FLAT SPACE CONGRUENCES OF ORDER ONE IN $[n]$

BY

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By an $[n-k]$ -congruence of order one in $[n]$, $k < n$, is meant an algebraic ∞^k -system of $[n-k]$'s in n -dimensional projective space such that one and only one $[n-k]$ of the system passes through an arbitrary point of $[n]$. A point on more than one $[n-k]$ of the congruence is called a *fundamental point*. A fundamental point fails to determine any $[n-k]$ of the congruence.

Although some special cases of flat space congruences have been studied, there is no general theory in the literature⁽¹⁾. Line congruences of order one in $[3]$ were investigated by E. Kummer⁽²⁾ and this was completed by R. Sturm⁽³⁾. G. Marletta^(4,5) classified line congruences of order one in $[4]$ but his methods were not suitable for extension to $[n]$. Some line congruences in $[5]$ were discussed by M. Sgroi⁽⁶⁾. G. Aprile⁽⁷⁾ collected the known examples of line congruences of order one in $[n]$; these are all special cases of the more general congruences defined in the present paper and it will be shown in §8 that his construction for a line congruence in $[n]$ with irreducible locus of fundamental points is a restricted case of one of ours⁽⁸⁾.

Probably not all possible flat space congruences of order one are included in what follows. The line congruence in $[4]$ consisting of the trisecants of the F^4 , projection of the Veronese surface in $[5]$, and the two line congruences in $[5]$ consisting of the bisecants of a ruled F^4 or of the bisecants of the del Pezzo surface, do not appear to belong to our class of congruences.

1. Classification. On each $[n-k]$ of our $[n-k]$ -congruences of order one

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⁽¹⁾ Encyklopädie der Mathematischen Wissenschaften vol. III C 7, p. 964.

⁽²⁾ E. Kummer, *Über die algebraischen Strahlensysteme, in besondere die der ersten und zweiten Ordnung*, Berlin, Abhandlungen der Preussischen Akademie der Wissenschaften, 1866.

⁽³⁾ R. Sturm, *Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung*, Leipzig, vol. 2, 1893.

⁽⁴⁾ G. Marletta, *Sui complessi di rette del primo ordine dello spazio a quattro dimensioni*, Rend. Circ. Mat. Palermo vol. 28 (1909).

⁽⁵⁾ Marletta, *Sopra i complessi di rette d'ordine uno dell' S_4* , Atti Accademia Gioenia Catania, (5) vol. 3 (1909).

⁽⁶⁾ M. Sgroi, *Sui 4-complessi di rette d'ordine uno nell' S_5* , Atti Accademia Gioenia Catania (5) vol. 15 (1925).

⁽⁷⁾ G. Aprile, *Giornale di Matematica* vol. 70 (1932) pp. 196–216.

⁽⁸⁾ After the present paper had been sent to the editor, I received from Professor Virgil Snyder an offprint of his *Cremona involutions belonging to the Bordiga surface in $[4]$* , Revista de Tucumán (A) vol. 2 (1941) pp. 203–210, in which he describes a line congruence of order one in $[n]$ consisting of the $(n-1)$ -secants of the generalized Bordiga. This is type $(n-1)_n$ in my classification.

$$\lambda_1:\lambda_2:\cdots:\lambda_{r+1} = P_1:P_2:\cdots:P_{r+1}.$$

Therefore

$$(2.4) \quad \begin{aligned} P_1x_{11} + \cdots + P_{r+1}x_{1\ r+1} &= 0, \\ &\cdots \\ P_1x_{r1} + \cdots + P_{r+1}x_{r\ r+1} &= 0, \end{aligned}$$

are the equations of the $[n-r]$ of system (2.1) through point P .

Now a necessary condition for any other point Q to lie on (2.4), the $[n-r]$ through P , is

$$(2.5) \quad \begin{aligned} P_1q_{11} + \cdots + P_{r+1}q_{1\ r+1} &= 0, \\ &\cdots \\ P_1q_{r1} + \cdots + P_{r+1}q_{r\ r+1} &= 0, \end{aligned}$$

where the q_{ij} are obtained by substituting the coordinates of Q in the x_{ij} . From which $P_1:P_2:\cdots:P_{r+1}=Q_1:Q_2:\cdots:Q_{r+1}$. Thus every point on (2.4) determines that same $[n-r]$, and the ratios $P_1:P_2:\cdots:P_{r+1}$ are unique for the $[n-r]$ of the system (2.1) passing through P .

Since (2.4) defines one $[n-r]$ through an arbitrary point P of $[n]$ and every point of (2.4) determines that same $[n-r]$, it follows that *the totality of $[n-r]$'s given by (2.4) for all points P of $[n]$ form an $[n-r]$ -congruence of order one in $[n]$.*

(2.3) is a determinantal locus⁽⁹⁾. Since it will appear that (2.3) serves as a carrier or base for flat space congruences of order one, (2.3) will be called a *fixed base*.

Indicate the fixed base (2.3) by $\mathcal{A}_1$. It is a variety of dimension $n-2$ and order $C_{r+1,2}$ ⁽¹⁰⁾. Degenerate cases, for which the order of the fixed base is lower, will be treated in a separate note.

The points of $\mathcal{A}_1$ are fundamental points for the $[n-r]$ -congruence of order one in $[n]$ whose generic $[n-r]$ is given by (2.4).

For a general point P not on $\mathcal{A}_1$, at least one of the P_j ($j=1, 2, \cdots, r+1$) is different from zero. For convenience, let $P_{r+1} \neq 0$. If all the other P_j are zero, it is obvious that the intersection of the $[n-r]$ through P , given by (2.4), with the fixed base (2.3), is the same as the intersection of (2.4) with

$$(2.6) \quad X_{r+1} \equiv \begin{vmatrix} x_{11} & \cdots & x_{1r} \\ \cdot & \cdots & \cdot \\ x_{r1} & \cdots & x_{rr} \end{vmatrix} = 0.$$

If some other P_i , say P_i , is also different from zero, (2.6) can be rewritten

⁽⁹⁾ T. G. Room, *The geometry of determinantal loci*, Cambridge, 1938, p. 33.

⁽¹⁰⁾ Room, loc. cit. pp. 34, 43.

such $[n-k]$'s, for all points P_j of $[n]$, form an $[n-k]$ -congruence of order one in $[n]$, whose type symbol is $(A_1 A_2 \cdots A_w)_n$.

5. Fundamental points. For the $[n-k]$ -congruence of order one in $[n]$ whose type symbol is $(A_1 A_2 \cdots A_w)_n$, there are w varieties $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_w$, each of dimension $n-2$, all of whose points are fundamental. Through each point of any $\mathcal{A}_i$ there pass infinitely many $[n-k]$'s.

$\mathcal{A}_1$ is the fixed base (4.1). We have seen (§2) that $\mathcal{A}_1$ is a variety of dimension $n-2$ and order $\alpha_1 = C_{r+1,2}$.

The dependent base associated with A_2 of the type symbol is

$$(5.1) \quad \left\| \begin{array}{cccc} x_{11}^{(2)} & \cdots & x_{1\ s+1}^{(2)} \\ \cdot & \cdots & \cdot \\ x_{s1}^{(2)} & \cdots & x_{s\ s+1}^{(2)} \end{array} \right\| = 0$$

where $x_{ij}^{(2)}$ are linear forms in $x_0, \cdots, x_n$, whose coefficients are forms of order σ_{21} in $P_1, \cdots, P_{r+1}$.

Denote by $x_{ij}^{(2)}$ and $F_j^{(2)}$ the result of changing to $x_0, \cdots, x_n$ the coordinates of P wherever they appear in $x_{ij}^{(2)}$ and $F_j^{(2)}$, respectively. Now

$$(5.2) \quad \left\| \begin{array}{cccc} x_{11}^{(2)} & \cdots & x_{1\ s+1}^{(2)} \\ \cdot & \cdots & \cdot \\ x_{s1}^{(2)} & \cdots & x_{s\ s+1}^{(2)} \end{array} \right\| = 0,$$

is a variety of dimension $n-2$. Its order is the number of points in which it intersects a general plane of $[n]$. Since (5.2) has no special position relative to the frame of reference, its order will be given by the number of points in which it intersects the particular plane π , whose equations are $x_3 = x_4 = \cdots = x_n = 0$. Since each element $x_{ij}^{(2)}$ of the matrix (5.2) is homogeneous of degree σ_{21} in $F_1, \cdots, F_{r+1}$, every point of $\mathcal{A}_1$ lies on (5.2). That is, (5.2) is a composite variety of dimension $n-2$, consisting of $\mathcal{A}_1$ and a residual which is $\mathcal{A}_2$. We seek the order of $\mathcal{A}_2$. The order of $\mathcal{A}_2$ is the number of points, not on $\mathcal{A}_1$, in which the plane π intersects (5.2).

Denote by $\xi_{ij}^{(2)}$, Φ_j , and $\Phi_j^{(2)}$, the result obtained by equating to zero $x_3, x_4, \cdots, x_n$ in $x_{ij}^{(2)}$, F_j , and $F_j^{(2)}$, respectively. Salmon⁽¹³⁾ has pointed out that the locus

$$(5.3) \quad \left\| \begin{array}{cccc} \xi_{11}^{(2)} & \cdots & \xi_{1\ s+1}^{(2)} \\ \cdot & \cdots & \cdot \\ \xi_{s1}^{(2)} & \cdots & \xi_{s\ s+1}^{(2)} \end{array} \right\| = 0$$

⁽¹³⁾ G. Salmon, *Modern higher algebra*, 4th edition, Dublin, 1885, p. 287.

is the intersection of $\Phi_1^{(2)}=0$ and $\Phi_2^{(2)}=0$ after rejecting those points of the locus

$$\left\| \begin{array}{ccc} \xi_{13}^{(2)} & \cdots & \xi_{s3}^{(2)} \\ \cdot & \cdots & \cdot \\ \xi_{1s+1}^{(2)} & \cdots & \xi_{s+1}^{(2)} \end{array} \right\| = 0$$

which do not lie on $\Phi_1^{(2)}=0$ and $\Phi_2^{(2)}=0$. (Notice that the rows of this latter matrix are the columns of (5.3) common to $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$.)

Let $s=1$ in (5.3). Then $\Phi_1^{(2)}=0$ is a plane curve of order $r\sigma_{21}+1$. Since $\Phi_1^{(2)}$ is homogeneous of degree σ_{21} in $\Phi_1, \Phi_2, \dots, \Phi_{r+1}$, and $\Phi_1=\Phi_2=\dots=\Phi_{r+1}=0$ is $C_{r+1,2}$ points, each of these points is σ_{21} -fold on the plane curve $\Phi_1^{(2)}=0$. Similarly, $\Phi_2^{(2)}=0$ is a plane curve of order $r\sigma_{21}+1$ on which each of the $C_{r+1,2}$ points, $\Phi_1=\Phi_2=\dots=\Phi_{r+1}=0$, is σ_{21} -fold. Therefore $\Phi_1^{(2)}=0$ and $\Phi_2^{(2)}=0$ intersect in $(r\sigma_{21}+1)^2$ points, among which the $C_{r+1,2}$ points count $C_{r+1,2}\sigma_{21}^2$. That is, when $s=1$, the number of points, not on $\mathcal{A}_1$, in which (5.1) intersects π is $(r\sigma_{21}+1)^2 - C_{r+1,2}\sigma_{21}^2$. This is the order of $\mathcal{A}_2$ when $s=1$.

Now consider the case $s=2$ in (5.3). $\Phi_1^{(2)}=0$ is a plane curve of order $2(r\sigma_{21}+1)$. Since $\Phi_1^{(2)}$ is homogeneous of degree $2\sigma_{21}$ in $\Phi_1, \dots, \Phi_{r+1}$, each of the $C_{r+1,2}$ points, $\Phi_1=\Phi_2=\dots=\Phi_{r+1}=0$, is $2\sigma_{21}$ -fold on the plane curve $\Phi_1^{(2)}=0$. Similarly, $\Phi_2^{(2)}=0$ is a plane curve of order $2(r\sigma_{21}+1)$, on which each of the $C_{r+1,2}$ points is $2\sigma_{21}$ -fold. $\Phi_1^{(2)}=0$ and $\Phi_2^{(2)}=0$ intersect in $4(r\sigma_{21}+1)^2$ points in which the $C_{r+1,2}$ points count as $4\sigma_{21}^2 C_{r+1,2}$. That is, $\Phi_1^{(2)}=0$ and $\Phi_2^{(2)}=0$ intersect in $4(r\sigma_{21}+1)^2 - 4\sigma_{21}^2 C_{r+1,2}$ points not on $\mathcal{A}_1$. We have already seen ($s=1$) that

$$\left\| \begin{array}{cc} \xi_{13}^{(2)} & \xi_{23}^{(2)} \end{array} \right\| = 0$$

consists of $(r\sigma_{21}+1)^2 - \sigma_{21}^2 C_{r+1,2}$ points not on $\mathcal{A}_1$. Therefore, for $s=2$, (5.3) consists of $3\{(r\sigma_{21}+1)^2 - C_{r+1,2}\sigma_{21}^2\}$ points not on $\mathcal{A}_1$. This is the order of $\mathcal{A}_2$ for $s=2$.

By induction it can be shown that the order of $\mathcal{A}_2$, for s any positive integer, is $\alpha_2 = C_{s+1,2}\{(r\sigma_{21}+1)^2 - C_{r+1,2}\sigma_{21}^2\}$.

The determination of the orders of $\mathcal{A}_3, \mathcal{A}_4, \dots, \mathcal{A}_w$ becomes increasingly tedious and does not differ essentially from the above.

When $\sigma_{hk} \neq 0$ in the dependent base (4.3), the $P_1^{(k)}, P_2^{(k)}, \dots$ appear homogeneously in the coefficients of the linear forms $x_{ij}^{(h)}$, and this causes $\mathcal{A}_h$ to intersect $\mathcal{A}_k$ in a variety of dimension $n-3$ ($k < h$). If, moreover, $\sigma_{kg} \neq 0$ ($g < k$), $\mathcal{A}_k$ intersects $\mathcal{A}_g$ in a variety of dimension $n-3$. But $P_1^{(g)}, P_2^{(g)}, \dots$ appear in the $P_1^{(h)}, P_2^{(h)}, \dots$, and therefore also in the coefficients of the linear forms $x_{ij}^{(h)}$, and consequently $\mathcal{A}_h$ intersects $\mathcal{A}_g$ in a variety of dimension $n-3$. If $\mathcal{A}_k$ intersects $\mathcal{A}_g$ in a variety of dimension $n-3$ and $\mathcal{A}_h$ intersects $\mathcal{A}_k$ in a variety of dimension $n-3$ ($g < k < h$), then $\mathcal{A}_h$ intersects $\mathcal{A}_g$ in a variety of dimension $n-3$. The three varieties of dimension $n-3$, just mentioned, are

distinct. Subject only to this restriction, we cause $\mathcal{A}_h$ to intersect $\mathcal{A}_g$ in a V_{n-3} or not, according as we choose σ_{hg} different from, or equal to, zero in constructing our flat space congruence of type $(A_1 \cdots A_w)_n$.

6. Coincident fundamental points. For convenience in the type symbol $(A_1 A_2 \cdots A_{h-1} A_h \cdots A_w)_n$, let $A_1 + A_2 + \cdots + A_{h-1} = u$. A unique $[n-u]$ of the congruence $(A_1 A_2 \cdots A_{h-1})_n$ passes through P . This $[n-u]$ intersects $\mathcal{A}_i$ in $\mathcal{D}_i$ ($i < h$). $\mathcal{D}_i$ is a determinantal primal of space $[n-u]$ and is of dimension $n-u-1$ and order A_i (§2).

By means of the dependent base (4.3) there is determined in $[n-u]$ a fixed base D , whose matrix contains s rows.

When $s = A_i$ ($i < h$), D can clearly lie on $\mathcal{A}_i$. When $s = A_i - 1$, there are two systems of fixed bases, whose matrices have s rows, on $\mathcal{D}_i$ ⁽¹⁴⁾ and thus D can lie on $\mathcal{D}_i$.

Therefore, *whenever in the type symbol an integer is equal to, or one less than, an integer further to the left, there exists an $[n-k]$ -congruence of order one in $[n]$ with coincidences among the fundamental points.*

In the case of a line congruence having $A_w = 1$ in its type symbol, the plane through P of the associated plane congruence $(A_1 A_2 \cdots A_{w-1})_n$ intersects $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{w-1}$ in plane curves $\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_{w-1}$, respectively, and $\mathcal{A}_w$ in a single point. This point can be made to lie on any of the plane curves, and therefore *for the type symbol $(A_1 A_2 \cdots A_{w-1})_n$, with $A_1 + A_2 + \cdots + A_{w-1} + 1 = n - 1$, there exist line congruences of order one in $[n]$ whose fundamental points associated with A_w are coincident with those associated with any earlier A_i .*

7. Ordinary space. In the interest of exposition, the foregoing will now be applied to the well known situation in three-dimensional space.

There is only one plane congruence⁽¹⁵⁾ of order one in $[3]$. Its type symbol is $(1)_3$ and its fixed base is

$$(7.1) \quad \begin{vmatrix} x_{11} & x_{12} \end{vmatrix} = 0,$$

where x_{1j} ($j=1, 2$) are linear forms in x_0, x_1, x_2, x_3 , with arbitrary constant coefficients.

(7.1) represents a fixed line in $[3]$. The unique plane through this line and a generic point P is

$$(7.2) \quad P_1 x_{11} + P_2 x_{12} = 0,$$

where P_1 is the result of substituting the coordinates of P in x_{12} and P_2 is the result of substituting the coordinates of P in $-x_{11}$. The totality of planes (7.2) form a plane congruence of order one in $[3]$.

The locus of fundamental points, $\mathcal{A}_1$, is the line (7.1).

⁽¹⁴⁾ Room, loc. cit. p. 106.

⁽¹⁵⁾ While a pencil of planes in $[3]$ is not ordinarily called a congruence, it fulfills our definition ($k=1$), and such usage makes for uniformity of language.

In [3] there are line congruences of order one with type symbols $(2)_3$ and $(11)_3$. Each gives rise to an additional line congruence having coincident fundamental points.

The fixed base for type $(2)_3$ is

$$(7.3) \quad \left\| \begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{array} \right\| = 0,$$

where the x_{ij} are linear forms in x_0, x_1, x_2, x_3 , having arbitrary constant coefficients. (7.3) is a twisted cubic curve whose unique bisecant through a generic point P of [3] is

$$(7.4) \quad \begin{aligned} P_1 x_{11} + P_2 x_{12} + P_3 x_{13} &= 0, \\ P_1 x_{21} + P_2 x_{22} + P_3 x_{23} &= 0. \end{aligned}$$

The P_j are obtained by substituting the coordinates of P in X_j ($j=1, 2, 3$), and X_j is $(-1)^{j+1}$ times the determinant formed from the matrix of (7.3) by omitting the j th column.

The totality of rays (7.4) is a line congruence of order one in [3] with type symbol $(2)_3$ and whose irreducible locus of fundamental points $\mathcal{A}_1$ is a twisted cubic curve (7.3).

If all the planes $x_{ij}=0$ ($i=1, 2; j=1, 2, 3$) pass through the same point O , the twisted cubic (7.3) degenerates into three lines through O . The congruence now consists of the bundle of lines through O . The three lines (7.3) through O are not fundamental but are members of the congruence. The only fundamental point is the vertex O . This oldest line congruence of order one in [3] is thus a special case of our type $(2)_3$.

In the line congruence of type $(11)_3$, the fixed base is the line (7.1). The terminal base is given by

$$(7.5) \quad \left\| x_{11}^{(2)} \quad x_{12}^{(2)} \right\| = 0,$$

where $x_{ij}^{(2)} \equiv \sum_{k=0}^3 a_{ijk}^{(2)} x_k$, and $a_{ijk}^{(2)}$ are forms of order τ_{ji} (any positive integer or zero) in P_1, P_2 of plane (7.2). For each plane (7.2), (7.5) represents a line in [3] intersecting that plane in a point O .

The equation

$$(7.6) \quad P_1^{(2)} x_{11}^{(2)} + P_2^{(2)} x_{12}^{(2)} = 0,$$

in which $P_1^{(2)}$ and $P_2^{(2)}$ are obtained by substituting the coordinates of P for the running coordinates in $x_{12}^{(2)}$ and $-x_{11}^{(2)}$, respectively, represents for each $P_1:P_2$ a plane through P .

Equations (7.2) and (7.6) together represent the ray through P , lying in plane (7.2) and through point O . The totality of rays given by (7.2) and (7.6) form a line congruence of order one in [3].

The locus of fundamental points is composite, consisting of $\mathcal{A}_1$, the fixed

are the equations of a unique $[n-2]$ through a generic point P of $[2n-3]$.

(8.2) intersects the normal curve C in the $n-1$ points whose parameters are given by

$$P_1\lambda^{n-1} + P_2\lambda^{n-2}\mu + P_3\lambda^{n-3}\mu^2 + \dots + P_n\mu^{n-1} = 0,$$

as may be seen by substituting the parametric equations of curve C in (8.2).

Thus Aprile's $[n-2]$ -congruence in $[2n-3]$ is the same as that given by (8.2), and the section of this congruence by an $[n]$ of $[2n-3]$ is Aprile's line congruence of order one in $[n]$.

The $[n]$ -section of (8.1) is a fixed base. But (8.1) is what Room calls a partly symmetric determinantal locus, as is also its $[n]$ -section. Hence⁽¹⁷⁾, the freedom of the fixed base of Aprile's line congruence of order one in $[n]$ is

$$\delta = (1/2)(n-1)(n-2)(n+1) - (n-1)n + 1$$

less than that of our fixed base, for $n > 3$.

⁽¹⁷⁾ Room, loc. cit. p. 160.