ON THE OSCILLATION OF THE DERIVATIVES OF A PERIODIC FUNCTION

BY

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1. *Let $f(x)$ be a real valued periodic function of period 2π defined for all real values of x and possessing derivatives of all orders. Let N_k denote the number of changes of sign of $f^{(k)}(x)$ in a period. We consider the order of magnitude of N_k as $k\to\infty$.*

(I) *If $N_k=O(1)$, $f(x)$ is a trigonometric polynomial.*

(II) *If $N_k=O(k^\delta)$ where δ is fixed, $0<\delta<1/2$, $f(x)$ is an entire function of finite order not exceeding $(1-\delta)/(1-2\delta)$.*

(III) *If $N_k=o(k^{1/2})$, $f(x)$ is an entire function.*

We prove this theorem by consideration of the Fourier series of $f(x)$

$$f(x) = \sum c_n e^{inx}, \tag{1}$$

$c_{-n}=\bar{c}_n$ $(n=0, 1, 2, \cdots)$. Here, as in what follows, the sign $\sum$ without explicitly stated limits means a summation from $-\infty$ to ∞. Under the present conditions, the series (1) is absolutely and uniformly convergent for real x, and so are the Fourier series of $f'(x), f''(x), \cdots$, obtained from (1) by term by term differentiation. If we focus our attention on the Fourier series, we may express the general trend of our theorem by saying that *a small amount of oscillation in the higher derivatives implies a rapid decrease in the coefficients*, this decrease being so extreme in case (I) that all coefficients from a certain point onward vanish.

The theorem we have to prove and a few analogous facts(1) point towards a general principle which cannot yet be stated in precise terms but which is not entirely unsuitably expressed by saying that *a small amount of oscillation in the higher derivatives indicates a great amount of simplicity in the analytic nature of the function.*

An analogous theorem may be formulated for almost periodic functions. As in other theorems of this kind, the number of changes of sign in a period is replaced by the density of these changes over the infinite line and a trigonometric polynomial is replaced by an entire function of exponential type. The extension of case (I) of our theorem offers the least difficulty.

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(1) See S. Bernstein, *Leçons sur les Propriétés Extrémales*, Paris, 1926, pp. 190–197 and Communications de la Société Mathématique de Kharkow, (4), vol. 2 (1928), pp. 1–11; R. P. Boas and G. Pólya, Proceedings of the National Academy of Sciences, vol. 27 (1941), pp. 323–325.

2. We start with a few preliminary remarks on changes of sign. We consider first a real-valued function $f(x)$ which is defined in an interval $a \leqq x \leqq b$. We say that this function has N changes of sign in this interval if it is possible to find $N+1$, and no more, abscissae $x_0, x_1, \cdots, x_N$ in the interval such that

$$x_0 < x_1 < \cdots < x_{N-1} < x_N, \tag{2}$$

$$f(x_{\nu-1})f(x_\nu) < 0, \qquad \nu = 1, 2, 3, \cdots, N. \tag{3}$$

If a function has N changes of sign in an interval, its derivative has there at least $N-1$ *changes of sign.* This variant of Rolle's theorem is easily proved by considering ξ_ν, such that

$$f(x_\nu) - f(x_{\nu-1}) = (x_\nu - x_{\nu-1})f'(\xi_\nu), \qquad x_{\nu-1} < \xi_\nu < x_\nu,$$

and observing that $f(x_\nu)f'(\xi_\nu) > 0$ and that therefore

$$f'(\xi_{\nu-1})f'(\xi_\nu) < 0, \qquad \nu = 2, 3, \cdots, N.$$

Applying this to the function $e^{ax}f(x)$ (where a is a real constant) and its derivative $[e^{ax}f(x)]' = e^{ax}[af(x) + f'(x)]$, we see that *the number of changes of sign of*

$$(a + D)f(x)$$

(where D is the symbol of differentiation) *is not inferior to* $N-1$, N *being the number of changes of sign of* $f(x)$.

Now let $f(x)$ be periodic with the period 2π. We say that the number of changes of sign of $f(x)$ *in a period* is N, if it is possible to find just $N+1$, and no more, abscissae $x_0, x_1, \cdots, x_N$ such that

$$x_N = x_0 + 2\pi,$$

and (2), (3) hold. Observe that $f(x_N) = f(x_0)$ and that, therefore, N is necessarily *even*. Hence it follows that the number of changes of sign of $(a+D)f(x)$ in a period is not inferior to that of $f(x)$. We defined N_k in our initial statement; now we see that

$$N_0 \leqq N_1 \leqq N_2 \leqq \cdots \leqq N_{k-1} \leqq N_k \leqq \cdots. \tag{4}$$

Observing that

$$(a + D)(a - D) \sum \frac{c_n e^{inx}}{a^2 + n^2} = \sum c_n e^{inx},$$

we obtain:

LEMMA I. *The number of changes of sign of the function* (1) *in a period is not inferior to that of*

$$\sum \frac{c_n e^{inx}}{a^2 + n^2}.$$

3. The series (1) represents a trigonometric polynomial of order m if $c_n=0$ for $n=m+1,\ m+2,\cdots$. If $f(x)$ is a trigonometric polynomial of order m, it cannot have more than $2m$ roots in a period; this is well known. Observe that for large k, $f^{(k)}(x)$ has actually $2m$ changes of sign, because as $k\to\infty$, $(im)^{-k}f^{(k)}(x)$ approaches the first or the second of the two expressions

$$-c_{-m}e^{-imx}+c_me^{imx},\qquad c_{-m}e^{-imx}+c_me^{imx}, \tag{5}$$

according as k is odd or even. The second of these expressions is of the form $2|c_m|\cos(mx-\gamma)$, with a certain real γ and the first is of the same form except for a factor i.

The case (I) of our theorem characterizes the trigonometric polynomials and can be stated as follows: *A real-valued periodic function $f(x)$ possessing derivatives of all orders is a trigonometric polydomial if and only if the number of changes of sign of $f^{(k)}(x)$ remains bounded for $k\to\infty$.*

In order to prove this we consider (1). We have to show that some $f^{(k)}(x)$ have an arbitrarily great number of changes of sign if there are $c_n\neq 0$ with arbitrarily large subscripts n. More precisely we shall show this:

If $m>0$ and $c_m\neq 0$, then all derivatives of (1), *from a certain stage onward, have not less than $2m$ changes of sign.*

In fact, by repeated application of Lemma I, we ascertain that

$$f^{(k)}(x)=i^k\sum n^kc_ne^{inx} \tag{6}$$

does not have fewer changes of sign than

$$i^k\sum\left(\frac{2mn}{m^2+n^2}\right)^kc_ne^{inx}. \tag{7}$$

But since it is given that $c_m\neq 0$ and that $\sum c_n$ is absolutely convergent, we have from a certain k onward

$$|c_m|>\left(\sum_{n=1}^{m-1}+\sum_{n=m+1}^{\infty}\right)\left(\frac{2mn}{m^2+n^2}\right)^k|c_n|. \tag{8}$$

Indeed we have for $n>0$, $n\neq m$, $0<2mn<m^2+n^2$, and therefore, each term tends to 0 on the right-hand side of (8) for $k\to\infty$.

But if (8) holds for a certain even k, the sum in (7) has the same sign as the second expression (5) in all those real points x in which this latter reaches $2|c_m|$, the maximum of its absolute value. This maximum is reached with alternating signs, in equidistant points, the distance of two consecutive points being π/m. Therefore (7) has not less than $2m$ changes of sign. We have proved this for even k but the same is true and the proof is nearly the same for odd k. Then, by Lemma I, (6) has not less than $2m$ changes of sign, and case (I) of our theorem is proved.

4. We consider the case (III) of our theorem before case (II).

If the periodic function $f(x)$ is analytic along the whole real axis, it is analytic in a certain horizontal strip bisected by the real axis and the Fourier series (1), which is a Laurent series in

$$z = e^{ix},$$

converges in the interior of the strip. Hence, by examining

$$\limsup_{n\to\infty} |c_n|^{1/n} \tag{9}$$

we can distinguish the following three cases:

If $f(x)$ is not analytic along the whole real axis, (9) has the value 1.

If $f(x)$ is analytic in a certain horizontal strip of width $2h$ bisected by the real axis, but in no wider horizontal strip, (9) has the value e^{-h}.

If $f(x)$ is an entire function, (9) has the value 0.

In order to prove case (III) of our theorem, we have to show that in the first two cases $N_k = o(k^{1/2})$ is excluded. We prove the following statement.

If there exists a positive number γ such that

$$\limsup_{n\to\infty} |c_n| e^{n\gamma} = \infty, \tag{10}$$

then there exists a positive number g such that $f^{(k)}(x)$ has, for an infinity of values of k, not less than $(k/g)^{1/2}$ changes of sign.

By the considerations of the foregoing section, $f^{(k)}(x)$ has certainly not less than $2m$ changes of sign if (8) holds. Using (10), we have to find an arbitrarily large m and a corresponding k such that (8) holds. We shall succeed in finding such an m by applying the following known lemma[2].

LEMMA II. *We consider two infinite sequences $l_1, l_2, \cdots, l_n, \cdots$ and $s_1, s_2, \cdots, s_n, \cdots$, and suppose that*

$$l_n \geqq 0, \qquad n = 1, 2, 3, \cdots, \tag{11}$$

$$0 < s_1 < s_2 < s_3 < \cdots, \tag{12}$$

$$\lim_{n\to\infty} l_n = 0, \tag{13}$$

$$\limsup_{n\to\infty} l_n s_n = \infty. \tag{14}$$

Then there exists an infinity of integers m such that

$$l_m \geqq l_{m+\mu}, \qquad \mu = 1, 2, 3, \cdots,$$

$$l_m s_m \geqq l_{m-\mu} s_{m-\mu}, \qquad \mu = 1, 2, \cdots, m-1.$$

We put

(2) See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 1, pp. 18 and 173, Problem 109.

$$|c_n| = l_n, \qquad e^{n\gamma} = s_n.$$

This choice satisfies (11), (12), (13), (14); in fact, (13) is satisfied because (1) is convergent, and (14) is satisfied because we have supposed (10). Thus we obtain an infinity of m such that

$$|c_{m+\mu}| \leqq |c_m|, \qquad \mu = 1, 2, 3, \cdots,$$

$$|c_{m-\mu}| e^{(m-\mu)\gamma} \leqq |c_m| e^{m\gamma}, \qquad \mu = 1, 2, \cdots, m-1.$$

This we use to estimate the following sum. (Our ultimate aim is to prove (8).)

$$\begin{aligned}&\sum_{\mu=1}^{m-1}\left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2}\right)^k \left|\frac{c_{m-\mu}}{c_m}\right| + \sum_{\mu=1}^{\infty}\left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2}\right)^k \left|\frac{c_{m+\mu}}{c_m}\right| \\ &\quad < \sum_{\mu=1}^{m-1}\left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2}\right)^k e^{\gamma\mu} + \sum_{\mu=1}^{\infty}\left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2}\right)^k \\ &\quad = S_1 + S_2.\end{aligned} \tag{15}$$

We introduced the abbreviations

$$S_1 = \sum_{\mu=1}^{m-1} e^{\gamma\mu}\left(1 + \frac{\mu^2}{2m(m-\mu)}\right)^{-k}, \tag{16}$$

$$S_2 = m\sum_{\mu=1}^{\infty}\left(\frac{2(1+\mu/m)}{1+(1+\mu/m)^2}\right)^k \frac{1}{m}, \tag{17}$$

and we shall consider S_1 and S_2 in turn.

(1) Split the sum S_1 in two parts, μ being less than or equal to $m/2$ in the first part and greater than $m/2$ in the second. Using the fact that

$$(1+x)^{-1} < e^{-x/2}, \qquad 0 < x < 1,$$

we obtain

$$\begin{aligned}S_1 &< \sum_{1}^{m/2} e^{\gamma\mu - k\mu^2/4m^2} + \sum_{m/2}^{m-1} e^{\gamma\mu}(4/5)^k \\ &< \sum_{1}^{m/2} e^{-[(k/4m^2)-\gamma]\mu} + me^{\gamma m}(4/5)^k \\ &< e^{-(g-\gamma)}/(1-e^{-(g-\gamma)}) + me^{\gamma m}(4/5)^{4gm^2}.\end{aligned}$$

We put

$$k = 4gm^2 \tag{18}$$

where g is a positive integer, $g>\gamma$. We choose a fixed g such that for sufficiently great m

$$S_1 < 1/2. \tag{19}$$

(2) The function $2x(1+x^2)^{-1}$ decreases for $x>1$. Therefore by (17)

$$S_2 < m\int_1^\infty \left(\frac{2x}{1+x^2}\right)^k dx \sim \frac{m}{2}\left(\frac{2\pi}{k}\right)^{1/2} = \frac{1}{2}\left(\frac{\pi}{2g}\right)^{1/2}.$$

We used a well known asymptotic evaluation of definite integrals(3) and (18). If $g \geqq 2$, which we assume, we obtain for sufficiently great m

$$S_2 < 1/2. \tag{20}$$

But (15), (19), (20) show that (8) is true so that $f^{(k)}(x)$ has not fewer changes of sign than

$$2m = (k/g)^{1/2}.$$

5. We now proceed to the proof of case (II).

LEMMA III. *The Fourier series* (1) *represents an entire function of the finite order* λ, $\lambda>1$, *if and only if*

$$\liminf_{n\to\infty} \frac{\log\log(1/|c_n|)}{\log n} = \frac{\lambda}{\lambda-1}. \tag{21}$$

The proof consists of two parts. Both parts follow familiar lines; so we do not give all the details.

(1) Assume that $f(x)$ is entire and of order λ. Then for a fixed positive ϵ and for sufficiently great $|x|$,

$$|f(x)| < e^{|x|^{\lambda+\epsilon}}. \tag{22}$$

If we evaluate c_n and shift the line of integration (periodicity and Cauchy's formula), we obtain as a result that if r is any positive number,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-ir-\pi}^{-ir+\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}e^{-nr}\int_{-\pi}^{\pi} f(-ir+u)e^{-inu}du, \\ |c_n| &\leqq e^{(r+\pi)^{\lambda+\epsilon}-nr}. \end{aligned} \tag{23}$$

Here we use (22). We choose r, for given n, so that this right-hand side of (23) shall be a minimum. It follows by straight-forward calculation that (21) holds with "$\geqq$" instead of "$=$."

(2) Assume that

$$\liminf_{n\to\infty} \frac{\log\log(1/|c_n|)}{\log n} = \kappa > 0.$$

(3) See, for example, G. Pólya and G. Szegö, loc. cit., vol. 1, pp. 78 and 244, Problem 201.

Therefore we have, for a given positive ϵ and all sufficiently great n

$$|c_n e^{inx}| < e^{-n^{\kappa-\epsilon}+|x|n}.$$

We choose n, for a given x, so that the right-hand side is a maximum. This maximum gives the right order of magnitude because the terms of (1) whose index surpasses a certain multiple of the index of the maximum term, yield a negligible contribution. We find that the order λ of $f(x)$ satisfies the inequality

$$\lambda \leqq \frac{\kappa}{\kappa - 1}\cdot$$

This gives (21) with "$\leqq$" instead of "$=$."

6. Now we are prepared to prove case (II) of our theorem. We have to show that *if the entire function $f(x)$ is of order λ, and $\epsilon > 0$ then, for an infinity of k,*

$$N_k > k^{(\lambda-1)/(2\lambda-1)-\epsilon}.$$

Put $\lambda/(\lambda-1)+\eta=\gamma$, η being positive and small. By Lemma III, the fact we have to show can be stated as follows:

If there exists a positive number γ, $\gamma > 1$, such that

$$\limsup_{n\to\infty} |c_n| e^{n^\gamma} = \infty,$$

then there exists a positive g such that $f^{(k)}(x)$ has, for an infinity of k, more than $(k/g)^{1/(\gamma+1)}$ changes of sign.

We apply Lemma II, whose conditions are satisfied by

$$l_n = |c_n|, \qquad s_n = e^{n^\gamma}.$$

We obtain the result that for an infinity of m,

$$|c_m| \geqq |c_{m+\mu}|, \qquad \mu = 1, 2, \cdots,$$
$$|c_m| e^{m^\gamma} \geqq |c_{m-\mu}| e^{(m-\mu)^\gamma}, \qquad \mu = 1, 2, \cdots, m-1.$$

Hence, using the fact that $\gamma > 1$, we obtain

$$\begin{aligned} &\sum_{\mu=1}^{m-1}\left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2}\right)^k\left|\frac{c_{m-\mu}}{c_m}\right| + \sum_{\mu=1}^{\infty}\left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2}\right)^k\left|\frac{c_{m+\mu}}{c_m}\right| \\ &< \sum_{\mu=1}^{m-1}\left(\frac{2m(m-\mu)}{m^2+(m-\mu)^2}\right)^k e^{\gamma m^{\gamma-1}\mu} + \sum_{\mu=1}^{\infty}\left(\frac{2m(m+\mu)}{m^2+(m+\mu)^2}\right)^k \qquad (24) \\ &= S_1' + S_2. \end{aligned}$$

S_2 has the same meaning as before (see (17)), and

$$
\begin{aligned}
S_1' &= \sum_{\mu=1}^{m-1} e^{\gamma m^{\gamma-1}\mu}\left(1 + \frac{\mu^2}{2m(m-\mu)}\right)^{-k} \\
&< \sum_{\mu=1}^{m/2} e^{\gamma m^{\gamma-1}\mu - k\mu^2/4m^2} + me^{\gamma m^\gamma}(4/5)^k \qquad (25)\\
&= \sum_{\mu=1}^{m/2} e^{-m^{\gamma-1}(g\mu^2-\gamma\mu)} + me^{\gamma m^\gamma}(4/5)^{4gm^{\gamma+1}}.
\end{aligned}
$$

We put

$$k = 4gm^{\gamma+1},$$

and we choose g so that

$$\gamma + 1 \leqq g < \gamma + 2,$$

and so that k is an integer. This choice assures that

$$S_1' \to 0, \qquad S_2 \to 0$$

for $m \to \infty$ (see (25) and the considerations preceding (20)). Therefore, by (24), (8) is true and $f^{(k)}(x)$ has not fewer changes of sign than

$$2m > 2[k/4(\gamma + 2)]^{1/(\gamma+1)}.$$

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