SURFACE AREA. II

BY

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1. Introduction. All definitions of §2 of our preceding paper, *Surface Area*. I⁽¹⁾, are again in force.

Throughout the present paper we fix two positive integers $m \leq n$. We are interested in the validity of the formula

$$\int N(f, T, y) d\Phi y = \int_T Jf(x) dx$$

where f is a function on E_m to E_n and $T \subset E_m$. The left-hand member of this equation is the integral, relative to our m -dimensional surface measure over E_n , of the multiplicity of f on T . We believe that this integral appropriately measures the area of that part of the surface f which corresponds to T . The integrand of the right-hand member is the Jacobian associated with f by means of its approximate differential.

Our main results are contained in the Theorems 5.1, 5.2, 6.1 and 7.2. The last of these, which is a specialization of the preceding theory, yields new information about Dini derivatives.

In case $m = 2$ it follows from the work of Radó, McShane, and Morrey⁽²⁾ that the right-hand member of the identity frequently equals the Lebesgue surface area of f .

If f is univalent, the left-hand integral is the Φ measure of the image of T under f .

2. Definition of the measure Φ .

2.1 Definition. We say P is a *projecting function* (on E_n to E_m) if and only if there is an orthogonal transformation R on E_n to E_n such that the relations

$$R(x) = y \quad \text{and} \quad P(x) = (y_1, y_2, \dots, y_m)$$

are equivalent whenever x and y are points in E_n .

For $S \subset E_n$ we define $\gamma(S)$ to be the supremum of numbers of the form $|P^*(S)|$ where P is a projecting function⁽³⁾.

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⁽¹⁾ Trans. Amer. Math. Soc. vol. 55 (1944) pp. 420-437. We hereafter refer to this paper as SA I.

⁽²⁾ T. Radó, *Über das Flächenmass rektifizierbarer Flächen*, Math. Ann. vol. 100, p. 445; *On the derivative of the Lebesgue area of continuous surfaces*, Fund. Math. vol. 30, p. 34.

E. J. McShane, *Integrals over surfaces in parametric form*, Ann. of Math. vol. 34, p. 815.

C. B. Morrey, *A class of representations of manifolds*, Amer. J. Math. vol. 55, p. 683.

⁽³⁾ $|P^*(S)|$ is the m -dimensional Lebesgue measure of $P^*(S)$.

Whenever $S \subset E_n$ and $r > 0$, we denote by $\gamma_r(S)$ the infimum of numbers of the form

$$\sum_{T \in F} \gamma(T)$$

where F is a countable family of open connected subsets of E_n , each of diameter less than r , such that $S \subset \sigma(F)$.

We now define the function Φ on the set of all subsets of E_n by the relation

$$\Phi(S) = \lim_{r \rightarrow 0+} \gamma_r(S).$$

This limit exists because $\gamma_r(S)$ is monotone in r for each fixed set S .

2.2 Remark. If $m=2$ and $n=3$, then the functions γ , γ_r , Φ of Definition 2.1 are identical with the functions bearing the same names in SA I.

In fact the reader will easily convince himself that the functions P_c introduced in §3 of SA I are projection functions in the sense of 2.1 of this paper; and each of our present projection functions is of the form $R:P_c$ where R is an orthogonal transformation of E_2 . Using the invariance of plane Lebesgue measure under such transformations it may be seen that the old and new meanings of γ are identical; consequently the same is true for γ_r and Φ .

2.3 Remark. Theorems 3.4, 3.5, 3.6, 4.4 of SA I are true in our present general setting with P_c replaced by any projecting function P . The proofs are exactly the same as before.

2.4 Remark. If $m=n$, then Φ is m -dimensional Lebesgue measure.

2.5 Remark. If $m=1$, then Φ is Carathéodory linear measure over E_n . In fact $\gamma(T) = \text{diam } T$ for every connected set $T \subset E_n$.

3. Further definitions.

3.1 Definition. We write $\det A$ for the determinant of a square matrix A . Now suppose p and q are positive integers.

A function L on E_p to E_q is said to be *linear* if and only if L is continuous and

$$L(x + y) = L(x) + L(y) \quad \text{for } x \in E_p, y \in E_p.$$

We make no distinction between L and its matrix. The j th column ($j=1, 2, \dots, p$) of L is a point of E_q and denoted by L^j . We further define the *norm* of L by the relation

$$\|L\| = \sup_{|x|=1} |L(x)|.$$

If $p \leq q$, then

$$\Delta(L) = \left\{ \sum_{x \in S} (\det M_x)^2 \right\}^{1/2}$$

where $x \in S$ if and only if x is a set of p integers between 1 and q ; and M_x is the minor of L which is made up from the p rows whose indices are elements of x .

3.2 Definition. Suppose g is a numerically valued function on E_p and $x \in E_p$. Then we define

$$\limsup_{z \rightarrow x} \operatorname{ap} g(z)$$

as the infimum of numbers of the form

$$\limsup_{z \rightarrow x, z \in A} g(z)$$

where A is a Lebesgue measurable subset of E_p with density 1 at x .

3.3 Definition. Suppose f is a function on E_p to E_q .

We say f is *differentiable at x* if and only if there is a linear function L on E_p to E_q such that

$$\limsup_{z \rightarrow x} \frac{|f(z) - f(x) - L(z - x)|}{|z - x|} = 0.$$

The (unique) linear function occurring in this definition is called the *differential of f at x* .

We say f is *approximately differentiable at x* if and only if there is a linear function L on E_p to E_q such that

$$\limsup_{z \rightarrow x} \operatorname{ap} \frac{|f(z) - f(x) - L(z - x)|}{|z - x|} = 0.$$

The (unique) linear function occurring in this definition is called the *approximate differential of f at x* .

If $p \leq q$ then we define the function Jf as follows: The domain of Jf is the set of all points of approximate differentiability of f and for each such point x we let

$$Jf(x) = \Delta(L)$$

where L is the approximate differential of f at x .

3.4 Remark. Suppose f is a function on E_m to E_n .

It may interest the reader, although we shall not use it, that in case $m = 2$ our definitions imply

$$Jf = (EG - F^2)^{1/2}$$

where the right-hand member has its classical meaning.

Furthermore in case $m = n$ our $Jf(x)$ equals the absolute value of the classical Jacobian determinant of f at x .

Similarly in case $m = 1$ our $Jf(x)$ is the absolute value of the approximate derivative of f at x .

3.5 Remark. If f is a univalent differentiable function on E_m to E_n , then f transforms E_m into a Riemannian manifold R_m . Fix a point $x \in E_m$ and let g be the matrix of the fundamental differential quadratic form associated with the point $f(x) \in R_m$ and the coordinate system defined by f . Denoting the

differential of f at x by L and its conjugate by $\bar{L}$, we have $g = \bar{L}:L$ and see from the proof of Lemma 4.2 below that $(\det g)^{1/2} = Jf(x)$.

No use is made of this fact in this paper.

4. Lipschitzian surfaces.

4.1 LEMMA. *If f is a function on $T \subset E_m$ to E_n and M is a number such that*

$$|f(z) - f(x)| \leq M |z - x| \quad \text{whenever } x \in T, z \in T$$

then

$$\Phi[f^*(T)] \leq (5M)^m |T|.$$

Proof. Let $r > 0$.

Let S be an open set for which $T \subset S$ and $|S| \leq |T| + r$. Whenever $x \in T$, $\rho > 0$ and

$$\alpha = E_m E_z [|z - x| \leq \rho]$$

we denote

$$\tilde{\alpha} = E_m E_z [|z - x| < 5\rho], \quad \bar{\alpha} = E_n E_y [|y - f(x)| < 5M\rho]$$

and readily check that

$$f^*(T\tilde{\alpha}) \subset \bar{\alpha}, \quad \gamma(\bar{\alpha}) = M^m |\tilde{\alpha}| = (5M)^m |\alpha|.$$

Now let F be the family of all closed spheres α with center in T and such that $\alpha \subset S$, $\text{diam } \bar{\alpha} < r$. Clearly F covers T and we use a well known covering theorem⁽⁴⁾ to select such a disjointed subfamily G of F that

$$T \subset \sum_{\alpha \in G} \tilde{\alpha}.$$

Hence $f^*(T) \subset \sum_{\alpha \in G} \bar{\alpha}$ and

$$\gamma_r[f^*(T)] \leq \sum_{\alpha \in G} \gamma(\bar{\alpha}) = (5M)^m \sum_{\alpha \in G} |\alpha| \leq (5M)^m |S| \leq (5M)^m (|T| + r).$$

Let $r \rightarrow 0$.

4.2 LEMMA. *If L is a linear function on E_m to E_n and R is an orthogonal transformation on E_n to E_n , then $\Delta(R:L) = \Delta(L)$.*

Proof. For each $n \times m$ matrix A we denote its conjugate $m \times n$ matrix by $\bar{A}$, that is $\bar{A}'_i = A'_i$; hence the Cauchy-Binet theorem⁽⁵⁾ implies

$$[\Delta(A)]^2 = \det (\bar{A}:A).$$

From this we infer

$$\begin{aligned} [\Delta(R:L)]^2 &= \det [(\bar{R}:\bar{L}):(R:L)] = \det [(\bar{L}:\bar{R}):(R:L)] \\ &= \det [\bar{L}:(\bar{R}:R):L] = \det [(\bar{L}:L)] = [\Delta(L)]^2. \end{aligned}$$

⁽⁴⁾ A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205-235, Theorem 3.5.

⁽⁵⁾ A. C. Aitken, *Determinants and matrices*, Edinburgh, 1942, p. 86.

4.3 LEMMA. If f is a function on E_m to E_n ; M is a number such that

$$|f(z) - f(x)| \leq M |z - x| \quad \text{for } z \in E_m, x \in E_m;$$

A is a Lebesgue measurable subset of E_m ; f is differentiable at each point of A and

$$0 < \lambda < Jf(x) < \mu < \infty \quad \text{for } x \in A;$$

then

$$\lambda |A| \leq \int N(f, A, y) d\Phi y \leq \mu |A|.$$

Proof. Denoting the family of all Lebesgue measurable subsets of A by K , we infer from 4.1 that

$$f^*(X) \text{ is } \Phi \text{ measurable for } X \in K.$$

For $x \in A$ let $Df(x)$ be the differential of f at x . For $k=1, 2, 3, \dots$ we define

$$\eta_k(x) = \sup_{0 < |z-x| < k^{-1}} \frac{|f(z) - f(x) - [Df(x)](z-x)|}{|z-x|}$$

and are told by 3.3 that

$$\eta_k(x) \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad \text{for each } x \in A.$$

Helped by Egoroff's theorem we next select such disjoint closed subsets $B_1, B_2, B_3, \dots$ of A that

$$(1) \quad |A - B| = 0, \quad \text{where } B = \sum_{s=1}^{\infty} B_s,$$

and

$$\eta_k(x) \rightarrow 0, \text{ uniformly for } x \in B_s, \quad \text{as } k \rightarrow \infty$$

for each positive integer s . Thus f is uniformly differentiable on B_s . Using the standard theorem on uniform convergence we also note that the partial derivatives $D_1f, D_2f, \dots, D_mf$ are continuous relative to B_s . In the terminology of Hassler Whitney⁽⁶⁾ all this implies that f is of class C' in B_s in terms of the functions f and $D_1f, D_2f, \dots, D_mf$ for each positive integer s .

Now Whitney's extension theorem⁽⁷⁾ gives us such functions $g_1, g_2, g_3, \dots$ with continuous partial derivatives on E_m to E_n that

$$g_s(x) = f(x) \quad \text{and} \quad D_j g_s(x) = D_j f(x)$$

for $x \in B_s$; $s=1, 2, 3, \dots$; $j=1, 2, \dots, m$. Since

$$|D_j g_s(x)| = |D_j f(x)| \leq M \quad \text{for } x \in B_s$$

⁽⁶⁾ H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. vol. 36 (1934) p. 64.

⁽⁷⁾ Loc. cit. ⁽⁶⁾, p. 69, Lemma 2.

and $D_s g_s$ is continuous, we can select open sets $Q_1, Q_2, Q_3, \dots$ such that $B_s \subset Q_s \subset E_m$ and

$$(2) \quad |g_s(x) - g_s(z)| \leq 2M|x - z| \quad \text{whenever } x \in Q_s, z \in Q_s.$$

The remainder of the proof is divided into three parts.

Part 1. If s is a positive integer, $x \in B_s$ and $r > 0$, then there is a family $F(x)$ of closed subsets of Q_s such that

$$0 = \inf_{\alpha \in F(x)} \text{diam } \alpha; \quad 0 < \inf_{\alpha \in F(x)} \frac{|\alpha|}{(\text{diam } \alpha)^m};$$

$\alpha \in F(x)$ implies $x \in \alpha$, $\lambda|\alpha| < \Phi[g_s^*(\alpha)]$ and associated with α is an open sphere $\bar{\alpha} \subset E_n$ for which $g_s^*(\alpha) \subset \bar{\alpha}$, $\text{diam } \bar{\alpha} < r$ and $\gamma(\bar{\alpha}) < \mu|\alpha|$.

Proof. Abbreviate $L = Df(x)$. Select an orthogonal transformation R on E_n to E_n such that

$$(3) \quad R(L^j) \in E_n \underset{y}{E} [y_i = 0 \text{ for } i = m+1, m+2, \dots, n]$$

for $j = 1, 2, \dots, m$; and let P be the projecting function such that the relations

$$R(y) = z \quad \text{and} \quad P(x) = (z_1, z_2, \dots, z_m)$$

are equivalent whenever y and z are points of E_n . Take

$$p = P \circ g_s, \quad T = P \circ L,$$

and note that L is the differential of g_s at x . Thus T is the differential of p at x and from (3) and Lemma 4.2 we infer

$$(4) \quad Jp(x) = |\det T| = \Delta(T) = \Delta(P \circ L) = \Delta(R \circ L) = \Delta(L) = Jf(x) > 0.$$

Using (3) again we verify

$$(5) \quad |T(w)| = |R[L(w)]| = |L(w)| \quad \text{for } w \in E_m.$$

By virtue of (4) there exists an inverse U of T .

Now choose ϵ so that $0 < \epsilon \|U\| < 1$ and

$$(6) \quad \lambda < Jf(x)(1 - \epsilon \|U\|)^m < Jf(x)(1 + \epsilon \|U\|)^m < \mu$$

and then select such a positive number δ that

$$(7) \quad 2\delta(1 + \epsilon \|U\|) < r \|U\|$$

and

$$(8) \quad |g_s(z) - g_s(x) - L(z - x)| < \epsilon |z - x| \quad \text{whenever } 0 < |z - x| < \delta.$$

Next the classical theorem on continuously differentiable functions with nonvanishing Jacobian yields in view of (4) a number t such that $0 < t < \delta$ and

$p^*(S)$ is an open subset of E_m ,

where

$$S = E_m E_z [|z - x| < t] \subset Q_s.$$

Now a typical set $\alpha \in F(x)$ is constructed as follows:

Choose a number ρ for which $0 < \rho \|U\| < t$ and

$$(9) \quad w \in p^*(S) \quad \text{whenever} \quad |w - p(x)| < (1 - \epsilon \|U\|)\rho.$$

Corresponding to this number ρ let

$$\alpha = E_m E_z [|T(z - x)| \leq \rho],$$

$$\bar{\alpha} = E_n E_y [|y - g_*(x)| < (1 + \epsilon \|U\|)\rho].$$

Clearly $x \in \alpha$, α is closed, $\bar{\alpha}$ is an open sphere of E_n , and $\text{diam } \bar{\alpha} = 2\rho(1 + \epsilon \|U\|) < 2\delta \|U\|^{-1}(1 + \epsilon \|U\|) < r$ by virtue of (7). A moment's thought convinces us that

$$(10) \quad |\alpha| \cdot |\det T| = |E_m E_z [|z| \leq \rho]|,$$

$$(11) \quad \gamma(\bar{\alpha}) = (1 + \epsilon \|U\|)^m |E_m E_z [|z| \leq \rho]|.$$

Next we check

$$(12) \quad |z - x| = |U[T(z - x)]| \leq \|U\|\rho < t < \delta \quad \text{for } z \in \alpha;$$

$$(13) \quad \alpha \subset S \subset Q_s \quad \text{and} \quad \text{diam } \alpha < 2\|U\|\rho.$$

Evidently $g_*(x) \in \bar{\alpha}$ and the relation $x \neq z \in \alpha$ implies via (12), (8), (5), (12) that

$$\begin{aligned} |g_*(z) - g_*(x)| &< |L(z - x)| + \epsilon |z - x| \leq |T(z - x)| + \epsilon \|U\|\rho \\ &\leq \rho(1 + \epsilon \|U\|), \end{aligned}$$

hence $g_*(z) \in \bar{\alpha}$. Accordingly

$$(14) \quad g_*^*(\alpha) \subset \bar{\alpha}.$$

Now let

$$C = E_m E_w [|w - p(x)| < (1 - \epsilon \|U\|)\rho].$$

If $w \in C$ we use (9) to pick $z \in S$ with $p(z) = w$. Since $t < \delta$ we may use (8) and (12) to check

$$\begin{aligned} |p(z) - p(x) - T(z - x)| &= |P[g_*(z)] - P[g_*(x)] - P[L(z - x)]| \\ &= |P[g_*(z) - g_*(x) - L(z - x)]| \\ &\leq |g_*(z) - g_*(x) - L(z - x)| \\ &\leq \epsilon |z - x| \leq \epsilon \|U\| \cdot |T(z - x)|, \end{aligned}$$

$$\begin{aligned}
 (1 - \epsilon \|U\|) \rho &\geq |p(z) - p(x)| \\
 &\geq |T(z - x)| - \epsilon \|U\| \cdot |T(z - x)| \\
 &= (1 - \epsilon \|U\|) \cdot |T(z - x)|,
 \end{aligned}$$

hence $\rho \geq |T(z - x)|$ and $z \in \alpha$, $w \in p^*(\alpha)$. Thus

$$(15) \quad C \subset p^*(\alpha).$$

From Theorem 3.6 of SA I (see Remark 2.3 of this paper) and the relations (15), (10), (4), (6) we infer

$$\begin{aligned}
 \Phi[g_s^*(\alpha)] &\geq |P[g_s^*(\alpha)]| = |p^*(\alpha)| \geq |C| \\
 &= (1 - \epsilon \|U\|)^m |E_m E_z [|z| \leq \rho]| = (1 - \epsilon \|U\|)^m |\alpha| \cdot |\det T| \\
 &= (1 - \epsilon \|U\|)^m Jf(x) |\alpha| > \lambda |\alpha|; \\
 (16) \quad &\lambda |\alpha| < \Phi[g_s^*(\alpha)].
 \end{aligned}$$

From (11), (10), (4), (6) we similarly obtain

$$(17) \quad \gamma(\bar{\alpha}) < \mu |\alpha|.$$

At last we use (10) and (13) to check that

$$\frac{|\alpha|}{(\text{diam } \alpha)^m} \geq \frac{|E_m E_z [|z| \leq \rho]|}{|\det T| \cdot (2\|U\|\rho)^m} = \frac{|E_m E_z [|z| \leq 1]|}{|\det T| \cdot (2\|U\|)^m}$$

and observe that the right-hand member of the last equation is a positive number independent of ρ and α .

We combine the last remark with the relations (13), (16), (14), (17) to complete the proof of Part 1.

Part 2. If $X \in K$ and $X \subset B_s$, then

$$\lambda |X| \leq V_K(f, X, \Phi) \quad \text{and} \quad \Phi[f^*(X)] \leq \mu |X|.$$

Proof. Let $r > 0$.

Choose an open set S such that $X \subset S$ and $|S - X| \leq r$.

For $x \in X$ define $F(x)$ as in Part 1 and let

$$G(x) = F(x) E_\alpha [\alpha \subset S],$$

$$U = \sum_{x \in X} G(x).$$

The existence of a countable disjointed family $H \subset U$ for which $|X - \sigma(H)| = 0$ may be deduced from the classical Vitali covering theorem⁽⁸⁾ or from the

(8) S. Saks, *Theory of the integral*, Warsaw, 1937, p. 109.

fact that G is a diametrically regular blanket⁽⁹⁾. Now use Part 1, Lemma 4.1 and (2) to obtain

$$\begin{aligned}
 \gamma_r[f^*(X)] &= \gamma_r[g_s^*(X)] \leq \gamma_r\{g_s^*[\sigma(H)]\} + \gamma_r\{g_s^*[X - \sigma(H)]\} \\
 &\leq \sum_{\alpha \in H} \gamma(\alpha) + \Phi\{g_s^*[X - \sigma(H)]\} \\
 &\leq \sum_{\alpha \in H} \mu|\alpha| + (10M)^m |X - \sigma(H)| \\
 &\leq \mu|S| \leq \mu(|X| + r); \\
 \lambda|X| &\leq \lambda \sum_{\alpha \in H} |\alpha| \leq \sum_{\alpha \in H} \Phi[g_s^*(\alpha)] \leq \sum_{\alpha \in H} \Phi[g_s^*(\alpha X)] + \Phi[g_s^*(\alpha - X)] \\
 &\leq \sum_{\alpha \in H} \Phi[f^*(\alpha X)] + (10M)^m |\alpha - X| \\
 &\leq V_K(f, X, \Phi) + (10M)^m |S - X| \leq V_K(f, X, \Phi) + (10M)^m r.
 \end{aligned}$$

Thus

$$\lambda|X| \leq V_K(f, X, \Phi) + (10M)^m r \quad \text{and} \quad \gamma_r[f^*(X)] \leq \mu(|X| + r)$$

for every $r > 0$. Let $r \rightarrow 0$.

Part 3. $\lambda|A| \leq \int N(f, A, y) d\Phi y \leq \mu|A|$.

Proof. We use (1), Lemma 4.1 and Theorem 4.1 of SA I to check:

$$\begin{aligned}
 0 &= |A - B| = \Phi[f^*(A - B)], \\
 \sum_{s=1}^{\infty} |B_s| &= |B| = |A|, \\
 \sum_{s=1}^{\infty} V_K(f, B_s, \Phi) &= \sum_{s=1}^{\infty} \int N(f, B_s, y) d\Phi y = \int \sum_{s=1}^{\infty} N(f, B_s, y) d\Phi y \\
 &= \int N(f, B, y) d\Phi y = \int N(f, A, y) d\Phi y.
 \end{aligned}$$

We shall accordingly complete the proof by showing that

$$(18) \quad \lambda|B_s| \leq V_K(f, B_s, \Phi) \leq \mu|B_s| \quad \text{for } s = 1, 2, 3, \dots$$

Fixing a positive integer s , we see that the first inequality in (18) follows immediately from Part 2. If, on the other hand, $G \subset K$ and G is a partition of B_s , then

$$\sum_{S \in G} \Phi[f^*(S)] \leq \mu \sum_{S \in G} |S| = \mu|B_s|$$

by virtue of Part 2. Considering the arbitrary nature of G , we are convinced of the second inequality in (18).

4.4 LEMMA. *If f is a function on E_m to E_n ; M is a number such that*

$$|f(z) - f(x)| \leq M|z - x| \quad \text{for } z \in E_m, x \in E_m;$$

(9) Loc. cit. (3), Definition 6.5 and Theorem 6.12.

A is a Lebesgue measurable subset of E_m ; f is differentiable at each point of A and $Jf(x)=0$ for $x \in A$; then $\Phi[f^*(A)]=0$.

Proof. We assume, without loss of generality, that $|A| < \infty$ and divide the remainder of the proof into two parts.

Part 1. If $x \in A$ and $r > 0$, then there is a closed sphere $\alpha \subset E_m$ and an open connected set $\bar{\alpha} \subset E_n$ such that $x \in \alpha$, $\text{diam } \alpha < r$, $\text{diam } \bar{\alpha} < r$, $f^*(\alpha) \subset \bar{\alpha}$, $\gamma(\bar{\alpha}) \leq r|\alpha|$.

Proof. Let L be the differential of f at x and let h be the function such that

$$h(z) = f(x) + L(z - x).$$

Choose ϵ so that

$$0 < 2\epsilon \{ \|L\| + \epsilon \}^{m-1} |E_{m-1} E_{\frac{r}{2}} [|z| \leq 1]| \leq r |E_m E_w [|w| \leq 1]|,$$

select ρ so that $0 < 2\rho \{ \|L\| + \epsilon + 1 \} < r$ and $|f(z) - h(z)| < \epsilon |z - x|$ whenever $0 < |z - x| \leq \rho$.

Denoting by $d(y, Y)$ the distance of the point y from the set Y , we define

$$\alpha = E_m E_{\frac{r}{2}} [|z - x| \leq \rho], \quad \bar{\alpha} = E_n E_y [d\{y, h^*(\alpha)\} < \epsilon\rho].$$

Of all the properties required for α and $\bar{\alpha}$, only the last is not quite obviously satisfied. In order to prove it, let P be any projection function on E_n to E_m . Denote

$$Q = E_m E_w [w_m = 0].$$

Since $\Delta(L) = Jf(x) = 0$, the points $L^1, L^2, \dots, L^m$ are linearly dependent; hence the same is true of the points $P(L^1), \dots, P(L^m)$. We may therefore select a distance preserving transformation T on E_m to E_m such that

$$[T:P:h]^*(\alpha) \subset Q.$$

Let

$$B = Q E_w [|w - [T:P:h](x)| \leq \|L\|\rho],$$

$$C = E_m E_w [|w - [T:P:h](x)| < \{ \|L\| + \epsilon \}\rho; |w_m| < \epsilon\rho]$$

and successively check the relations

$$\begin{aligned} h^*(\alpha) &\subset E_n E_y [|y - h(x)| \leq \|L\|\rho], & [T:P:h]^*(\alpha) &\subset B, \\ [T:P]^*(\bar{\alpha}) &\subset E_m E_w [d\{w, [T:P:h]^*(\alpha)\} < \epsilon\rho] \subset E_m E_w [d(w, B) < \epsilon\rho] \subset C, \\ |P^*(\bar{\alpha})| &= |[T:P]^*(\bar{\alpha})| \leq |C| \\ &= 2\epsilon\rho \{ \|L\| + \epsilon \}^{m-1} \rho^{m-1} |E_{m-1} E_{\frac{r}{2}} [|z| \leq 1]| \\ &= 2\epsilon \{ \|L\| + \epsilon \}^{m-1} \frac{|E_{m-1} E_{\frac{r}{2}} [|z| \leq 1]|}{|E_m E_w [|w| \leq 1]|} |\alpha| \leq r |\alpha|. \end{aligned}$$

Thus $|P^*(\bar{\alpha})| \leq r|\alpha|$ and from the arbitrary nature of P we infer $\gamma(\bar{\alpha}) \leq r|\alpha|$.

Part 2. $\Phi[f^*(A)] = 0$.

Proof. Let $r > 0$.

Choose an open set S such that $A \subset S \subset E_m$, $|S| < |A| + r$. Let F be the family of all closed spheres $\alpha \subset S$ which satisfy all the conditions of Part 1. Clearly F covers A in the sense of Vitali and there is a countable disjointed subfamily G of F such that $|A - \sigma(G)| = 0$. Use Part 1 and 4.1 to infer

$$\begin{aligned} \gamma_r[f^*(A)] &\leq \gamma_r\{f^*[\sigma(G)]\} + \gamma_r\{f^*[A - \sigma(G)]\} \\ &\leq \sum_{\alpha \in G} \gamma(\bar{\alpha}) + \Phi\{f^*[A - \sigma(G)]\} \leq r \sum_{\alpha \in G} |\alpha| + (5M)^m |A - \sigma(G)| \\ &\leq r|S| < r\{|A| + r\}; \\ \gamma_r[f^*(A)] &< r\{|A| + r\}. \end{aligned}$$

Let $r \rightarrow 0$.

4.5 THEOREM. If f is a function on E_m to E_n ; M is a number such that

$$|f(z) - f(x)| \leq M|z - x| \quad \text{for } z \in E_m, x \in E_m;$$

T is a Lebesgue measurable subset of E_m ; then

$$\int N(f, T, y) d\Phi y = \int_T Jf(x) dx.$$

Proof. Let A be the set of those points of T at which f is not differentiable. From Rademacher's theorem⁽¹⁰⁾ we know that $|A| = 0$. Hence $\Phi[f^*(A)] = 0$ by 4.1. Denote

$$B = (T - A) \ E_x [Jf(x) = 0], \quad C = (T - A) \ E_x [Jf(x) > 0],$$

and observe that the theorem follows immediately from the following

Statement. If $t > 1$, then

$$t^{-1} \int_T Jf(x) dx \leq \int N(f, T, y) d\Phi y \leq t \int_T Jf(x) dx.$$

Proof. Let α be the set of all integers. For $j \in \alpha$ let

$$C_j = C \ E_x [t^j < Jf(x) \leq t^{j+1}],$$

and infer from 4.3, taking $\lambda = t^j$ and letting $\mu \rightarrow t^{j+1} +$, that

$$t^j |C_j| \leq \int N(f, C_j, y) d\Phi y \leq t^{j+1} |C_j|$$

⁽¹⁰⁾ H. Rademacher, *Über partielle und totale Differenzierbarkeit*, I, Math. Ann. vol. 79 (1919).

because C_j is a Borel set. Note that

$$T = A + B + \sum_{j \in \alpha} C_j,$$

and accordingly

$$\begin{aligned} t^{-1} \int_T Jf(x) dx &= t^{-1} \int_A Jf(x) dx + t^{-1} \int_B Jf(x) dx + \sum_{j \in \alpha} t^{-1} \int_{C_j} Jf(x) dx \\ &\leq \sum_{j \in \alpha} t^j |C_j| \leq \sum_{j \in \alpha} \int N(f, C_j, y) d\Phi y = \int \sum_{j \in \alpha} N(f, C_j, y) d\Phi y \\ &= \int N(f, C, y) d\Phi y = \sum_{j \in \alpha} \int N(f, C_j, y) d\Phi y \leq \sum_{j \in \alpha} t^{j+1} |C_j| \\ &\leq \sum_{j \in \alpha} t \int_{C_j} Jf(x) dx \leq t \int_T Jf(x) dx; \\ t^{-1} \int_T Jf(x) dx &\leq \int N(f, C, y) d\Phi y \leq t \int_T Jf(x) dx. \end{aligned}$$

But $\Phi[f^*(A+B)] \leq \Phi[f^*(A)] + \Phi[f^*(B)] = 0$ in view of our remark at the beginning of the proof and of Lemma 4.4. Thus

$$N(f, C, y) = N(f, T, y)$$

for Φ almost all y in E_n and we complete the proof by substitution in the last integral inequality.

5. Approximately differentiable surfaces.

5.1 THEOREM. *If f is a Lebesgue measurable function on E_m to E_n , T is a Lebesgue measurable subset of E_m and if there are sets $T_1, T_2, T_3, \dots$ and numbers $M_1, M_2, M_3, \dots$ such that*

$$T = \sum_{i=1}^{\infty} T_i$$

and

$$|f(z) - f(x)| \leq M_i |z - x| \quad \text{whenever } z \in T_i, x \in T_i,$$

then

$$\int N(f, T, y) d\Phi y = \int_T Jf(x) dx.$$

Proof. Let C be the set of points of approximate continuity of f . For each positive integer j let D_j be the set of points of density of T_j and define

$$S_j = TCD_j, \quad A_j = S_j - \sum_{i=1}^{j-1} S_i.$$

We note that S_j is a Lebesgue measurable set with

$$\begin{aligned}
 |T_j - S_j| &= |T_j - TCD_j| = |T_j - CD_j| \\
 &\leq |T_j - C| + |T_j - D_j| = 0, \\
 (1) \quad |T_j - S_j| &= 0 \quad \text{for } j = 1, 2, 3, \dots
 \end{aligned}$$

The remainder of the proof is divided into four parts.

Part 1. If $x \in S_j, z \in S_j$, then $|f(z) - f(x)| \leq M_j |z - x|$.

Proof. Select a measurable set A with density 1 at x such that f is continuous relative to A at x . Since A is measurable it is easily seen⁽¹¹⁾ that x is a point of density of the set AT_j , even though T_j may be nonmeasurable. Hence there are points $u^1, u^2, u^3, \dots$ such that

$$\lim_{k \rightarrow \infty} u^k = x, \quad u^k \in AT_j \quad \text{for } k = 1, 2, 3, \dots$$

It follows that

$$\lim_{k \rightarrow \infty} f(u^k) = f(x), \quad u^k \in T_j \quad \text{for } k = 1, 2, 3, \dots$$

Applying the same reasoning to z we find points $v^1, v^2, v^3, \dots$ for which

$$\lim_{k \rightarrow \infty} v^k = z, \quad \lim_{k \rightarrow \infty} f(v^k) = f(z), \quad v^k \in T_j \quad \text{for } k = 1, 2, 3, \dots$$

Accordingly

$$\begin{aligned}
 |f(z) - f(x)| &= \left| \lim_{k \rightarrow \infty} f(v^k) - \lim_{k \rightarrow \infty} f(u^k) \right| = \lim_{k \rightarrow \infty} |f(v^k) - f(u^k)| \\
 &\leq M_j \limsup_{k \rightarrow \infty} |v^k - u^k| = M_j |z - x|.
 \end{aligned}$$

Part 2. If j is a positive integer, then

$$\int N(f, A_j, y) d\Phi y = \int_{A_j} Jf(x) dx.$$

Proof. In view of Part 1 there exists⁽¹²⁾ a function g on E_m to E_n such that g satisfies a Lipschitz condition on E_m and

$$(2) \quad g(x) = f(x) \quad \text{for } x \in S_j.$$

It follows from Theorem 4.5 that

$$(3) \quad \int N(g, A_j, y) d\Phi y = \int_{A_j} Jg(x) dx.$$

⁽¹¹⁾ The relation $|XA| + |XT_j| = |X(A + T_j)| + |XAT_j|$ holds for every $X \subset E_m$. See C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1927, p. 252.

⁽¹²⁾ H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. vol. 36 (1934). Apply the formula given in the last footnote on page 63 to each of the n coordinate functions.

But $A_j \subset S_j$; hence (2) implies

$$(4) \quad N(g, A_j, y) = N(f, A_j, y) \quad \text{for } y \in E_n.$$

Moreover at each point of density of S_j at which g is differentiable, the differential of g is the approximate differential of f . Accordingly

$$(5) \quad Jg(x) = Jf(x) \quad \text{for almost all } x \text{ in } S_j.$$

Substitute (4) and (5) in (3) to establish Part 2.

Part 3. If j is a positive integer, then $\Phi[f^*(T_j - S_j)] = 0$.

Proof. Remember that f satisfies a Lipschitz condition on T_j and apply (1) and 4.1.

Part 4. $\int N(f, T, y) d\Phi y = \int_T Jf(x) dx$.

Proof. Let $S = \sum_{j=1}^{\infty} S_j$; note that $A_1, A_2, A_3, \dots$ are disjoint and

$$(6) \quad S = \sum_{j=1}^{\infty} A_j, \quad (T - S) \subset \sum_{j=1}^{\infty} (T_j - S_j).$$

Hence (1) and Part 3 imply

$$(7) \quad |T - S| = \Phi[f^*(T - S)] = 0,$$

$$(8) \quad N(f, T, y) = N(f, S, y) \quad \text{for } \Phi \text{ almost all } y \text{ in } E_n.$$

We now use (7), Part 2, (6), (8) to conclude

$$\begin{aligned} \int_T Jf(x) dx &= \int_S Jf(x) dx = \sum_{j=1}^{\infty} \int_{A_j} Jf(x) dx \\ &= \sum_{j=1}^{\infty} \int N(f, A_j, y) d\Phi y = \sum_{j=1}^{\infty} \int N(f, A_j, y) d\Phi y \\ &= \int N(f, S, y) d\Phi y = \int N(f, T, y) d\Phi y. \end{aligned}$$

5.2 THEOREM. If f is a Lebesgue measurable function on E_m to E_n , T is a Lebesgue measurable subset of E_m and

$$\limsup_{z \rightarrow x} \frac{|f(z) - f(x)|}{|z - x|} < \infty \quad \text{for every } x \in T,$$

then

$$\int N(f, T, y) d\Phi y = \int_T Jf(x) dx.$$

Proof. Let

$$K_0 = E_m \underset{w}{E} [|w| \leq 1],$$

$$K_1 = E_m \underset{w}{E} [|w - (1, 0, \dots, 0)| \leq 1]$$

and denote

$$\alpha = \frac{|K_0 K_1|}{|K_0| + |K_1|}.$$

Since $K_0 K_1$ has interior points, we know that $\alpha > 0$.

For each positive integer j , we let T_j be the set of all points x for which $x \in T$, $|f(x)| \leq j$, and $0 < r \leq j^{-1}$ implies

$$\begin{aligned} |E_w[|w - x| \leq r, |f(w) - f(x)| \leq j|w - x|]| \\ > (1 - \alpha) |E_w[|w - x| \leq r]|. \end{aligned}$$

Evidently

$$T = \sum_{j=1}^{\infty} T_j$$

and use of 5.1 will complete the proof as soon as we verify the following

Statement. If $x \in T_j$, $z \in T_j$, then

$$|f(z) - f(x)| \leq 2j^2 |z - x|.$$

Proof. If $|z - x| > j^{-1}$, we have $|f(z) - f(x)| \leq 2j < 2j^2 |z - x|$. We henceforth assume $|z - x| = r \leq j^{-1}$ and define

$$C_0 = E_w[|w - x| \leq r], \quad C_1 = E_w[|w - z| \leq r];$$

note that

$$\alpha = \frac{|C_0 C_1|}{|C_0| + |C_1|}$$

and let

$$\begin{aligned} B_0 &= C_0 E_w[|f(w) - f(x)| \leq j|w - x|], \\ B_1 &= C_1 E_w[|f(w) - f(z)| \leq j|w - z|]. \end{aligned}$$

Since f is a measurable function we know that B_0 and B_1 are measurable sets and

$$\begin{aligned} |B_0 B_1| &= |B_0| + |B_1| - |B_0 + B_1| \\ &> (1 - \alpha) |C_0| + (1 - \alpha) |C_1| - |C_0 + C_1| \\ &= (1 - \alpha) \{ |C_0| + |C_1| \} - \{ |C_0| + |C_1| - |C_0 C_1| \} \\ &= |C_0 C_1| - \alpha \{ |C_0| + |C_1| \} = 0. \end{aligned}$$

Thus $|B_0 B_1| > 0$ and we can pick a point $w \in B_0 B_1$. Then

$$\begin{aligned} |f(z) - f(x)| &\leq |f(z) - f(w)| + |f(w) - f(x)| \leq j|z - w| + j|z - x| \\ &= 2jr = 2j|z - x| \leq 2j^2 |z - x|. \end{aligned}$$

5.3 THEOREM. If f is a continuous function on E_m to E_n , T is an analytic

subset of E_m and the approximate partial derivatives of f exist (finite) at almost all points of T , then

$$\int_T Jf(x)dx \leq \int N(f, T, y)d\Phi y.$$

Proof. Let S be the subset of T on which f is approximately differentiable. From Stephanoff's theorem⁽¹³⁾ we see that $|T - S| = 0$ and combine this with 5.2 to infer

$$\int_T Jf(x)dx = \int_S Jf(x)dx = \int N(f, S, y)d\Phi y.$$

But $N(f, S, y) \leq N(f, T, y)$ for $y \in E_n$ and completion of the proof depends merely on showing that the last expression is Φ measurable. For this purpose we use Theorem 4.1 of SA I, taking F = the family of all analytic subsets of T .

5.4 Remark. Suppose $\int_T Jf(x)dx < \infty$ in addition to the hypotheses of 5.3. Then equality holds in the conclusion of that theorem if and only if

$$\Phi[f^*(X)] = 0 \quad \text{whenever} \quad X \subset T, \quad |X| = 0.$$

6. Two-dimensional surfaces in n -dimensional space.

6.1 THEOREM. If f is a Lebesgue measurable function on E_2 to E_n ($2 \leq n$); T is a Lebesgue measurable subset of E_2 ; corresponding to each $x \in T$ there are three distinct points $\bar{x}^1, \bar{x}^2, \bar{x}^3$ of E_2 such that $|\bar{x}^i| = 1$ and

$$\limsup_{t \rightarrow 0+} \left| \frac{f(x + t\bar{x}^j) - f(x)}{t} \right| < \infty \quad \text{for } j = 1, 2, 3;$$

then

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Let $C = E_2 E_z [|z| = 1]$.

Let W be the set of all matrices whose columns are three distinct points of C . For each $x \in T$, the points $\bar{x}^1, \bar{x}^2, \bar{x}^3$ are evidently the columns of the matrix $\bar{x} \in W$. Each matrix $w \in W$ determines a subdivision of C into three arcs as follows: $C_1(w)$ leads from w^2 to w^3 ; $C_2(w)$ from w^3 to w^1 ; $C_3(w)$ from w^1 to w^2 . Further we denote by $\delta(w)$ the smallest of the distances from w^i to $C_j(w)$, where $j = 1, 2, 3$, and observe that $\delta(w) > 0$ for $w \in W$.

Now let D be a countable subset of C which is dense in C and define

$$Z = W \bigcap_{w \in W} [w^j \in D \text{ for } j = 1, 2, 3].$$

We write $a \cdot b = a_1 b_1 + a_2 b_2$ for $a \in E_2, b \in E_2$ and note that $|\bar{x}^i \cdot \bar{x}^j| < 1$ whenever $x \in T; i = 1, 2, 3; j = 1, 2, 3; i \neq j$.

⁽¹³⁾ Loc. cit. (7), p. 300. The extension to m dimensions offers no difficulty.

For each positive integer k and each matrix $w \in Z$, we let T_k^w be the set of all points x for which:

$$\begin{aligned} x &\in T; & |f(x)| &\leq k; & \delta(\bar{x}) &> 4k^{-1}. \\ |f(x + t\bar{x}^j) - f(x)| &\leq kt & \text{whenever } 0 \leq t \leq 2k^{-1}, j = 1, 2, 3. \\ 1 - x^i \cdot x^j &> 3k^{-1} & \text{whenever } i = 1, 2, 3; j = 1, 2, 3; i \neq j. \\ |\bar{x}^j - w^j| &\leq k^{-1} & \text{for } j = 1, 2, 3. \end{aligned}$$

Since Z is countable and

$$T = \sum_{k=1}^{\infty} \sum_{w \in Z} T_k^w$$

our theorem is a consequence of 5.1 and the following

Statement. If $x \in T_k^w$ and $z \in T_k^w$, then

$$|f(z) - f(x)| \leq 2k^3 |z - x|.$$

Proof. Let $r = k^{-1}$. In case $|z - x| \geq r^2$, we have

$$|f(z) - f(x)| \leq 2k = 2k^3 r^2 \leq 2k^3 |z - x|$$

and we henceforth assume

$$(1) \quad |z - x| < r^2.$$

We define

$$K = E_u [|u - x| = r],$$

$$S_j = E_u \left[|u - x| \leq r, \left| \frac{u - x}{u - x} \right| \in C_j(\bar{x}) \right], \quad K_j = KS_j \quad \text{for } j = 1, 2, 3;$$

and let L_j be the line segment joining x and $x + r\bar{x}^j$. Since (1) implies $z \in \sum_{j=1}^3 S_j$, we may, without loss of generality, assume $z \in S_3$.

The half ray issuing from z with direction $\bar{z}^3$ intersects the boundary of S_3 in a point u ; that is, a number $\lambda \geq 0$ can be found such that

$$(2) \quad z + \lambda \bar{z}^3 = u \in (L_1 + L_2 + K_3).$$

We have $\lambda = |\lambda \bar{z}^3| = |u - z| \leq 2r$ since z and u are inside K ; hence

$$(3) \quad 0 \leq \lambda \leq 2r.$$

We shall prove next (by contradiction) that

$$(4) \quad u \in L_1 + L_2.$$

By virtue of (2), the denial of (4) implies $u \in K_3$. From the definition of K_3 we infer

$$|u - x| = r, \quad (u - x)/r \in C_3(x),$$

and use the definitions of $\delta(\bar{x})$ and T_k^w as well as the relations (2), (1) to obtain

$$\begin{aligned}\delta(\bar{x}) &\leq |(u-x)/r - \bar{x}^3| = k|z + \lambda\bar{z}^3 - x - r\bar{x}^3| \\ &\leq k\{|z-x| + |\lambda\bar{z}^3 - r\bar{z}^3| + |r\bar{z}^3 - rw^3| + |rw^3 - r\bar{x}^3|\} \\ &< k\{r^2 + |\lambda-r| + r^2 + r^2\} = 3r + k|\lambda-r| \\ &= 3r + k||u-z| - |u-x|| \leq 3r + k|x-z| < 3r + r < \delta(\bar{x});\end{aligned}$$

hence $\delta(x) < \delta(\bar{x})$, which is false. Thus we have verified (4) and we may, without loss of generality, assume

$$u \in L_1.$$

We infer the existence of a number μ such that

$$(5) \quad 0 \leq \mu \leq r, \quad x + \mu\bar{x}^1 = u.$$

Use (2), (5) and the definition of T_k^w to compute

$$\begin{aligned}z-x &= \mu\bar{x}^1 - \lambda\bar{z}^3, \quad (z-x) \cdot (\bar{x}^1 - \bar{z}^3) = (\mu+\lambda)(1 - \bar{z}^3 \cdot \bar{x}^1), \\ 1 - \bar{z}^3 \cdot \bar{x}^1 &= 1 - \bar{x}^3 \cdot \bar{x}^1 + (\bar{x}^3 - \bar{z}^3) \cdot \bar{x}^1 \geq 3r - |\bar{x}^3 - \bar{z}^3| \geq r, \\ \lambda + \mu &\leq \frac{|z-x| \cdot |\bar{x}^1 - \bar{z}^3|}{1 - \bar{z}^3 \cdot \bar{x}^1} \leq (2/r)|z-x|; \quad \mu + \lambda \leq 2k|z-x|.\end{aligned}$$

The last relation combines with (2), (5), (3) and the definition of T_k^w to give us the result:

$$\begin{aligned}|f(z) - f(x)| &\leq |f(z) - f(z + \lambda\bar{z}^3)| + |f(x + \mu\bar{x}^1) - f(x)| \\ &\leq k\lambda + k\mu = k(\lambda + \mu) \leq 2k^2|z-x| \leq 2k^3|z-x|.\end{aligned}$$

6.2 Remark. It is easy to show by an example that the three directions associated with each point in 6.1 cannot be replaced by two directions.

6.3 THEOREM. *If the Lebesgue measurable function f on E_2 to E_n ($2 \leq n$) has finite partial derivatives at each point of the Lebesgue measurable set T , then*

$$\int N(f, T, y) d\Phi y = \int_T Jf(x) dx.$$

Proof. Take $\bar{x}^1 = (1, 0)$; $\bar{x}^2 = (0, 1)$; $\bar{x}^3 = (-1, 0)$ for $x \in T$ and apply 6.1.

7. Functions on the line to the line.

7.1 Definition. If f is a function on E_1 to E_1 , then D^+f is the function such that

$$D^+f(x) = \limsup_{z \rightarrow x+} \frac{f(z) - f(x)}{z - x}.$$

7.2 THEOREM. *If f is a Lebesgue measurable function on E_1 to E_1 , T is a Lebesgue measurable set of numbers and*

then $|D^+f(x)| < \infty$ for $x \in T$,

$$\int N(f, T, y) dy = \int_T |D^+f(x)| dx.$$

Proof. From Theorem 9.9 of Saks⁽¹⁴⁾, chap. 9, we infer the existence of a set $S \subset T$ such that

$$|T - S| = 0, \quad Jf(x) = |D^+f(x)| \quad \text{for } x \in S.$$

Hence Theorem 5.2 of this paper yields

$$\int N(f, S, y) dy = \int_S Jf(x) dx = \int_S |D^+f(x)| dx = \int_T |D^+f(x)| dx.$$

But Theorem 4.6 of Saks⁽¹⁴⁾, chap. 9, tells us

$$|f^*(T - S)| = 0$$

and we complete the proof by observing that the last relation implies $N(f, S, y) = N(f, T, y)$ for almost all numbers y .

⁽¹⁴⁾ Loc. cit. (?).