less than 2δ. This completes the proof of the theorem.

The result of Marcinkiewicz and Zygmund [6], namely that $\sigma(f)$ is restrictedly summable (C, α, β), $\alpha, \beta > 0$, almost everywhere to f, follows immediately from Theorem 5.

10. **Lemma for square partial sums.** Turning now to the almost everywhere summability of the square partial sums we require the following lemma.

LEMMA 10. *Let N_p be a regular Nörlund method of summability. Suppose there exists a constant $a > 0$ such that*

$$(10.01)\qquad \sum_{j=1}^{n} j \, | p_j - p_{j-1} | \left(\frac{n}{j} \right)^a = O(| P_n |)$$

and

$$(10.02)\qquad \sum_{j=1}^{n} \frac{| P_j |}{j} \left(\frac{n}{j} \right)^a = O(| P_n |).$$

Let

$$(10.03)\quad h^*(x, y; f) = \sup_{1 \leqq n < \infty} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f(x+t, y+u)K_n(t, u) | \, dtdu.$$

Then for any $\xi > 0$

$$(10.04)\quad \left| \underset{(x,y)}{E} \{ [(x, y) \in Q][h^*(x, y; f) > \xi] \} \right| \leqq \frac{A(a)}{\xi} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f(x, y) | \, dxdy$$

where $A(a)$ *depends only on* a.

Proof. As in Lemma 9 we may suppose $0 < a < 1$. Let k be an integer such that $2^k \leqq n < 2^{k+1}$. Let D be the part of Q $(-\pi, \pi; -\pi, \pi)$ in which $t, u \geqq 0$. Let $D^{(0)}$ be the part of Q in which $t, u > \pi/2$. Divide $D - D^{(0)}$ into 9 domains $D_k^{(i)}$ $(i = 1, 2, 3, \cdots, 9)$ as in the proof of (3.16), the only difference being that in all the inequalities defining the regions $1/n$ is replaced by $\pi 2^{-k-1}$. We shall evaluate separately the integrals

$$(10.05)\quad \begin{aligned} A_k^{(i)} &= \iint_{D_k^{(i)}} | f(x+t, y+u)K_n(t, u) | \, dtdu, \quad i = 1, 2, 3, \cdots, 9, \\ A^{(0)} &= \iint_{D^{(0)}} | f(x+t, y+u)K_n(t, u) | \, dtdu. \end{aligned}$$

This may be done by methods similar to those used in the evaluation of $P_{jk}(x, y)$ and so on in Lemma 9. First of all from (3.01), (7.01) and (7.04) we get

$$(10.06)\quad A_k^{(1)} \leqq Af^{*a}(x, y).$$

In $D_k^{(2)}$ we note that $u \leqq t/2$, $t - u \geqq t/2 \geqq \pi 2^{-k-1} > 1/n$, $1/n \leqq t + u \leqq 2t$. Applying (3.11) and using the relations (8.07)–(8.09) we easily get

$$(10.07)\quad A_k^{(2)} \leqq Af^{*a}(x, y).$$

In $D_k^{(4)}$, $t \geqq u \geqq t/2$, $t + u \leqq 2t \leqq 4u$. Applying (3.02) and making the transformation $t = 2^{-1/2}(t' - u')$, $u = 2^{-1/2}(t' + u')$ we have

$$\begin{aligned} A_k^{(4)} \leqq{} & A \sum_{r=0}^{k} \int_{-\pi 2^{-k-1}}^{0} du \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} \left| f\left(x + \frac{t-u}{2^{1/2}}, y + \frac{t+u}{2^{1/2}}\right) \right| t^{-2} dt \\ & + A \int_{-\pi 2^{-k-1}}^{0} du \int_{\pi}^{\pi 2^{1/2}} \left| f\left(x + \frac{t-u}{2^{1/2}}, y + \frac{t+u}{2^{1/2}}\right) \right| t^{-2} dt. \end{aligned}$$

By (7.02) and (7.05) and taking account of (8.08) we have

$$(10.08)\quad A_k^{(4)} \leqq Af^{**a}(x, y).$$

In $D_k^{(6)}$, $u \leqq t/2$, $t-u \geqq t/2 \geqq \pi 2^{-k-1} > 1/n$, $1/n \leqq t+u \leqq 2t$. In $D_k^{(8)}$, $t-u \geqq 1/n$, $t \geqq u \geqq t/2$, $1/n \leqq t+u \leqq 2t \leqq 4u$. Then by (3.10) we have

$$A_k^{(6)} + A_k^{(8)} \leqq J_1 + J_2 + J_3$$

where

$$J_1 = \frac{A}{R_n} \iint_{D_k^{(6)}+D_k^{(8)}} \frac{|f(x+t, y+u)|}{tu} \left\{ R\left(\frac{1}{t}\right) + \frac{1}{t}\left[r_n + V_n - V\left(\frac{1}{2t}\right)\right]\right\} dtdu,$$

$$J_2 = \frac{A}{R_n} \iint_{D_k^{(6)}} \frac{|f(x+t, y+u)|}{tu} \left\{ R\left(\frac{2}{t}\right) + \frac{2}{t}\left[r_n + V_n - V\left(\frac{1}{t}\right)\right]\right\} dtdu,$$

$$J_3 = \frac{A}{R_n} \iint_{D_k^{(8)}} \frac{|f(x+t, y+u)|}{(t+u)^2} \left\{ R\left(\frac{1}{t-u}\right) + \frac{1}{t-u}\left[r_n + V_n - V\left(\frac{1}{t-u}\right)\right]\right\} dtdu.$$

Now it is clear that

$$\iint_{D_k^{(6)}+D_k^{(8)}} (\cdots) dtdu \leqq \sum_{s=1}^{k} \sum_{r=0}^{s} \int_{\pi 2^{-s-1}}^{\pi 2^{-s}} du \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} (\cdots) dt,$$

$$\iint_{D_k^{(6)}} (\cdots) dtdu \leqq \sum_{s=1}^{k} \sum_{r=0}^{s-1} \int_{\pi 2^{-s-1}}^{\pi 2^{-s}} du \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} (\cdots) dt.$$

Using these facts and (8.07)–(8.09) we find that $J_1 + J_2 \leqq Af^{*a}(x, y)$. Transforming J_3 by the substitution $t = 2^{-1/2}(t'-u')$, $u = 2^{-1/2}(t'+u')$, noting that

$$\iint_{D_k^{(8)}} (\cdots) dt'du' \leqq \sum_{s=1}^{k+1} \sum_{r=0}^{s-1} \int_{-\pi 2^{-s}}^{-\pi 2^{-s-1}} du' \int_{\pi 2^{-r-1}}^{\pi 2^{-r}} (\cdots) dt'$$
$$+ \int_{-\pi/2}^{-\pi/4} du' \int_{\pi}^{\pi 2^{1/2}} (\cdots) dt',$$

and using (8.07)–(8.09) we find that $J_3 \leqq Af^{**a}(x, y)$. Thus

(10.09) $$A_k^{(6)} + A_k^{(8)} \leqq A\{f^{*a}(x, y) + f^{**a}(x, y)\}.$$

Considering now the symmetric domains we have also

(10.10) $$A_k^{(3)} + A_k^{(5)} + A_k^{(7)} + A_k^{(9)} \leqq A\{f^{*a}(x, y) + f^{**a}(x, y)\}.$$

In $D^{(0)}$, $t, u > \pi/2$. Applying (3.02), (7.01) and (7.04) we get

(10.11) $$A^{(0)} \leqq Af^{*a}(x, y).$$

Combining (10.06)–(10.11) and noting that Q is the sum of four domains like D we have

(10.12) $$h^*(x, y; f) \leqq A\{f^{*a}(x, y) + f^{**a}(x, y)\}.$$

(10.04) now follows immediately from Lemma 7.

11. **Summability of the square partial sums almost everywhere.**

THEOREM 6. *Let N_p be a regular Nörlund method of summability. Suppose there exists a constant $a>0$ such that* (10.01) *and* (10.02) *are satisfied. Then the sequence* $\{s_{nn}(x, y; f)\}$ *is summable N_p almost everywhere to* $f(x, y)$.

Proof. This theorem follows immediately from Lemma 10 just as Theorem 5 follows from Lemma 9.

We easily obtain the following corollary of Theorem 6.

COROLLARY 6.1. *The sequence* $\{s_{nn}(x, y; f)\}$ *is summable* (C, α), $\alpha>0$, *almost everywhere to* $f(x, y)$.

In conclusion let us note that conditions (5.02) and (6.02) do not in general hold almost everywhere. Hence it was not possible to deduce any results concerning almost everywhere summability from Theorems 1 and 3. However (C, α) applied to the square partial sums of the double Fourier series is effective almost everywhere if $\alpha>0$ but possesses the localization property if and only if $\alpha\geqq 1$. Also restricted (C, α, β) summability of the double Fourier series is effective almost everywhere if $\alpha, \beta>0$, but possesses the localization property if and only if $\alpha\geqq 1$, $\beta\geqq 1$.

BIBLIOGRAPHY

1. S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

2. G. Grünwald, *Zur Summabilitätstheorie der Fourierschen Doppelreihe*, Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), pp. 343–350.

3. ———, *Über die Summabilität der Fourierschen Reihe*, Acta Litterarum ac Scientiarum Szeged, Sectio Scientiarum Mathematicarum, vol. 10 (1941), pp. 55–63.

4. E. Hille and J. D. Tamarkin, *On the summability of Fourier series*. I, these Transactions, vol. 34 (1932), pp. 757–783.

5. J. Marcinkiewicz, *Sur une méthode remarquable de sommation des séries doubles de Fourier*, Annali della R. Scuola Normale Superiore di Pisa, (2), vol. 8 (1939), pp. 149–160.

6. J. Marcinkiewicz and A. Zygmund, *On the summability of double Fourier series*, Fundamenta Mathematicae, vol. 32 (1939), pp. 122–132.

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