ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE AND THEIR RADII OF UNIVALENCE AND OF p-VALENCE

BY

W. SEIDEL AND J. L. WALSH

TABLE OF CONTENTS

Presented to the Society, February 26, 1938; received by the editors July 7, 1941. A short abstract of this paper under a slightly different title without details of proofs has already been published, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 337–340.

CHAPTER V. MISCELLANEOUS

1. **Introduction.** Various results are known concerning the order of growth of the first and higher derivatives of univalent and of bounded functions analytic in the unit circle, in the plane of the complex variable z. Among these may be mentioned Koebe's distortion theorem (Verzerrungssatz) in the univalent case, and Schwarz's lemma and the results of O. Szász(1) in the bounded case. A consequence of these results for a function $f(z)$ analytic in $|z|<1$ is $|f'(z)|=O((1-|z|)^{-3})$ in the case that $f(z)$ is univalent and $|f^{(n)}(z)|=O((1-|z|)^{-n})$ in the case that $f(z)$ is bounded. Various distortion theorems for bounded univalent functions were found by G. Pick and R. Nevanlinna(2). H. Frazer and more recently M. L. Cartwright have obtained results on the order of growth of p-valent functions(3) in a complete form.

All these investigations, however, fail to give an adequate description of the behavior of $|f'(z)|\ (1-|z|)$ as $|z|\to 1$ from the interior of the unit circle $|z|<1$. In the univalent case an answer to this question is contained in the following result due to J. E. Littlewood without the precise constant involved and to A. J. Macintyre(4) in the precise form stated here.

THEOREM 1. *Let $f(z)$ be analytic and univalent in $|z|<1$ and let it omit there the value ω. Then, in $|z|<1$ the following inequality is satisfied:*

$$|f'(z)|\,(1-|z|^2) \leqq 4\,|\omega - f(z)|. \tag{1.1}$$

Theorem 1 is in fact essentially one form of Koebe's distortion theorem, as we indicate below.

The object of the present paper is to study in some detail the behavior of expressions of the form $|f^{(p)}(z)|\ (1-|z|)^p$ for various classes of functions $f(z)$ analytic in the unit circle $|z|<1$, especially the behavior as $|z|\to 1$. We thus obtain results which can be interpreted as new distortion theorems. In

(1) O. Szász, Mathematische Zeitschrift, vol. 8 (1920), pp. 303–309.

(2) G. Pick, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Vienna, Abteilung IIa, vol. 126 (1917), pp. 247–263; R. Nevanlinna, Översigt af Finska Vetenskaps Societetens Förhandlingar, vol. 62 (1919).

(3) M. L. Cartwright, Mathematische Annalen, vol. 111 (1935), pp. 98–118.

(4) J. E. Littlewood, Proceedings of the London Mathematical Society, vol. 23 (1924) p. 507; A. J. Macintyre, Journal of the London Mathematical Society, vol. 11 (1936), pp. 7–11.

particular, the expression $|f'(z)|\ (1-|z|^2)$ is found to be closely connected with the radius of univalence, which is now to be defined.

DEFINITION 1. *Let $w=f(z)$ be analytic in $|z|<1$ and let R denote the Riemann configuration*(5) *over the w-plane onto which this function maps the region $|z|<1$. Let w_0 be an arbitrary point, not a branch point, of R. Then the radius of the largest smooth circle (boundary not included) with center at w_0 and wholly contained in R is called the radius of univalence of R at w_0 and will be denoted by $D_1(w_0)$. At a branch point w_0 of R we define $D_1(w_0)$ as zero.*

In this definition w_0 refers to an actual point of R and not merely to any point of R whose affix is the complex number w_0; the notation $D_1(w_0)$ is thus not fully explicit. The reader will easily verify that the largest smooth circle whose existence is asserted in the definition does exist and is unique.

This terminology differs from that of Montel(6), who uses the term modulus of univalence for our radius of univalence. A similar comment applies to the terminology radius of p-valence which we define in §14.

Explicit inequalities connecting $|f'(z)|\ (1-|z|^2)$ and $D_1(w)$ are obtained for the class of functions $f(z)$ univalent in $|z|<1$ in Theorem 3, Chapter I, for functions $f(z)$ bounded in $|z|<1$ in Theorem 3, Chapter II, and for functions $f(z)$ omitting two values in $|z|<1$ in Theorems 2 and 4 of Chapter IV. Analogous to the inequalities connecting $|f'(z)|\ (1-|z|^2)$ and $D_1(w)$ we determine inequalities connecting $|f^{(k)}(z)|\ (1-|z|^2)^k$ for $k=1, 2, \cdots, p$ and $D_p(w)$, where $D_p(w)$ is the radius of the largest p-sheeted circle with center in the point w contained in R. For the precise definitions the reader may be referred to Chapter II, §§13, 14. We obtain such inequalities on higher derivatives for the class of univalent functions in Theorem 5, Chapter I, for bounded functions in Theorems 1 and 2 of Chapter III, and for the functions omitting two values in Theorem 5, Chapter IV. For the detailed analysis of the paper the reader is referred to the Table of Contents.

Applications of the results just mentioned occur throughout the paper, particularly in Chapter V.

CHAPTER I. UNIVALENT FUNCTIONS

2. **Preliminary identities.** In the sequel we shall make extensive use of a lemma due to O. Szász(7).

LEMMA 1. *Let $f(z)$ be a function analytic in the circle $|z|<1$. Let*

---

(5) We use the term *Riemann configuration* on which the function $w=f(z)$ regular in $|z|<1$ maps the circle $|z|<1$ to denote that subregion of the Riemann surface of the inverse function of $w=f(z)$ which corresponds to the circle $|z|<1$.

(6) *Leçons sur les Fonctions Univalentes ou Multivalents*, Paris, 1933, pp. 22 and 110.

(7) O. Szász, Mathematische Zeitschrift, vol. 8 (1920), pp. 306–307.

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right). \tag{2.1}$$

Then $g(\zeta)$ is a function regular in $|\zeta| < 1$ for every value of z in $|z| < 1$ and

$$\frac{(1 - |z|^2)^n}{n!} f^{(n)}(z) = \frac{g^{(n)}(0)}{n!} + C_{n-1,1}\bar{z}\frac{g^{(n-1)}(0)}{(n-1)!} + C_{n-1,2}\bar{z}^2\frac{g^{(n-2)}(0)}{(n-2)!} + \cdots + \bar{z}^{n-1}g'(0). \tag{2.2}$$

We omit the proof of Lemma 1 and proceed to the proof of

LEMMA 2. *Let $f(z)$ be a function analytic in the circle $|z| < 1$. Let*

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right).$$

Then, for every fixed value of z in $|z| < 1$, $g(\zeta)$ is a function of ζ regular in $|\zeta| < 1$, and

$$\begin{aligned}\frac{g^{(n)}(0)}{n!} = \frac{(1 - |z|^2)^n f^{(n)}(z)}{n!} &- C_{n-1,1}\bar{z}\frac{(1 - |z|^2)^{n-1} f^{(n-1)}(z)}{(n-1)!} \\ &+ C_{n-1,2}\bar{z}^2\frac{(1 - |z|^2)^{n-2} f^{(n-2)}(z)}{(n-2)!} - \cdots \\ &+ (-1)^{n-1}\bar{z}^{n-1}(1 - |z|^2) f'(z).\end{aligned} \tag{2.3}$$

Let us write equation (2.2) for $n = k$ and allow k to assume the values $1, 2, \cdots, n$:

$$\frac{(1 - |z|^2)^k f^{(k)}(z)}{k!} = \frac{g^{(k)}(0)}{k!} + C_{k-1,1}\bar{z}\frac{g^{(k-1)}(0)}{(k-1)!} + C_{k-1,2}\bar{z}^2\frac{g^{(k-2)}(0)}{(k-2)!} + \cdots + \bar{z}^{k-1}g'(0). \tag{2.4}$$

Let us proceed similarly with (2.3):

$$\begin{aligned}\frac{g^{(k)}(0)}{k!} = \frac{(1 - |z|^2)^k f^{(k)}(z)}{k!} &- C_{k-1,1}\bar{z}\frac{(1 - |z|^2)^{k-1} f^{(k-1)}(z)}{(k-1)!} \\ &+ C_{k-1,2}\bar{z}^2\frac{(1 - |z|^2)^{k-2} f^{(k-2)}(z)}{(k-2)!} - \cdots \\ &+ (-1)^{k-1}\bar{z}^{k-1}(1 - |z|^2) f'(z).\end{aligned} \tag{2.5}$$

The lemma will be proved if it can be shown that (2.5) is obtained from (2.4) by solving the latter system for $g^{(k)}(0)/k!$ $(k = 1, 2, \cdots, n)$. To do that it suffices to prove that the matrix of the coefficients of (2.4),

$$\Delta = \left\| \begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ \bar{z} & 1 & 0 & \cdots & 0 \\ \bar{z}^2 & 2\bar{z} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{z}^{n-1} & C_{n-1,n-2}\bar{z}^{n-2} & C_{n-1,n-3}\bar{z}^{n-3} & \cdots & 1 \end{array} \right\|$$

and the matrix of the coefficients of (2.5),

$$\Delta' = \left\| \begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ -\bar{z} & 1 & 0 & \cdots & 0 \\ \bar{z}^2 & -2\bar{z} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (-1)^{n-1}\bar{z}^{n-1} & (-1)^{n-2}C_{n-1,n-2}\bar{z}^{n-2} & (-1)^{n-3}C_{n-1,n-3}\bar{z}^{n-3} & \cdots & 1 \end{array} \right\|$$

are inverse matrices, or that $\Delta \cdot \Delta' = I$, I being the unit matrix. Now, it is immediately evident that the elements in the principal diagonal of the product matrix are 1, while the element in the kth row and lth column, where $k > l$, is

$$\bar{z}^{k-l}[C_{k-1,k-l} - C_{k-1,k-l-1}C_{l,1} + C_{k-1,k-l-2}C_{l+1,2} - \cdots \pm C_{k-1,2}C_{k-3,k-l-2} \mp C_{k-1,1}C_{k-2,k-l-1} \pm C_{k-1,k-l}]. \tag{2.6}$$

The sum (2.6) may be written as follows:

$$\bar{z}^{k-l}C_{k-1,k-l}[1 - C_{k-l,1} + C_{k-l,2} - C_{k-l,3} + \cdots \pm 1] = C_{k-1,k-l}(1-1)^{k-l}\bar{z}^{k-l}$$

which is zero. The case $k < l$ may be treated similarly. This proves that $\Delta \cdot \Delta'$ is the unit matrix. Thus, Lemma 2 is established.

3. **Littlewood-Macintyre theorem.** We proceed to prove Theorem 1; this method is different from those of Littlewood and Macintyre. Indeed, form the function

$$\phi(\zeta) = \frac{f((\zeta + z)/(1 + \bar{z}\zeta)) - f(z)}{(1 - |z|^2)f'(z)} \tag{3.1}$$

for a fixed value of z in $|z| < 1$(8). This function is evidently regular and univalent in $|\zeta| < 1$ and omits there the value $(\omega - f(z))/(1 - |z|^2)f'(z)$. Since, furthermore, $\phi(0) = 0$ and $\phi'(0) = 1$, we may apply a well known result(9) of Koebe in the theory of univalent functions, according to which

$$\left| \frac{\omega - f(z)}{(1 - |z|^2)f'(z)} \right| \geqq \frac{1}{4}.$$

(8) The function $\phi(\zeta)$ plays an important role in the theory of univalent functions, cf. P. Montel, *Leçons Sur Les Fonctions Univalentes ou Multivalentes*, Paris, 1933, p. 51.

(9) See, for example, P. Montel, loc. cit., p. 50.

This proves the theorem. Direct computation shows that the limit is attained for the univalent function $z/(1-z)^2$ and $\omega=-1/4$. Of course Koebe's theorem is the special case $z=0$ of Theorem 1.

4. **Inequalities concerning** D_1. For the sequel it is desirable to restate Theorem 1 in a more geometric form. If we set $w=f(z)$, the right side of inequality (1.1) attains its least value when ω is one of those boundary points of the region R onto which $f(z)$ maps the circle $|z|<1$ which are nearest the point w. In that case $|\omega-f(z)|=D_1(w)$, as defined in the introduction, and Theorem 1 becomes

THEOREM 1'. *Let $f(z)$ be analytic and univalent in $|z|<1$. Then the inequality*

$$(1-|z|^2)|f'(z)| \leqq 4D_1(w) \tag{4.1}$$

is satisfied for all values of z in $|z|<1$, where $D_1(w)$ is the radius of univalence at the point $w=f(z)$ of the region R onto which $f(z)$ maps the circle $|z|<1$.

It may be of some interest to point out a geometric interpretation of the left side of inequality (4.1). Denote by $\rho(w)$ the "inner radius" of R with respect to a fixed interior point w(10). Then $\rho(w)$ can be expressed in terms of $f(z)$ as follows

$$\rho(w) = |f'(z)|(1-|z|^2), \tag{4.2}$$

where z is the point corresponding to w. Inequality (4.1) may, therefore, be written in the geometric form (11)

$$\rho(w) \leqq 4D_1(w). \tag{4.3}$$

Theorem 1' gives an upper bound for $|f'(z)|(1-|z|^2)$. It is desirable also to obtain a lower bound for this expression.

THEOREM 2. *Let $f(z)$ be analytic in $|z|<1$, let z_0 be any point of $|z|<1$, and $w_0=f(z_0)$. Then*

$$D_1(w_0) \leqq |f'(z_0)|(1-|z_0|^2). \tag{4.4}$$

We notice that unlike (4.1), the relation (4.4) holds without any restriction other than analyticity on the function $f(z)$. Denote by R the Riemann surface over the w-plane onto which $w=f(z)$ maps the circle $|z|<1$. If w_0 is

(10) The "inner radius" of a simply connected region R with respect to an interior point w_0 is the radius of the circle on which the region R can be mapped conformally by a function $f(w)$ so that $f(w_0)=0$ and $f'(w_0)=1$. Cf. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, vol. II, Berlin, 1925, pp. 16–21.

(11) Inequalities (4.1) and (4.3) together with Corollary 2 below were first proved by J. L. Walsh, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 520–523. In the same paper the author suggests the use of the present method in the study of higher derivatives of univalent functions, which is one of the principal topics taken up in the present chapter.

a branch point of R, (4.4) is trivial, for in that case both sides of the inequality reduce to zero. Otherwise, let

$$g(\zeta) = f\left(\frac{\zeta + z_0}{1 + \bar{z}_0\zeta}\right).$$

This function is also analytic in $|\zeta| < 1$ and maps the circle onto R. Furthermore, $g(0) = w_0$. If we denote by $\zeta = h(w)$ the inverse function of $w = g(\zeta)$, the function $h(w)$ is defined, regular, and single-valued on R. In particular, a suitable branch of $h(w)$ will be regular and single-valued on the single-sheeted circle C with center at w_0 and radius $D_1(w_0)$. The values which this branch assumes in C all lie in the circle $|\zeta| < 1$. Hence, in C: $|h(w)| < 1$, $h(w_0) = 0$. Consequently, applying Schwarz's lemma

$$|h'(w_0)| \leqq \frac{1}{D_1(w_0)}\cdot$$

Hence, $|g'(0)| \geqq D_1(w_0)$ and the evaluation of $g'(0)$ in terms of $f(z)$ yields (4.4).

The inequality in (4.4) is sharp, reducing to an equality when

$$f(z) = \frac{z - z_1}{1 - \bar{z}_1 z}, \qquad |z_1| < 1.$$

Combining Theorems 1′ and 2, we obtain

THEOREM 3. *Let $f(z)$ be regular and univalent in $|z| < 1$, let z_0 be any point of $|z| < 1$, and $w_0 = f(z_0)$. Then,*

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq 4D_1(w_0). \tag{4.5}$$

We remark that Theorems 1 and 1′ can be somewhat improved if we assume $f(z)$ not merely analytic and univalent in $|z| < 1$, but also bounded there: $|f(z)| \leqq M$. Under those conditions the function $\phi(z)$ defined by (3.1) is also analytic and univalent there, with $\phi(0) = 0$, $\phi'(0) = 1$,

$$|\phi(\zeta)| \leqq \frac{2M}{|f'(z)|(1 - |z|^2)}\cdot$$

Since $\phi(\zeta)$ in $|\zeta| < 1$ omits the value

$$\frac{\omega - f(z)}{f'(z)(1 - |z|^2)}$$

provided the function $f(z)$ omits the value ω, the inequality of Pick([12]) yields

([12]) That is to say, under a smooth map of the region $|z| < 1$ by a function $w = f(z)$ with $f(0) = 0$, $f'(0) = 1$, $|f(z)| < M$, every boundary point of the image in the w-plane satisfies the inequality $|w| \geq [M - (M^2 - M)^{1/2}]^2$. See Pick, and R. Nevanlinna, loc. cit.

$$D_1(w_0) \geqq \left[\frac{2M}{|f'(z_0)|^{1/2}(1-|z_0|^2)^{1/2}} - \left(\frac{4M^2}{|f'(z_0)|(1-|z_0|^2)} - 2M\right)^{1/2}\right]^2,$$

$$\frac{4M[D_1(w_0)]^{1/2}}{D_1(w_0)+2M} \geqq |f'(z_0)|^{1/2}(1-|z_0|^2)^{1/2}.$$

It may be noted that as M becomes infinite this last inequality approaches the form (4.1).

5. **Applications.** From Theorem 3 various corollaries may be immediately deduced.

COROLLARY 1. *Let $f(z)$ be regular and univalent in $|z|<1$, $\{z_n\}$ any sequence of points in $|z|<1$, and $w_n=f(z_n)$. Then, a necessary and sufficient condition that*

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|) = 0$$

is that

$$\lim_{n\to\infty} D_1(w_n) = 0,$$

and a necessary and sufficient condition that $|f'(z_n)|(1-|z_n|)$ remain bounded is that $D_1(w_n)$ remain bounded.

COROLLARY 2. *Let $f(z)$ be regular, univalent, and bounded in $|z|<1$, $\{z_n\}$ any sequence of points in $|z|<1$ for which $\lim_{n\to\infty}|z_n|=1$. Then*

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|)=0.$$

The proof of Corollary 1 follows directly from the inequalities (4.5), while Corollary 2 follows from Corollary 1 if one remarks that under the hypotheses of Corollary 2 we have $D_1(w_n)\to 0$[13]. Another consequence of (4.5) is the following:

COROLLARY 3. *Let $f(z)$ be regular and univalent in $|z|<1$, let z_0 be any point of $|z|=1$. Then there exists a sequence of points $\{z_n\}$ $(|z_n|<1)$ converging to z_0 such that*

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|) = 0.$$

In accordance with Corollary 1 it suffices to find a sequence $\{z_n\}$ converging to z_0 for which the points $w_n=f(z_n)$ satisfy the relation $D_1(w_n)\to 0$. Such a

(13) As was pointed out by Walsh (loc. cit.), Corollary 2 may also be proved by Carathéodory's method of the conformal mapping of variable regions, cf. C. Carathéodory, *Conformal Representation*, Cambridge, 1932, p. 75.

sequence may be found as follows. It is well known(14) that a univalent function has finite limit values on almost all radii. These limit values are boundary points of the region onto which $f(z)$ maps the circle $|z|<1$. Choose a sequence of such radii r_n which converges to the radius joining z_0 with the origin. On the radius r_n choose a point z_n ($|z_n|<1$) so near to the circumference $|z|=1$ that

$$D_1(w_n) < 1/n.$$

This sequence $\{z_n\}$ fulfills the necessary requirements.

6. **Inequalities for higher derivatives.** We now turn to the corresponding study of the higher derivatives of univalent functions. In particular, we shall determine upper bounds for expressions of the form

$$|f^{(n)}(z_0)|(1-|z_0|^2)^n. \tag{6.1}$$

It is clear immediately that lower bounds for these expressions in terms of $D_1(w)$ cannot be obtained even in the case $n=2$. For the expression (6.1) is identically zero for $n\geqq 2$ when $f(z)\equiv z$. Even for the upper bounds of (6.1) the sharp inequalities will now be obtained only in the case $n=2$, 3. For higher values of n the corresponding inequalities depend on the assumption of the truth of Bieberbach's conjecture, which up to the present has not been established.

We begin by proving the following inequalities

THEOREM 4. *Let $f(z)$ be regular and univalent in $|z|<1$, let z_0 be any point of $|z|<1$, and let $w_0=f(z_0)$. Then,*

$$|f''(z_0)|(1-|z_0|^2)^2 \leqq 8(|z_0|+2)D_1(w_0) \tag{6.2}$$

and

$$|f'''(z_0)|(1-|z_0|^2)^3 \leqq 24(|z_0|^2+4|z_0|+3)D_1(w_0). \tag{6.3}$$

These inequalities are sharp, reducing to equalities for $f(z)=z/(1+z)^2$ for real negative values of z.

To prove (6.2) and (6.3) compute the second and third Taylor coefficients, b_2 and b_3, of the function (3.1) where we set $z=z_0$. By direct computation (or by §2, Lemma 2) we find that

$$\begin{aligned} b_2 &= \frac{1}{2}\frac{f''(z_0)}{f'(z_0)}(1-|z_0|^2)-\bar{z}_0, \\ b_3 &= \frac{1}{6}\frac{f'''(z_0)}{f'(z_0)}(1-|z_0|^2)^2-\frac{f''(z_0)}{f'(z_0)}\bar{z}_0(1-|z_0|^2)+\bar{z}_0^2. \end{aligned} \tag{6.4}$$

(14) See, for example, W. Seidel, Mathematische Annalen, vol. 104 (1931), p. 191.

Now, according to Bieberbach's theorem and Löwner's theorem[15] $|b_2| \leqq 2$ and $|b_3| \leqq 3$. Hence

$$\left| \frac{1}{2} \frac{f''(z_0)}{f'(z_0)} (1 - |z_0|^2) - \bar{z}_0 \right| \leqq 2$$

and

$$(6.2') \qquad |f''(z_0)(1 - |z_0|^2)^2 - 2\bar{z}_0(1 - |z_0|^2)f'(z_0)| \leqq 4(1 - |z_0|^2)|f'(z_0)|.$$

Applying (4.1) we obtain at once inequality (6.2). To obtain (6.3) we use the evaluation of b_3 in (6.4) and write

$$\left| \frac{1}{6} \frac{f'''(z_0)}{f'(z_0)} (1 - |z_0|^2)^2 - \frac{f''(z_0)}{f'(z_0)} \bar{z}_0(1 - |z_0|^2) + \bar{z}_0^2 \right| \leqq 3$$

and

$$|f'''(z_0)(1 - |z_0|^2)^3 - 6\bar{z}_0 f''(z_0)(1 - |z_0|^2)^2 + 6\bar{z}_0^2 f'(z_0)(1 - |z_0|^2)| \leqq 18|f'(z_0)|(1 - |z_0|^2).$$

It follows that

$$\begin{aligned} |f'''(z_0)|(1 - |z_0|^2)^3 &\leqq |6\bar{z}_0 f''(z_0)(1 - |z_0|^2)^2 - 6\bar{z}_0^2 f'(z_0)(1 - |z_0|^2)| \\ &\quad + 18|f'(z_0)|(1 - |z_0|^2) \\ &\leqq 6|\bar{z}_0 f''(z_0)(1 - |z_0|^2)^2 - 2\bar{z}_0^2 f'(z_0)(1 - |z_0|^2)| \\ &\quad + 6|z_0|^2|f'(z_0)|(1 - |z_0|^2) + 18|f'(z_0)|(1 - |z_0|^2). \end{aligned}$$

Applying now inequalities (6.2′) and (4.1) we obtain inequality (6.3).

If now Bieberbach's conjecture concerning the coefficients of univalent were known to be true[16], one could write

$$\frac{|\phi^{(n)}(0)|}{n!} \leqq n.$$

With the aid of a little algebraic manipulation (see below) this would lead to the sharp inequality

$$(6.5) \qquad |f^{(n)}(z_0)|(1 - |z_0|^2)^n \leqq 4n!(n + |z_0|)(1 + |z_0|)^{n-2} D_1(w_0),$$

which becomes an equality for $f(z) = z/(1+z)^2$ for real negative values of z. Unfortunately, however, the inequality $|b_n| \leqq n$ has been proved only for

(15) L. Bieberbach, Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin, vol. 38 (1916), pp. 940–955; K. Löwner, Mathematische Annalen, vol. 89 (1923), pp. 103–121.

(16) See, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 2d edition, 1931, p. 80, Footnote 4.

$n=2$ and 3, so that the validity of inequality (6.5) has been established for $n=2$ and 3 only. Weaker inequalities have actually been proved by various authors, in particular, J. E. Littlewood[17] who showed

$$\frac{|\phi^{(n)}(0)|}{n!} < en \tag{6.6}$$

and E. Landau[18] who showed

$$\frac{|\phi^{(n)}(0)|}{n!} \leqq \left(\frac{1}{2}+\frac{1}{\pi}\right)en.$$

Making use of (6.6) and Lemma 1 of §2 we find

$$\frac{(1-|z|^2)^n|f^{(n)}(z)|}{n!} \leqq (1-|z|^2)|f'(z)| \sum_{\nu=0}^{n-1} C_{n-1,\nu}|z|^\nu \frac{|\phi^{(n-\nu)}(0)|}{(n-\nu)!}$$

and using (4.1)

$$\frac{(1-|z|^2)^n|f^{(n)}(z)|}{n!} \leqq 4eD_1(w)\sum_{\nu=0}^{n-1}(n-\nu)C_{n-1,\nu}|z|^\nu.$$

Since, however,

$$\sum_{\nu=0}^{n-1} C_{n-1,\nu}|z|^\nu = (1+|z|)^{n-1}$$

and

$$\sum_{\nu=0}^{n-1} \nu C_{n-1,\nu}|z|^\nu = (n-1)|z|(1+|z|)^{n-2},$$

we obtain

$$(1-|z|^2)^n|f^{(n)}(z)| \leqq 4e\cdot n!D_1(w)[n(1+|z|)^{n-1}+(n-1)|z|(1+|z|)^{n-2}]$$

and finally

$$(1-|z|^2)^n|f^{(n)}(z)| \leqq 4e\cdot n!(n+|z_0|)(1+|z_0|)^{n-2}D_1(w_0).$$

This clearly is not a sharp inequality. We thus obtain

THEOREM 5. *Let $f(z)$ be regular and univalent in $|z|<1$, let z_0 be any point in $|z|<1$, and let $w_0=f(z_0)$. Then*

$$|f^{(n)}(z_0)|(1-|z_0|^2)^n \leqq 4e\cdot n!(|z_0|+n)(1+|z_0|)^{n-2}D_1(w_0). \tag{6.7}$$

From this inequality we obtain again two corollaries analogous to those of Theorem 3.

(17) J. E. Littlewood, loc. cit., p. 498.

(18) E. Landau, Mathematische Zeitschrift, vol. 30 (1929), p. 635.

COROLLARY 4. *Let $f(z)$ be regular and univalent in $|z|<1$, $\{z_n\}$ any sequence of points in $|z|<1$ and $w_n=f(z_n)$. Then, if*

$$\lim_{n\to\infty} D_1(w_n) = 0,$$

all the derivatives of $f(z)$ will satisfy the relation

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k = 1, 2, 3, \cdots.$$

Clearly the converse of the theorem is false since taking $f(z)\equiv z$, $z_n=0$ $(n=1, 2, \cdots)$, we have $f^{(k)}(z_n)=0$ for all $k\geqq 2$ and all n while $D_1(w_n)=1$.

COROLLARY 5. *Let $f(z)$ be regular, univalent, and bounded in $|z|<1$, $\{z_n\}$ any sequence of points in $|z|<1$ for which $\lim_{n\to\infty}|z_n|=1$. Then*

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k = 1, 2, 3, \cdots.$$

7. **Applications.** A few remarks concerning Theorem 3 will now be made. Koebe's "Verzerrungssatz" can be written in the form[19]

$$\frac{1-|z|}{(1+|z|)^3} \leqq |f'(z)| \leqq \frac{1+|z|}{(1-|z|)^3}.$$

If we combine this inequality with (4.5) we obtain

$$\frac{1}{4}\left(\frac{1-|z_0|}{1+|z_0|}\right)^2 \leqq D_1(w_0) \leqq \left(\frac{1+|z_0|}{1-|z_0|}\right)^2.$$

We may state this result as follows:

COROLLARY 6. *Let $f(z)$ be regular and univalent in $|z|<1$ with $f(0)=0$, $f'(0)=1$, let z_0 be any point of $|z|<1$, and let $w_0=f(z_0)$. Then the radius of univalence $D_1(w_0)$ at the point w_0 satisfies the inequality*

$$\frac{1}{4}\left(\frac{1-|z_0|}{1+|z_0|}\right)^2 \leqq D_1(w_0) \leqq \left(\frac{1+|z_0|}{1-|z_0|}\right)^2.$$

The lower bound of $D_1(w_0)$ was obtained in less precise form by W. E. Sewell[20]. The first inequality is sharp, becoming an equality for $f(z)=z/(1+z)^2$ along the positive real axis. The second inequality is probably not sharp.

Another application of Theorem 3 concerns infinite regions. Suppose that R is a simply connected region of the w-plane for which $w=\infty$ is an accessible boundary point, let

[19] See, for instance, Paul Montel, loc. cit., p. 52.

[20] W. E. Sewell, these Transactions, vol. 41 (1937), p. 90.

$$\limsup_{w\to\infty} D_1(w) = D,$$

where w is an interior point of R, and let $w=f(z)$ map R on the interior of the circle $|z|<1$; suppose that $z=\alpha$, $(|\alpha|=1)$, corresponds to $w=\infty$. From Theorem 3 it follows that

$$\limsup_{z\to\alpha} |f'(z)|(1-|z|^2) \leqq 4D.$$

For an arbitrary infinite region the relations $w_n=f(z_n)\to\infty$, $\limsup_{n\to\infty}D_1(w_n)=D$, $\liminf_{n\to\infty}D_1(w_n)=d$ clearly imply that $\limsup_{n\to\infty}|f'(z_n)|(1-|z_n|^2)\leqq 4D$, $\liminf_{z\to\alpha}|f'(z_n)|(1-|z|^2)\geqq d$.

The final remark concerns an inequality derived by G. Szegö([21]) on the difference quotient of a univalent function. His inequality is as follows: Let $f(z)$ be regular and univalent in $|z|<1$, let z_1 and z_2 be any two points of the circle $|z|<1$. Then,

$$\begin{aligned} &|f'(z_2)|(1-|z_2|^2)\frac{|1-\bar z_2 z_1|}{(|z_1-z_2|+|1-\bar z_2 z_1|)^2} \\ &\leqq \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \leqq |f'(z_2)|(1-|z_2|^2)\frac{|1-\bar z_2 z_1|}{(|z_1-z_2|-|1-\bar z_2 z_1|)^2}. \end{aligned} \tag{7.1}$$

Let us introduce the non-euclidean distance $\rho(z_1, z_2)$ between the points z_1 and z_2 by means of the following relations

$$\rho(z_1, z_2) = \log\frac{1+r}{1-r}, \qquad r = \left|\frac{z_1-z_2}{1-\bar z_2 z_1}\right|.$$

By virtue of (7.1) and (4.5) we obtain the inequalities

$$\begin{aligned} D_1(w_2)\frac{|1-\bar z_2 z_1|}{(|z_1-z_2|+|1-\bar z_2 z_1|)^2} &\leqq \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \\ &\leqq 4D_1(w_2)\frac{|1-\bar z_2 z_1|}{(|z_1-z_2|-|1-\bar z_2 z_1|)^2}, \end{aligned}$$

where $w_2=f(z_2)$. In terms of $\rho(z_1, z_2)$ the inequalities become

$$(1/4)D_1(w_2)(1-e^{-2\rho(z_1,z_2)}) \leqq |f(z_1)-f(z_2)| \leqq (e^{2\rho(z_1,z_2)}-1)D_1(w_2). \tag{7.2}$$

From the inequalities (7.2) we obtain the corollary:

COROLLARY 7. *Let $f(z)$ be regular and univalent in $|z|<1$, let $\{z_n\}$ and $\{z_n'\}$ be two sequences of points in $|z|<1$, such that $\rho(z_n, z_n')$ is bounded and let $w_n'=f(z_n')$. Then $\lim_{n\to\infty}|f(z_n)-f(z_n')|=0$ if, and only if,*

$$\lim_{n\to\infty}(e^{2\rho(z_n,z_n')}-1)D_1(w_n') = 0.$$

([21]) G. Szegö, Mathematische Annalen, vol. 100 (1928), pp. 190–191.

8. **Behavior of the first derivative almost everywhere.** Corollary 2 may be stated as asserting that for a regular, univalent, and bounded function in the circle $|z| < 1$ the first derivative is of order $o((1-r)^{-1})$ on all radii of the circle. The next theorem shows, however, that this order of growth can be attained only on a small number of radii and that on most radii the order of growth is considerably smaller. Indeed, we prove the following

THEOREM 6. *Let $f(z)$ be regular and univalent in the circle $|z| < 1$. Then*

$$\lim_{z \to e^{i\alpha}} |f'(z)| (1 - |z|)^{1/2} = 0 \tag{8.1}$$

for all points $e^{i\alpha}$ of the circumference $|z| = 1$ with the exception of at most a set of measure zero, where z in the above limit is taken in any angle less than π with vertex in $e^{i\alpha}$ and bisected by the radius joining $z=0$ with $z=e^{i\alpha}$. Furthermore, in any such angle the above limit is uniform.

The proof depends on a number of lemmas.

LEMMA 3. *If $f(z)$ is univalent in the circle $|z| < 1$, then on almost all radii*

$$|f'(z)| = O((1 - |z|)^{-1/2}), \tag{8.2}$$

where the symbol O does not necessarily indicate uniformity for the different radii. The relation (8.2) *holds also in any angle of the type described in Theorem* 6 *which corresponds to a radius for which* (8.2) holds.

If we set $w=f(z)$, then the function maps $|z| < 1$ on a simply connected region R of the w-plane. Now, this region R possesses at least two distinct boundary points $w=a$ and $w=b$, $(a \neq b)$. Indeed, if R were the entire plane then the inverse function $z=g(w)$ of $w=f(z)$ would map the plane on the interior of $|z| < 1$. It would, therefore, be bounded in the whole plane and by Liouville's theorem be identically a constant, which is contrary to our assumption. If R were the whole plane with the exception of one point, $w=a$, then $g(w)$ would be regular and bounded in the whole plane with the exception of the one point, $w=a$. This point, by Riemann's theorem, would be a removable singularity, and again $z=g(w)$ would be identically constant. Now, by a familiar argument the function

$$t = \frac{1}{((w-a)/(w-b))^{1/2} - c} = \lambda(w),$$

where the constant c is suitably chosen, maps the region R conformally on a bounded region of the t-plane.

The function

$$h(z) = \lambda(f(z))$$

is regular, univalent and bounded in $|z| < 1$. Let us suppose that Lemma 3 has already been proved for $h(z)$. Then, it will also hold for $f(z)$. Indeed,

$$f'(z) = \frac{h'(z)}{\lambda'(f(z))}\cdot$$

Since we have assumed that $\limsup_{z\to e^{i\alpha}} |h'(z)| (1-|z|)^{1/2} < \infty$ for almost all points $z=e^{i\alpha}$ on $|z|=1$, where z lies in corresponding angles as described in Theorem 6, the asserted lemma will follow for $f'(z)$ provided that $\liminf_{z\to e^{i\alpha}} |\lambda'(f(z))| > 0$ for almost all $e^{i\alpha}$ in the corresponding angles. But now

$$\lambda'(w) = -\frac{a-b}{2}\frac{[\lambda(w)]^2}{(w-b)^{3/2}(w-a)^{1/2}},$$

which shows that $\liminf_{z\to e^{i\alpha}} |\lambda'(f(z))| = 0$ only if there exists a sequence of points $z_n \to e^{i\alpha}$ for which $f(z_n)\to b$ or $f(z_n)\to a$. This, however, can only happen for a set of $e^{i\alpha}$ of measure zero(22).

It suffices, therefore, to prove Lemma 3 for a bounded univalent function $f(z)$. Now, $w=f(z)$ maps the circle $|z|<1$ on a bounded region of the w-plane. Denote the area of this region by A. We have, setting $z=re^{i\theta}$,

$$\int_0^{2\pi}\int_0^{\rho} |f'(re^{i\theta})|^2 r\,dr\,d\theta < A \tag{8.3}$$

for every $0\leqq\rho<1$. The function

$$\Phi(z) = \int_0^z z[f'(z)]^2 dz \tag{8.4}$$

is regular in $|z|<1$ and we shall perform the integration along the radius joining $z=0$ and $z=re^{i\theta}$ so that

$$\Phi(\rho e^{i\theta}) = \int_0^{\rho} re^{2i\theta}[f'(re^{i\theta})]^2 dr.$$

Hence,

$$|\Phi(\rho e^{i\theta})| \leqq \int_0^{\rho} r|f'(re^{i\theta})|^2 dr.$$

Integration of the last inequality with respect to θ together with (8.3) yields

$$\int_0^{2\pi} |\Phi(\rho e^{i\theta})|\,d\theta < A \tag{8.5}$$

for every $0\leqq\rho<1$.

Now it is a familiar fact that if a function $\Phi(z)$ is regular in $|z|<1$ and

(22) F. and M. Riesz, Compte Rendu du Quatrième Congrès des Mathématiciens Scandinaves, 1920, pp. 28–30.

satisfies the condition (8.5) it may be represented in the following form([23]):

$$\Phi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) + i\beta, \tag{8.6}$$

where the integral is a Stieltjes integral, $\mu(t)$ is a function of bounded variation in the interval $0 \leqq t \leqq 2\pi$ and β is a constant. Equations (8.4) and (8.6) permit us to express $[f'(z)]^2$ in the form

$$[f'(z)]^2 = \frac{1}{\pi z}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}\, d\mu(t).$$

Hence,

$$(1-r^2)\,|f'(z)|^2 \leqq \frac{1}{\pi r}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\, dM(t), \tag{8.7}$$

where $M(t)$ denotes the total variation of the function $\mu(t)$ in the interval $(0, t)$. The right-hand side of this inequality approaches a definite finite limit as $z = re^{i\theta} \to e^{i\alpha}$ in an angle of the type described in Theorem 6 for almost all $e^{i\alpha}$([24]). Hence, the right-hand side remains bounded in such angles. Thus,

$$|f'(z)| \leqq C_\alpha \cdot (1 - |z|)^{-1/2} \tag{8.8}$$

in the angular neighborhood of almost all points $e^{i\alpha}$, where C_α is a constant independent of z, but in general depending on α. This proves the lemma.

COROLLARY 8. *Let $w = f(z)$ be regular and univalent in $|z| < 1$. Then, for almost all points $e^{i\alpha}$ on $|z| = 1$ every line segment joining an interior point of $|z| < 1$ with $e^{i\alpha}$ is mapped on a rectifiable arc by the function $w = f(z)$.*

This follows readily by integrating (8.8) along such a line segment([25]).

If we restrict ourselves to radial approach in Corollary 8, it is possible to state a sharper result which will be used in the proof of Theorem 6:

LEMMA 4. *Let $w = f(z)$ be regular and univalent in $|z| < 1$. If*

$$l_{\rho,\theta} = \int_\rho^1 |f'(re^{i\theta})|\, dr, \qquad z = re^{i\theta}, \tag{8.9}$$

then for almost all values of θ in $0 \leqq \theta \leqq 2\pi$ and for all values of ρ in $0 \leqq \rho < 1$, $l_{\rho,\theta}$ is finite and

$$\lim_{\rho \to 1} l_{\rho,\theta}(1-\rho)^{-1/2} = 0. \tag{8.10}$$

([23]) See, for example, R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936, p. 185.

([24]) After integration by parts the integral in (8.7) becomes one of the type considered in Carathéodory's proof of Fatou's theorem. Cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 2d edition, (1931), pp. 148–151.

([25]) The corollary includes a result stated by M. Lavrentieff, Physico-Mathematical Institute of Stekloff, vol. 5 (1934), p. 207.

The formula in (8.9) represents the length of the image of the radial segment joining the points $\rho e^{i\theta}$ and $e^{i\theta}$.

One may assume without loss of generality, for the same reasons as in the proof of Lemma 3, that $f(z)$ is bounded in $|z|<1$. Then, inequality (8.3) holds for some A. The total area A of the image of $|z|<1$ is given by

$$A = \int_0^{2\pi}\int_0^1 |f'(re^{i\theta})|^2 r\,dr\,d\theta.$$

Hence, by Fubini's theorem, for almost all θ in $0\leqq\theta\leqq 2\pi$

$$\int_0^1 r|f'(re^{i\theta})|^2 dr$$

has a finite value. Hence, for almost all θ

$$\lim_{\rho\to 1}\int_\rho^1 r|f'(re^{i\theta})|^2 dr = 0.$$

Thus to any $\epsilon>0$ one may assign a number $\delta=\delta(\epsilon,\theta)$ so that $1-\rho<\delta$ implies

$$\int_\rho^1 r|f'(re^{i\theta})|^2 dr < \epsilon$$

for almost all θ. Hence, by Schwarz's inequality

$$\int_\rho^1 |f'(re^{i\theta})|\,dr \leqq \left[(1-\rho)\int_\rho^1 |f'(re^{i\theta})|^2 dr\right]^{1/2} < \left[\frac{\epsilon(1-\rho)}{\rho}\right]^{1/2}$$

for almost all θ and $1>\rho>1-\delta(\epsilon,\theta)$. This proves (8.10).

Using this lemma, one can now prove

LEMMA 5. *Let $w=f(z)$ be regular and univalent in $|z|<1$. Then on almost all radii*

$$|f'(z)| = o((1-r)^{-1/2}), \qquad |z| = r,$$

where the symbol o is not intended to indicate uniformity for the different radii.

We know that on almost all radii (8.2) and (8.10) hold and $\lim_{r\to 1} f(re^{i\theta})=\omega$ exists and is finite(26). Choose any one of these radii $\theta=\theta_0$ and on it an arbitrary point z_0. Let $f(z_0)=w_0$. The segment of the radius between the points z_0

(26) For the proof of the last statement one need merely apply the fact that the integral in (8.9) remains finite for almost all θ. Indeed, take any such θ_0. Then,

$$|f(r_1e^{i\theta_0}) - f(r_2e^{i\theta_0})| \leqq \int_{r_1}^{r_2} |f'(re^{i\theta_0})|\,dr, \qquad r_1<r_2,$$

and the last integral may be made smaller than any preassigned $\epsilon>0$ provided that r_1 and r_2 are both chosen sufficiently near unity.

and $e^{i\theta_0}$ is carried into a rectifiable arc joining the points w_0 and ω. Its length l_{z_0} is given by

$$l_{z_0} = \epsilon_{z_0}(1 - |z_0|)^{1/2}, \tag{8.11}$$

where by (8.10)

$$\lim_{z_0 \to e^{i\theta_0}} \epsilon_{z_0} = 0, \tag{8.12}$$

the approach being taken radially. Now draw a circle K_{z_0} about the point z_0 as center with radius equal to $1-|z_0|$. The interior of the circle K_{z_0} is carried by $w=f(z)$ into a region R_{z_0} of the w-plane.

According to Koebe's "Verzerrungssatz" the region R_{z_0} contains the circle $|w-w_0| < ((1-|z_0|)/4)\ |f'(z_0)|$.

Now if we set

$$|f'(z_0)| = C_{z_0}(1 - |z_0|)^{-1/2},$$

according to (8.2) C_{z_0} is bounded along the radius $\theta=\theta_0$. Thus R_{z_0} contains the circle $|w-w_0| < (1/4)C_{z_0}\ (1-|z_0|)^{1/2}$. In view of (8.11) this may also be written $|w-w_0| < C_{z_0}l_{z_0}/4\epsilon_{z_0}$. Denoting by ρ_{z_0} the radius of this circle, we have on the one hand

$$\rho_{z_0} = \frac{C_{z_0}}{4\epsilon_{z_0}} l_{z_0}$$

and on the other $\rho_{z_0} \leqq l_{z_0}$. Hence,

$$C_{z_0} \leqq 4\epsilon_{z_0}.$$

Together with (8.12) this implies that

$$\lim_{z_0 \to e^{i\theta_0}} C_{z_0} = 0$$

with radial approach. This proves the lemma.

We are now ready for the proof of Theorem 6. Let $\theta=\theta_0$ be a radius for which (8.2) holds in any angle as asserted in Lemma 3 and also

$$\lim_{r\to 1} |f'(re^{i\theta_0})|\ (1 - r)^{1/2} = 0. \tag{8.13}$$

By Lemmas 3 and 5 the set of such θ_0 is of measure 2π.

Consider the function

$$g(z) = f'(z)(e^{i\theta_0} - z)^{1/2},$$

where we choose that branch of the square root which is positive for real positive values of the radicand. This function is regular and single-valued in $|z| < 1$. Now, take a fixed angle of opening less than π with vertex in $e^{i\theta_0}$. In this angle

$$\frac{1}{M} < \frac{|e^{i\theta_0} - z|}{1 - |z|} < M$$

for a suitable positive constant M. Hence, by (8.13)

$$\lim_{r\to 1} g(re^{i\theta_0}) = 0,$$

while by (8.2) the function $g(z)$ is bounded in the fixed angle. By Lindelöf's theorem ([27]) $\lim_{z\to e^{i\theta_0}} g(z) = 0$ uniformly in every angle contained in the fixed angle. This proves the theorem.

9. **Example on the slowness of approach of $|f^{(k)}(z)|\ (1-|z|)^k$.** We have shown in Corollary 2, §5, that if the function $f(z)$ is bounded and univalent for $|z| < 1$, and also under various alternative conditions, then we have

$$\lim_{|z_n|\to 1} f'(z_n)(1 - |z_n|) = 0, \qquad |z_n| < 1. \tag{9.1}$$

Even for the class of bounded univalent functions, continuous in $|z| \leqq 1$, equation (9.1) cannot be improved by establishing results on rate of approach in equation (9.1) or by replacing the second factor by that factor raised to a suitable power. Indeed we shall prove that the limit in (9.1) can be approached arbitrarily slowly, in the sense of

THEOREM 7. *Let the function $Q(r)$ be defined and positive for $0 < r < 1$, with $\lim_{r\to 1} Q(r) = 0$. Then there exists a function $F(z)$ analytic and univalent interior to γ: $|z| = 1$, continuous for $|z| \leqq 1$, and there exists a sequence of points $z_1, z_2, \cdots$ interior to γ with $|z_n| = r_n \to 1$, such that we have*

$$\lim_{n\to\infty} \frac{F'(z_n)(1 - |z_n|)}{Q(|z_n|)} = \infty. \tag{9.2}$$

In fact, we shall choose $F(z)$ real for real z, and z_n real.

As a matter of convenience, we establish first Theorem 7 and then an extension of Theorem 7 to higher derivatives. The ensuing proof is given in preparation for the more general theorem, and is somewhat more complicated than is necessary for the proof of Theorem 7 alone.

We shall find useful a function analytic and univalent for $|z| < 1$ whose Taylor expansion about the origin has all of its coefficients positive. Such a function is

$$w_1 = f_1(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

which maps the region $|z| < 1$ smoothly onto the w_1-plane slit along the axis of reals from $-1/4$ to $-\infty$. The function

([27]) E. Lindelöf, Acta Societatis Scientiarum Fennicae, vol. 46 (1915).

$$w_2 = f_2(z) = \frac{f_1(\rho z)}{\rho} = z + 2\rho z^2 + 3\rho^2 z^3 + \cdots, \qquad 0 < \rho < 1,$$

then maps $|z| < 1$ smoothly onto a *Jordan region*(28) symmetric in the axis of reals. For definiteness we choose $\rho = 1/2$, and denote by J_0 the Jordan region of the w-plane which is the image of $|z| < 1$ under the map(29)

$$w = F_0(z) = z + \frac{2}{2} z^2 + \frac{3}{2^2} z^3 + \frac{4}{2^3} z^4 + \cdots. \tag{9.3}$$

Construct in the w-plane new Jordan regions $J_1, J_2, \cdots$ with the same shape and orientation as J_0, mutually exterior and exterior to J_0, with the analogue B_k for J_k of the point $w=0$ for J_0 lying on the axis of reals, so that the sequence $B_0 = 0, B_1, B_2, \cdots$ forms a monotonically increasing sequence. Choose moreover the region J_k just $(1/2^k)$th the size of J_0 in linear dimensions, and locate (as is possible) the sequence of regions J_k in such a way that their totality lies in some circle $|w| \leqq D$.

The region J_0 is symmetric in the axis of reals, so its boundary (an analytic Jordan curve) cuts that axis in precisely two points A_0 (to the left of the origin) and C_0 (to the right of the origin). Denote the analogous points for J_k by A_k and C_k. The boundary of J_k has a vertical tangent at both A_k and C_k.

A Jordan region R is to be constructed in the w-plane from the regions $J_0, J_1, J_2, \cdots$ by connecting each region to the preceding region by a canal; each of the two banks of such a canal shall be a segment of one of the lines $y = \pm d_k$, $d_k > 0$. Each point interior to J_k shall lie interior to R. The first canal, whose boundaries are segments of $y = \pm d_1$, joins J_0 in the neighborhood of C_0 with J_1 in the neighborhood of A_1; the second canal, whose boundaries are segments of $y = \pm d_2$, joins J_1 in the neighborhood of C_1 with J_2 in the neighborhood of A_2, and so on. The choice of the numbers d_k is now to be made more precise.

Denote by $w = F(z)$ the function which maps $|z| < 1$ onto R with $F(0) = 0$, $F'(0) > 0$; of course $F(z)$ depends on the numbers $d_1, d_2, \cdots$. Choose d_1 independently of $d_2, d_3, \cdots$ so small that the subset R_1 composed of all points of R not in J_0 corresponds under the transformation $w = F(z)$ to a set of points z interior to γ: $|z| = 1$ at which we have

$$Q(|z|) < 1/3. \tag{9.4}$$

(28) A Jordan region is any region bounded by a Jordan curve.

(29) It is sufficient for the purpose of both Theorem 7 and Theorem 8 to choose here a function $F_0(z)$ which maps $|z| < 1$ smoothly onto a Jordan region with $F_0(0) = 0$, $F_0'(0) = 1$, and has all of the coefficients of its Taylor expansion about the origin positive. For instance we may also choose

$$w = F_0(z) = \frac{2z}{2 - z} = z + \frac{1}{2} z^2 + \frac{1}{2^2} z^3 + \cdots$$

which maps $|z| < 1$ onto the interior of the circle $|w - 2/3| = 4/3$.

Such choice of d_1 is possible. For under the map $w=F(z)$ it follows from a theorem due to Lindelöf[30] that the subset R_1 is mapped into a set bounded in part by an arc of γ and whose remaining boundary (a Jordan arc) can be made as near to γ as desired. For the boundary points of R_1 not boundary points of R are the points of the boundary of J_0 in the neighborhood of the point C_0 between the lines $y=\pm d_1$; by choosing d_1 sufficiently small all such points can be made uniformly as near as desired to the boundary of R; so by Lindelöf's theorem all points of the boundary of the transform of R_1 (and hence all points of the transform of R_1 itself) can be made as near to γ as desired, and (9.4) is justified.

Similarly the number d_2 is to be chosen so small that all points of R not in J_0 or J_1 or in the canal joining J_0 and J_1 correspond under the map $w=F(z)$ to points interior to γ at which we have $Q(|z|)<1/9$; more generally the number d_k is to be chosen so that all points of R not in $J_0 \cdot J_1, \cdots, J_{k-1}$ or in the canals joining successive regions $J_0, J_1, \cdots, J_{k-1}$, correspond under the map $w=F(z)$ to points interior to γ at which we have

$$Q(|z|) < 1/3^k; \tag{9.5}$$

such successive choice of the numbers d_k is possible, again by Lindelöf's theorem. There are no further restrictions on the numbers d_k so far as the requirements of Theorem 7 itself are concerned. We now introduce the inner radius $\rho(w_0)$ of the region R with respect to the arbitrary point w_0 of R[31]. It is well known that $\rho(w_0)$ has a monotonic character with respect to R: if R is increased so also is $\rho(w_0)$; if R is stretched uniformly in the linear ratio $1:m$ with w_0 fixed, then $\rho(w_0)$ is multiplied by m; if R is the interior of a circle with center at w_0, the inner radius is the usual radius of this circle.

The inner radius of R with respect to the point B_k is greater than $1/2^k$, for it follows from (9.3) that the inner radius of J_0 with respect to B_0 is unity, so the inner radius of J_k with respect to B_k is $1/2^k$. On the other hand, if z_k denotes the point of $|z|<1$ which corresponds to the point B_k under the transformation $w=F(z)$, the inner radius of R with respect to B_k is $|F'(z_k)|(1-|z_k|^2)$, so we may write $|F'(z_k)|(1-|z_k)^2)>1/2^k$. From inequality (9.5) we have $Q(|z_k|)<1/3^k$, whence

$$\frac{|F'(z_k)|(1-|z_k|^2)}{Q(|z_k|)} > \frac{3^k}{2^k}, \tag{9.6}$$

from which (9.2) follows[32].

(30) Acta Societatis Scientiarum Fennicae, vol. 46 (1915). Or see Walsh, *Interpolation and Approximation*, §2.1. In applying Lindelöf's result it is essential to notice that the region R is bounded independently of the numbers d_k.

(31) Cf. §4, Footnote 10.

(32) In the proof of Theorem 7 we might equally well have used an example due to Szegö, Mathematische Zeitschrift, vol. 23 (1925), pp. 45–61; pp. 57–59. Szegö does not mention the

Under the present circumstances the region R is symmetric in the axis of reals, the numbers z_k are real, and $F'(z_k)$ is positive, so the absolute value signs may be removed from (9.6). Of course $F(z)$ is continuous in $|z|\leq 1$ (when suitably defined on $|z|=1$), as the mapping function for a Jordan region. The points B_k are real and positive and approach the boundary of R, so the points z_k are real and positive and approach the point $z=1$.

Theorem 7 shows that the limit in (9.1) can be approached arbitrarily slowly; by virtue of §4, Theorem 3, we may also say that $\lim_{|z_n|\to 1} D_1[f(z_n)]$ considered as a function of $1-|z_n|$ can also be approached arbitrarily slowly.

We now consider the generalization of Theorem 7 to higher derivatives:

THEOREM 8. *Let the function $Q(r)$ be defined and positive for $0<r<1$, with $\lim_{r\to 1} Q(r)=0$. Let the positive integer m be given. Then there exists a function $F(z)$ analytic and univalent interior to γ: $|z|=1$, continuous for $|z|\leq 1$, and a sequence of points $z_1, z_2, \cdots$ interior to γ with $|z_n|=r_n\to 1$, such that we have*

$$\lim_{n\to\infty} \frac{F^{(m)}(z_n)(1-|z_n|)^m}{Q(|z_n|)} = \infty. \tag{9.7}$$

Indeed, we shall choose $F(z)$ real for real z, and z_n real.

In the proof of Theorem 8 we use precisely the region R introduced in the proof of Theorem 7, with further restrictions on the numbers d_k; the function $F(z)$ is, as before, the mapping function.

It follows from equation (9.3) that the function

$$w = F_k(z) = b_k + \frac{1}{2^k}\left[z + \frac{2}{2}z^2 + \frac{3}{2^2}z^3 + \frac{4}{2^3}z^4 + \cdots\right] \tag{9.8}$$

maps $|z|<1$ onto the region J_k in such a way that the point $z=0$ corresponds to the point B_k: $w=b_k$ with the axis of reals in one plane corresponding to the axis of reals in the other plane. The function

$$w(\zeta) = F\left(\frac{z_k+\zeta}{1+z_k\zeta}\right) = b_k + a_1^{(k)}\zeta + a_2^{(k)}\zeta^2 + a_3^{(k)}\zeta^3 + \cdots, \tag{9.9}$$

where $F(z_k)=b_k$, maps $|\zeta|<1$ onto R so that $\zeta=0$ corresponds to the point B_k with the axis of reals in the one plane corresponding to the axis of reals in the other. When d_k and d_{k+1} approach zero, the kernel in the sense of Cara-

property (9.2), nor does Sewell, but the latter (these Transactions, vol. 41 (1937), pp. 84–123) mentions for Szegö's region the relation (notation of §1) $\lim_{k\to\infty} D_1(w_k)/Q(|z_k|)=\infty$, $w_k=F(z_k)$, which by virtue of the inequality $|F'(z_n)|(1-|z_n|^2)\geq D_1(w_n)$ implies (9.2). Szegö's example does not seem to apply at once to higher derivatives.

The method of proof of Theorem 7 has also been employed by Walsh, Bulletin of the American Mathematical Society, vol. 46 (1940), pp. 101–108, for a somewhat different purpose.

théodory(33) of the variable region R, considered with B_k as central point (that is, Aufpunkt) is precisely the region J_k. It follows from the results of Carathéodory (loc. cit.) that the corresponding mapping function $w(\zeta)$ defined by (9.9) approaches the function $F_k(\zeta)$ defined by (9.8), throughout the interior of $|\zeta|<1$, uniformly on any closed point set interior to $|\zeta|<1$. Indeed, such uniform approach of $w(\zeta)$ defined by (9.9) to $F_k(\zeta)$ is a consequence of the approach to zero of d_k and d_{k+1}, *independently of the behavior of* $d_1, d_2, \cdots, d_{k-1}, d_{k+2}, d_{k+3}, \cdots$. Otherwise there would exist a sequence of sequences of numbers $d_1, d_2, \cdots$ with d_k and d_{k+1} approaching zero and the corresponding function $w(\zeta)$ in (9.9) not approaching $F_k(\zeta)$ as defined by (9.8); this is impossible. Thus the coefficient $a_j^{(k)}$, considered as a function of d_k and d_{k+1} alone, approaches the corresponding coefficient $j/2^{j+k-1}$.

The inner radius $\rho(b_k)$ of R with respect to the point B_k is greater than $1/2^k$, so in (9.9) we have

$$a_1^{(k)} > 1/2^k. \tag{9.10}$$

We have already made restrictions on the numbers d_k in connection with Theorem 7. We now impose the further restriction that $d_1, d_2, \cdots$ are to be chosen in pairs $(d_1, d_2), (d_2, d_3), (d_3, d_4), \cdots$ successively so small that we always have the inequalities $(k=1, 2, 3, \cdots)$

$$a_2^{(k)} > 0,\ a_3^{(k)} > 0, \cdots, a_m^{(k)} > 0; \tag{9.11}$$

this choice of the d_k is possible. We have no other restrictions to be placed on the numbers d_k.

By Lemma 1 of §2 we now have

$$\frac{(1-|z_k|^2)^m}{m!}F^{(m)}(z_k) = \sum_{\nu=0}^{m-1} C_{m-1,\nu}\bar{z}_k^{\nu}\frac{w^{(m-\nu)}(0)}{(m-\nu)!},$$

where $w(\zeta)$ is defined by (9.9). Inequalities (9.11) and (9.10) now yield $(z_k=\bar{z}_k>0)$

$$\frac{(1-|z_k|^2)^m}{m!}F^{(m)}(z_k) > z_k^{m-1}a_1^{(k)} > \frac{z_k^{m-1}}{2^k}, \qquad k>1,$$

so, as in (9.6), we write from (9.5)

$$\frac{(1-|z_k|^2)^m F^{(m)}(z_k)}{Q(|z_k|)} > m!\cdot z_k^{m-1}\cdot\frac{3^k}{2^k}, \qquad z_k=|z_k|.$$

When k becomes infinite, the point z_k approaches the point $z=1$, so equation (9.7) and Theorem 8 follow.

As will be seen, this function $F(z)$ is significant as a "Gegenbeispiel" also in some of our later theorems.

(33) Cf. Footnote 13.

CHAPTER II. BOUNDED FUNCTIONS: CONFIGURATIONS C_p AND D_p

The problem which will occupy us in this chapter and the next is to what extent the results of the first chapter can be extended to the class of bounded functions.

It should be remarked at the start that in §5, Corollary 2, it is not possible to drop the condition of univalence. Indeed we have

THEOREM 1. *There exists a function $f(z)$ regular and bounded in $|z|<1$ and a sequence of points z_n ($|z_n|<1$), $|z_n|\to 1$, for which*

$$\liminf_{n\to\infty} |f'(z_n)|(1-|z_n|) > 0.$$

That $|f'(z_n)|(1-|z_n|)$ is always bounded when $f(z)$ is regular and bounded in $|z|<1$, follows from an easy application of Schwarz's lemma([34]).

To prove Theorem 1 we consider the function([35])

$$f(z) = \exp\left[\frac{z+1}{z-1}\right].$$

It is clear that since $\Re[(z+1)/(z-1)]<0$ in $|z|<1$, we have $|f(z)|<1$ in $|z|<1$. Now,

$$|f'(re^{i\theta})|(1-r^2) = 2\frac{1-r^2}{1-2r\cos\theta+r^2}\cdot\exp\left[\frac{-1+r^2}{1-2r\cos\theta+r^2}\right].$$

Along the curve $r=\cos\theta$ which passes through the point $z=1$ and is tangent there to the unit circle

$$|f'(re^{i\theta})|(1-r^2) = 2/e$$

so that as $\theta\to 0$, the corresponding limit is $2/e>0$.

10. **A lower bound on $D_1(w)$.** In order to obtain the conclusion $\lim_{n\to\infty}|f'(z_n)|(1-|z_n|)=0$, it is necessary to limit oneself to particular sequences $\{z_n\}$ in the circle $|z|<1$. By Theorem 1 our result is as follows:

THEOREM 2. *Let $f(z)$ be regular and bounded in $|z|<1$:*

$$|f(z)| \leqq M,$$

let $\{z_n\}$ be any sequence of points in $|z|<1$, and let $w_n=f(z_n)$. Then, a necessary and sufficient condition for

$$\lim_{n\to\infty}|f'(z_n)|(1-|z_n|) = 0$$

is that $\lim_{n\to\infty} D_1(w_n)=0$.

([34]) Cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 2d edition, 1931, p. 112.

([35]) For this particularly simple example the authors are indebted to Professor G. Szegö.

This condition will follow directly from the more precise

THEOREM 3. *Let $f(z)$ be regular and bounded in $|z|<1$:*

$$|f(z)| \leqq M,$$

let z_0 be any point in $|z|<1$, and let $w_0=f(z_0)$. Then, the following inequality

$$D_1(w_0) \leqq |f'(z_0)|(1-|z_0|^2) \leqq [8MD_1(w_0)]^{1/2} \tag{10.1}$$

is always satisfied.

The first inequality in (10.1) is simply a particular case of §4, Theorem 2. It, therefore, remains to prove the second inequality alone.

It was proved by Landau and Dieudonné[36] that if

$$w = g(z) = z + \cdots$$

is a regular function in $|z|<1$ satisfying the inequality

$$|g(z)| \leqq M \quad \text{for} \quad |z| < 1,$$

then $g(z)$ is univalent in the circle $|z|<1/2M$ and covers simply the circle $|w| \leqq 1/4M$.

Consider now the function

$$\phi(z) = \frac{f((z+z_0)/(1+\bar{z}_0z)) - f(z_0)}{f'(z_0)(1-|z_0|^2)} = z + \cdots .$$

In $|z|<1$ the function $\phi(z)$ is regular and satisfies the inequality

$$|\phi(z)| \leqq \frac{2M}{|f'(z_0)|(1-|z_0|^2)}.$$

Hence, in accordance with the theorem of Landau and Dieudonné $w=\phi(z)$ covers simply the circle

$$|w| \leqq \frac{|f'(z_0)|(1-|z_0|^2)}{8M}.$$

The function $w=f(z)$, therefore, covers simply the circle

$$|w-w_0| \leqq \frac{|f'(z_0)|^2(1-|z_0|^2)^2}{8M}, \qquad w_0 = f(z_0).$$

From this it follows that

(36) E. Landau, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin, Physikalisch-Mathematische Klasse, (1926), pp. 467–474; J. Dieudonné, Annales de l'École Normale Supérieure, (3), vol. 48 (1931), pp. 247–358.

$$D_1(w_0) \geqq \frac{|f'(z_0)|^2(1-|z_0|^2)^2}{8M},$$

which is merely another form of the second inequality (10.1)([37]).

While the constant 8 in (10.1) is not the best possible, the order $[D_1(w_0)]^{1/2}$ as $D_1(w)\to 0$ cannot be improved, as may be seen from a study in $|z|<1$ of the function

$$f(z) = \frac{Mz(1-Mz)}{M-z}, \qquad M > 1,$$

previously considered by J. Dieudonné([38]) in the neighborhood of the point $z=M-[M^2-1]^{1/2}$. Indeed, let z be any point of the unit circle, lying on the real axis, such that $0<z<M-[M^2-1]^{1/2}$. It is seen by direct computation that

$$|f'(z)|(1-z^2) = M^2\frac{(1-2Mz+z^2)(1-z^2)}{(M-z)^2}$$

$$(10.2)\qquad = \frac{M^2(1-z^2)}{(M-z)^2}[z-(M-(M^2-1)^{1/2})][z-(M+(M^2-1)^{1/2})].$$

We set $w_0=f(M-(M^2-1)^{1/2})=M(M-(M^2-1)^{1/2})^2$. Hence, since $D_1(w)=w_0-w$,

$$(10.3)\qquad D_1(w) = \frac{M^2}{M-z}[z-(M-(M^2-1)^{1/2})]^2.$$

Comparison of the equations (10.2) and (10.3) shows that as $w\to w_0$ $|f'(z)|(1-z^2)=O((D_1(w))^{1/2})$, but $|f'(z)|(1-z^2)\neq o((D_1(w))^{1/2})$.

11. **Irregular sequences.** The question now arises whether one may generalize Theorem 3 to higher derivatives in the same manner as Theorems 4 and 5 generalize Theorem 3 in Chapter I. In the present case, however, the situation is more complicated than in the case of univalent functions, as examples (§12) will show. Before giving the examples it will be desirable to give some definitions and prove two theorems. Being given two points z_1 and z_2 of the unit circle $|z|<1$, we define as in §7 the non-euclidean distance $\rho(z_1, z_2)$ between them([39]).

DEFINITION 1. *A sequence of points* $\{z_n\}$, $(|z_n|<1)$, $z_n\to 1$, *will be called a regular sequence for a function* $f(z)$ *analytic in* $|z|<1$ *if there exists a number*

([37]) It will be observed from the above that it might be of advantage sometimes to replace the right-hand side of (10.1) by $[4M'D_1(w_0)]^{1/2}$ where M' is the least upper bound of $|f((z+z_0)/(1+\bar{z}_0z))-f(z_0)|$ for $|z|<1$ and z_0 fixed.

([38]) J. Dieudonné, ibid.

([39]) For the notions of non-euclidean geometry particularly in their relation to the theory of functions, cf. G. Julia, *Principes Géométriques d'Analyse*, Première Partie, 1930, especially Chapters II and IV.

$\lambda > 0$ *such that for any sequence of points* $\{z_n'\}$ *whose non-euclidean distance* $\rho(z_n, z_n')$ *is less than* λ *for all n we have*

$$\lim_{n\to\infty} [f(z_n) - f(z_n')] = 0.$$

A sequence of points $\{z_n\}$ *which is not regular will be called irregular.*

DEFINITION 2. *A sequence of points* $\{z_n\}$, $(|z_n| < 1)$, $z_n \to 1$, *will be called a quasi-regular sequence of order m for a function* $f(z)$ *analytic in* $|z| < 1$ *if*

$$\lim_{n\to\infty} |f^{(k)}(z_n)| (1 - |z_n|)^k = 0, \quad \textit{for } k = 1, 2, \cdots, m,$$

while

$$\limsup_{n\to\infty} |f^{(m+1)}(z_n)| (1 - |z_n|)^{m+1} > 0.$$

The case $m = \infty$ *is allowed and means that* $\lim_{n\to\infty} |f^{(k)}(z_n)| (1-|z_n|)^k = 0$ for $k = 1, 2, \cdots$.

Denote by Γ_n^λ the non-euclidean circle of non-euclidean radius λ and non-euclidean center z_n. We prove now the following

THEOREM 4. *An irregular sequence* $\{z_n\}$ *for a function* $f(z)$ *regular and bounded in* $|z| < 1$ *is quasi-regular of order m if to every sufficiently small positive* λ *there corresponds an integer* $N(\lambda) > 0$ *such that for all* $n > N(\lambda)$ *the function* $f(z)$ *assumes the value* $f(z_n)$ *exactly* $m+1$ *times in the circle* Γ_n^λ *(counting multiplicities).*

Consider the function

$$g_n(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n\zeta}\right) - f(z_n). \tag{11.1}$$

By hypothesis, for $n > N(\lambda)$ the function $g_n(\zeta)$, which is regular and bounded in $|\zeta| < 1$, assumes the value 0 exactly $m+1$ times in the circle $|\zeta| < (e^\lambda - 1)/(e^\lambda + 1)$. Now, the sequence $\{z_n\}$ is assumed to be irregular. In accordance with Definition 1 this means that for any $\lambda > 0$ we can find a subsequence of the $\{z_n\}$, which we shall denote by $\{z_{n_k}\}$, and a sequence $\{z_{nk}'\}$ such that $\rho(z_{n_k}, z_{n_k}') < \lambda$ and for some $\delta > 0$ we have $|f(z_{n_k}) - f(z_{n_k}')| \geqq \delta$. This implies, however, that the sequence (11.1) cannot tend uniformly to zero in every closed subregion of $|\zeta| < 1$. Indeed, suppose that $\lim_{n\to\infty} g_n(\zeta) = 0$ uniformly in every closed subregion of $|\zeta| < 1$. To any preassigned $\epsilon > 0$ there would correspond a positive integer $n(\epsilon)$ so that for $n > n(\epsilon)$ we would have $|g_n(\zeta)| < \epsilon$ in $|\zeta| < (e^\lambda - 1)/(e^\lambda + 1)$. Setting $\zeta_{n_k} = (z_{nk}' - z_{n_k})/(1 - \bar{z}_{n_k} z_{nk}')$ we would infer that $|g_{n_k}(\zeta_{n_k})| < \epsilon$ for $n > n(\epsilon)$. Replacing this inequality in (11.1), we find $|f(z_{n_k}') - f(z_{n_k})| < \epsilon$ for $n > n(\epsilon)$. If we choose $\epsilon < \delta$, we arrive at a contradiction.

Hence, there exists([40]) a subsequence of the sequence $\{g_n(\zeta)\}$, which we shall denote by $\{g_{n_k}(\zeta)\}$, which converges uniformly in every closed subregion of $|\zeta|<1$ to a function $G(\zeta)$ which is not identically zero, and (since $G(0)=0$) is not identically a constant. The function $G(\zeta)$ is regular in $|\zeta|<1$.

Since $G(\zeta)$ is not identically zero, there must exist a $0<\lambda_1<\lambda$ so that $G(\zeta)\neq 0$ on the circle $|\zeta|=(e^{\lambda_1}-1)/(e^{\lambda_1}+1)$. Since, furthermore, the sequence $g_{n_k}(\zeta)$ converges uniformly to $G(\zeta)$ on that circle, for sufficiently large values of n_k we have $g_{n_k}(\zeta)\neq 0$ on $|\zeta|=(e^{\lambda_1}-1)/(e^{\lambda_1}+1)$. Now, by hypothesis $g_{n_k}(\zeta)$ vanishes precisely $m+1$ times in the circle $|\zeta|<(e^{\lambda_1}-1)/(e^{\lambda_1}+1)$ provided $n_k>N(\lambda_1)$. Hence, by Hurwitz's theorem $G(\zeta)$ vanishes precisely $m+1$ times in the circle $|\zeta|<(e^{\lambda_1}-1)/(e^{\lambda_1}+1)$. But since λ may be taken arbitrarily small, $G(\zeta)$ must have a zero of order $m+1$ at the origin. Hence, $G'(0)=0$, $G''(0)=0, \cdots, G^{(m)}(0)=0$, $G^{(m+1)}(0)\neq 0$. In view of (11.1) and (2.2), we see from the relations $g'_{n_k}(0)\to 0$, $g''_{n_k}(0)\to 0, \cdots, g^{(m)}_{n_k}(0)\to 0$, $g^{(m+1)}_{n_k}(0)\to G^{(m+1)}(0)$ that

$$\limsup_{n\to\infty} |f^{(m+1)}(z_n)|(1-|z_n|)^{m+1} > 0.$$

On the other hand, suppose that for some integer $0<k<m+1$ and for some subsequence $\{z_{n'}\}$ of the $\{z_n\}$

$$|f^{(k)}(z_{n'})|(1-|z_{n'}|)^k > \eta > 0 \tag{11.2}$$

for a suitable positive η, independent of n'. Consider the corresponding subsequence $\{g_{n'}(\zeta)\}$ of the sequence (11.1). By selecting a further subsequence, if necessary, we may assume that the sequence $\{g_{n'}(\zeta)\}$ is a uniformly convergent one in every closed subregion of $|\zeta|<1$. Two cases are possible according as $\{g_{n'}(\zeta)\}$ converges to zero or to some function not identically a constant. In the first case, the derivatives of all orders of $g_{n'}(\zeta)$ also converge to zero and application of formula (2.2) for the case $n=k$ shows that $|f^{(k)}(z_{n'})|(1-|z_{n'}|)^k\to 0$, which is a contradiction of (11.2). In the second case, the nonconstant limit function $G(\zeta)$ of the sequence $g_{n'}(\zeta)$ by the argument already given must have a zero of order $m+1$ at the origin, so that all its derivatives up to the $(m+1)$st must vanish at the origin. Application of formula (2.2) again contradicts (11.2). Thus, in both cases (11.2) yields a contradiction. Hence,

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k=1, 2, \cdots, m,$$

and the sequence $\{z_n\}$ is quasi-regular of order m.

We proceed to prove some related results.

([40]) This follows from the fact that the functions $g_n(\zeta)$, being uniformly bounded in their totality in $|\zeta|<1$, form a normal family, cf. P. Montel, *Leçons Sur les Familles Normales de Fonctions Analytiques*, 1927, p. 21.

THEOREM 5. *A necessary and sufficient condition that* $\{z_n\}$ *be a regular sequence for a function* $f(z)$ *regular in* $|z|<1$ *and bounded there:* $|f(z)|\leqq M$, *is that it be quasi-regular of infinite order.*

The condition is necessary. Indeed, form the functions

$$g_n(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n\zeta}\right) - f(z_n). \tag{11.3}$$

This sequence of functions is uniformly bounded in $|\zeta|<1$. From every subsequence can be extracted a new subsequence whose uniform limit is zero in every circle $\rho(\zeta,0)<\lambda$, where λ is the number of Definition 1. It follows that the sequence $g_n(\zeta)$ converges uniformly to zero in the circle $|\zeta|\leqq(e^\rho-1)/(e^\rho+1)$, $\rho<\lambda$. Lemma 1 of §2 shows that $|f^{(k)}(z_n)|\ (1-|z_n|)^k\to 0$ for $k=1, 2, \cdots$.

The condition is sufficient. Again form the functions (11.3). Since $|g_n(\zeta)|\leqq 2M$ in $|\zeta|<1$, the functions form a normal family. A suitable subsequence converges uniformly in every circle $|\zeta|\leqq d<1$ to some function $G(\zeta)$ which is regular and bounded in $|\zeta|<1$ and $G(0)=0$. Expanding $G(\zeta)$ in a Taylor series about $\zeta=0$:

$$G(\zeta) = c_1\zeta + c_2\zeta^2 + \cdots, \tag{11.4}$$

applying Lemma 2 of §2 and the hypothesis that $\{z_n\}$ is quasi-regular of infinite order, we see that all the coefficients in the expansion (11.4) are zero and that therefore $G(\zeta)\equiv 0$.

Since we may repeat this argument starting with any subsequence of the family $\{g_n(\zeta)\}$, it follows that $g_n(\zeta)\to 0$ uniformly in any circle $|\zeta|\leqq d<1$. From this follows at once the fact that $\{z_n\}$ is a regular sequence for $f(z)$.

A type of converse of Theorem 4 may be stated in the following form:

THEOREM 6. *Let* $f(z)$ *be regular and bounded in the unit circle* $|z|<1$: $|f(z)|\leqq M$. *Let the sequence* $\{z_n\}$ *be quasi-regular of order* m. *Then for every subsequence of the* $\{z_n\}$ *there exists a new subsequence* $\{z_{n_k}\}$ *with the property that to every* $\rho>0$ *which is sufficiently small there corresponds an integer* $N(\rho)>0$ *such that for all* $n_k>N(\rho)$ *the function* $f(z)$ *assumes the value* $f(z_{n_k})$ *precisely* $m+1$ *times in the circle* Γ_n^ρ *(counting multiplicities).*

Again we form the functions (11.3). In view of Lemma 2 of §2 we have

$$\frac{g_n^{(p)}(0)}{p!} = \sum_{\nu=0}^{p-1}(-1)^\nu C_{p-1,\nu}\bar{z}_n^\nu\frac{(1-|z_n|^2)^{p-\nu}f^{(p-\nu)}(z_n)}{(p-\nu)!}.$$

The hypothesis that $\{z_n\}$ is a quasi-regular sequence of order m for $f(z)$ implies that

$$\lim_{n\to\infty} g_n^{(p)}(0) = 0, \qquad \text{for } p = 1, 2, \cdots, m, \tag{11.5}$$

while

$$\limsup_{n\to\infty} \left| g_n^{(m+1)}(0) \right| > 0. \tag{11.6}$$

Let us select a subsequence of the family $\{g_n(\zeta)\}$ for which the actual limit in (11.6) exists and is positive and denote this subsequence for simplicity by $\{g_n(\zeta)\}$ again. Since for all n we have $|g_n(\zeta)| \leqq 2M$ in $|\zeta| < 1$, the sequence $\{g_n(\zeta)\}$ is a normal family. We may therefore extract a further subsequence $\{g_{n_k}(\zeta)\}$ which in every closed subregion of the circle $|\zeta| < 1$ converges uniformly to a function $G(\zeta)$. According to (11.5) and (11.6) we obtain $G^{(p)}(0) = 0$ for $p = 1, 2, \cdots, m$ and $G^{(m+1)}(0) \neq 0$. Since $G(0) = 0$, it follows that for every $\rho > 0$ which is sufficiently small the function $G(\zeta)$ has precisely $m+1$ zeros in the circle $|\zeta| < \rho$ and is different from zero on the circumference $|\zeta| = \rho$. Let us fix a definite value of ρ. By Hurwitz's theorem it follows that there exists an integer $N(\rho) > 0$ so that each function $g_{n_k}(\zeta)$ for which $n_k > N(\rho)$ has precisely $m+1$ zeros in the circle $|\zeta| < \rho$. The theorem then follows immediately from the definition (11.3) of $g_n(\zeta)$.

12. **Counterexamples (Gegenbeispiele).** Theorem 4 may be used to obtain an example in which $D_1(w_n) = 0$, while $|f''(z_n)| (1 - |z_n|)^2$ does not tend to zero. Indeed, consider the Blaschke product(41)

$$\phi(z) = \prod_{n=1}^{\infty} z_n \frac{1 - z/z_n}{1 - z_n z}, \qquad z_n = \frac{n! - 1}{n! + 1}. \tag{12.1}$$

As is well known(42), since $\prod_{n=1}^{\infty}(n!-1)/(n!+1)$ converges, the product (12.1) represents in the circle $|z| < 1$ an analytic function whose absolute value is less than unity. As was shown by one of the authors(43), the sequence $\{z_n\}$ is an irregular sequence for $\phi(z)$. Now, form

$$f(z) = [\phi(z)]^2.$$

Again, the sequence $\{z_n\}$ is an irregular sequence for $f(z)$. Furthermore, since the z_n are zeros of order 2 and the only zeros of $f(z)$, we have $D_1(0) = 0$ when the point $w = 0$ is considered in any sheet of the Riemann configuration for $w = f(z)$. On the other hand, the non-euclidean distance $\rho(z_n, z_{n+1}) = \log(n+1) \to \infty$. Hence, for any $\lambda > 0$ and for sufficiently large values of n the function $f(z)$ vanishes precisely twice in Γ_n^λ. Applying Theorem 4, therefore, we find

$$\limsup_{n\to\infty} |f''(z_n)| (1 - |z_n|)^2 > 0.$$

(41) Such products were first introduced by W. Blaschke, Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften, Mathematisch-Physische Klasse, Leipzig, vol. 67 (1915), pp. 194–200.

(42) Cf. G. Julia, ibid., pp. 65–66.

(43) W. Seidel, these Transactions, vol. 34 (1932), pp. 14–15. Equation (7.2) there should read

$$\phi(z) = \prod_{n=1}^{\infty} t_n \frac{1 - z/t_n}{1 - t_n z}.$$

Let us state this example as a theorem:

THEOREM 7. *There exists a bounded regular function* $f(z)$ *in* $|z|<1$ *and a sequence of points* $\{z_n\}$ $(|z_n|<1, |z_n|\to 1)$ *in* $|z|<1$ *such that, setting* $w_n=f(z_n)$, $\lim_{n\to\infty} D_1(w_n)=0$, *while* $\limsup_{n\to\infty} |f''(z_n)|\,(1-|z_n|)^2>0$.

Indeed, for the specific example already given we may assert $f'(z_n)=0$, $D_1(w_n)=0$.

The converse situation may also arise:

THEOREM 8. *There exists a bounded regular function* $f(z)$ *in* $|z|<1$ *and a sequence of points* $\{z_n\}$ $(|z_n|<1, |z_n|\to 1)$ *in* $|z|<1$ *such that, setting* $w_n=f(z_n)$, *we have* $\liminf_{n\to\infty} D_1(w_n)>0$, *while* $\lim_{n\to\infty} |f''(z_n)|\,(1-|z_n|)^2=0$.

Let

$$\phi(z) = (1 - e^{-W+1})^2, \qquad W = \frac{1+z}{1-z}\cdot$$

The function $W=(1+z)/(1-z)$ maps the circle $|z|<1$ on the half-plane $\Re W>0$. Now, for $\Re W>0$

$$|e^{-W+1}| = \exp(-\Re W + 1) < e.$$

Hence, for $|z|<1$, $|\phi(z)|<(1+e)^2$.

Direct computation shows that

$$\phi'(z) = -4e^{-W+1}(1-e^{-W+1})\cdot\frac{1}{(1-z)^2},$$

$$\phi''(z) = \frac{8}{(1-z)^3}e^{-W+1}(e^{-W+1}-1) + \frac{8}{(1-z)^4}e^{-W+1}(1-2e^{-W+1}).$$

We now choose the points $z_n=n\pi i/(n\pi i+1)$. It is clear that $|z_n|<1$ and $\lim_{n\to\infty} z_n=1$. Setting $W_n=(1+z_n)/(1-z_n)$, we find $W_n=1+2n\pi i$. Hence

$$|\phi'(z_n)|\,(1-|z_n|) = 0, \quad \phi''(z_n)(1-|z_n|^2)^2 \to 8 \quad \text{as} \quad n\to\infty. \tag{12.2}$$

This incidentally gives another example for the proof of Theorem 7, since the relation $D_1(w_n)\to 0$ follows from (12.2) and (10.1).

Next, we introduce the function

$$\psi(z) = e^{-W+1}, \qquad W = \frac{1+z}{1-z}\cdot$$

Again, we observe that in $|z|<1$ the function $\psi(z)$ is bounded:

$$|\psi(z)| < e.$$

Choosing again $z_n=n\pi i/(n\pi i+1)$, we find

$$|\psi'(z_n)|(1-|z_n|^2) = 2, \qquad \text{for all } n, \tag{12.3}$$

and

$$\psi''(z_n)\cdot(1 - |z_n|^2)^2 \to 4 \quad \text{as} \quad n \to \infty. \tag{12.4}$$

Finally, we introduce the function

$$f(z) = \phi(z) + 2\psi(z).$$

It is clear that $f(z)$ is bounded in the circle $|z| < 1$, satisfying there the inequality

$$|f(z)| < (1 + e)^2 + 2e.$$

For the sequence of points $z_n = n\pi i/(n\pi i + 1)$ by virtue of (12.2), (12.3), (12.4) we have the relations

$$|f'(z_n)|(1 - |z_n|^2) = 4 \neq 0, \qquad \text{for all } n,$$

and

$$f''(z_n)(1 - |z_n|^2)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

It follows from Theorem 3 that $D_1(w_n)$ has a positive lower bound. The function $f(z)$ is therefore an example of a function with the properties asserted in Theorem 8, and Theorem 8 is established.

By forming the function $f(z) = a\phi(z) + b\psi(z)$ with arbitrary constants a and b, one can now obtain arbitrary limits for $|f'(z_n)|(1 - |z_n|^2)$ and $|f''(z_n)|(1 - |z_n|^2)^2$ as $n \to \infty$.

It may be observed that if the condition $|z_n| \to 1$ in Theorems 7 and 8 were dropped, one might take as examples to prove the theorems the simple functions $f(z) = z^2$ and $f(z) = z$, $z_n = 0$, respectively.

Finally, it may be noted that in the Theorems 4–6, the boundedness of $f(z)$ was assumed merely in order to ensure the normality of the family $g_n(\zeta)$. Thus, it would have sufficed to assume that $f(z)$ has 2 exceptional values and $f(z_n)$ is bounded.

13. Definition and some properties of C_p.

DEFINITION 3. *Let C_p be a simply connected Riemann configuration containing the point w_0, lying over the circle $|w - w_0| < \rho$ and covering it precisely p times. Such a region C_p will be called a p-sheeted circle of center w_0 and radius ρ.*

We shall exclude the case $\rho = \infty$ (called an improper p-sheeted circle) for a reason that will be given a little later. It should be observed that the center of a p-sheeted circle is not uniquely defined.

The necessity of assuming explicitly (rather than proving) in Definition 3 that C_p shall be simply connected may be seen from the following example. Consider in the w-plane the (simply connected) Riemann surface of the function $((w - \alpha)/(w - \beta))^{1/2}$ where α and β are two complex numbers, with the branch line the rectilinear segment $\alpha\beta$. Let us now cut this surface by a circular biscuit-cutter which includes the two points α and β. The resulting circular region cut out of the surface satisfies all the requirements in Definition 3 except the condition of simple connectivity. In fact, every region lying over a circle $|w-w_0|<\rho$ and covering it precisely twice ceases to be simply connected as soon as it has two branch points or more. Indeed, in such a case it is clearly possible to find a cut joining two boundary points and crossing a branch line which will not sever the surface. In general, by applying the theorem of Bôcher and Walsh (as in the proof of Theorem 13 below) one may easily show that every region lying over a circle $|w-w_0|<\rho$ and covering it precisely p times ceases to be simply connected as soon as the sum of the multiplicities of its branch points exceeds $p-1$. The multiplicity of a branch point is to be understood as one less than the number of sheets which come together at that point. An algebraic branch point (but not a transcendental one) is to be considered as belonging to the Riemann configuration.

One may prove some immediate consequences of Definition 3.

THEOREM 9. *Any p-sheeted circle over the w-plane can be mapped in a one-to-one conformal manner on the unit circle $|z|<1$.*

According to the fundamental theorem of uniformization the p-sheeted circle C_p, being simply connected, may be mapped in a one-to-one conformal manner either on a circle, or on the full plane, or on the full plane from which the point at infinity is excluded. Denote the mapping function by $w=f(z)$. Since C_p is a bounded region, the function $f(z)$ must be bounded. This is certainly not possible in the two latter cases. Thus, C_p can be mapped only on a circle.

THEOREM 10. *A p-sheeted circle C_p with center w_0 and radius ρ can be mapped in a one-to-one conformal manner on the unit circle $|z|<1$ by means of a function of the form*

$$f(z) = w_0 + \rho e^{i\theta} z^k \prod_{j=1}^{p-k} \frac{z - z_j}{1 - \bar{z}_j z}, \tag{13.1}$$

where θ is an arbitrary real number, k an integer satisfying the inequality $0<k\leqq p$, and where $z_1, z_2, \cdots, z_{p-k}$ are points of the unit circle $|z|<1$. Conversely, every function of the form (13.1) *realizes a one-to-one conformal map of the unit circle $|z|<1$ on some p-sheeted circle with center at w_0 and radius ρ.*

In speaking of conformality, it must be remembered that it will break down at a branch point. To prove the first part of the theorem introduce a similarity transformation in the w-plane with center in w_0 which transforms the circle C_p into a p-sheeted circle C_p' of radius 1. By means of a translation we can always bring the point w_0 into the origin. The resulting one-to-one map of $|z|<1$ on C_p' can be interpreted as a $(1, p)$ conformal corre-

[44] T. Radó, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Szeged, vol. 1 (1922), p. 55. Preliminary related work is due also to Fatou and Julia.

spondence of a unit circle on itself. By applying Radó's theorem([44]) on the representation of such correspondences we obtain the expression (13.1). The converse may also be derived from Radó's theorem together with a translation and similarity transformation in the w-plane.

A remark will now be made to justify the exclusion of the case $\rho=\infty$ in the definition of C_p. An improper p-sheeted circle could be interpreted as the w-plane covered precisely p times. If such a circle belonged to a simply connected Riemann surface, the surface could not be of hyperbolic type and consequently Theorems 9 and 10 would no longer apply. Suppose first that a simply connected Riemann surface which contains an improper p-sheeted circle could be mapped conformally on the unit circle. Thereby the p-sheeted circle would be transformed into a simply connected subregion of the unit circle. Now if the improper p-sheeted circle has no boundary points such a transformation is clearly impossible. Suppose then that the p-sheeted circle has the point $w=\infty$ as a boundary point. Then, the mapping function in the unit circle approaches infinity whenever the point z approaches the boundary of the subregion. This is again impossible.

THEOREM 11. *Let C_p be a p-sheeted circle with center at w_0 and radius R. Let c_p be a subregion of C_p which lies over a circle $|w-w_0|<r$, where $r<R$, and covers it precisely p times. Then, c_p is also simply connected.*

We can map C_p on the unit circle $|z|<1$ in accordance with Theorem 9. The mapping function $w=f(z)$ is regular in $|z|<1$ and maps c_p on a certain subregion B of $|z|<1$. On the boundary Γ of B we have $|f(z)-w_0|=r$, while in the interior of B we have $|f(z)-w_0|<r$. From the maximum modulus principle it follows that B is simply connected. Since the map defined by $w=f(z)$ is topological, the image of c_p of B must likewise be simply connected.

In order to establish the uniqueness in Definition 3, we shall prove

THEOREM 12. *Let R be a simply connected Riemann surface of hyperbolic type. Let w_0 be a point of R. Let C_p and C_p' be two p-sheeted circles with center at w_0 and radius ρ. Then, C_p and C_p' are identical.*

If we map R on the unit circle $|z|<1$ by means of the function $w=f(z)$ so that $f(0)=w_0$, the two circles C_p and C_p' will be mapped on two regions B and B' belonging to the circle $|z|<1$. In the interiors of B and B' we have $|f(z)-w_0|<\rho$ and on the boundaries $|f(z)-w_0|=\rho$. Furthermore, both regions B and B' contain the origin. Thus, unless B and B' are identical at least one boundary point of one region, say B, will be interior to the other region B'. This, however, constitutes a contradiction.

14. **Definition of D_p.**

DEFINITION. *Let $w=f(z)$ regular in the unit circle $|z|<1$ map the circle on a Riemann configuration R. That is to say, R is an arbitrary simply connected Riemann configuration of hyperbolic type over the finite w-plane. Let w_0 be an* *arbitrary point belonging to R. A non-negative number $D_p(w_0)$, called the radius of p-valence of R at the point w_0, shall be associated with the point w_0 in the following manner:*

(a) *For $p=1$, we define $D_p(w_0)=D_1(w_0)$ (see §1).*

(b) *If there exists a p-sheeted circle with center w_0 contained in R, there exists a largest such circle, and the radius of this largest circle is defined as $D_p(w_0)$.*

(c) *If $p>1$, and if w_0 is a branch point of order greater than $p-1$, then $D_p(w_0)=0$.*

(d) *If there exists no p-sheeted circle ($p>1$) with center w_0 contained in R, and if w_0 is not a branch point of order greater than $p-1$, then we define $D_p(w_0)$ as $D_{p-1}(w_0)$.*

It should be observed that in the definition in part (b) the existence of a largest p-sheeted circle with center in w_0 contained in R is asserted and still requires some justification. From Theorem 12 it follows that if such a circle exists, it must be unique. Furthermore, as one starts with a p-sheeted circle with center in w_0 contained in R and proceeds to enlarge its radius, it can never happen that it becomes multiply connected and on enlarging the radius still more, finally again becomes simply connected. This possibility is ruled out by Theorem 11. Finally, the existence of a p-sheeted circle with center in w_0 and contained in R whose radius is the least upper bound of the radii of all p-sheeted circles with center in w_0 and contained in R can be established by simple considerations of continuity, which are left to the reader.

The number $D_p(w_0)$ is not, as the notation would seem to indicate, a function merely of w_0, a value of w, but is rather a function of a specific point of R whose affix is w_0; thus $D_p(w_0)$ is precisely a function of z_0, where R is determined by the transformation $w=f(z)$. However, no confusion is likely to result from the slight lack of definiteness in the notation $D_p(w_0)$. We denote by $R_p(w_0)$ the unique region of R which is a q-sheeted circle C_q ($q\leqq p$) whose center is w_0 and radius $D_p(w_0)$.

For the sake of clearness, we present now a numerical illustration of the definition of $D_p(w_0)$. Let R consist of the doubly-carpeted unit circle $|w|<1$ with branch point of the first order at the origin $w=0$, except that in the second sheet there is deleted the subregion of $|w|<1$ contained in the region $|w+1|<1/3$; for definiteness choose the branch line as the segment $0\leqq w<1$; of course this configuration R can be mapped in a one-to-one manner on $|z|<1$ by a single-valued function $w=f(z)$, as can be seen at once by use of the auxiliary transformation $w=z_1^2$, which maps R onto a smooth Jordan region of the z_1-plane. We obviously have $D_2(0)=2/3$, for the doubly-carpeted (that is, two-sheeted) circle $|w|<2/3$ is contained in R, and that is true of no larger concentric doubly-carpeted circle. When w_0 is positive, and in either sheet of R, we have

$$D_2(w_0) = w_0 + 2/3, \qquad 0 \leqq w_0 \leqq 1/6,$$

$$D_2(w_0) = 1 - w_0, \qquad 1/6 \leqq w_0 < 1;$$

for positive w_0, the size of the region $R_2(w_0)$ is limited by the nearer of the two points $-2/3$, $+1$. When w_0 moves from the origin to the left in the first sheet, the size of $R_2(w_0)$ continues to be limited by the point $w=-2/3$:

$$D_2(w_0) = w_0 + 2/3, \qquad -1/3 \leqq w_0 < 0.$$

But when the point w_0 continues to the left from the point $w_0=-1/3$, the size of $R_2(w_0)$ is now no longer limited by the point $-2/3$, but is conditioned by the necessity of including no point of $|w+1|<1/3$, hence is limited by the origin; the corresponding region cut out of R is smooth, merely the region $|w-w_0|<|w_0|$:

$$D_2(w_0) = -w_0, \qquad -1/2 \leqq w_0 \leqq -1/3.$$

As w_0 moves further to the left from $w=-1/2$, still in the first sheet of R, the region $R_2(w_0)$ is now limited only by the point $w=-1$:

$$D_2(w_0) = 1 + w_0, \qquad -1 < w_0 \leqq -1/2.$$

When w_0 moves from the origin to the left in the second sheet of R, the size of $R_2(w_0)$ is also limited by the point $w=-2/3$:

$$D_2(w_0) = w_0 + 2/3, \qquad -2/3 < w < 0;$$

this situation continues as w_0 moves from the value zero to the value $-2/3$, but the region $R_2(w_0)$ is a doubly-carpeted circle for $-1/3<w<0$, and is singly-carpeted (smooth) for $-2/3<w\leqq -1/3$. This completes the study of our numerical case.

Let us now discuss the manner in which $D_p(w_0)$ and $R_p(w_0)$ vary on the general Riemann configuration R, the image of $|z|<1$ under the arbitrary map $w=f(z)$, where $f(z)$ is analytic in $|z|<1$. The various possibilities that arise are illustrated by the example just given. We cut all the sheets of R through with a circular biscuit-cutter whose center is w_0 and whose radius is the variable r. One of the connected sets thus cut out of R contains w_0 and is denoted by R_1. When r is small it follows from the usual implicit function theorem that if w_0 is not a branch point of R the region R_1 is smooth, and if w_0 is a q-fold point of R, then R_1 consists of a q-sheeted circle whose only branch point is w_0. As r is gradually increased, this situation continues until the boundary of R_1 reaches either a boundary point of R or a branch point of R. In the former case we have $D_p(w_0)$ equal to this particular value r_1 of r, and R_1 is $R_p(w_0)$. In the latter case if r is further increased, it may be that R_1 becomes a q'-sheeted circle with $q<q'\leqq p$, in which case we have $D_p(w_0)\geqq r>r_1$. But it may occur that whenever r is near to but greater than r_1 the region R_1 is a q''-sheeted circle, $q''>p$, in which case we have $D_p(w_0)=r_1$; it may also occur that whenever r is near to but greater than r_1 the region R_1 has boundary points in common with R, in which case we have also $D_p(w_0)=r_1$. If we have $D_p(w_0)>r_1$, the radius r can be perhaps increased until still further branch points of R lie interior to R_1, while R_1 remains a q_1-sheeted circle whose center is w_0, with $q_1 \leqq p$. In any case the radius r can be increased from zero to such a value r_2 that: (i) either a boundary point of R lies on the boundary of R_1, (ii) or there lie on the boundary of R_1 branch points of R of such multiplicities that for all values of r slightly greater than r_2 the region R_2 containing w_0 and cut out of R by the biscuit-cutter with center w_0 and radius r is a q''-sheeted circle with $q''>p$, (iii) or there lie on the boundary of R_1 branch points of R of such nature that for all values of r slightly greater than r_2 this region R_2 has boundary points which satisfy the relation $|w-w_0|<r_2$. It is to be noted that if the biscuit-cutter of radius r cuts from R the region R_1 containing w_0, and if R_1 has a boundary point w_1 (necessarily a boundary point of R) for which $|w_1-w_0|<r$, then we must have $D_p(w_0)<r$. For under these conditions R_1 cannot be a q-sheeted circle; the point w_1 of the w-plane may be covered by R_1 precisely q times (not necessarily by q sheets meeting at w_1), but then (by the implicit function theorem) a suitably chosen neighborhood of w_1 is also covered precisely q times by the sheets of R_1 that cover w_1, and suitable points w in this neighborhood are covered more than q times in all, for they are covered also by R_1 in the neighborhood of the boundary point w_1.

It is of interest to trace also the situation in the z-plane corresponding to the preceding discussion. When r is sufficiently small, $r>0$, the locus $|f(z)-w_0|=r$ consists (in addition to possible other arcs or curves) of a Jordan curve $J(r)$ in the neighborhood of the point z_0, where $w_0=f(z_0)$; for r sufficiently small, interior to $J(r)$ the function $f(z)$ takes on every value that it assumes (by Theorem 13 below) precisely a number of times equal to the multiplicity q of z_0 as a zero of the function $f(z)-w_0$; the image of the interior of $J(r)$ over the w-plane is a q-sheeted circle of radius r whose only branch point is $w=w_0$. As r now increases, this situation continues until $J(r)$ reaches $|z|=1$ or until at least one multiple point of $J(r)$ appears (at a multiple point the tangents to $J(r)$ are equally spaced); in the former case we simply have $D_p(w_0)$ equal to the corresponding value r_1 of r; in the latter case for values of r near to but slightly greater than r_1, the locus $|f(z)-w_0|=r$ consists of a Jordan arc J_1 near but exterior to $J(r_1)$ plus other Jordan arcs forming with J_1 a maximal connected set which we denote by $J(r)$; still other Jordan arcs may belong to the locus and not be connected with J_1, but such arcs do not concern us at present. If for every r near to but slightly greater than r_1 the set $J(r)$ has a boundary point on $|z|=1$, then we have $D_p(w_0)=r_1$; in the contrary case $J(r)$ consists of a Jordan curve in $|z|<1$ containing $J(r_1)$ in its interior; the function $f(z)$ takes on interior to $J(r)$ all the values that it takes on there the same number of times, say q'. If q' is greater than p we have $D_p(w_0)=r_1$, but if q' is not greater than p we have $D_p(w_0)>r_1$, and the process of enlarging $J(r)$ can continue beyond $r=r_1$. The process continues as r increases, and $J(r)$ may pass through multiple points, thereby increasing not merely r but also the number of times (the same for all values) that $f(z)$ takes on interior to $J(r)$ values that it takes on there. The process eventually comes to an end at some value $r=r_2=D_p(w_0)$, either because $J(r_2)$ reaches the boundary $|z|=1$ and hence is no longer a Jordan curve in $|z|<1$, or because the locus $|f(z)-w_0|=r_2$ has a multiple point, and for every $r>r_2$ but near to r_2 the locus $|f(z)-w_0|=r$ either fails now to separate z_0 from $|z|=1$ or divides $|z|<1$ into regions of which the one containing z_0 is a Jordan region in which each value assumed is assumed more than p times.

15. **Some properties of D_p.** We return now to the general theory of $D_p(w_0)$; an important tool is[45]

THEOREM 13. *Let $f(z)$ not identically constant be analytic in the simply connected region B, let $|f(z)|$ be continuous in the corresponding closed region and have the constant value b on the boundary C of B. Then all values w taken on by $f(z)$ in B are taken on there the same number of times q, and $f'(z)$ has precisely $q-1$ zeros interior to B.*

The region B cannot be the entire plane or the entire plane with the omission of a single point, so B can be mapped conformally onto the interior of the unit circle γ. It is sufficient to establish the theorem where B is the interior of γ, which we shall now do. We must have $b>0$, so by the well known properties of the maxima and minima of $|f(z)|$, the zeros of $f(z)$ interior to γ are finite in number, $\beta_1, \beta_2, \cdots, \beta_q$ with $q>0$. The function

$$f(z)\cdot\prod_{k=1}^{q}\frac{1-\bar{\beta}_k z}{z-\beta_k},$$

when suitably defined in the points β_k, is analytic and different from zero at every point interior to γ; its modulus is continuous in the corresponding closed region and takes the constant value b on γ. Hence this function itself is a constant of modulus b, and we have

$$f(z) = \omega b\prod_{k=1}^{q}\frac{z-\beta_k}{1-\bar{\beta}_k z}, \qquad |\omega| = 1.$$

The first part of Theorem 13 now follows from Rouché's theorem, for if we have $|c|<b$ we have on γ the inequality $|c|<|f(z)|$. The latter part of Theorem 13 follows from a theorem due to Bôcher and Walsh[46].

THEOREM 14. *Let the function $w=f(z)$ analytic for $|z|<r$ with $f(0)=0$ map $|z|<r$ onto a Riemann configuration R such that no point of the boundary of R*

[45] The part of this theorem which refers to the zeros of $f'(z)$ is not new, if q is defined as the number of zeros of $f(z)$ in B, and has been considered by de Boer, Macdonald, de la Vallée Poussin, Whittaker and Watson, Denjoy, Lange-Nielsen, and Ålander. See for instance Denjoy, Comptes Rendus de l'Académie des Sciences, Paris, vol. 166 (1918), pp. 31–33; Ålander, Comptes Rendus de l'Académie des Sciences, Paris, vol. 184 (1927), pp. 1411–1413.

[46] J. L. Walsh, these Transactions, vol. 19 (1918), pp. 291–298, especially p. 297.

satisfies the inequality $|w| < \rho > 0$. *Then the connected region* R_1 *of* R *which contains the transform of* $z=0$ *and which is cut out of* R *by a biscuit-cutter whose center is* $w=0$ *and radius* ρ *is simply connected; and each point* w *of the* w-*plane with* $|w| < \rho$ *is covered by* R_1 *the same number of times.*

The region R_1 corresponds to some region R_2 in $|z| < r$ containing $z=0$. The function $|f(z)|$ is continuous in the closed region consisting of R_2 plus its boundary, and assumes the constant value ρ on the boundary; of course the boundary of R_2 may coincide in whole or in part with $|z| = r$. It follows from the principle of maximum modulus applied to $f(z)$ in $|z| < r$ that the boundary of R_2 cannot fall into two or more continua, one of which would necessarily lie in a simply connected region interior to $|z| = r$ bounded by another continuum belonging to the boundary of R_2. Then R_2 is simply connected, and so consequently is R_1. The remainder of Theorem 14 follows from Theorem 13.

THEOREM 15. *Let* $w=f(z)$ *be analytic for* $|z| < 1$ *and map* $|z| < 1$ *onto the Riemann configuration* R *with* $w_0=f(z_0)$, $|z_0| < 1$. *Let* $f(z)$ *take on in* $|z| < 1$ *every value* w *in the region* $|w-w_0| < \rho > 0$ *precisely* p *times. Then we have* $D_p(w_0) \geqq \rho$.

No boundary point w_1 of R can satisfy the inequality $|w_1 - w_0| < \rho$; for if it did the point w_1 of the w-plane would be covered by R a totality of p times, and by the implicit function theorem a suitably chosen neighborhood of w_1 would also be covered by R precisely p times by the sheets of R covering w_1. Some values w in every neighborhood of w_1 are covered also by the sheet (or sheets) of R of which w_1 is a boundary point; so some points w with $|w-w_0| < \rho$ are covered more than p times, contrary to hypothesis.

We have now shown that no boundary point of R satisfies the inequality $|w-w_0| < \rho$; so it follows from Theorem 14 that the region containing w_0 cut out of R by a biscuit-cutter of center w_0 and radius ρ covers each point of $|w-w_0| < \rho$ the same number of times, a number which by the hypothesis of Theorem 15 cannot exceed p; hence Theorem 15 is established.

Still another result related to Theorems 14 and 15 follows easily:

THEOREM 16. *Let the function* $w=f(z)$ *be analytic for* $|z| < 1$ *and map* $|z| < 1$ *onto the Riemann configuration* R *with* $w_0=f(z_0)$, $|z_0| < 1$. *Suppose* $\liminf_{|z|\to 1} |f(z)-w_0| \geqq \rho$, *and suppose no value in* $|w-w_0| < \rho$ *is taken on by* $f(z)$ *in* $|z| < 1$ *more than* p *times. Then we have* $D_p(w_0) \geqq \rho$.

It follows from our hypothesis that no boundary point of R lies in $|w-w_0| < \rho$; so Theorem 16 follows from Theorems 14 and 15.

COROLLARY. *Let* $w=f(z)$ *be analytic for* $|z| < 1$ *and map* $|z| < 1$ *onto the Riemann configuration* R *with* $w_0=f(z_0)$, $|z_0| < 1$. *Let* R_1 *be a subregion of* $|z| < 1$ *containing* z_0, *whose boundary* B *satisfies the condition* $\lim_{z\to B, |z|<1} |f(z)-w_0|$

$=\rho>0$, *and suppose no value* w *is taken on by* $f(z)$ *in* R_1 *more than* p *times. Then we have* $D_p(w_0)\geqq\rho$.

It follows from the principle of maximum modulus that R_1 is simply connected. If R_1 is mapped smoothly and conformally onto $|\zeta|<1$, and if Theorem 16 is applied to the function which maps $|\zeta|<1$ onto R_1, we obtain the corollary.

Although the following theorem is not needed in the sequel, it is of some interest in itself.

THEOREM 17. *Let* R *be a simply connected Riemann configuration of hyperbolic type, and let* w_0 *be any point of* R. *Then* $D_p(w_0)$ *is a continuous function of* w_0.

We need to define what we shall mean by the continuity of $D_p(w_0)$ on R. If w_0 is a branch point of order greater than $p-1$, $(p>1)$, we shall say that $D_p(w_0)$ is continuous at w_0 if to any $\epsilon>0$ we can assign a number $\delta>0$ so that for any point w_0' at a distance not greater than δ from w_0 and lying on one of the sheets that come together at w_0 the relation $|D_p(w_0')-D_p(w_0)|<\epsilon$ holds; here $D_p(w_0)=0$. If w_0 is a branch point of order q, where $0\leqq q\leqq p-1$, we shall say that $D_p(w_0)$ is continuous at w_0 if to any $\epsilon>0$ we can assign a number $\delta>0$ so that for any point w_0' within the q-sheeted circle C_q with center at w_0 and radius δ the relation $|D_p(w_0')-D_p(w_0)|<\epsilon$ holds. The proof of this theorem is left to the reader.

16. The limit property of D_p for continuously convergent sequences.

THEOREM 18. *Let* $\{f_n(z)\}$ *be a sequence of functions analytic in the unit circle* $|z|<1$, *and converging uniformly in every closed subregion of* $|z|<1$ *to an analytic function* $f(z)$. *Let* z_0 *be any point in the circle* $|z|<1$ *and set* $w_n=f_n(z_0)$, $w_0=f(z_0)$. *Denoting by* $D_p(w_n)$ *the radius of* p*-valence at the point* w_n *of the Riemann configuration* R_n *on which* $f_n(z)$ *maps the circle* $|z|<1$ *and by* $D_p(w_0)$ *the radius of* p*-valence at the point* w_0 *of the Riemann configuration* R_0 *on which* $f(z)$ *maps the circle* $|z|<1$, *we have*

$$\lim_{n\to\infty} D_p(w_n) = D_p(w_0), \qquad p = 1, 2, 3, \cdots . \tag{16.1}$$

The proof of the theorem will be based on two lemmas:

LEMMA 1. *Under the conditions of Theorem* 18,

$$\liminf_{n\to\infty} D_p(w_n) \geqq D_p(w_0). \tag{16.2}$$

The lemma is clearly trivial if $D_p(w_0)=0$.

Let us assume, therefore, that $D_p(w_0)>0$ and choose any positive number ρ so that $\rho<D_p(w_0)$. Hence, the Riemann configuration R_0 contains in its interior some q-sheeted circle $C_q(w_0)$, $1\leqq q\leqq p$, of center w_0 and radius ρ, to-

gether with its boundary. Denote the region in $|z|<1$ on which the function $w=f(z)$ maps $C_q(w_0)$ by R_0. The boundary B_0 of R_0 must consequently lie wholly in the interior of $|z|<1$. In R_0 we have $|f(z)-w_0|<\rho$ and on B_0 we have $|f(z)-w_0|=\rho$. Let $\epsilon>0$ be any number such that $\rho+\epsilon<D_p(w_0)$. Due to the uniform convergence of the sequence $f_n(z)$ on B_0, there exists a positive integer $n(\epsilon)$ such that for all integers $n>n(\epsilon)$ the inequality $|f_n(z)-w_n|>\rho-\epsilon$ holds on B_0. Hence, that region R_n in the circle $|z|<1$ which contains the point z_0 and on which $|f_n(z)-w_n|<\rho-\epsilon$ lies wholly interior to R_0. On the boundary B_n of R_n we have $|f_n(z)-w_n|=\rho-\epsilon$. In accordance with Theorem 13 the function $f_n(z)$ takes on all its values the same number of times q_n in R_n. By Hurwitz's theorem, since $f(z)$ is at most p-valent([47]) in R_0, for sufficiently large values of n we have $q_n\leqq p$. Hence, by the corollary to Theorem 16 we have $D_p(w_n)\geqq D_{q_n}(w_n)\geqq\rho-\epsilon$. Hence, $\liminf_{n\to\infty} D_p(w_n)\geqq\rho$. But ρ is an arbitrary positive number less than $D_p(w_0)$. The relation (16.2) follows at once.

LEMMA 2. *Under the conditions of Theorem* 18,

$$\limsup_{n\to\infty} D_p(w_n) \leqq D_p(w_0). \tag{16.3}$$

If this lemma is false there must exist a positive constant a such that for infinitely many values of n

$$D_p(w_n) > a > D_p(w_0). \tag{16.4}$$

We shall neglect all those functions $f_n(z)$ for which the above inequality fails and assume that (16.4) holds for all n.

Consider that largest region R_n in the circle $|z|<1$ which contains the point z_0, for which $|f_n(z)-w_n|<a$. Then in R_n the function $f_n(z)$ is q-valent $(q\leqq p)$. According to (16.4) the boundary C_n of the region R_n lies wholly in the circle $|z|<1$. Furthermore, by the principle of maximum modulus we conclude that R_n is simply connected. Clearly, on the curve C_n the relation $|f_n(z)-w_n|=a$ is satisfied. Every value taken on by $f_n(z)$ in R_n is taken on the same number of times.

Denote by $z=\phi_n(t)$ a function which maps the region R_n on the circle $|t|<1$ in such a manner that $\phi_n(0)=z_0$. Since the curve C_n is a Jordan curve, by a well known theorem of Osgood-Carathéodory the function $\phi_n(t)$ is continuous in the closed circle $|t|\leqq 1$([48]). The function $f_n(\phi_n(t))=g_n(t)$ is analytic in $|t|<1$, continuous in $|t|\leqq 1$ and $|g_n(t)-w_n|=a$ on $|t|=1$. By Schwarz's reflection principle([49]), we infer that $g_n(t)$ is analytic in the closed circle

([47]) We shall say that a function $f(z)$ is *p-valent in a region* R if it assumes no value more than p times in R and at least one value precisely p times. A function $f(z)$ will be called *at most p-valent in* R if it is q-valent in R for some $q\leqq p$.

([48]) W. F. Osgood and E. H. Taylor, these Transactions, vol. 14 (1913), pp. 277–298; C. Carathéodory, Mathematische Annalen, vol. 73 (1913), pp. 305–320.

([49]) Cf. G. Julia, loc. cit., p. 44 ff.

$|t| \leqq 1$. Finally, $g_n(t)$ is precisely q-valent in $|t| < 1$ since $f_n(z)$ possesses the same property in R_n. By the theorem of Radó, referred to earlier, we may represent $g_n(t)$ in the following manner:

$$g_n(t) = w_n + ae^{i\theta_n}t^{k_n} \prod_{j=1}^{q-k_n} \frac{t - t_j^{(n)}}{1 - \bar{t}_j^{(n)}t}, \quad k_n \geqq 1; \ |t_j^{(n)}| \leqq 1. \tag{16.5}$$

Since the $g_n(t)$ are uniformly bounded, they form a normal family and we may select a subsequence, which for simplicity will again be denoted by $\{g_n(t)\}$, converging uniformly in every closed subregion of $|t| < 1$ to a function $G(t)$ analytic in $|t| < 1$. On account of (16.5) $G(t)$ has itself a representation of the form

$$G(t) = w_0 + ae^{i\theta_0}t^k \prod_{j=1}^{q-k} \frac{t - t_j}{1 - \bar{t}_j t}, \qquad k \geqq 1. \tag{16.6}$$

Just as in (16.5) some of the t_j here may have the absolute value 1.

Now consider that largest region R_0 in $|z| < 1$ which contains the point z_0 and in which $|f(z) - w_0| < a$. According to the maximum modulus principle R_0 is simply connected and we may map it on the circle $|t| < 1$ by means of a function $z = \phi_0(t)$ so that $\phi_0(0) = z_0$. On that part of the boundary B_0 of R_0 which lies interior to the circle $|z| < 1$ if it exists we have $|f(z) - w_0| = a$. We shall now show that R_0 is the kernel of the sequence of regions $\{R_n\}$ [50]. Indeed, consider any region R_0' which together with its boundary lies interior to R_0 and contains the point z_0. By the definition of R_0, in the region R_0' and on its boundary we have $|f(z) - w_0| < a$. Since the functions $f_n(z) - w_n$ converge uniformly to $f(z) - w_0$ in the closure of R_0', for n sufficiently large we have $|f_n(z) - w_n| < a$ in the closure of R_0', and therefore R_0' belongs to all R_n for sufficiently large values of n. Next, choose any point z' of the circle $|z| < 1$ exterior to R_0 (if such a point exists). Connect the point z' with the point z_0 by any Jordan arc L which lies wholly in the circle $|z| < 1$. Since z' is exterior to R_0, there must exist on the arc L at least one point Z at which $|f(Z) - w_0| > a$. For sufficiently large values of n we must have $|f_n(Z) - w_n| > a$, and consequently Z is exterior to R_n. Thus, on any Jordan arc joining the points z_0 and z' there exists a point exterior to R_n for all sufficiently large values of n. Consequently R_0 is the kernel of the sequence of regions $\{R_n\}$. Hence, by a well known theorem of Carathéodory[51] the sequence of functions $\phi_n(t)$ converges uniformly in every closed subregion of $|t| < 1$ to the function $\phi_0(t)$, provided merely we have chosen $\phi_n'(0) > 0$, $\phi_0'(0) > 0$.

If we form the function $g_0(t) = f(\phi_0(t))$, it follows that the sequence of func-

---

([50]) For the notion of kernel of a sequence of domains cf. C. Carathéodory, *Conformal Representation*, Cambridge Tract in Mathematics and Mathematical Physics, no. 28, (1932), pp. 74–77.

([51]) C. Carathéodory, loc. cit., particularly p. 76.

tions $\{g_n(t)\}$ converges uniformly in every closed subregion of $|t|<1$ to the function $g_0(t)$. We have shown earlier, however, that the sequence $\{g_n(t)\}$ converges to the function $G(t)$ whose representation is given in (16.6). We thus find that $g_0(t)=G(t)$ identically in $|t|<1$. From (16.6) it follows therefore that $g_0(t)$ is analytic in $|t|\leqq 1$, is q'-valent ($q'\leqq p$) in $|t|<1$, and on the circumference $|t|=1$ satisfies the relation $|g_0(t)-w_0|=a$.

Consider now an arbitrary positive number ϵ such that $a-\epsilon>D_p(w_0)$. Denote by R_ϵ the largest region in $|t|<1$ which contains the origin and throughout which $|g_0(t)-w_0|<a-\epsilon$. The boundary C_ϵ of this region lies wholly interior to $|t|<1$ and in R_ϵ the function $g_0(t)$ is q''-valent ($q''\leqq p$). The function $z=\phi_0(t)$ maps the region R_ϵ on a region P_ϵ in the z-plane which is together with its boundary Γ_ϵ interior to R_0. In P_ϵ we have $|f(z)-w_0|<a-\epsilon$ and on Γ_ϵ we have $|f(z)-w_0|=a-\epsilon$. Since the region P_ϵ contains the point z_0 and since $f(z)$ is q''-valent in P_ϵ, it follows that $a-\epsilon<D_p(w_0)$. This contradicts our assumption concerning ϵ.

Since the assumption (16.4) leads to a contradiction, the relation (16.3) is true.

We are now ready to prove the theorem. Lemmas 1 and 2 together yield the inequalities

$$\limsup_{n\to\infty} D_p(w_n) \leqq D_p(w_0) \leqq \liminf_{n\to\infty} D_p(w_n).$$

Since, however, we always have $\liminf_{n\to\infty} D_p(w_n)\leqq\limsup_{n\to\infty} D_p(w_n)$, it follows that $\limsup_{n\to\infty} D_p(w_n)=\liminf_{n\to\infty} D_p(w_n)=\lim_{n\to\infty} D_p(w_n)=D_p(w_0)$, which proves the theorem.

17. $\lim_{n\to\infty} D_p(w_n)=0$ **is a necessary and sufficient condition for** $\lim_{n\to\infty}|f^{(k)}(z_n)|(1-|z_n|)^k=0$ ($k=1, 2, \cdots, p$). An immediate consequence of Theorem 18 is the following extension of Theorem 2, Chapter II, to the higher derivatives of bounded functions.

THEOREM 19. *Let $f(z)$ be regular and bounded in $|z|<1$:*

$$|f(z)| \leqq M,$$

let $\{z_n\}$ be any sequence of points in $|z|<1$, and let $w_n=f(z_n)$. Then, a necessary and sufficient condition for

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k = 1, 2, \cdots, p,$$

is that $\lim_{n\to\infty} D_p(w_n)=0$.

We first prove the sufficiency of the condition. We assume that $\lim_{n\to\infty} D_p(w_n)=0$. In accordance with the definition of the radius of p-valence it follows that

$$(17.1) \qquad \lim_{n\to\infty} D_k(w_n) = 0, \qquad k = 1, 2, \cdots, p.$$

By virtue of Theorem 2, Chapter II, the condition is sufficient for $p=1$. Let us assume that the condition is sufficient for $p-1$ and prove it to be sufficient for p. We assume therefore that (17.1) implies

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \quad k = 1, 2, \cdots, p-1. \tag{17.2}$$

If the condition is not sufficient for p, we could find a positive constant δ and a subsequence of $\{z_n\}$, which for simplicity will again be denoted by $\{z_n\}$ for which

$$|f^{(p)}(z_n)|(1-|z_n|)^p \geqq \delta > 0 \tag{17.3}$$

and at the same time the relations (17.1) and (17.2) hold.

Now if we introduce the sequence of functions

$$\phi_n(\zeta) = f\left(\frac{\zeta + z_n}{1+\bar{z}_n\zeta}\right)$$

which are bounded and regular in $|\zeta|<1$: $|\phi_n(\zeta)| \leqq M$, we obtain by virtue of the expression (2.3)

$$\frac{\phi_n^{(p)}(0)}{p!} = \sum_{\nu=0}^{p-1} (-1)^{\nu} C_{p-1,\nu} \bar{z}_n^{\nu} \frac{(1-|z_n|^2)^{p-\nu} f^{(p-\nu)}(z_n)}{(p-\nu)!}.$$

The relations (17.2) and (17.3) imply

$$\liminf_{n\to\infty} |\phi_n^{(p)}(0)| \geqq \delta > 0, \tag{17.4}$$

while the relation (2.3) written out for $n=1, 2, \cdots, p-1$ together with (17.2) shows that

$$\lim_{n\to\infty} |\phi_n^{(k)}(0)| = 0, \qquad \text{for } k = 1, 2, \cdots, p-1. \tag{17.5}$$

The sequence of functions $\{\phi_n(\zeta)\}$ forms a normal family in $|\zeta|<1$. We may, therefore, extract a convergent subsequence which for simplicity will again be denoted by $\{\phi_n(\zeta)\}$

$$\lim_{n\to\infty} \phi_n(\zeta) = \phi(\zeta).$$

The relations (17.4) and (17.5) imply

$$\phi^{(k)}(0) = 0 \quad \text{for} \quad k = 1, 2, \cdots, p-1; \qquad |\phi^{(p)}(0)| \geqq \delta. \tag{17.6}$$

The equations in (17.6), however, imply that the radius of p-valence $D_p[\phi(0)]$ of the Riemann surface on which $\phi(\zeta)$ maps the circle $|\zeta|<1$ is positive at the point $\phi(0)$ of the surface $D_p[\phi(0)]>0$. According to Theorem 18 if we denote by $D_p[\phi_n(0)]$ the radius of p-valence at the point $\phi_n(0)$ of the Riemann surface R_n on which $\phi_n(\zeta)$ maps the circle $|\zeta|<1$ and observe that $\phi_n(0)=w_n$, we obtain

$$\lim_{n\to\infty} D_p(w_n) = D_p[\phi(0)] > 0.$$

But R_n is precisely the Riemann surface R on which $f(z)$ maps the circle $|z|<1$. Hence, the last relation contradicts (17.1) for $k=p$. This proves that the assumption (17.3) is false and the sufficiency of our condition is established.

We now turn to the proof of the necessity of the condition in Theorem 19. Let us assume that

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \quad \text{for } k = 1, 2, \cdots, p.$$

Forming again the functions $\phi_n(\zeta)$, we see that

$$\lim_{n\to\infty} |\phi_n^{(k)}(0)| = 0 \qquad \text{for } k = 1, 2, \cdots, p.$$

Let us assume that we have already selected a uniformly convergent subsequence of the $\{\phi_n(\zeta)\}$, which, because of the normality of the family, is always possible. The limit function $\phi(\zeta)$ of the sequence has the property that $\phi^{(k)}(0) = 0$ for $k = 1, 2, \cdots p$. Consequently, $D_p[\phi(0)]=0$ and by Theorem 18

$$\lim_{n\to\infty} D_p(w_n) = 0.$$

The last relation has been proved only for a subsequence of the original sequence. But since from every sequence we may select a subsequence with this property, it must also hold for the whole sequence. Theorem 19 is now established.

It will be noticed that Theorem 19 is unsatisfactory in that no indication is given of the manner in which expressions of the type $|f^{(k)}(z_n)|(1-|z_n|)^k$ depend on the radii of p-valence $D_p(w_n)$. In the case $p=1$ we have already given inequalities which bring out this dependence (Theorem 3, Chapter II). Our next task will be to extend Theorem 3, Chapter II, to the higher derivatives of bounded functions. The constants that we shall obtain will, however, not be precise. We shall first study upper bounds for the derivatives of bounded functions. The inequalities that we shall obtain will, of course, yield a new proof of Theorem 19 by quantitative methods rather than the purely qualitative methods that we used in the present proof.

CHAPTER III. BOUNDED FUNCTIONS; INEQUALITIES ON D_p

18. **A preliminary lower bound for D_p.** For our purpose in the use of $D_p(w_0)$ for the study of such relations as $|f^{(p)}(z_k)|(1-|z_k|)^p\to 0$, it is desir-

able to have explicit numerical inequalities connecting $D_p(w_k)$ and the derivatives $f'(z_k), f''(z_k), \cdots, f^{(p)}(z_k)$. We first prove regarding this relationship

THEOREM 1. *Suppose the function $f(z)$ analytic for $|z|<1$ with $f(0)=0$, $f^{(p)}(0)=p!$, and with $|f(z)| \leqq M$ for $|z|<1$. Then we have*

$$D_p(0) \geqq M_p > 0, \tag{18.1}$$

where $M_p = M_p(M)$ is a suitably chosen constant depending on M and p but not on $f(z)$.

Our proof of Theorem 1 is a direct generalization of Landau's proof([52]) for the case $p=1$. For the case $p=1$, Landau's method yields the inequality

$$D_1(0) \geqq 1/(6M), \tag{18.2}$$

a special case of inequality (18.9) to be proved below. But other related methods([53]) yield the inequality

$$D_1(0) \geqq 1/(4M), \tag{18.3}$$

which is somewhat sharper than (18.2) and which we shall therefore take as point of departure.

We remark that if $f(z)$ is analytic for $|z|<1$ with $f(0)=0$, $f'(0)=m\neq 0$, with $|f(z)| \leqq M$ for $|z|<1$, then the function $f(z)/m$ has the derivative unity at the origin and modulus in $|z|<1$ not greater than $M/|m|$. Consequently under the transformation $w=f(z)/m$ we have from (18.3) the result $D_1(0) \geqq |m|/(4M)$, and under the transformation $w=f(z)$ we have

$$D_1(0) \geqq \frac{|m^2|}{4M}. \tag{18.4}$$

Let us now suppose Theorem 1 established with p replaced by j for $j=1, 2, \cdots, p-1$; we proceed to prove by induction the theorem as stated. The cases

$$|f'(0)| \geqq \frac{1}{(12M)^{p-1}}, \tag{18.5'}$$

$$\frac{|f''(0)|}{2!} \geqq \frac{1}{(12M)^{p-2}}, \tag{18.5''}$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{|f^{(p-1)}(0)|}{(p-1)!} \geqq \frac{1}{12M} \tag{18.5^{(p-1)}}$$

are all handled in a manner similar to the proof of (18.4). Thus in case

([52]) E. Landau, Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften, Berlin, 1926, pp. 467–474.

([53]) J. Dieudonné, Annales de l'École Normale Supérieure, vol. 48 (1931), pp. 247–358. Or see Montel, *Fonctions Univalentes*, §37.

$(18.5^{(j)}), j=1, 2, \cdots, p-1$, the function

$$\frac{f(z)}{f^{(j)}(0)/j!} \tag{18.6}$$

has the jth derivative $j!$ at the origin, and modulus in $|z|<1$ not greater than $j!M/|f^{(j)}(0)|$, so by our assumption that Theorem 1 with p replaced by j is established, we have under the transformation $w=j!f(z)/f^{(j)}(0)$, $D_j(0) \geqq M_j(j!M/|f^{(j)}(0)|)$, and we have under the transformation $w=f(z)$

$$D_j(0) \geqq \frac{|f^{(j)}(0)|}{j!} \cdot M_j\left[\frac{j!M}{|f^{(j)}(0)|}\right]; \tag{18.7}$$

hence by the relation $D_p(0) \geqq D_j(0)$ the theorem may be considered to be proved. It remains to study the case that we have simultaneously

$$\frac{|f^{(j)}(0)|}{j!} < \frac{1}{(12M)^{p-j}}, \qquad j = 1, 2, \cdots, p-1, \tag{18.5$^{(p)}$}$$

with, of course, the relation $f^{(p)}(0) = p!$.

Suppose r can be chosen $(0<r<1)$ so that the expression

$$R = r^p - \max_{|z|=r} |f(z) - z^p| \tag{18.8}$$

is positive. Then we have $R \leqq r^p < r$, and for $|w| < R$ the inequality

$$\left|\frac{f(z) - z^p}{z^p - w}\right| < 1$$

holds on the circle $|z| = r$. Of course, $z^p - w$ cannot vanish on $|z| = r$. Then by Rouché's theorem the function $f(z) - w$ has precisely as many zeros in $|z| < r$ as does the function $z^p - w$, namely p. Then the transformation $w = f(z)$ maps $|z| < r$ onto a Riemann configuration which contains the region $|w| < R$ with each point covered precisely p times. Thus (Theorem 15, Chapter II), we have $D_p(0) \geqq R$, whether $D_p(0)$ refers to the Riemann configuration which is the image of $|z| < r$ or to the configuration which is the image of $|z| < 1$ under the transformation $w = f(z)$.

It remains to show that r can be chosen in such a way that R as defined by (18.8) is positive. If we set $f(z) \equiv \sum_{n=1}^{\infty} a_n z^n$, Cauchy's inequality is $|a_n| \leqq M$, and in particular $a_p = 1 \leqq M$. Consequently we may write on $|z| = r$ by the use of $(18.5^{(p)})$

$$\left|\sum_{n=p+1}^{\infty} a_n z^n\right| \leqq \frac{Mr^{p+1}}{1 - r},$$

$$\left|\sum_{n=1}^{p-1} a_n z^n\right| \leqq \sum_{n=1}^{p-1} \frac{r^n}{(12M)^{p-n}} = \frac{r}{(12M)^{p-1}} \frac{1 - (12M)^{p-1} r^{p-1}}{1 - 12Mr},$$

and with the choice $r=1/(4M)$,

$$
\begin{aligned}
R &= r^p - \max_{|z|=r} |f(z) - z^p| \\
&\geqq r^p - \frac{Mr^{p+1}}{1-r} - \frac{r}{(12M)^{p-1}} \frac{1-(12M)^{p-1}r^{p-1}}{1-12Mr} \\
&\geqq r^p - \frac{4}{3}Mr^{p+1} - \frac{r}{(12M)^{p-1}} \frac{1-(12M)^{p-1}r^{p-1}}{1-12Mr} \\
&= \frac{1+3^{p-2}}{2\cdot 3^{p-1}\cdot 4^p\cdot M^p}.
\end{aligned}
\tag{18.9}
$$

We have now proved the desired inequality $R>0$ and thus completed the proof of Theorem 1, and we also have material for obtaining an explicit inequality for $M_p(M)$ in inequality (18.1).

19. **Numerical lower bounds for D_p.** When $p=2$, relation (18.7) [or (18.4)] becomes in case (18.5′)

$$D_1(0) \geqq \frac{1}{24^2M^3},$$

whereas in case (18.5″) we have from (18.9)

$$D_2(0) \geqq \frac{1}{48M^2},$$

so in either case we may write

$$D_2(0) \geqq \frac{1}{24^2M^3}. \tag{19.1}$$

Inequality (19.1) is to be generalized by proving

$$D_p(0) \geqq M_p(M) \equiv \frac{1}{4\cdot 12^{2^p-2}M^{2^p-1}}. \tag{19.2}$$

We remark that $M_p(M)$, as thus defined, decreases monotonically as M increases. It is to be noticed that (19.2) holds for $p=1$, by inequality (18.3), and for $p=2$, by inequality (19.1); we assume (19.2) to hold with p replaced by j for $j=2, 3, \cdots, p-1$, and shall establish (19.2) as written. In case (18.5$^{(j)}$) we find from (18.7) and (19.2) the inequality ($p>2$)

$$D_j(0) \geqq \frac{1}{(12M)^{p-j}} \frac{1}{4\cdot 12^{2^j-2}[M(12M)^{p-j}]^{2^j-1}}. \tag{19.3}$$

Direct comparison of the right-hand members of (19.2) and (19.3) now shows, by virtue of the inequality $2^{q-1} \geqq q$, q a positive integer, and by virtue of $D_p(0) \geqq D_j(0)$, that (19.2) holds in each of the cases $(18.5^{(j)})$, $j=1, 2, \cdots, p-1$. Also in case $(18.5^{(p)})$ inequality (19.2) is valid, as we find from (18.9), so we have established.

COROLLARY 1. *Under the hypothesis of Theorem* 1, *we have inequality* (19.2).

Needless to say, the numerical results contained in some of the preceding inequalities can be improved, and it is to be supposed that those contained in inequality (19.2) can be greatly improved.

Inequality (18.7) is valid under the assumption $f^{(j)}(0) \neq 0$ instead of $f^{(j)}(0) = j!$, so by using $M_p(M)$ as defined by (19.2) we may formulate:

COROLLARY 2. *Suppose the functions* $f_k(z)$ *analytic for* $|z| < 1$, *with* $f_k(0) = 0$ *and* $|f_k(z)| \leqq M$ *for* $|z| < 1$. *If as k becomes infinite the corresponding sequence* $D_p(0)$ *approaches zero, then we have also*

$$\lim_{k\to\infty} f_k^{(p)}(0) = 0.$$

Under the conditions of Corollary 2 we have $D_p(0) \geqq D_{p-1}(0) \geqq \cdots \geqq D_1(0)$, from which follows for $j=1, 2, \cdots, p$ the relation

$$\lim_{k\to\infty} f_k^{(j)}(0) = 0. \tag{19.4}$$

A specific inequality for the direct proof of (19.4) is useful. A consequence of (19.2) and (18.7) for $j=1, 2, \cdots$, p, with the omission of the requirement $f^{(j)}(0) = j!$, is

$$D_j(0) \geqq \frac{|f^{(j)}(0)|}{j!} \; \frac{1}{4 \cdot 12^{2^j-2}(j!M/|f^{(j)}(0)|)^{2^j-1}} \cdot$$

The inequality $D_j(0) \leqq M$ is obvious, so we have

$$\begin{aligned} \frac{|f^{(j)}(0)|}{j!} &\leqq 4^{2^{-j}} \cdot 12^{1-2^{1-j}} M^{1-2^{-j}} [D_j(0)]^{2^{-j}} \\ &\leqq 24M \left[\frac{D_j(0)}{M}\right]^{2^{-j}} \\ &\leqq 24M \left[\frac{D_j(0)}{M}\right]^{2^{-p}} . \end{aligned} \tag{19.5'}$$

By virtue of the inequalities $D_j(0) \geqq D_{j-1}(0)$ we may now write[54]

$$|f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)| \leqq 24p(1+M)[D_p(0)]^{2^{-p}}. \tag{19.5}$$

[54] For $0 < \alpha \leqq 1$, $M > 0$, we have $M^\alpha \leqq 1 + M$.

We state explicitly a major result:

COROLLARY 3. *If $f(z)$ is analytic and in modulus not greater than M for $|z|<1$, with $f(0)=0$, then inequality* (19.5) *is valid for every positive integer p.*

For the purpose of Corollary 3, the factor $1+M$ in the right-hand member of (19.5) may of course be replaced by $M^{1-2^{-p}}$.

20. **A lower bound for the derivative of a circular product.** The converse of Corollary 2 is false, as is illustrated by the sequence $f_k(z)\equiv z$, with $p=2$. The second derivative $f_k''(0)$ vanishes for every k, yet $D_2(0)$ has the constant value unity, so the relation $D_2(0)\to 0$ is not satisfied. Indeed, in the general situation that $f_k(z)$ is analytic for $|z|<1$ with $f_k(0)=0$ and $|f_k(z)|\leqq M$ for $|z|<1$, it is not to be expected that $f_k^{(p)}(0)\to 0$ should imply $D_p(0)\to 0$, for the latter relation by virtue of $D_l(0)\geqq D_{l-1}(0)$ implies also $D_l(0)\to 0$, $l=1, 2, \cdots, p-1$ which by Corollary 2 implies (19.4), a relation which is not implied by the hypothesis and is indeed completely independent of the hypothesis. We should expect, then, that a relation in the opposite sense to Corollary 2 would necessarily involve the lower derivatives. We shall proceed to prove

THEOREM 2. *Let the function $w=f(z)$ analytic for $|z|<1$ map $|z|<1$ onto a Riemann configuration with $f(0)=0$. Then there exists a positive constant γ_p depending on p but not on $f(z)$ such that we have*

$$(20.1)\qquad D_p(0)\leqq\frac{1}{\gamma_p}\left[\,|f'(0)|+\frac{1}{2!}|f''(0)|+\cdots+\frac{1}{p!}|f^{(p)}(0)|\,\right].$$

The proof of Theorem 2 is to be carried out in several steps, of which the first is

THEOREM 3. *Let $w=g(z)$ analytic for $|z|<1$ map $|z|<1$ onto $|w|<1$ counted precisely p times, or precisely $m<p$ times, with $g(0)=0$. Then we have*

$$(20.2)\qquad |g'(0)|+\frac{1}{2!}|g''(0)|+\cdots+\frac{1}{p!}|g^{(p)}(0)|\geqq c_p>0,$$

where c_p is a suitably chosen number depending on p but not on $g(z)$.

To be explicit, we prove (20.2) with $c_p=2^{-(p+1)!}$.

The most general function $g(z)$ is of the form[55]

$$(20.3)\qquad w=g_p(z)=z\prod_{j=1}^{p-1}\frac{z-\beta_j}{1-\bar{\beta}_jz},\qquad |\beta_j|\leqq 1,$$

except for a constant factor of modulus unity which does not affect the left-hand member of (20.2) and which we therefore suppress. In the case $p=1$, the

[55] T. Radó, ibid.

form (20.3) breaks down, but we have $g_1(z) \equiv z$, and (20.2) is fulfilled with c_p replaced by unity, which is greater than $c_1 = 1/4$. Henceforth, we suppose $p \geqq 2$.

We prove (20.2) by induction, assuming the validity of (20.2) with p replaced by $p-1$ and proving (20.2) as written[56]. Equation (20.3) can be expressed in the equivalent form

$$g_p(z) = g_{p-1}(z) \frac{z - \alpha}{1 - \bar{\alpha} z}, \qquad |\alpha| \leqq 1, \tag{20.4}$$

where we have also

$$g_{p-1}(z) = a_1 z + a_2 z^2 + \cdots, \qquad |z| < 1,$$
$$g_p(z) = b_1 z + b_2 z^2 + \cdots, \qquad |z| < 1.$$

The power series expansions of the second factor in the right-hand member of (20.4) and of its reciprocal yield by direct comparison of coefficients the two sets of equations

$$\begin{aligned} b_1 &= - a_1 \alpha, \\ b_2 &= a_1(1 - \alpha\bar{\alpha}) - a_2 \alpha, \\ b_3 &= a_1 \bar{\alpha}(1 - \alpha\bar{\alpha}) + a_2(1 - \alpha\bar{\alpha}) - a_3 \alpha, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ b_k &= a_1 \bar{\alpha}^{k-2}(1 - \alpha\bar{\alpha}) + a_2 \bar{\alpha}^{k-3}(1 - \alpha\bar{\alpha}) + \cdots + a_{k-1}(1 - \alpha\bar{\alpha}) - a_k \alpha; \end{aligned} \tag{20.5}$$

$$\begin{aligned} a_1 &= - \frac{b_1}{\alpha}, \qquad \alpha \neq 0, \\ a_2 &= - b_1 \frac{1 - \alpha\bar{\alpha}}{\alpha^2} - \frac{b_2}{\alpha}, \\ a_3 &= - b_1 \frac{1 - \alpha\bar{\alpha}}{\alpha^3} - b_2 \frac{1 - \alpha\bar{\alpha}}{\alpha^2} - \frac{b_3}{\alpha}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ a_k &= - b_1 \frac{1 - \alpha\bar{\alpha}}{\alpha^k} - b_2 \frac{1 - \alpha\bar{\alpha}}{\alpha^{k-1}} - \cdots - b_{k-1} \frac{1 - \alpha\bar{\alpha}}{\alpha^2} - \frac{b_k}{\alpha}. \end{aligned} \tag{20.6}$$

---

(56) The succeeding proof can be considerably shortened if no numerical estimate for c_p is desired. The left-hand member of (20.2) is a continuous function of the numbers β_j in the closed limited point set $|\beta_j| \leqq 1$, hence takes on a minimum value c_p; we must prove $c_p > 0$. By the hypothesis in the induction, the minimum value zero cannot be taken on when one or several numbers β_j vanish, for then by (20.3) the left-hand member of (20.2) equals the corresponding sum with p replaced by some $m < p$ for some function $g_m(z)$: $g_p(z) = z^{p-m} g_m(z)$. The minimum value zero cannot be taken on when all of the numbers β_j are different from zero $c_p \geqq |g'(0)| = |\beta_1 \beta_2 \cdots \beta_{p-1}| > 0$. Thus (20.2) is established.

The following series of steps is a consequence of equations (20.5):

$$
\begin{aligned}
b_1 &= -a_1\alpha, & b_1 &= -a_1\alpha,\\
b_2 - b_1\bar{\alpha} &= a_1 - a_2\alpha, & b_2 - a_1 &= b_1\bar{\alpha} - a_2\alpha,\\
b_3 - b_2\bar{\alpha} &= a_2 - a_3\alpha, & b_3 - a_2 &= b_2\bar{\alpha} - a_3\alpha,\\
&\cdots\cdots, & &\cdots\cdots,\\
b_k - b_{k-1}\bar{\alpha} &= a_{k-1} - a_k\alpha, & b_k - a_{k-1} &= b_{k-1}\bar{\alpha} - a_k\alpha,\\
&\cdots\cdots; & &\cdots\cdots;
\end{aligned}
$$

$$
\begin{aligned}
&|b_2 - a_1| + |b_3 - a_2| + \cdots + |b_p - a_{p-1}|\\
&\qquad \leqq [|b_1| + |b_2| + \cdots + |b_{p-1}|]\cdot|\alpha|\\
&\qquad\quad + [|a_2| + |a_3| + \cdots + |a_{p-1}|]\cdot|\alpha| + |a_p|\cdot|\alpha|;\\
&[|a_1| + |a_2| + \cdots + |a_{p-1}|] - [|b_2| + |b_3| + \cdots + |b_p|]\\
&\qquad \leqq [|b_1| + |b_2| + \cdots + |b_{p-1}|]\cdot|\alpha|\\
&\qquad\quad + [|a_2| + |a_3| + \cdots + |a_{p-1}|]\cdot|\alpha| + |a_p|\cdot|\alpha|.
\end{aligned}
$$

Cauchy's inequality for the function $g_{p-1}(z)$ informs us that $|a_p| \leqq 1$, so we may write

$$
\begin{aligned}
&|b_1| + |b_2| + \cdots + |b_p|\\
&\qquad \geqq [|a_1| + |a_2| + \cdots + |a_{p-1}|]\frac{1-|\alpha|}{1+|\alpha|} - |a_p|\frac{|\alpha|}{1+|\alpha|}\\
&\qquad \geqq c_{p-1}\frac{1-|\alpha|}{1+|\alpha|} - \frac{|\alpha|}{1+|\alpha|}.
\end{aligned}
\tag{20.7}
$$

Case I. $|\alpha| \leqq c_{p-1}/2$. For $p \geqq 2$ we have $c_{p-1} \leqq 1/4$, $|\alpha| \leqq 1/8$; so the last member of (20.7) is not less than

$$c_{p-1}\left[\frac{7}{9} - \frac{1}{2}\right] = \frac{5}{18}c_{p-1} > \frac{c_{p-1}}{2^{p\cdot p!}} = c_p.$$

Case II. $|\alpha| > c_{p-1}/2$. Here we replace each term of each of equations (20.6) by the corresponding absolute value. The resulting *inequalities* when added member for member with $k = p-1$ become (for abbreviation we write $|\alpha| = a$)

$$
\begin{aligned}
&\left(\frac{1}{a^{p-1}} + \frac{1}{a^{p-2}} - 1\right)|b_1| + \left(\frac{1}{a^{p-2}} + \frac{1}{a^{p-3}} - 1\right)|b_2| + \cdots + \frac{1}{a}|b_{p-1}|\\
&\qquad\qquad \geqq |a_1| + |a_2| + \cdots + |a_{p-1}| \geqq c_{p-1}.
\end{aligned}
$$

The coefficient of $|b_1|$ is here not less than the coefficients of $|b_2|$, $|b_3|$, $\cdots$, $|b_{p-1}|$; so we obtain at once from $a > c_{p-1}/2$

$$
\begin{aligned}
|b_1| + |b_2| + \cdots + |b_p| \\
&\geqq \frac{c_{p-1}}{\dfrac{1}{a^{p-1}} + \dfrac{1}{a^{p-2}} - 1} \geqq \frac{a^{p-1}}{2} c_{p-1} \geqq \frac{1}{2}\left(\frac{c_{p-1}}{2}\right)^{p-1} c_{p-1} \\
&= \left(\frac{c_{p-1}}{2}\right)^{p} \geqq \frac{1}{2^{(p+1)!}} = c_p.
\end{aligned}
$$

Theorem 3 is completely established.

It is obvious that the choice $c_p = 2^{-(p+1)!}$ can be considerably improved by the present method alone.

It is quite natural to divide the proof of Theorem 3 into two cases depending on the size of $|\alpha|$, comparing b_j with a_{j-1} when $|\alpha|$ is small and comparing b_j with a_j when $|\alpha|$ is large. For it follows from (20.4) that $b_j = a_{j-1}$ when $\alpha = 0$ and that $|b_j| = |a_j|$ when $|\alpha| = 1$.

21. **Numerical upper bound for D_p.** Theorem 3, of some interest in itself, is an important step in the proof of Theorem 2. Another preliminary proposition is

THEOREM 4. *Let the function $w = f(z)$ analytic in $|z| < 1$ with $f(0) = 0$ map a smooth region R interior to $|z| = 1$ onto the unit circle $|w| < 1$ covered precisely p times or precisely m times, $m < p$. Then we have*

$$
(21.1) \qquad |f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)| \geqq \gamma_p > 0,
$$

where the number γ_p depends on p but not on R or $f(z)$. To be explicit, we shall establish (21.1) *with $\gamma_p = 2^{-(p+1)!-p}$.*

We shall make use of the analyticity of $f(z)$ only in R, not throughout the entire region $|z| < 1$.

Denote by $z = h(Z)$ a function which maps the region $|Z| < 1$ smoothly onto the region R of the z-plane, with $h(0) = 0$. Then the function $w = g(Z) = f[h(Z)]$ maps the region $|Z| < 1$ onto the unit circle $|w| < 1$ covered precisely p times or precisely $m < p$ times, with $g(0) = 0$, so $g(Z)$ satisfies the hypothesis of Theorem 3.

Let us introduce the notation

$$
\begin{aligned}
g(Z) &= a_1 Z + a_2 Z^2 + \cdots, \\
f(z) &= b_1 z + b_2 z^2 + \cdots, \\
h(Z) &= d_1 Z + d_2 Z^2 + \cdots.
\end{aligned}
$$

We note that Cauchy's inequality for the function $h(Z)$ yields

$$
(21.2) \qquad |d_k| \leqq 1, \qquad k = 1, 2, \cdots.
$$

The coefficients of $f(z)$ and $g(Z)$ are related by equations that we now need to consider:

$$
\begin{aligned}
g(Z) &= f[h(Z)] \\
&= b_1[d_1Z + d_2Z^2 + d_3Z^3 + \cdots] \\
&\quad + b_2[d_1Z + d_2Z^2 + d_3Z^3 + \cdots]^2 \\
&\quad + b_3[d_1Z + d_2Z^2 + d_3Z^3 + \cdots]^3 \\
&\quad + \cdots\cdots\cdots \\
&= a_1Z + a_2Z^2 + a_3Z^3 + \cdots .
\end{aligned} \tag{21.3}
$$

By equating coefficients of corresponding powers of Z we obtain

$$
\begin{aligned}
a_1 &= b_1d_1, \\
a_2 &= b_1d_2 + b_2d_1^2, \\
a_3 &= b_1d_3 + 2b_2d_1d_2 + b_3d_1^3, \\
a_4 &= b_1d_4 + b_2(d_2^2 + 2d_1d_3) + 3b_3d_1^2d_2 + b_4d_1^4, \\
a_5 &= b_1d_5 + b_2(2d_1d_4 + 2d_2d_3) + b_3(3d_1d_2^2 + 3d_1^2d_3) + b_4(4d_1^3d_2) + b_5d_1^5, \\
&\cdots\cdots\cdots
\end{aligned} \tag{21.4}
$$

The law of the coefficients of the b_k in equations (21.4) is relatively simple, and is readily formulated in terms of the subscripts of the numbers a_j and b_k, and involves primarily the partitions of the subscripts of the numbers a_j. The precise law would be a needless refinement for our present relatively rough purposes. If we replace each b_k by unity, it is obvious from (21.2) that the function $g(Z)$ in (21.3) is dominated by

$$
\begin{aligned}
&[Z + Z^2 + Z^3 + \cdots] + [Z + Z^2 + Z^3 + \cdots]^2 \\
&+ [Z + Z^2 + Z^3 + \cdots]^3 + \cdots \\
&= \frac{Z}{1 - 2Z} = Z + 2Z^2 + 4Z^3 + 8Z^4 + \cdots .
\end{aligned}
$$

Then the sum of the absolute values of all the coefficients of all the numbers b_j in the first p of equations (21.4) is not greater than $1+2+4+ \cdots +2^{p-1}$, which is less than 2^p. Insertion in each of equations (21.4) of absolute value signs on the numbers a_j, on the numbers b_j, and on the coefficients of the numbers b_j yields a corresponding inequality. When the first p of these inequalities are added member for member, there results the inequality

$$
|a_1| + |a_2| + \cdots + |a_p| \leqq 2^p[|b_1| + |b_2| + \cdots + |b_p|],
$$

so (21.1) with $\gamma_p = 2^{-(p+1)!-p}$ is a consequence of Theorem 3.

We are now in a position to prove Theorem 2; the trivial case $D_p(0) = 0$ needs no further discussion and is henceforth excluded. Under the hypothesis of Theorem 2 the function

$$w_1(z) = \frac{f(z)}{D_p(0)} \tag{21.5}$$

is analytic for $|z| < 1$ and maps a smooth region R interior to $|z| = 1$ onto the region $|w_1| < 1$ covered precisely p times or precisely $m < p$ times, with $w_1(0) = 0$. Theorem 4 applied to the function (21.5) yields at once inequality (20.1). Theorem 2 is established, and we may state the

COROLLARY. *In Theorem 2 we may take* $\gamma_p = 2^{-(p+1)!-p}$.

The number $2^{-(p+1)!-p}$ can obviously be greatly improved, even without change of method.

Theorem 2 is stated in the form convenient for applications, but we have used in the proof the analyticity of $f(z)$ not in the entire region $|z| < 1$, only in a neighborhood of the origin. However, if $f(z)$ is analytic in a region containing points for which $|z| \geqq 1$, the number $D_p(0)$ is to be defined as referring to the Riemann configuration which is the image of $|z| < 1$ under the transformation $w = f(z)$. Theorem 2 is false if the points $|z| \geqq 1$ are not excluded, as is shown by the example $p = 1, f(z) \equiv z$.

It is clear now that from Theorem 1 (with Corollary 1) and Theorem 2 of the present chapter, Theorem 19 of Chapter II may be obtained in the explicit form of inequalities. Indeed, we have

THEOREM 5. *Let $f(z)$ be regular in $|z| < 1$ and bounded there:*

$$|f(z)| < M.$$

Let $\{z_n\}$ ($|z_n| < 1$) be a sequence of points in $|z| < 1$ and let $w_n = f(z_n)$. Then, there exist two constants λ_p and Λ_p of which λ_p depends on p alone, while Λ_p depends on p and M so that

$$\begin{aligned} \lambda_p \cdot D_p(w_n) &\leqq \sum_{k=1}^{p} \left| \sum_{\nu=0}^{k-1} (-1)^{\nu} C_{k-1,\nu} \bar{z}_n^{\nu} \frac{(1 - |z_n|^2)^{k-\nu} f^{(k-\nu)}(z_n)}{(k-\nu)!} \right| \\ &\leqq \Lambda_p [D_p(w_n)]^{2^{-p}}, \end{aligned} \tag{21.6}$$

where $D_p(w_n)$ is the radius of p-valence at the point w_n of the Riemann surface on which $w = f(z)$ maps the circle $|z| < 1$.

The writers are not informed as to whether the exponent 2^{-p} in (21.6) is the best possible one. Here, and in improving the constants λ_p and Λ_p already obtained, lie a number of interesting open problems.

As a consequence of the second half of inequality (21.6) and the example of §9, Theorem 8 we may state

THEOREM 6. *Let the function $Q(r)$ be defined and positive for $0 < r < 1$, with* $\lim_{r\to 1} Q(r) = 0$. *Let the positive integer m be given. Then there exist a function*

$w=f(z)$ analytic and univalent in $|z|<1$, continuous in $|z|\leqq 1$, and a sequence of points $z_1, z_2, \cdots$ with $|z_n|<1$, $|z_n|\to 1$, such that we have

$$\lim_{n\to\infty} \frac{D_m(w_n)}{Q(|z_n|)} = \infty,$$

where $w_n=f(z_n)$.

Chapter IV. Functions which omit two values

22. **Inequalities for $D_p(w_n)$ when $|f(z_n)|$ is bounded.** Practically all the results of the Chapters II and III may be extended to the class of functions $f(z)$ regular in the circle $|z|<1$ which in that circle differ from 0 and 1([57]). To be more specific, suppose that $f(z)$ is regular in the circle $|z|<1$ and that $f(z)\neq 0, 1$ in $|z|<1$. Let $\{z_n\}$ ($|z_n|<1$) be an arbitrary sequence of points in the circle so that $|f(z_n)|$ remains bounded for all n. Under these assumptions what is a necessary and sufficient condition that $(1-|z_n|)^k f^{(k)}(z_n)\to 0$ ($k=1, 2, \cdots, p$)? If we examine the proof of Theorem 19, Chapter II, we notice that absolutely no modification is necessary in order to extend this theorem to the case under consideration since we are again dealing with a normal family $\{\phi_n(\zeta)\}$ which, due to the condition that $|f(z_n)|$ is bounded, does not contain the infinite constant. The proof of Theorem 19, therefore, may be repeated verbatim to yield

Theorem 1. *Let $f(z)$ be regular in $|z|<1$ and $f(z)\neq 0$, 1 there. Let $\{z_n\}$ ($|z_n|<1$) be a sequence of points in $|z|<1$ such that $|f(z_n)|<M$ for all n. Then, a necessary and sufficient condition that*

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k = 1\ 2\ \cdots, p,$$

for a fixed positive integer p is that

$$\lim_{n\to\infty} D_p(w_n) = 0,$$

where $w_n=f(z_n)$.

Again as in the case of Theorem 19 it is desirable to give explicitly the relation between $|f^{(p)}(z_n)|\ (1-|z_n|^2)^p$ and $D_p(w_n)$. In view of §21, Theorem 5 and Schottky's theorem this relation is easily obtained. We use Schottky's theorem in the following form([58]): *If $f(z)$ is regular in $|z|<1$ and omits there the values zero and one, if $f(z)=a_0+a_1z+\cdots$, then there exists a positive constant Δ, independent of a_0, θ, $a_1, \cdots$ so that*

$$|f(z)| < [|a_0|+2]^{\Delta/(1-\theta)} \tag{22.1}$$

in the circle $|z|<\theta<1$.

([57]) The case $f(z)\neq a, b$ may always be reduced to the above case by considering $\phi(z) =(f(z)-a)/(b-a)$.

([58]) Cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 2d. edition, 1931, p. 224.

Let us assume now that the hypotheses of Theorem 1 are satisfied, and form the functions

$$\phi_n(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n\zeta}\right). \tag{22.2}$$

These functions are all regular in $|\zeta| < 1$ and omit there the two values 0 and 1. Furthermore, $\phi_n(0) = f(z_n) = w_n$ are bounded in absolute value by the constant M:

$$|\phi_n(0)| < M, \qquad n = 1, 2, \cdots.$$

Applying Schottky's theorem in the form (22.1) to the functions $\phi_n(\zeta)$, we find that

$$|\phi_n(\zeta)| < [M + 2]^{\Delta/(1-\theta)} = M_\theta$$

in the circle $|\zeta| < \theta < 1$. If we set now

$$g_n(\zeta) = \phi_n(\theta\zeta), \tag{22.3}$$

we obtain a regular function $g_n(\zeta)$ in the circle $|\zeta| < 1$ which satisfies the inequality $|g_n(\zeta)| < M_\theta$ in the whole circle $|\zeta| < 1$. Finally we set

$$h_n(\zeta) = g_n(\zeta) - g_n(0), \tag{22.4}$$

so that $h_n(\zeta)$ is regular in $|\zeta| < 1$, $h_n(0) = 0$, and

$$|h_n(\zeta)| < 2M_\theta.$$

Now, according to §21, Theorem 5, we have

$$\lambda_p \cdot D_p(0) \leqq |h_n'(0)| + \frac{1}{2!}|h_n''(0)| + \cdots + \frac{1}{p!}|h_n^{(p)}(0)| \leqq \Lambda_p \cdot [D_p(0)]^{2^{-p}},$$

where $D_p(0)$ is the radius of p-valence (§14) at the point $w = 0$ of the Riemann configuration R_n on which $w = h_n(z)$ maps the circle $|z| < 1$. From (22.4) we obtain

$$\lambda_p \cdot D_p(0) \leqq |g_n'(0)| + \frac{1}{2!}|g_n''(0)| + \cdots + \frac{1}{p!}|g_n^{(p)}(0)| \leqq \Lambda_p \cdot [D_p(0)]^{2^{-p}}.$$

But $g_n(\zeta)$ maps $|\zeta| < 1$ on a Riemann configuration R_n' obtained from R_n by translating it along the vector $g_n(0)$. Therefore, $D_p(0)$ is equal to the radius of p-valence of R_n' at the point $w = g_n(0)$. This radius we shall denote by $D_p[g_n(0)]$. We thus obtain

$$\lambda_p \cdot D_p[g_n(0)] \leqq |g_n'(0)| + \frac{1}{2!}|g_n''(0)| + \cdots + \frac{1}{p!}|g_n^{(p)}(0)|$$
$$\leqq \Lambda_p[D_p[g_n(0)]]^{2^{-p}}.$$

By virtue of (22.3) this becomes

$$\begin{aligned}\lambda_p\cdot D_p[g_n(0)] &\leqq \theta\,|\phi_n'(0)| + \frac{\theta^2}{2!}|\phi_n''(0)| + \cdots + \frac{\theta^p}{p!}|\phi_n^{(p)}(0)| \\ &\leqq \Lambda_p\cdot [D_p[g_n(0)]]^{2^{-p}}.\end{aligned} \tag{22.5}$$

Now, the Riemann surface R_n' can simply be considered as the surface on which the function $w=\phi_n(\zeta)$ maps the circle $|\zeta|<\theta$. It is, therefore, merely a part of the surface R on which $\phi_n(\zeta)$, and by (22.2) $w=f(z)$, maps the circle $|z|<1$. If we denote by $D_p(w_n)$ the radius of p-valence of R at the point $w=w_n$, we clearly must have

$$D_p[g_n(0)] \leqq D_p(w_n).$$

We may, therefore, infer the inequality

$$\theta\,|\phi_n'(0)| + \frac{\theta^2}{2!}|\phi_n''(0)| + \cdots + \frac{\theta^p}{p!}|\phi_n^{(p)}(0)| \leqq \Lambda_p\cdot [D_p(w_n)]^{2^{-p}}.$$

Now, since $0<\theta<1$, we find

$$|\phi_n'(0)| + \frac{1}{2!}|\phi_n''(0)| + \cdots + \frac{1}{p!}|\phi_n^{(p)}(0)| \leqq \frac{\Lambda_p}{\theta^p}[D_p(w_n)]^{2^{-p}}.$$

According to (22.2) and (2.3) we obtain

$$\sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^\nu C_{k-1,\nu}\bar{z}_n^\nu\frac{(1-|z_n|^2)^{k-\nu}f^{(k-\nu)}(z_n)}{(k-\nu)!}\right| \leqq \frac{\Lambda_p}{\theta^p}[D_p(w_n)]^{2^{-p}}. \tag{22.6}$$

This gives us the desired inequality from above. The corresponding inequality from below, is contained in §20, Theorem 2:

$$\lambda_p\cdot D_p(w_n) \leqq \sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^\nu C_{k-1,\nu}\bar{z}_n^\nu\frac{(1-|z_n|^2)^{k-\nu}f^{(k-\nu)}(z_n)}{(k-\nu)!}\right|. \tag{22.7}$$

We may, therefore, state the following

THEOREM 2. *Let $f(z)$ be regular in $|z|<1$ and $f(z)\neq 0$, 1 there. Let $\{z_n\}$ ($|z_n|<1$) be a sequence of points in $|z|<1$ such that $|f(z_n)|<M$ for all n. Then, for any $0<\theta<1$ there exist two constants λ_p and Λ_p of which λ_p depends on p alone, while Λ_p depends on p, M, and θ, so that*

$$\begin{aligned}\lambda_p\cdot D_p(w_n) &\leqq \sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^\nu C_{k-1,\nu}\bar{z}_n^\nu\frac{(1-|z_n|^2)^{k-\nu}f^{(k-\nu)}(z_n)}{(k-\nu)!}\right| \\ &\leqq \frac{\Lambda_p}{\theta^p}[D_p(w_n)]^{2^{-p}},\end{aligned} \tag{22.8}$$

where $D_p(w_n)$ is the radius of p-valence at the point $w_n=f(z_n)$ of the Riemann surface on which $w=f(z)$ maps the circle $|z|<1$.

Since from the form of Λ_p it is evident that it tends to infinity as θ tends to 1, the best value for the right side of (22.8) is obtained for that value of θ for which Λ_p/θ^p attains its minimum. That value may be readily computed from the expression for Λ_p. It is evident also that Theorem 2 implies Theorem 1.

We remark that under the conditions of Theorem 2 we have $D_p(w)\leqq|w|$, so that (22.8) gives an inequality on the approach to zero of $(1-|z|^2)^k f^{(k)}(z)$ as w tends to zero, for every k.

A further consequence of Theorem 2 is that under the hypothesis of that theorem, an additional inequality of the form $|f(z)|\leqq M$ implies inequalities $|f^{(k)}(z)|(1-|z|^2)^k\leqq M_k$, where M_k depends only on k and M. Indeed, we have $D_p(w)\leqq M$; our conclusion[59] follows from (22.8).

23. **Counterexamples.** In Theorems 1 and 2 an important part of the hypothesis was the fact that $|f(z_n)|<M$ for all n. Since any sequence $\{z_n\}$ can be decomposed into sequences on which $|f(z_n)|$ is bounded and those on which $|f(z_n)|$ tends to infinity, it is natural to inquire how far Theorem 2 can be extended to sequences $\{z_n\}$ for which $|f(z_n)|\to\infty$.

That the conclusion of Theorem 2 as a proposition is false for such sequences is a theorem which we shall establish:

THEOREM 3. *There exists a function $f(z)$ with two omitted values and regular in $|z|<1$ and there exists a sequence of points $\{z_n\}$ ($|z_n|<1$, $|z_n|\to 1$) such that, setting $w_n=f(z_n)$, we have $D_1(w_n)\to 0$, $w_n\to\infty$, and yet* $\lim_{n\to\infty}|f'(z_n)|(1-|z_n|^2)=8\pi$.

In the half-plane $\Re W>0$, where $W=u+iv$,

$$\Re(W+e^{-W+1}) = u+e^{-u+1}\cos v \geqq -e^{-u+1}\geqq -e.$$

Consequently, in $\Re W>0$ the function $W+e^{-W+1}+3$ omits all values in some neighborhood of the origin, as does the function

$$w=f(z)=(W+e^{-W+1}+3)^2=F(W), \tag{23.1}$$

where we set $W=(1+z)/(1-z)$, so that z is a point of the unit circle $|z|<1$. We choose $W_n=1+2n\pi i+1/n$, whence $e^{-W_n+1}=e^{-1/n}$ and find

$$\frac{df(z)}{dW}=2(W+e^{-W+1}+3)(1-e^{-W+1}). \tag{23.2}$$

Thus, $f'(z)$ vanishes in the points where $1-e^{-W+1}=0$, namely $W=1+2n\pi i$, $n=0,\pm 1,\pm 2,\cdots$. If we define z_n by the relation $W_n=(1+z_n)/(1-z_n)$, we find from (23.2)

(59) More precise inequalities of this type were developed by O. Szász, loc. cit.

$$\frac{df(z_n)}{dW} = 2\left(4 + 2n\pi i + \frac{1}{n} + e^{-1/n}\right)(1 - e^{-1/n}),$$

so that $df(z_n)/dW \to 4\pi i$. We next compute $|1-z|^2 = 4/|W+1|^2$ and $|dW/dz| = |W+1|^2/2$. Hence,

$$\left|\frac{dW}{dz}\right|_{z=z_n} (1 - |z_n|^2) = \frac{1}{2}\left[|W_n + 1|^2 - |W_n - 1|^2\right] = 2 + \frac{2}{n} \to 2.$$

Thus, we obtain finally

$$|f'(z_n)|(1 - |z_n|^2) \to 8\pi.$$

It now remains to be shown that $D_1(w_n) \to 0$. This may be shown as follows. In the W-plane consider the two points $W = 1 + 2n\pi i$ and $W_n = 1 + 2n\pi i + 1/n$. Join these two points by a rectilinear segment, necessarily horizontal. This segment is mapped by the function $w = F(W)$ on a certain arc lying on the corresponding Riemann configuration and joining the points $w = (5 + 2n\pi i)^2$ and $w_n = (4 + e^{-1/n} + 1/n + 2n\pi i)^2$, of which the first is a branch point of the Riemann configuration in question. It is clear, therefore, that this arc emanates from the center of the circle $|w - w_n| \leqq D_1(w_n)$ and terminates in a point lying exterior to or on the boundary of that circle[60]. Hence, the length of this arc cannot be less than $D_1(w_n)$. But the length can be estimated directly. Indeed, it is equal to

$$\int_1^{1+1/n} |F'(2n\pi i + u)|\, du.$$

From (23.2) we find

$$D_1(w_n) \leqq 2\int_1^{1+1/n} |2n\pi i + u + e^{-u+1} + 3|(1 - e^{-u+1})du.$$

Now, in the interval $1 \leqq u \leqq 1 + 1/n$, we have $1 - e^{-u+1} \leqq 1 - e^{-1/n}$ and $e^{-u+1} \leqq 1$, so that

$$D_1(w_n) \leqq 2(1 - e^{-1/n})(2n\pi + 5 + 1/n)\cdot 1/n. \tag{23.3}$$

Hence, as $n \to \infty$, we have $D_1(w_n) \to 0$, which completes the proof of the theorem.

In connection with the present example one may make two remarks.

Remark 1. If one replaces the function $f(z)$ in (23.1) by the function

$$f(z) = (W + e^{-W+1} + 3)^4, \qquad W = (1 + z)/(1 - z), \tag{23.4}$$

---

(60) Study of the variation of arg (dw) on the arc shows that the arc lies wholly in the circle in question, and hence that $D_1(w_n) = |F(W) - F(W_n)|$, where $W = 1 + 2n\pi i$. A similar fact holds under Remarks 1 and 2.

with $W_n+1=2n\pi i+1/n^2$, clearly the relation $D_1(w_n)\to 0$ still holds, while $|f'(z_n)|(1-|z_n|^2)\to\infty$. Thus, $D_1(w_n)\to 0$ *does not even imply the boundedness* of $|f'(z_n)|\ (1-|z_n|^2)$.

Remark 2. Let α be any real number in the interval $0<\alpha<1$. Choose an integer k so that $k>\alpha/(1-\alpha)$. Then, the choice

$$f(z) = (W + e^{-W+1} + 3)^{k+1}, \qquad W = (1+z)/(1-z),$$

with $W_n = 1+2n\pi i+1/n^k$ yields $D_1(w_n)\to 0$. Indeed, a computation analogous to the one in the preceding example shows that $D_1(w_n)=O(1/n^k)$. On the other hand, $|w_n| = O(n^{k+1})$. Hence $|w_n|^\alpha\cdot D_1(w_n)=O(n^{\alpha k+\alpha-k})$ and this expression tends to zero. Furthermore, it is easily seen that $|f'(z_n)|\ (1-|z_n|^2)>c>0$, where c is a certain positive constant. Thus, *for the class of functions with a region of omitted values no relation* $|w_n|^\alpha\cdot D_1(w_n)\to 0$ *with* $0<\alpha<1$ *can imply* $|f'(z_n)|\ (1-|z_n|^2)\to 0$.

In the example of Theorem 3 and the examples in the two remarks it will be noticed that $|w_n|\cdot D_1(w_n)$ does not tend to zero. The case that $|w_n|\cdot D_1(w_n)$ tends to zero will not be treated in its full generality in this paper. A special case is considered in §25. The case $|w_n|^\alpha\cdot D_1(w_n)\to 0$ for $\alpha>1$ will be considered in the next section.

24. **Case:** $\lim_{n\to\infty} |w_n|^{(1+\epsilon)(2^p-1)}D_p(w_n)=0$. The following extension of Theorem 1 for $p=1$ to the case $|w_n|\to\infty$ will now be proved:

THEOREM 4. *Let* $f(z)$ *be analytic in* $|z|<1$ *and omit two values there. Let* $\{z_n\}$ $(|z_n|<1)$ *be a sequence of points in* $|z|<1$ *such that, setting* $w_n=f(z_n)$, *we have* $\lim_{n\to\infty}|w_n|=\infty$. *Then, the condition*

$$\lim_{n\to\infty} |w_n|^{1+\epsilon}D_1(w_n) = 0 \tag{24.1}$$

for any positive ϵ *implies*

$$\lim_{n\to\infty} |f'(z_n)|\,(1-|z_n|^2) = 0. \tag{24.2}$$

It is clear that the sequence of functions

$$\phi_n(\zeta) = f\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right)$$

regular in $|\zeta|<1$ is normal. Since by hypothesis $\lim_{n\to\infty}|w_n|=\infty$, we have

$$\lim_{n\to\infty} |\phi_n(0)| = \infty,$$

so that

$$\lim_{n\to\infty} |\phi_n(\zeta)| = \infty \tag{24.3}$$

uniformly in every closed subregion of $|\zeta|<1$.

Choose a positive number

$$\rho < \frac{\epsilon}{2+\epsilon} < 1,$$

whence

$$\frac{1+\rho}{1-\rho} < 1+\epsilon.$$

It follows from (24.3) that for n sufficiently large the function $1/\phi_n(\zeta)$ is regular in the circle $|\zeta| \leqq \rho_1$, where ρ_1 is any number such that $\rho < \rho_1 < 1$. Furthermore, n may be chosen so large that $1/|\phi_n(\zeta)| < 1$ in $|\zeta| \leqq \rho_1$, which implies that $\log |\phi_n(\zeta)|$ is harmonic and positive in $|\zeta| \leqq \rho_1$. Then, using Poisson's integral for the region $|\zeta| < \rho_1$, one sees immediately that

$$\log |\phi_n(\zeta)| \leqq \frac{\rho_1+\rho}{\rho_1-\rho} \log |\phi_n(0)| \tag{24.4}$$

in the circle $|\zeta| \leqq \rho < \rho_1$. Now, by taking ρ_1 so near to unity that $\rho_1+\rho/\rho_1-\rho < 1+\epsilon$ and then by choosing n sufficiently large, the inequality (24.4) implies $|\phi_n(\zeta)| < |\phi_n(0)|^{1+\epsilon}$ in the circle $|\zeta| \leqq \rho$([61]).

Now, according to Theorem 3 of Chapter II,

$$D_1(w_n) \geqq \frac{|\phi_n'(0)|^2 r^2}{8M_n},$$

where $M_n \geqq \max_{|\zeta| \leqq r} |\phi_n(\zeta)|$. If we set $r=\rho$, $M_n = |\phi_n(0)|^{1+\epsilon}$, $\phi_n(0) = w_n$, we obtain for n sufficiently large

$$D_1(w_n) \geqq \frac{|f'(z_n)|^2(1-|z_n|^2)^2\rho^2}{8|w_n|^{1+\epsilon}}, \tag{24.5}$$

from which the theorem follows at once.

The treatment of the case for general p is quite analogous:

THEOREM 5. *Let $f(z)$ be analytic in $|z| < 1$ and omit two values there. Let $\{z_n\}$ ($|z_n| < 1$) be a sequence of points in $|z| < 1$ such that, setting $w_n = f(z_n)$, $\lim_{n\to\infty} |w_n| = \infty$. Then, the condition*

$$\lim_{n\to\infty} |w_n|^{(1+\epsilon)(2^p-1)} D_p(w_n) = 0 \tag{24.6}$$

for any positive ϵ implies

$$\lim_{n\to\infty} |f^{(j)}(z_n)| (1-|z_n|^2)^j = 0, \qquad j = 1, 2, \cdots, p. \tag{24.7}$$

([61]) The reasoning employed in the proof of this inequality is well known. Cf. A. Ostrowski, Abhandlungen des Mathematischen Seminars der Hamburgischen Universität, vol. 1 (1922), pp. 327–350; S. Mandelbrojt, Comptes Rendus de l'Académie des Sciences, Paris, vol. 185 (1927), pp. 1098–1100; H. Cartan, Annales de l'École Normale Supérieure, (3), vol. 45 (1928), pp. 255–346; J. L. Walsh, Tôhoku Mathematical Journal, vol. 38 (1933), pp. 375–389.

The proof of Theorem 4 is repeated verbatim, and we find, as before, that for any positive ϵ there exists a positive number $\rho<1$ such that for all sufficiently large values of n

$$|\phi_n(\zeta)| < |w_n|^{1+\epsilon}$$

in the circle $|\zeta| \leqq \rho$.

Now, according to inequality (19.5′), we obtain

$$\sum_{j=1}^{p} \frac{\rho^j}{j!} |\phi_n^{(j)}(0)| \leqq 24 |w_n|^{1+\epsilon} \sum_{j=1}^{p} \left(\frac{D_j(w_n)}{|w_n|^{1+\epsilon}} \right)^{1/2^j}$$

$$= 24 \sum_{j=1}^{p} (|w_n|^{(1+\epsilon)(2^j-1)} D_j(w_n))^{1/2^j},$$

whence, applying (2.3), we find for n sufficiently large

$$\sum_{j=1}^{p} \rho^j \left| \sum_{\nu=0}^{j-1} (-1)^\nu C_{j-1,\nu} \bar{z}_n^\nu \frac{(1-|z_n|^2)^{j-\nu} f^{(j-\nu)}(z_n)}{(j-\nu)!} \right|$$

$$\leqq 24 \sum_{j=1}^{p} (|w_n|^{(1+\epsilon)(2^j-1)} D_j(w_n))^{1/2^j}.$$

Since (24.6) implies the relation

$$\lim_{n\to\infty} |w_n|^{(1+\epsilon)(2^j-1)} D_j(w_n) = 0, \qquad j = 1, 2, \cdots, p,$$

we obtain (24.7).

25. **Mandelbrojt's theorem.** The following theorem is due to S. Mandelbrojt(62):

THEOREM A. *Let $f_n(z)$ be a sequence of functions analytic in a region R and tending uniformly in R to infinity. If there exists a positive constant M such that for all n and for all z in R*

$$(25.1) \qquad |\arg f_n(z)| < M$$

with some determination of the argument, then to every closed region R_1 wholly interior to R there corresponds a finite positive number α $(1<\alpha<+\infty)$ and a positive integer n_0 such that for every pair of points z_0 and z_1 in R_1 and for every $n>n_0$, the inequality

$$(25.2) \qquad \frac{1}{\alpha} < \left| \frac{f_n(z_1)}{f_n(z_0)} \right| < \alpha$$

holds.

(62) S. Mandelbrojt, loc. cit.

We indicate a proof of Theorem A[63]. Let us first prove the assertion of the theorem in the special case that R_1 is the circle C: $|z-a| \leqq \rho$ lying wholly in R. It is clear by hypothesis that for sufficiently large values of n the functions $f_n(z) \neq 0$ in C, and henceforth we shall consider only such values of n. Hence, the functions $-i \log f_n(z)$ will be regular in C, single-valued in C after a particular determination of the logarithm is selected. We choose that determination for which $\mathfrak{R}[-i \log f_n(z)] = \arg f_n(z)$, where the argument is the one asserted in (25.1). Now, take a circle C': $|z-a| \leqq \rho'$ for which $\rho' > \rho$ and which also lies wholly in R; choose n so large that $f_n(z) \neq 0$ in C'.

In C' we have the representation

$$(25.3)\qquad \begin{aligned} &-\log |f_n(a+re^{i\theta})| + \log |f_n(a)| \\ &= \frac{1}{2\pi}\int_0^{2\pi} \arg f_n(a+\rho' e^{i\phi}) \frac{2\rho' r \sin(\theta-\phi)}{\rho'^2 + r^2 - 2\rho' r\cos(\theta-\phi)}\, d\phi. \end{aligned}$$

Let $z_0 = a + r_0 e^{i\theta_0}$ and $z_1 = a + r_1 e^{i\theta_1}$ be any two points of C. We may, then, write (25.3) for the points z_0 and z_1 and subtract the second equation from the first. Thus, we obtain the equation

$$(25.4)\qquad \begin{aligned} \log\left|\frac{f_n(z_1)}{f_n(z_0)}\right| = \frac{1}{\pi}\int_0^{2\pi} \arg f_n(a+\rho' e^{i\phi}) \Bigg[&\frac{\rho' r_0 \sin(\theta_0-\phi)}{\rho'^2 + r_0^2 - 2\rho' r_0 \cos(\theta_0-\phi)} \\ &- \frac{\rho' r_1 \sin(\theta_1-\phi)}{\rho'^2 + r_1^2 - 2\rho' r_1 \cos(\theta_1-\phi)}\Bigg] d\phi. \end{aligned}$$

Taking absolute values in (25.4) and observing (25.1), we find

$$\begin{aligned} &\left|\log\left|\frac{f_n(z_1)}{f_n(z_0)}\right|\right| \\ &\leqq \frac{M}{\pi}\int_0^{2\pi}\left| \frac{\rho' r_0 \sin(\theta_0-\phi)}{\rho'^2 + r_0^2 - 2\rho' r_0 \cos(\theta_0-\phi)} - \frac{\rho' r_1 \sin(\theta_1-\phi)}{\rho'^2 + r_1^2 - 2\rho' r_1 \cos(\theta_1-\phi)}\right| d\phi. \end{aligned}$$

But since z_0 and z_1 lie in the circle C, an easy calculation shows that

$$(25.5)\qquad \left|\log\left|\frac{f_n(z_1)}{f_n(z_0)}\right|\right| \leqq \frac{4\rho\rho' M}{(\rho'-\rho)^2}.$$

The right-hand side in (25.5) is independent of the pair of points z_0 and z_1. From (25.5) follows at once the assertion of Theorem A in the case that R_1 is a circle.

We now pass to the general case. Let R_1' be any closed region contained in R and itself containing R_1 in its interior. Consider the class of all open

[63] The proof of this theorem given by Mandelbrojt, loc. cit., is not clear to the writers. The proof given in the text was suggested to the authors by Professor S. E. Warschawski.

circles with centers in R_1' contained together with their boundaries in R. In accordance with the Heine-Borel theorem one may select out of this class a finite number of circles which cover R_1'. Denote this number by N. By the first part of the proof with each one of these circles there is associated a number α_ν $(1<\alpha_\nu<\infty)$ and a positive integer λ_ν, such that for any pair of points z_0 and z_1 in that circle and for $n>\lambda_\nu$

$$\frac{1}{\alpha_\nu}<\left|\frac{f_n(z_1)}{f_n(z_0)}\right|<\alpha_\nu.$$

Let β and n_0 be the largest of the numbers α_ν and λ_ν, respectively. Then, for $n>n_0$ and for any pair of points in any one of those circles we have

$$\frac{1}{\beta}<\left|\frac{f_n(z_1)}{f_n(z_0)}\right|<\beta. \tag{25.6}$$

Now consider any two points z_0 and z_1 in R_1. Connect z_0 and z_1 by a simple polygonal line P lying wholly in R_1' and so chosen as not to be tangent to any circle of the above class. Denote by C_1 any circle of the above class which contains the point z_0. As one travels along P from z_0 to z_1, there will be a last point of intersection ζ_1 of P with the circumference of C_1. Denote by C_2 any circle of the above class which contains ζ_1. Between z_0 and ζ_1 on P choose any point ξ_1 common to both C_1 and C_2. Now, starting with the point ξ_1 which belongs to C_2, repeat the argument. We obtain in this manner a point ξ_2 of P which is common to two circles C_2 and C_3 of the above family. Proceeding in this manner, after a finite number of steps we come to a first circle C_k which contains the point z_1. It is clear from (25.6) that for $n>n_0$

$$\frac{1}{\beta^N}\leqq\frac{1}{\beta^k}<\left|\frac{f_n(z_1)}{f_n(z_0)}\right|=\left|\frac{f_n(\xi_1)}{f_n(z_0)}\right|\cdot\left|\frac{f_n(\xi_2)}{f_n(\xi_1)}\right|\cdots\left|\frac{f_n(z_1)}{f_n(\xi_{k-1})}\right|<\beta^k\leqq\beta^N.$$

Setting $\beta^N=\alpha$, we obtain the constant asserted in Theorem A.

As Mandelbrojt himself points out, these results may be readily extended to the case of a sequence of functions $f_n(z)$ regular in R which converges uniformly in R to an analytic function $f(z)$ in such a manner that the differences $f_n(z)-f(z)$ do not vanish in R.

Theorem A may be used to obtain a result related to Theorem 4.

THEOREM 6. *Let $f(z)$ be analytic in $|z|<1$ and omit two values there, including the value $w=a$. Let $\{z_n\}$ $(|z_n|<1)$ be a sequence of points in $|z|<1$ such that, setting $w_n=f(z_n)$, we have* $\lim_{n\to\infty}|w_n|=\infty$. *If* arg $[f(z)-a]$ *is uniformly bounded in $|z|<1$*(64)*, then the condition*

$$\lim_{n\to\infty}|w_n|\cdot D_1(w_n)=0$$

(64) Geometrically, this condition means that the Riemann surface on which $w=f(z)$ maps the unit circle $|z|<1$ does not wind infinitely many times about the point $w=a$.

implies the relation

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|) = 0.$$

The boundedness of arg $[f(z)-a]$ implies the boundedness in $|\zeta|<1$ of arg $[\phi_n(\zeta)-a]$, where

$$\phi_n(\zeta) = f\left(\frac{\zeta + z_n}{1+\bar{z}_n\zeta}\right).$$

Just as in the proof of Theorem 4 we infer that

$$\lim_{n\to\infty} |\phi_n(\zeta)| = \infty$$

uniformly in every closed subregion of $|\zeta|<1$. Hence,

$$\lim_{n\to\infty} |\phi_n(\zeta) - a| = \infty$$

uniformly in every closed subregion of $|\zeta|<1$. We may therefore apply Theorem A of Mandelbrojt to the sequence of functions $\phi_n(\zeta)-a$ in the circle $|\zeta|<\rho$, where ρ is any fixed positive number less than unity. It follows that corresponding to any circle $|\zeta|\leqq\rho_1<\rho$ one may assign a finite positive number α and a positive integer n_0 such that for any pair of points ζ, ζ_0 in $|\zeta|\leqq\rho_1$

$$\frac{1}{\alpha} < \left|\frac{\phi_n(\zeta)-a}{\phi_n(\zeta_0)-a}\right| < \alpha$$

for every $n>n_0$. In particular, choosing $\zeta_0=0$, we obtain the inequality

$$|\phi_n(\zeta)-a| < \alpha|\phi_n(0)-a|$$

and

$$|\phi_n(\zeta)| < \alpha|\phi_n(0)| + (\alpha+1)|a| = \alpha|w_n| + (\alpha+1)|a| \tag{25.7}$$

in $|\zeta|\leqq\rho_1$ provided $n>n_0$.

Thus, one may apply Theorem 3 of Chapter II where $M=M_n=\alpha|w_n|+(\alpha+1)|a|$. The theorem follows at once. Conditions more delicate than those given in Theorems 4 and 5 may be obtained by different methods. Thus, it may be shown that the condition (24.1) may be replaced by the less stringent condition

$$\lim_{n\to\infty} |w_n|(\log|w_n|)^{1+\epsilon}D_1(w_n) = 0.$$

This result, and other analogous ones, will be developed in a later joint paper of A. S. Galbraith, W. Seidel, and J. L. Walsh.

26. **Counterexample for unrestricted functions.** In obtaining relations between $|f'(z)|(1-|z|)$ and $D_1(w)$ we have always restricted the class of func-

tions $f(z)$. We have thus far considered univalent functions, bounded functions, and functions omitting two values. That these or similar restrictions are essential is shown by the following example.

THEOREM 7. *There exists a function* $f(z)$ *analytic in the unit circle* $|z|<1$ *and a sequence of points* $\{z_n\}$ $(|z_n|<1,\ |z_n|\to 1)$ *such that, setting* $w_n=f(z_n)$, *we have* $D_1(w_n)\to 0$, $|w_n|$ *bounded, and*

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|) = 4\pi.$$

Consider the function

$$w = f(z) = \sin^2 W,$$

where $W=(1+z)/(1-z)$. It follows that

$$f'(z)(1-z) = \frac{2\sin 2W}{1-z}\cdot$$

Let us set

$$z_n = \frac{1/n+2n\pi-1}{1/n+2n\pi+1}, \qquad \zeta_n = \frac{2n\pi-1}{2n\pi+1}, \qquad W_n = \frac{1}{n}+2n\pi.$$

We find

$$f'(z_n)(1-z_n) = \left(1+\frac{1}{n}+2n\pi\right)\sin\frac{2}{n},$$

$$\lim_{n\to\infty} f'(z_n)(1-z_n) = 4\pi.$$

On the other hand, setting $w_n=f(z_n)$, it is clear that $D_1(w_n)$ cannot exceed the length of the image of the segment joining the points ζ_n and z_n, since the point ζ_n is mapped onto a branch point of the Riemann surface. The length of this image is given by the integral

$$\int_{\zeta_n}^{z_n}|f'(z)||dz| = \int_{\zeta_n}^{z_n}\left|\frac{2\sin 2W}{(1-z)^2}\right||dz| \leqq \frac{2}{(1-z_n)^2}(z_n-\zeta_n)$$
$$= \frac{(1/n+2\pi n+1)^2}{n(1/n+2\pi n+1)(2n\pi+1)}\cdot$$

Hence,

$$D_1(w_n) \leqq \frac{1/n+2\pi n+1}{n(2\pi n+1)}, \qquad \lim_{n\to\infty} D_1(w_n) = 0.$$

Finally $w_n=\sin^2(1/n+2\pi n)=\sin^2 1/n$, so that $\lim_{n\to\infty} w_n=0$. This completes the proof of the theorem.

The idea of this example, as well as of the examples of §§12 and 23 is the following. It is not true, as is well known(65), that $f_n(z)$ analytic for $|z|<1$, $f_n'(0)=1$, $f_n(0)=0$, implies that $w=f_n(z)$ maps $|z|<1$ onto a Riemann configuration which contains in its interior a fixed smooth circle whose center is at the origin. The simplest counterexample is perhaps

$$f_n(z) = z - nz^2.$$

The derivative $f_n'(z)=1-2nz$ vanishes for $z=1/2n$ and the corresponding value of w is $f_n(1/2n)=1/4n$, which approaches zero.

This example indicates that the phenomenon of a branch point's approaching the origin is not dependent on the transcendentality of $f_n(z)$, or even on the possibility that an ever-increasing number of sheets of the image of $|z|<1$ should come together. It is a matter primarily of having the image of a point at which $f_n'(z)$ vanishes approach the origin. The examples mentioned above were constructed with this idea in mind.

Chapter V. Miscellaneous

27. **Limit values of analytic functions.** The methods developed in the present paper have close connections with the general subject of limit values of functions analytic in the unit circle, including various theorems due to Lindelöf and to Montel. We proceed now to discuss such connections.

Theorem 1. *Let the function $f(z)$ be analytic for $|z|<1$ and omit two values there. Suppose for the sequence $\{z_n\}$ with $|z_n|<1$ we have $\lim_{n\to\infty} f(z_n)=\alpha$, where α is finite or infinite. Let the non-euclidean distance $\rho(z_n, z_n')$ between z_n and z_n' approach zero as n becomes infinite, with $|z_n'|<1$. Then we have $\lim_{n\to\infty} f(z_n')=\alpha$.*

We define as usual the functions $g_n(\zeta)$:

$$g_n(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n\zeta}\right), \tag{27.1}$$

whence $g_n(0)=f(z_n)$. If we set

$$z_n' = \frac{\zeta_n' + z_n}{1 + \bar{z}_n\zeta_n'}, \tag{27.2}$$

we have $g_n(\zeta_n')=f(z_n')$, and the non-euclidean distance

$$\rho(0, \zeta_n') = \rho(z_n, z_n') \tag{27.3}$$

approaches zero as n becomes infinite. The family $g_n(\zeta)$ omits two values in $|\zeta|<1$, hence is normal there. Given any infinite sequence of indices n, there can be extracted a subsequence for which the corresponding functions $g_n(\zeta)$ converge for $|\zeta|<1$, uniformly in every closed subregion, to some limit func-

(65) See, for example, P. Montel, *Leçons sur les Fonctions Univalentes ou Multivalentes*, Paris, 1933, p. 121, where a different example is given.

tion $g(\zeta)$, with $g(0)=\lim_{n\to\infty} g_n(0)=\alpha$. The approach of zero to $\rho(0, \zeta_n')$ implies the approach to zero of ζ_n'; so for the subsequence of indices considered the uniformity of convergence yields $\lim_{n\to\infty} g_n(\zeta_n')=\alpha$. Thus from any subsequence of the sequence $\{f(z_n')\}$ can be extracted a new subsequence converging to the limit α, which implies the conclusion of Theorem 1.

THEOREM 2. *Let $f(z)$ be analytic for $|z|<1$ and omit two values there. Let the sequence $\{z_n\}$ with $|z_n|<1$ have the property that $\lim_{n\to\infty} f(z_n)=\alpha$, where α is finite. Then a necessary and sufficient condition that the sequence $\{z_n\}$ be regular*[66] *is*

$$\lim_{n\to\infty} g_n(\zeta) = \alpha \qquad \textit{for } |\zeta| < 1, \tag{27.4}$$

uniformly in every closed subregion, where $g_n(\zeta)$ is defined by (27.1).

Let a sequence $\{z_n'\}$ be given for which $\rho(z_n, z_n')$ is bounded.

Again we define by ζ_n' equation (27.2), from which it follows that (27.3) is valid, and the non-euclidean distance $\rho(0, \zeta_n')$ is bounded. The sufficiency of (27.4) is obvious, for (27.4) implies that $g_n(\zeta_n')\to\alpha$, which is the conclusion to be established; we note that here the λ of §11, Definition 1, can be taken arbitrarily large. We proceed to show the necessity of (27.4).

If the sequence $\{z_n\}$ is regular but (27.4) is not satisfied, there exists a sequence of indices n_k such that $\lim_{k\to\infty} g_{n_k}(\zeta)=g_0(\zeta)$ for $|\zeta|<1$, uniformly in every closed subregion, where $g_0(\zeta)$ is analytic but not identically equal to α in $|\zeta|<1$. Suppose for definiteness $g_0(\zeta_0)\neq\alpha$, where the non-euclidean distance $\rho(0, \zeta_0)$ is less than the λ of §11, Definition 1. If we define z_n' by the equation

$$z_n' = \frac{\zeta_0 + z_n}{1 + \bar{z}_n\zeta_0},$$

we have

$$f(z_{n_k}') = g_{n_k}(\zeta_0) \to g_0(\zeta_0) \neq \alpha, \quad \rho(z_n, z_n') = \rho(0, \zeta_0) < \lambda,$$

contrary to hypothesis.

In Theorem 2 we have for simplicity assumed that $f(z)$ omits two values in $|z|<1$. It is obviously sufficient if $f(z)$ omits two values in the non-euclidean circle with non-euclidean center z_n and non-euclidean radius ρ_n, where ρ_n has a positive lower bound as n becomes infinite. A similar remark applies to the later results of the present section.

A consequence of the foregoing remark is that if $f(z)$ is analytic for $|z|<1$, if $|z_n|<1$, if $\lim_{n\to\infty} f(z_n)=\alpha$, where α is finite, and if the sequence $\{z_n\}$ is irregular, then $f(z)$ has at most one omitted value in each set of non-euclidean circles with non-euclidean radius ρ_n, where ρ_n has a positive lower bound. We

---

(66) For the definition of regularity see Definition 1 of §11, Chapter II.

consider pathology in more detail in §30. In Theorem 2, we have assumed the finiteness of α. A result without this restriction appears in

COROLLARY 1. *Let $f(z)$ be analytic for $|z|<1$ and omit two values there. Let the sequence $\{z_n\}$ with $|z_n|<1$ have the property that* $\lim_{n\to\infty} f(z_n)=\infty$. *Then if $|z_n'|<1$ and if the non-euclidean distance $\rho(z_n, z_n')$ is bounded, we have also* $\lim_{n\to\infty} f(z_n')=\infty$.

Since the functions $g_n(\zeta)$ form a normal family in $|\zeta|<1$ and since $g_n(0)\to\infty$, we have $\lim_{n\to\infty} g_n(\zeta)=\infty$ in $|\zeta|<1$, uniformly in every closed subregion. Our conclusion is an immediate consequence. We turn to another result.

COROLLARY 2. *Let $f(z)$ be analytic for $|z|<1$ and omit there two values including the value α. Let the sequence $\{z_n\}$ with $|z_n|<1$ have the property that* $\lim_{n\to\infty} f(z_n)=\alpha$. *Then if $|z_n'|<1$ and if the non-euclidean distance $\rho(z_n, z_n')$ is bounded, we have also* $\lim_{n\to\infty} f(z_n')=\alpha$[67].

From every infinite subsequence of the set $g_n(\zeta)$ defined by (27.1) can be extracted a new subsequence which converges for $|\zeta|<1$, uniformly in every closed subregion. The limit of this new subsequence is α in the point $\zeta=0$, hence by Hurwitz's theorem is identically α in $|\zeta|<1$. Then we have $\lim_{n\to\infty} g_n(\zeta)=\alpha$ for $|\zeta|<1$, uniformly in every subregion. Our conclusion follows as in the first part of the proof of Theorem 2. Thus the sequence $\{z_n\}$ is regular, and the number λ of §11, Definition 1, may be chosen arbitrarily. A generalization of Theorem 2 is

COROLLARY 3. *Let $f(z)$ be analytic for $|z|<1$ and omit two values there, and let the sequence $w_n=f(z_n)$ with $|z_n|<1$ be bounded. Then a necessary and sufficient condition that the sequence $\{z_n\}$ be regular is*

$$\lim_{n\to\infty} [g_n(\zeta) - g_n(0)] = 0 \qquad \text{for } |\zeta| < 1, \tag{27.5}$$

uniformly in every closed subregion, where $g_n(\zeta)$ is defined by (27.1).

If the sequence $\{z_n\}$ is regular, it follows that from any subsequence of the $g_n(\zeta)$ can be extracted a subsequence such that $\lim_{n\to\infty} g_n(\zeta)$ exists for $|\zeta|<1$, uniformly in every closed subregion; for this subsequence $\lim_{n\to\infty} g_n(0)=\alpha$ exists and by Theorem 2 the relation (27.5) holds for that subsequence. Thus, from any subsequence of the $g_n(\zeta)$ can be extracted a new subsequence such that (27.5) holds for that subsequence; so (27.5) itself is satisfied.

Conversely, if (27.5) is satisfied, and if $\rho(z_n, z_n')=\rho(0, \zeta_n')$ is bounded, it follows that $\lim_{n\to\infty} [g_n(\zeta_n')-g_n(0)]=0$, so the sequence $\{z_n\}$ is regular. Of

[67] This result for bounded functions was established by a different method by one of the present authors: W. Seidel, these Transaction, vol. 34 (1932), pp. 1–21; especially Theorem 3, p. 10.

course, this latter conclusion is independent of any assumption that $f(z)$ omit two values.

Two further propositions relate Theorem 2 to the results of §§17 and 22.

COROLLARY 4. *Let the function $f(z)$ be analytic in $|z|<1$ and omit two values there, and let the sequence $\{f(z_n)\}$ be bounded, $|z_n|<1$. A necessary and sufficient condition that $\{z_n\}$ be a regular sequence for $f(z)$ is*

$$\lim_{n\to\infty} |f^{(k)}(z_n)|(1-|z_n|)^k = 0, \qquad k = 1, 2, 3, \cdots. \tag{27.6}$$

From the sequence $f(z_n)$ can be extracted a subsequence $f(z_{n_j})$ which approaches a limit α. A necessary and sufficient condition for (27.4) for the sequence $\{n_j\}$ is

$$\lim_{n_j\to\infty} g_{n_j}^{(k)}(0) = 0, \qquad k = 1, 2, 3, \cdots, \tag{27.7}$$

since the functions $g_n(\zeta)$ form a normal family in $|\zeta|<1$. Equations (27.6) are equivalent to equations (27.7) if the latter are assumed to hold for a suitable subsequence $\{n_j\}$ of an arbitrary sequence of indices.

COROLLARY 5. *Let the function $f(z)$ be analytic and omit two values in $|z|<1$. A necessary and sufficient condition that $\{z_n\}$ be a regular sequence for $f(z)$, where we assume $w_n = f(z_n)$ bounded, is*

$$\lim_{n\to\infty} D_p(w_n) = 0, \qquad p = 1, 2, 3, \cdots.$$

Corollary 5 follows from Corollary 4 by virtue of our fundamental Theorem 1 of §22.

A further consequence of Corollary 5 is

COROLLARY 6. *Let the function $f(z)$ be analytic in $|z|<1$ and omit two values there. If the sequence of points $w_n = f(z_n)$, with $|z_n|<1$, approaches a finite boundary point of the Riemann configuration on which $w = f(z)$ maps $|z|<1$, then the sequence $\{z_n\}$ is regular.*

It is worth remarking that Corollary 2 is a consequence of Corollary 5 or Corollary 6, without the use of Hurwitz's theorem.

Theorems 1 and 2 are of particular interest if the function $f(z)$ approaches a limit along an arc.

THEOREM 3. *Let $f(z)$ be analytic in $|z|<1$ and omit two values there. Let the Jordan arc C lie in $|z|<1$ except for the end point $z=1$. Suppose*

$$\lim_{z\to 1,\, z \text{ on } C} f(z) = \alpha, \tag{27.8}$$

where α is finite. Then any sequence $\{z_n\}$ on C for which $z_n \to 1$ is regular.

From any subsequence of the sequence $g_n(\zeta)$ defined by (27.1) can be extracted a new subsequence converging to some function $g_0(\zeta)$ for $|\zeta|<1$, uniformly for $|\zeta|\leqq d<1$. Let h be arbitrary, $0<h<\infty$. Let z_n' be a point of C between the points z_n and $z=1$ with $\rho(z_n, z_n')=h$; such a point z_n' exists with $\lim_{n\to\infty} z_n'=1$. If ζ_n' is defined by (27.2), we have $\rho(0, \zeta_n')=h$. The equation (27.8) implies $\lim_{n\to\infty} f(z_n')=\alpha$, whence $\lim_{n\to\infty} g_n(\zeta_n')=\alpha$. On each circle $|\zeta|=d<1$ lies a sequence of points ζ_n' for which $g_n(\zeta_n')$ approaches α, so on each such circle lies at least one point ζ at which $g_0(\zeta)=\alpha$. Consequently $g_0(\zeta)\equiv\alpha$ in $|\zeta|<1$, every limit function of the sequence $g_n(\zeta)$ is identically α, this limit is approached by $g_n(\zeta)$ itself throughout $|\zeta|<1$, uniformly in any closed subregion; our conclusion follows from Theorem 2.

The method of proof of Theorem 3 establishes also the following: *Let $f(z)$ be analytic in $|z|<1$ and omit two values. Let $z_n\to 1$, with $|z_n|<1$, and let the non-euclidean distance $\rho(z_n, z_{n+1})$ approach zero. If $\lim_{n\to\infty} f(z_n)$ exists, then the sequence $\{z_n\}$ is regular.* By way of proof, we need merely modify the proof of Theorem 3 by considering instead of the arbitrary circle $|\zeta|=d<1$ an arbitrary annulus $0<d_1\leqq|\zeta|\leqq d_2<1$; each such annulus contains a sequence of points ζ_n' for which $g_n(\zeta_n')$ approaches α, so each closed annulus contains at least one point ζ in which $g_0(\zeta)=\alpha$.

The method of proof of Theorem 3 can be used to prove still another proposition: *Let $f(z)$ be analytic in $|z|<1$ and omit there two values. Suppose for real z we have $\lim_{z\to 1} f(z)=\alpha$. Then we have uniformly for approach within any triangle in $|z|<1$*

$$\lim_{z\to 1} f^{(k)}(z)(1-|z|^2)^k = 0.$$

In this proof, we need merely choose r, $0<r<1$, and the sequence of real z_n in such a way that under the transformation $z=(\zeta+z_n)/(1+\bar{z}_n\zeta)$ each point z of the given triangle corresponds to some ζ in $|\zeta|<r$. Various extensions of the proposition by the present methods suggest themselves, and are left to the reader.

We shall introduce the notion of the *non-euclidean Fréchet distance between two curves*. Let C_1 and C_2 be two open Jordan arcs lying in $|z|<1$. Consider a topological map T of C_1 on C_2. Denote by $F_T(C_1, C_2)$ the least upper bound (finite or infinite) of the non-euclidean distances between points of C_1 and C_2 which correspond in the map T. The greatest lower bound (finite or infinite) of the quantities $F_T(C_1, C_2)$ for all possible maps T will be called the *non-euclidean Fréchet distance $F(C_1, C_2)$ between C_1 and C_2*. With this definition we prove

THEOREM 4. *Let $f(z)$ be analytic in $|z|<1$ and omit two values there. Let C_1 and C_2 be Jordan arcs which, except for the common end point $z=1$, lie in $|z|<1$, and let $F(C_1, C_2)$ be finite. If*

$$\lim_{z\to 1;\, z \text{ on } C_1} f(z) = \alpha,$$

where α is finite or infinite, then also

$$\lim_{z\to 1;\, z \text{ on } C_2} f(z) = \alpha.$$

To any sequence z_n' on C_2 which approaches $z=1$ corresponds a sequence z_n on C_1 such that $\rho(z_n, z_n')$ is bounded. If α is finite, our conclusion follows from Theorem 3. If α is infinite, it follows from Corollary 1 to Theorem 2.

If the two Jordan arcs C_1 and C_2 of Theorem 4 are tangent and have the same order of contact with $|z|=1$ at $z=1$, then $F(C_1, C_2)$ is finite. For transform by a linear transformation of the complex variable the region $|z|<1$ onto the upper half of the w $(=x+iy)$-plane, so that $z=1$ corresponds to $w=0$. We shall assume that in the neighborhood of $w=0$ we may set up a one-to-one correspondence between the arcs C_1: $y=y_1(x)$ and C_2: $y=y_2(x)$ by means of the ordinates $x=$constant. The non-euclidean distance between corresponding points of the two curves reduces to

$$\left| \log \frac{y_1(x)}{y_2(x)} \right|,$$

which by the assumption on order of contact is bounded. In studying the finiteness of $F(C_1, C_2)$, we may confine ourselves to the neighborhood of the point $z=1$, so under the present hypothesis $F(C_1, C_2)$ is finite.

If the two Jordan arcs C_1 and C_2 of Theorem 4 are, except for the point $z=1$, contained in the lens-shaped region between two hypercycles through $z=\pm 1$, and possess tangents at the point $z=1$, we may set up a one-to-one correspondence between their points by the circles of the coaxial family determined by $z=\pm 1$ as null circles. Transformation of an arbitrary circle of that family into the axis of imaginaries by a transformation which leaves invariant $z=1$, $z=-1$, and $|z|=1$, as well as the two given hypercycles, shows that $F(C_1, C_2)$ is finite. Thus we have the

COROLLARY. *The condition of Theorem 4 that $F(C_1, C_2)$ be finite is satisfied if C_1 and C_2 are tangent and have contact of the same order with $|z|=1$ at $z=1$, or if C_1 and C_2 possess tangents at $z=1$ but neither is tangent to $|z|=1$ at $z=1$.*

The foregoing discussion has intimate connections with well known results on the limit values of analytic functions. The proof of Theorem 2 establishes the uniformity for all z_n' of $\lim_{n\to\infty} f(z_n')$ provided merely $\rho(z_n, z_n')$ is uniformly bounded. With this addition, Theorem 4 and its corollary include the theorem of Lindelöf that if $f(z)$ is analytic in $|z|<1$ and omits two values there, and if $\lim_{z\to 1} f(z)$ exists for approach along a line segment in $|z|\leqq 1$, then that limit exists uniformly for approach within an arbitrary triangle contained in

$|z| \leqq 1$. Likewise the corollary to Theorem 4 includes the theorem of Montel that if $f(z)$ is analytic in $|z| < 1$ and omits two values there, and if $\lim_{z \to 1} f(z)$ exists for approach along the arc of an oricycle, then that limit exists uniformly for approach between any two arcs of oricycles tangent at $z = 1$ to the original arc.

We add the general remark that the method of the present section seems to have further wide use in the study of limit values of analytic functions; for instance this method easily proves that *if $f(z)$ is analytic and bounded in $|z| < 1$, continuous on $|z| = 1$ or an open arc A of $|z| = 1$ with $z = 1$ as an end point, and if on this arc* $\lim_{z \to 1,\ z \text{ on } A} f(z) = \alpha$, *then also the limit of $f(z)$ is α uniformly as $|z| \to 1$ between A and the axis of reals.*

28. **Extension of Bloch's theorem.** Another application of the results of Chapter III deals with an extension of Bloch's theorem. We prove the following, which for $p = 1$ reduces to Bloch's theorem.

THEOREM 5. *Let $w = f(z)$ be regular in $|z| < 1$ and let $f^{(p)}(0) = 1$. There exists an absolute positive constant B_p, independent of the function $f(z)$ so that the Riemann configuration R_f on which $w = f(z)$ maps the circle $|z| < 1$ contains at least one point w_0 for which $D_p(w_0) \geqq B_p$. The constant B_p may be taken equal to $2^{-p}\lambda_p$, where $\lambda_p = M_p/p!$, $M_p = M_p(M)$ being the constant of Theorem* 1, *Chapter* III, *taken for $M = 2^p \cdot p!$.*

We assume first that $f(0) = 0$ and that $f(z)$ is regular in $|z| \leqq 1$. Let

$$M_p(r) = \max_{|z| \leqq r} |f^{(p)}(z)|.$$

We have $M_p(0) = 1$ and the function $M_p(r)$ is continuous and non-decreasing in the interval $0 \leqq r \leqq 1$. The function

$$\phi(r) = (1 - r)^p M_p(r)$$

is also continuous in $0 \leqq r \leqq 1$ and $\phi(0) = 1$, $\phi(1) = 0$. Hence, there exists a number r_0 ($0 \leqq r_0 < 1$) such that $\phi(r_0) = 1$ and $\phi(r) < 1$ for $r_0 < r \leqq 1$. The function $|f^{(p)}(z)|$ attains the value $M_p(r_0)$ at a point z_0 of modulus r_0:

$$|f^{(p)}(z_0)| = M_p(r_0) = \frac{1}{(1 - r_0)^p} \cdot \tag{28.1}$$

Consider a circle γ of center z_0 and radius $\rho = (1 - r_0)/2$ and the function

$$g(\zeta) = \frac{f(z_0 + \rho\zeta) - f(z_0)}{\rho^p f^{(p)}(z_0)} = a_1\zeta + a_2\zeta^2 + \cdots + \frac{\zeta^p}{p!} + \cdots$$

for suitably chosen constants $a_1, a_2, \cdots$. It is regular in $|\zeta| \leqq 1$ and

$$g^{(p)}(\zeta) = \frac{f^{(p)}(z_0 + \rho\zeta)}{f^{(p)}(z_0)} \cdot$$

Now, in the circle $|\zeta| \leqq 1$ we have $|z_0+\rho\zeta| \leqq r_0+(1/2)(1-r_0)=(1/2)(1+r_0)$ and therefore in $|\zeta| \leqq 1$

$$|f^{(p)}(z_0+\rho\zeta)| \leqq M_p\left(\frac{1+r_0}{2}\right) < \frac{1}{(1-(1/2)(1+r_0))^p} = \frac{2^p}{(1-r_0)^p}.$$

Hence, in view of (28.1)

$$|g^{(p)}(\zeta)| < 2^p$$

for $|\zeta| \leqq 1$. Successive integration shows that

$$|g(\zeta)| < 2^p \tag{28.2}$$

for $|\zeta| \leqq 1$ and we also have

$$g(0) = 0, \qquad g^{(p)}(0) = 1. \tag{28.3}$$

Now it was shown in Theorem 1, Chapter III, that for the class of functions satisfying the conditions (28.2) and (28.3)

$$D_p(0) \geqq \frac{M_p(2^p \cdot p!)}{p!} = \lambda_p. \tag{28.4}$$

Consequently, by the definition of $g(\zeta)$ it follows that for the function $f(z)$, setting $w_0=f(z_0)$,

$$D_p(w_0) \geqq \lambda_p \cdot \rho^p |f^{(p)}(z_0)| = \lambda_p/2^p \cdot$$

The condition that $f(z)$ be analytic in the closed circle $|z| \leqq 1$ may now be lifted. Indeed, let $f(z)$ be assumed to be analytic in $|z| < 1$. Then, if r is a value in the interval $0 < r < 1$, the function

$$F(z) = \frac{1}{r^p} f(rz)$$

is analytic for $|z| \leqq 1$ with $F^{(p)}(0)=1$. Furthermore, we have

$$D_p[F(z)] \leqq \frac{1}{r^p} D_p[f(rz)].$$

Hence, since the theorem applies to $F(z)$, there exists a z_0 ($|z_0| < 1$) so that

$$B_p \leqq \frac{1}{r^p} D_p[f(rz_0)].$$

Now allowing r to approach one, we obtain the theorem in the general case.

A lower bound for B_p may be obtained from the estimate in (19.2). This value, however, is certainly not sharp.

As a matter of record, we formulate without proof the

COROLLARY. *Let the function $w=f(z)$ be analytic in $|z|<1$, with*

$$|f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)| = m.$$

There exists a positive constant B_p' independent of m and $f(z)$ such that the Riemann configuration R_f onto which $w=f(z)$ maps the region $|z|<1$ contains at least one point for which $D_p(w_0) \geqq mB_p'$. In fact, we may choose B_p' as the smallest of the numbers $j!\cdot B_j/p$, $j=1, 2, \cdots, p$, in the notation of Theorem 5.

29. **Unrestricted functions; properties of $\Delta(z)$.** From the example of §26 it is clear at once that one cannot obtain a relation between $D_1(w_0)$ and $|f'(z_0)|(1-|z_0|^2)$ without some restriction on the class of functions $f(z)$ to be considered. It is perhaps not without interest to remark that by introducing a new quantity $\Delta(z_0)$ one may obtain relations of the desired kind without imposing any restriction on $f(z)$ other than analyticity in the unit circle $|z|<1$. In fact, we prove

THEOREM 6. *Let $w=f(z)$ analytic for $|z|<1$ map $|z|<1$ onto a Riemann configuration S. Let z_0 be any point of the circle $|z|<1$ which is mapped by $w=f(z)$ onto a point w_0 of S which is not a branch point of S. Denoting by $\Delta(z_0)$ the radius of the largest circle of the ζ-plane with center $\zeta=0$ in which the function*

$$\phi(\zeta) = f\left(\frac{\zeta + z_0}{1 + \bar{z}_0\zeta}\right)$$

is univalent, the inequality

$$\frac{1}{4}\,\frac{D_1(w_0)}{\Delta(z_0)} \leqq |f(z_0)|(1-|z_0|^2) \leqq 4\,\frac{D_1(w_0)}{\Delta(z_0)} \tag{29.1}$$

holds. In particular, for a sequence of points $\{z_n\}$ ($|z_n|<1$) a necessary and sufficient condition for

$$\lim_{n\to\infty} |f'(z_n)|(1-|z_n|^2) = 0 \tag{29.2}$$

is

$$\lim_{n\to\infty} \frac{D_1(w_n)}{\Delta(z_n)} = 0, \tag{29.3}$$

where $w_n=f(z_n)$.

The function $\zeta=\psi(w)$ inverse to $w=\phi(\zeta)$ is univalent on S, in particular univalent for $|w-w_0|<D_1(w_0)$. Therefore, by Koebe's distortion theorem it must map the circle $|w-w_0|<D_1(w_0)$ onto some region of the ζ-plane within

which $\phi(\zeta)$ is univalent and which contains in its interior the circle

$$|\zeta| < (1/4)\,|\psi'(w_0)|\cdot D_1(w_0),$$

whence

$$\Delta(z_0) \geqq (1/4)\,|\psi'(w_0)|\cdot D_1(w_0).$$

By the relation $1/\psi'(w_0)=\phi'(0)=f'(z_0)\;(1-|z_0|^2)$ we obtain the left-hand side of inequality (29.1).

Similarly, the function $w=\phi(z)$ is univalent for $|\zeta|<\Delta(z_0)$, hence again by Koebe's distortion theorem maps smoothly $|\zeta|<\Delta(z_0)$ onto a region containing the circle $|w-w_0|<(1/4)\;|\phi'(0)|\cdot\Delta(z_0)$. Hence,

$$D_1(w_0) \geqq (1/4)\,|\phi'(0)|\cdot\Delta(z_0),$$

and the right-hand side of inequality (29.1) follows directly.

Next, the equivalence of the relations (29.2) and (29.3) follows from (29.1) provided w_n are not branch points of S. Indeed, if w_0 is a branch point of S, the expression $D_1(w_0)/\Delta(z_0)$ has no meaning since both numerator and denominator are zero. We observe, however, that if a sequence of points z_n for which the corresponding points w_n are not branch points of S converge to a point z_0 (with $|z_0|<1$) for which the corresponding point w_0 is a branch point of S, then by the first inequality of (29.1)

$$\lim_{n\to\infty}\frac{D_1(w_n)}{\Delta(z_n)} = 0.$$

Hence, it is reasonable to define $D_1(w_0)/\Delta(z_0)$ as zero when w_0 is a branch point of S. With this convention the equivalence of (29.2) and (29.3), as well as the inequality (29.1), remain valid even in the case of branch points.

30. **Pathology.** There are several fairly obvious extensions of our fundamental Theorem 2 of Chapter IV to the effect that if $f(z)$ is analytic and omits two values in $|z|<1$, if $\{z_n\}$ is a sequence of points in $|z|<1$, and if the numbers $w_n=f(z_n)$ are bounded, then the two conditions

$$D_1(w_n) \to 0, \tag{30.1}$$

$$f'(z_n)(1-|z_n|^2) \to 0, \tag{30.2}$$

are equivalent in the sense that each implies the other. The mere analyticity of $f(z)$ insures that (30.2) implies (30.1); so we are concerned at present only with the condition that (30.1) shall imply (30.2). Thus it is sufficient for (30.1) to imply (30.2) if we replace the condition that $f(z)$ omits two values in $|z|<1$ by the condition that

$$\phi_n(\zeta) = f\left(\frac{\zeta + z_n}{1+\bar{z}_n\zeta}\right) \tag{30.3}$$

shall omit two values in $|\zeta| < r < 1$, where r is independent of n; no essential change in the original reasoning is necessary; compare §27, Corollaries 4 and 5. It is obvious too that (30.1) implies (30.2) provided from each subsequence z_{n_k} of the z_n can be extracted a new subsequence z_{m_k} for which there exists a positive number $r < 1$ such that the corresponding functions $\phi_{m_k}(\zeta)$ defined by (30.3) have two omitted values in $|\zeta| < r$; for under such circumstances the fulfillment of condition (30.1) implies that for no subsequence z_{n_k} does the expression

$$f'(z_{n_k})(1 - |z_{n_k}|)$$

approach a limit different from zero, whence (30.2) is satisfied. For instance, it may occur that the functions $\phi_{2\mu}(\zeta)$ have the exceptional values 0 and 1 in $|\zeta| < 1/2$, and that the functions $\phi_{2\mu+1}(\zeta)$ have the exceptional values 2 and 3 in $|\zeta| < 1/4$.

DEFINITION. *Let the function $f(z)$ be analytic for $|z| < 1$, let $\{z_n\}$ be a sequence of points in $|z| < 1$, let $w_n = f(z_n)$ approach a finite limit, let* (30.1) *be satisfied but suppose*

$$\lim_{n\to\infty} f'(z_n)(1 - |z_n|) = \alpha \neq 0; \tag{30.4}$$

then we shall say that $\{z_n\}$ is a q-sequence.

The discussion we have already given yields

THEOREM 7. *Under the hypothesis of the italicized definition, let $\{z_n\}$ be a q-sequence. Then from no subsequence $\{z_{n_k}\}$ of the $\{z_n\}$ can there be extracted a new subsequence $\{z_{m_k}\}$ such that the functions $\phi_{m_k}(\zeta)$ defined by* (30.3) *have two exceptional values in any region $|\zeta| < r < 1$, where r is independent of m_k.*

In other words, if $\{z_n\}$ is a q-sequence, then for every r, $0 < r < 1$, and for every infinite sequence of subscripts $\{n_k\}$, the functions $\phi_{n_k}(\zeta)$ have at most one exceptional value in $|\zeta| < r$.

Some consequences of Theorem 7 are more conveniently described after transformation of $|z| < 1$ onto a half-plane $\Re(z') > 0$.

THEOREM 8. *Under the hypothesis of the italicized definition, let $\{z_n\}$ be a q-sequence having as limit the point z_0, with $|z_0| = 1$. Let the region $|z| < 1$ be transformed by a linear transformation onto $\Re(z') > 0$ so that $z = z_0$ corresponds to $z' = 0$. Then there exists a half-line L from $z' = 0$ in the closed region $\Re(z') \geqq 0$ possessing the property that if S is a sector (of a circle) containing L in its interior and with vertex in $z' = 0$, of arbitrarily small radius, then in S the transform of the function $f(z)$ has at most one exceptional value.*

Let the points z_n' (necessarily approaching $z' = 0$) be the transforms in the

z'-plane of the points z_n. The numbers

$$\theta_n = \arg z_n', \qquad -\pi < \theta n < \pi$$

have at least one limit value, say $\theta = \theta_0$; *the half-line L may be chosen as $\theta = \theta_0$, as we shall proceed to prove.*

A non-euclidean circle in the z'-plane whose non-euclidean center is $z' = \alpha$, $\Re(\alpha) > 0$, is transformed by shrinking or stretching the plane with $z' = 0$ fixed into a non-euclidean circle of the same radius, for the transformation leaves the region $\Re(z') > 0$ invariant. Let S be given, and let S' be a sector interior to S whose sides are also interior to S, likewise having $z' = 0$ as vertex, and containing L in its interior. Then an infinity of points z_n' lie interior to S'. Let ρ denote the smaller of the two non-euclidean radii of the two circles whose euclidean centers lie on the respective rays bounding S' and which are tangent to S; the circles are not uniquely determined but their non-euclidean radii are uniquely determined; there is an exceptional situation here, which presents no inherent difficulty and whose treatment is left to the reader, if the half-line $\theta = \pi/2$ or $\theta = -\pi/2$ lies in or on the boundary of S. The non-euclidean circles whose common non-euclidean radius is ρ and whose euclidean centers are the infinity of points z_n' interior to S' all of whose interior points are interior points of S. Theorem 8 now follows from Theorem 7.

It is obviously true that in S the function $f(z)$ takes on every value with at most one exception *an infinite number of times.*

Theorem 8 obviously bears a close analogy to Julia's theorems on entire functions[68]. The analogy can be pursued still more closely as we now indicate.

In the z'-plane used in Theorem 8 let C be an arbitrary curve (not necessarily a Jordan curve) joining the unit circle to the origin.

$$C: \quad z' = \sigma(t), \qquad 0 \leqq t \leqq 1,$$
$$\sigma(0) = 0, \qquad |\sigma(1)| = 1,$$

where $\sigma(t)$ is a continuous complex-valued function of the real parameter t. From C is found by rotation about the origin a curve which we denote by $C(\omega): z' = \omega \cdot \sigma(t)$, $|\omega| = 1$. We shall call a ***horn*** the set $H(\omega, \epsilon)$ of points each of which lies interior to at least one of the circles having its center in a point z' on $C(\omega)$ and of radius $\epsilon \cdot |z'|$. It will be noted that the horn $H(\omega, \epsilon)$ is then a region, and that each of its boundary points except $z' = 0$ is on the circumference of a circle of center z' on $C(\omega)$ and radius $\epsilon |z'|$. But of course the curve C and the horn $H(\omega, \epsilon)$ need not lie entirely in the closed region $\Re(z') \geqq 0$.

We now prove a generalization of Theorem 8.

THEOREM 9. *Under the hypothesis of Theorem* 8 *for arbitrary C there exists*

([68]) G. Julia, *Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé*, Paris, 1924, p. 105 ff.

a curve $C(\omega_0)$ *such that in every horn* $H(\omega_0, \epsilon)$ *the transform of the function* $f(z)$ *takes on every value an infinite number of times, with the exception of at most one value.*

Let the numbers ϵ and ϵ_1 be given, $1>\epsilon>\epsilon_1>0$. Consider all circles γ and γ_1 of radii $r\epsilon$ and $r\epsilon_1$ with variable common center (r, θ), where r is bounded and θ is arbitrary. Then the non-euclidean distance from a point of γ_1 in the region $\Re(z')>0$ to the nearest point of γ is bounded from zero, say is greater than or equal to some positive δ independent of r and θ. This conclusion follows from the fact that in studying the non-euclidean distance it is no loss of generality to take $r=1$.

As an application of this remark, since each boundary point of the horn $H(\omega, \epsilon_1)$, lies on a circle γ_1 with center z' on $C(\omega)$ and radius $\epsilon_1|z'|$, and since all points interior to the circle γ with center z' and radius $\epsilon|z'|$ belong to $H(\omega, \epsilon)$, it follows that the non-euclidean distance from each boundary point of $H(\omega, \epsilon_1)$ in $\Re(z')>0$ to the boundary of $H(\omega, \epsilon)$ is greater than or equal to δ. *If all points of a set* $\{z'_{n_k}\}$ *in* $\Re(z')>0$ *lie in* $H(\omega, \epsilon_1)$, *then each point whose non-euclidean distance from some* z_{n_k} *is less than* δ *lies in* $H(\omega, \epsilon)$.

Suppose now the points

$$z_n' = r_n e^{i\theta_n}, \qquad 0 < r_n \leqq 1;\ n = 1, 2, \cdots,$$

are the transforms in the z'-plane of the given q-sequence. Each z_n' lies on some curve $C(\omega_n)$; in fact, the continuous function $|\sigma(t)|$ must take on the value r_n for some value of t, say t_n, $0<t_n\leqq 1$, whence

$$z_n' = |\sigma(t_n)|\, e^{i\theta_n}, \qquad r_n = |\sigma(t_n)|,$$

so z_n' lies on the curve

$$C(\omega_n):\quad z' = \omega_n\cdot\sigma(t), \qquad \omega_n = \frac{e^{i\theta_n}\,|\sigma(t_n)|}{\sigma(t_n)}\cdot$$

Of course t_n and ω_n need not be uniquely defined, but we choose a specific determination.

Let the set $\omega_1, \omega_2, \cdots$ on the unit circle have the limit point ω_0. *Then for every* $\epsilon_1>0$, *the horn* $H(\omega_0, \epsilon_1)$ *has an infinity of the points* z_n' *in its interior.* For on the arc of $|z'|=1$ in the circle $|z'-\omega_0|=\epsilon_1$ lie an infinity of points ω_n, say $\omega_{n_1}, \omega_{n_2}, \cdots$. Then of the circle $r=r_{n_k}$ the entire arc which lies in the circle

$$|z' - \omega_0\cdot\sigma(t_{n_k})| = \epsilon_1\cdot r_{n_k}$$

lies on $H(\omega_0, \epsilon_1)$, and this arc of the circle $r=r_{n_k}$ contains the point

$$z'_{n_k} = |\sigma(t_{n_k})|\, e^{i\theta_{n_k}}$$

by virtue of the inequality for ω_{n_k}

$$\left| \frac{e^{i\theta_{n_k}} |\sigma(t_{n_k})|}{\sigma(t_{n_k})} - \omega_0 \right| \leqq \epsilon_1.$$

We are now in a position to prove Theorem 9. Let C be given. The number ω_0 is to be determined as just indicated, and thus $C(\omega_0)$ is defined. Then $\epsilon > 0$ is arbitrary, and we choose ϵ_1, $0 < \epsilon_1 < \epsilon$. The points z'_{n_k} already defined are the transforms of a q-sequence z_{n_k}; it follows from Theorem 7 that in the set of circles having the z'_{n_k} as non-euclidean centers and with a common non-euclidean radius the function $f_1(z')$ [transform of $f(z)$] takes on every value with at most one exception an infinite number of times. The points z'_{n_k} lie in $H(\omega_0, \epsilon_1)$, and these non-euclidean circles (chosen with common non-euclidean radius less than the number δ previously defined) all lie in $H(\omega_0, \epsilon)$. The proof is complete.

It is also true that in every $H(\omega_0, \epsilon)$ *in every neighborhood of the origin* the function $f_1(z')$ takes on every value with at most one exception an infinite number of times.

31. **Functions with bounded** D_1. In studying the relation between $D_1(w)$ and $|f'(z)|(1-|z|^2)$ we have restricted the class of functions $f(z)$ in such a manner that the associated functions $\phi_n(\zeta)$ should form a normal family. For this reason we considered the class of univalent functions, the class of bounded functions, and the class of functions omitting two values. There is, however, another criterion of normality, which was discovered by Bloch(69). It is the class of functions for which the radius of univalence $D_1(w)$ is bounded. The desired relations may be easily obtained for this class. Indeed, we have

THEOREM 10. *Let $w = f(z)$ be analytic for $|z| < 1$, and let $D_1(w)$ be uniformly bounded: $D_1(w) \leqq D$. Setting $w_0 = f(z_0)$, where z_0 is an arbitrary point of $|z| < 1$, the inequality*

$$|f'(z_0)|(1 - |z_0|^2) \leqq [K \cdot D_1(w_0)]^{1/2}, \tag{31.1}$$

holds, where K may be taken equal to $20D/B$, B being Bloch's constant.

We begin by using the method of proof (Montel, ibid.) of Bloch's theorem on normal families. If we set

$$\phi(z) = f\left(\frac{z + z_0}{1 + \bar{z}_0 z}\right), \qquad g(\zeta) = \frac{\phi[z_1 + (1 - |z_1|)\zeta]}{(1 - |z_1|)\phi'(z_1)}, \tag{31.2}$$
$$|z_0| < 1, \ |z_1| < 1, \ \phi'(z_1) \neq 0,$$

we note that $g(\zeta)$ is analytic in $|\zeta| < 1$, with $g'(0) = 1$. Then it follows from Bloch's theorem (§28, Theorem 5 for $p = 1$) that for the function $g(\zeta)$ and for some w we have $D_1(w) \geqq B$, where B is Bloch's constant; hence if $D_1(w)$ refers now to the function $\phi[z_1 + (1 - |z_1|)\zeta]$ we have for some w

(69) Cf. P. Montel, ibid., p. 115.

$$\frac{D_1(w)}{(1-|z_1|)\,|\phi'(z_1)|} \geqq B.$$

But by our hypothesis we have $D_1(w) \leqq D$, whence

$$|\phi'(z_1)| \leqq \frac{D}{B(1-|z_1|)};$$

this inequality is valid in the case $\phi'(z_1)=0$, exceptional for (31.2).

If we introduce the notation

$$\Phi(\zeta) = \phi(\zeta) - \phi(0) = \int_0^\zeta \phi'(\zeta)d\zeta, \tag{31.3}$$

where the integral is taken along a line segment, we have for $|\zeta| \leqq \rho < 1$

$$|\Phi(\zeta)| \leqq \int_0^\rho \frac{Dd\rho}{B(1-\rho)} = -\frac{D}{B}\log(1-\rho).$$

The inequality of Theorem 2, §10, can be written in the present case

$$D_1(w_0) \geqq \frac{|\phi'(0)|^2\rho^2}{-(4D/B)\log(1-\rho)}, \qquad w_0 = f(z_0).$$

It is seen immediately that the maximum of the function

$$-\frac{\rho^2}{\log(1-\rho)}, \qquad 0 < \rho < 1,$$

occurs when

$$-\log(1-\rho) = \frac{\rho}{2(1-\rho)}$$

which is approximately $\rho = .72$, so we may take

$$D_1(w_0) \geqq \frac{B}{10D}|f'(z_0)|^2(1-|z_0|^2)^2.$$

This proves the theorem.

As a corollary, it is seen that under the hypothesis of Theorem 10 the condition $D_1(w_n) \to 0$ is a sufficient condition for $|f'(z_n)|\ (1-|z_n|) \to 0$ even when $w_n \to \infty$. As a further remark it may be observed that the class of functions considered in Theorem 10 includes the case that the area of the image of $|z| < 1$ under the transformation $w = f(z)$ is finite.

It is clear that analogous inequalities could be obtained for the higher derivatives. We proceed instead to the analogous theorem for $D_p(w)$ in general:

THEOREM 11. *Let $w=f(z)$ be analytic for $|z|<1$, and let $D_p(w)$ be uniformly bounded: $D_p(w)\leqq D_p$, where p is given and D_p is independent of w. If we set $w_0=f(z_0)$, where $|z_0|<1$, we have*

$$\sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^{\nu}C_{k-1,\nu}\bar{z}_0^{\nu}\frac{(1-|z_0|^2)^{k-\nu}f^{(k-\nu)}(z_0)}{(k-\nu)!}\right|$$

$$\leqq 24pK_p\left(\frac{D_p}{B_p'}\right)^{1-2^{-p}}[D_p(w_0)]^{2^{-p}},$$

where B_p' is the constant of the corollary of §28, *and where K_p is a constant depending only on p; indeed we may set*

$$K_p=\min\left\{\rho^{-p}[-\log(1-\rho)]^{1-2^{-p}},\ 0<\rho<1\right\},$$

or we may set $K_p=2^p$.

Of course the boundedness of $D_p(w)$, as in Theorem 11, is a stronger condition than the boundedness of $D_1(w)$, as in Theorem 10, for we have $D_p(w)\geqq D_1(w)$.

As before, we introduce $\phi(z)$ by the first of equations (31.2), but set now $G(\zeta)=\phi[z_1+(1-|z_1|)\zeta]$, where z_1 is arbitrary provided $|z_1|<1$. Thus $G(\zeta)$ is analytic in $|\zeta|<1$. Then if $D_p(w_0)$ refers to $G(\zeta)$ or to $f(z)$, we have by the corollary, §28

$$D_p(w_0)\geqq B_p'\left[|\Phi'(0)|+\frac{1}{2!}|\Phi''(0)|+\cdots+\frac{1}{p!}|\Phi^{(p)}(0)|\right].$$

By virtue of the inequality $D_p(w_0)\leqq D_p$, we may now write

$$\frac{D_p}{B_p'}\geqq(1-|z_1|)|\phi'(z_1)|.$$

In the notation of (31.3) we have for $|\zeta|\leqq\rho<1$

$$|\Phi(\zeta)|\leqq-\frac{D_p}{B_p'}\log(1-\rho). \qquad (31.4)$$

The function $\Phi(\rho\zeta)$ is analytic in $|\zeta|<1$ and has there the bound indicated by (31.4). By §19, Corollary 3, we may write,

$$\rho|\Phi'(0)|+\frac{\rho^2}{2!}|\Phi''(0)|+\cdots+\frac{\rho^p}{p!}|\Phi^{(p)}(0)|$$

$$\leqq 24p\left[-\frac{D_p}{B_p'}\log(1-\rho)\right]^{1-2^{-p}}\cdot[D_p(0)]^{2^{-p}},$$

and this inequality is valid whether $D_p(0)$ refers to $\Phi(\rho\zeta)$ in $|\zeta|<1$, to $\Phi(\zeta)$

in $|\zeta|<\rho$, or to $\Phi(\zeta)$ in $|\zeta|<1$. The first part of Theorem 11 follows at once, where $D_p(w)$ refers now to $f(z)$, by §2, Lemma 2. The latter part of Theorem 11 follows from the inequality for $\rho=1/2$

$$\rho^{-p}[-\log(1-\rho)]^{1-2^{-p}} < 2^p.$$

An obvious consequence of Theorem 11 is that under the conditions of that theorem $D_p(w_n)\to 0$ implies

$$f^{(k)}(z_n)(1-|z_n|^2)^k \to 0, \qquad k = 1, 2, \cdots, p,$$

where $w_n=f(z_n)$, $|z_n|<1$; this conclusion is valid even if $w_n\to\infty$.

32. **Comments on condition** $|z_n|\to 1$. In the major part of the present paper, so far as it deals with $D_1(w)$, we are concerned with a function $f(z)$ analytic for $|z|<1$ and the two conditions

$$(32.1) \qquad D_1(w_n) \to 0, \qquad w_n = f(z_n),$$

$$(32.2) \qquad f'(z_n)(1-|z_n|^2) \to 0.$$

In the present section we propose to study the further condition

$$(32.3) \qquad |z_n| \to 1$$

in its relation to (32.1) and (32.2). To some extent, our remarks will be a recapitulation of material already developed.

The relation (32.2) implies (32.1) with no further restriction on $f(z)$, as follows from §4, Theorem 2.

For a univalent function $f(z)$, relation (32.1) implies (32.2) by §4, Theorem 1′. For such a function each of the conditions (32.1) and (32.2) implies (32.3), because $f'(z)$ has a positive lower bound in the closed region $|z|\leqq r<1$; but (32.3) does not imply (32.1) or (32.3), as is illustrated by the function $f(z)=z/(1-z)^2$, when real $z\to 1$; nevertheless(32.3) combined with the boundedness of w_n implies (32.1) and (32.3), as follows by the kind of reasoning about to be given.

However, if $f(z)$ is both univalent and bounded, each of the conditions (32.1), (32.2), (32.3) implies all those conditions; it is sufficient now to show that (32.3) implies (32.1). The plane region R which is the image of $|z|<1$ under the map $w=f(z)$ can be considered the sum of the plane regions R_ν, the respective images of $|z|<1-1/\nu$, $\nu=1, 2, \cdots$, under the map $w=f(z)$. The regions R_ν increase monotonically; given an arbitrary $\delta>0$, there exists an index N_δ such that every point of R_{N_δ} lies within a distance less than δ of the boundary of R; the inequality $|z|>1-1/N_\delta$ implies $D_1(w)<\delta$; thus (32.3) implies (32.1) and hence (32.2).

Let now $f(z)$ be bounded in $|z|<1$; we have already indicated (§10) that the conditions (32.1) and (32.2) are equivalent. Nevertheless it is obvious that (32.1) does not imply (32.3); whenever z_n approaches a point z_0 with $|z_0|<1, f'(z_0)=0$, the relation (32.1) is satisfied without (32.3); nevertheless, if $D_1(w_n) \to 0$, there exists a subsequence of the z_n which approaches a point z_0, with either $f'(z_0)=0$, $|z_0|<1$, or $|z_0|=1$. Reciprocally, Szegö's example (introduction to Chapter II) shows that (32.3) may be satisfied without (32.1).

Let us suppose now $f(z)$ bounded in $|z|<1$, $|f(z)| \leqq M$, $|z_n| \to 1$, $w_n = f(z_n) \to w_0$, $D_1(w_n) \geqq \delta > 0$; we shall derive some geometric properties of the Riemann configuration R onto which the transformation $w=f(z)$ maps $|z|<1$. By inequality (4.4) we may write also

$$|f'(z_n)|(1-|z_n|^2) \geqq \delta. \tag{32.4}$$

Let r be arbitrary, $0<r<1$. The function

$$\phi(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n \zeta}\right)$$

is analytic in $|\zeta|<r$, has a modulus there not greater than M, with $|\phi'(0)| = |f'(z_n)|(1-|z_n|^2) \geqq \delta$. It follows from the Landau-Dieudonné theorem (§10) that the image of $|\zeta|<r$ under the transformation $w=f(z)$ contains a smooth circle whose center is w_n and whose radius is at least $r^2\delta^2/8M = \delta_1$. By virtue of the relation $|z_n| \to 1$, it is possible to choose a subsequence z_{n_k} having the property that the circle whose non-euclidean center is z_{n_k} and non-euclidean radius $2 \log [(1+r)/(1-r)]$ contains on or within it none of the points $z_{n_{k+j}}$, $j>0$; as a consequence it follows from the triangle inequality that the circles γ_{n_k} whose non-euclidean centers are the points z_{n_k} having the common non-euclidean radius $\log [(1+r)/(1-r)]$ are mutually exterior; this circle γ_{n_k} is the image of $|\zeta| = r$ under the transformation $z = (\zeta + z_{n_k})/(1+\bar{z}_{n_k}\zeta)$. *Then the closed interiors of the smooth circles C_{n_k} on R whose centers are the respective points w_{n_k} having the common radius δ_1 are mutually disjoint.* By virtue of our assumption $w_n \to w_0$, it appears that the configuration R has an infinity of separate sheets over the point $w=w_0$, each sheet containing a circle of center w_0 and radius $\delta_1 - \eta$, where η is arbitrary. We shall prove

THEOREM 12. *Let the function $f(z)$ analytic and bounded in $|z|<1$ admit a sequence z_n with $|z_n|<1$, $|z_n| \to 1$,*

$$D_1(w_n) \geqq \delta > 0, \qquad w_n = f(z_n). \tag{32.5}$$

Then there exists a value $w=w_0$ such that the Riemann configuration R onto which the transformation $w=f(z)$ maps $|z|<1$ has an infinity of separate sheets over the point $w=w_0$, each sheet containing a smooth circle whose center lies over the point $w=w_0$ and whose radius is $\delta_2>0$, where δ_2 is suitably chosen.

In Theorem 12, the condition (32.5) may of course be replaced by the condition that $|f'(z_n)|(1-|z_n|^2)$ should be bounded from zero, a condition that implies (32.5).

To prove Theorem 12 it suffices to apply the reasoning already given to a subsequence of the w_n possessing a limit. Of course it is not possible to assert here that the original sequence of circles of radii $D_1(w_n)$ corresponds to separate sheets of R; if the circles of radii $D_1(w_{2n})$ are given arbitrarily, corresponding to separate sheets of R, the point z_{2n+1} can be chosen so near z_{2n} that the corresponding circles overlap, while an inequality of form (32.5) persists.

Conversely, let $f(z)$ now be analytic and bounded for $|z|<1$, and let $w=f(z)$ map $|z|<1$ onto a Riemann configuration which has an infinity of separate sheets over some point w_0, each sheet containing a smooth circle γ_n whose center lies over the point $w=w_0$ and whose radius is $\delta_2>0$; it is obvious that the centers of these smooth circles can be chosen as points w_n so that the relation $D_1(w_n)\geqq\delta_2$ is fulfilled. The relation $|z_n|\to 1$ follows because otherwise a subsequence z_{n_k} has a limit point z_0, with $|z_0|<1$; we have $w_0=f(z_{n_k})$, hence $w_0=f(z_0)$; an infinity of points z_{n_k} lie in an arbitrary neighborhood of z_0; an infinity of the points $w_{n_k}=f(z_{n_k})$ on R lie on R in each C_p whose center is $w_0=f(z_0)$, where $p-1$ is the order of z_0 as a zero of $f(z)$; this is in contradiction to our hypothesis that the γ_n lie in distinct sheets of R; the converse of Theorem 12 is established.

In Theorem 12 and its converse, we have supposed $f(z)$ to be bounded; it also sufficient if $f(z)$ has two exceptional values in $|z|<1$; compare §22.

We add one further remark, in a somewhat different order of ideas. Let $w=f(z)$ be analytic in $|z|<1$, and let us suppose

$$\limsup_{z\to 1} D_1(w) < \infty; \tag{32.6}$$

this condition is a consequence of

$$\limsup_{z\to 1} |f'(z)|(1-|z|^2) < \infty, \tag{32.7}$$

if (32.7) itself is valid. It follows from (32.6) that $D_1(w)$ is uniformly bounded in $|z|<1$. Hence (32.1) and (32.2) are equivalent. Moreover, the discussion of Theorem 12 and its converse applies here. But even under these circumstances it is not true that (32.3) implies (32.1) or (32.2); this is shown by the function $w=f(z)$ with $f(0)=0$, $f'(0)>0$, which maps $|z|<1$ onto the strip $|v|<\pi$, where $w=u+iv$; we have $D_1(w)\leqq\pi$. But when z_n is positive, $z_n\to 1$, we have $D_1(w_n)=\pi$, so neither (32.1) nor (32.2) is satisfied.

33. **p-valent functions.** For p-valent functions we can obtain results analogous to those for bounded functions and for functions which have two exceptional values.

THEOREM 13. *Let the function $f(z)$ be analytic and p-valent in the region $|z|<1$. Then we have*

$$|f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)| \leqq A_p\cdot D_p(0), \tag{33.1}$$

where A_p is a numerical constant depending only on p.

We assume $f(0)=0$, which obviously involves no loss of generality. We write for reference the inequality

$$
\begin{aligned}
&\mu_p = \max\,[\,|a_1|,\ |a_2|,\cdots,\ |a_p|\,] \leqq |a_1| + |a_2| + \cdots + |a_p|,\\
(33.2)\qquad &a_k = \frac{1}{k!} f^{(k)}(0).
\end{aligned}
$$

A theorem due to M. L. Cartwright[70] asserts that under the conditions of Theorem 1, since we have $f(0)=0$, we have

$$
(33.3)\qquad |f(z)| \leqq A_p' \cdot \mu_p \cdot (1-r)^{-2p}, \qquad |z| \leqq r < 1,
$$

where A_p' is a number depending only on p and where μ_p is defined by (33.2). We shall use (33.3) for the particular value $r=1/2$:

$$
(33.4)\qquad |f(z)| \leqq 2^{2p} \cdot A_p' \cdot \mu_p, \qquad |z| \leqq 1/2.
$$

The function $F(z) \equiv f(z/2)$ is analytic in the region $|z|<1$ and has there the bound $2^{2p} \cdot A_p' \cdot \mu_p$. If $D_p(0)$ refers to $F(z)$ or to $f(z)$ we have by §19, Corollary 3,

$$
\begin{aligned}
(33.5)\qquad &|F'(0)| + \frac{1}{2!}|F''(0)| + \cdots + \frac{1}{p!}|F^{(p)}(0)|\\
&\qquad \leqq B_p[2^{2p}\cdot A_p' \cdot \mu_p]^{1-2^{-p}} \cdot [D_p(0)]^{2^{-p}}
\end{aligned}
$$

where B_p may be chosen as $24p$. The first member of (33.5) can be written

$$
\frac{1}{2}|f'(0)| + \frac{1}{2^2\cdot 2!}|f''(0)| + \cdots + \frac{1}{2^p\cdot p!}|f^{(p)}(0)|,
$$

which is not greater than

$$
\frac{1}{2^p}\left[\,|f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)|\,\right].
$$

A consequence of (33.5) and (33.2) is then the inequality

$$
\begin{aligned}
&\left[\,|f'(0)| + \frac{1}{2!}|f''(0)| + \cdots + \frac{1}{p!}|f^{(p)}(0)|\,\right]^{2^{-p}}\\
&\qquad \leqq 2^p \cdot B_p[2^{2p}\cdot A_p']^{1-2^{-p}} \cdot [D_p(0)]^{2^{-p}},
\end{aligned}
$$

which can be put into the form (33.1).

By virtue of §2, Lemma 2 and §20, Theorem 2, we can formulate from Theorem 1

THEOREM 14. *Let the function $f(z)$ be analytic and p-valent in the region $|z|<1$. Then with the conditions $|z_0|<1$, $w_0=f(z_0)$, we have*

[70] Mathematische Annalen, vol. 111 (1935), pp. 98–118.

$$\gamma_p \cdot D_p(w_0) \leqq \sum_{k=1}^{p} \left| \sum_{\nu=0}^{k-1} (-1)^{\nu} C_{k-1,\nu} \bar{z}_0^{\nu} \frac{(1-|z_0|^2)^{k-\nu}}{(k-\nu)!} \cdot f^{(k-\nu)}(z_0) \right| \leqq \Theta_p \cdot D_p(w_0),$$

where γ_p *is the number of* §20, *Theorem* 2, *and* Θ_p *is a number depending only on* p *which may be chosen as* A_p *in Theorem* 1.

Consequently if we have $|z_n| < 1$, $w_n = f(z_n)$, *a necessary and sufficient condition for*

$$\lim_{n\to\infty} f^{(k)}(z_n)(1-|z_n|^2)^k = 0, \qquad k = 1, 2, \cdots, p,$$

is the condition $\lim_{n\to\infty} D_p(w_n) = 0$.

The case $p=1$ brings us back to §4, Theorem 3.

34. **Some extensions to meromorphic functions.** Let us consider a class of functions $f(z)$ meromorphic in $|z|<1$, omitting there the three distinct values a, b, c, and such that $f(0)=A$, $|A| \leqq A_0$, where A_0 is a positive constant independent of the particular function of the class. Corresponding to this class there exists a number θ $(0<\theta<1)$ such that we have

$$|f(z)| \leqq \Omega(A_0, \theta) \tag{34.1}$$

for $|z|<\theta$, where Ω is independent of any particular function of the class.

Indeed, suppose no such value of θ existed. On the circle $|z| = 1/n$ some function $f_n(z)$ would attain a value of modulus exceeding n. From the sequence of functions $f_n(z)$ one can extract a subsequence converging uniformly([71]) in every closed subregion of $|z|<1$ either to a meromorphic function or to the infinite constant. The second alternative cannot take place since by hypothesis $|f_n(0)| \leqq A_0$ for all n. But, on the other hand, if the sequence $f_n(z)$ converges to a meromorphic function, the latter must have a pole at the origin which is not possible on account of the condition $|f_n(0)| \leqq A_0$. Hence, the asserted existence of θ has been established.

Let z_0 $(|z_0|<1)$ be a point such that, setting $w_0 = f(z_0)$, we have $|w_0| \leqq A_0$. Consider the function

$$\phi(\zeta) = f\left(\frac{\zeta + z_0}{1 + \bar{z}_0\zeta}\right)$$

which is meromorphic in $|\zeta|<1$, omits there the values a, b, c, and for which $\phi(0) = w_0$. In accordance with (34.1) we have

$$|\phi(\zeta)| \leqq \Omega(A_0, \theta)$$

in $|\zeta|<\theta$. Hence, in $|\zeta|<1$ we have

$$|\phi(\theta\zeta)| \leqq \Omega(A_0, \theta).$$

([71]) Defined, for instance, as by Montel, *Leçons sur les Familles Normales de Fonctions Analytiques*, Paris, 1927, p. 124.

Now, applying Theorem 5, Chapter III, we obtain the inequality

$$(34.2)\qquad \theta|\phi'(0)| + \frac{\theta^2}{2!}|\phi''(0)| + \cdots + \frac{\theta^p}{p!}|\phi^{(p)}(0)| \leqq \Lambda_p'[D_p(w_0)]^{2-p},$$

where Λ_p' depends on p, θ, and A_0. It is clear, furthermore, that θ depends on a, b, c, A_0 but not on $\phi(\zeta)$ and consequently may be omitted by modifying Λ_p' properly. It is also to be noted that $D_p(w_0)$ in (34.2) is the radius of p-valence at the point w_0 of the Riemann surface on which the function $\phi(\theta\zeta)$ maps $|\zeta|<1$ which is the same as the radius of p-valence at the point w_0 of the Riemann surface on which $\phi(\zeta)$ maps $|\zeta|<\theta$. This radius of p-valence is not greater than the radius of p-valence at the point w_0 of the Riemann surface on which $\phi(\zeta)$ maps $|\zeta|<1$. Hence, if in (34.2) we return to the function $f(z)$ we obtain the inequality

$$(34.3)\qquad \sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^{\nu}C_{k-1,\nu}\bar{z}_0^{\nu}\frac{(1-|z_0|^2)^{k-\nu}f^{(k-\nu)}(z_0)}{(k-\nu)!}\right| \leqq \Lambda_p'[D_p(w_0)]^{2-p},$$

where $D_p(w_0)$ is now the radius of p-valence at the point w_0 of the Riemann surface on which $f(z)$ maps the circle $|z|<1$.

Now, as is remarked in §21 after the proof of Theorem 2, that theorem requires analyticity only in the neighborhood of the origin, which $f(z)$ possesses in $|z|<\theta$. Hence, applying Theorem 2 we find that

$$(34.4)\qquad \lambda_p D_p(w_0) \leqq \sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^{\nu}C_{k-1,\nu}\bar{z}_0^{\nu}\frac{(1-|z_0|^2)^{k-\nu}f^{(k-\nu)}(z_0)}{(k-\nu)!}\right|,$$

where λ_p depends on p alone. Thus, we may state

THEOREM 15. *Let $f(z)$ be a function meromorphic in $|z|<1$, omitting there the three distinct values a, b, c. Let z_0 ($|z_0|<1$) be a point such that, setting $w_0=f(z_0)$, $|w_0|\leqq A_0$ where A_0 is a positive constant. Then, the inequalities* (34.3) *and* (34.4) *hold, where Λ_p' depends on A_0, a, b, c, but not on z_0 or $f(z)$.*

It follows that if under the hypotheses of Theorem 15 for a sequence of points z_n ($|z_n|<1$) the sequence $w_n=f(z_n)$ is bounded, then a necessary and sufficient condition for $\lim_{n\to\infty} f^{(k)}(z_n)\,(1-|z_n|^2)^k=0$ ($k=1, 2, \cdots, p$) is $\lim_{n\to\infty} D_p(w_n)=0$.

It will be noted that under the conditions of Theorem 15 we have $D_p(w)\leqq|w-a|$ so that inequality (34.3) gives an inequality on the approach to zero of $(1-|z|^2)^k|f^{(k)}(z)|$ as w tends to zero, for every k.

We add the remark that much of the discussion of §27 can be carried over to meromorphic functions which omit three values; this development is left to the reader.

UNIVERSITY OF ROCHESTER,
ROCHESTER, N. Y.
HARVARD UNIVERSITY,
CAMBRIDGE, MASS.