ON THE DISTRIBUTION OF QUADRATIC NON-RESIDUES AND THE EUCLIDEAN ALGORITHM IN REAL QUADRATIC FIELDS. II

BY

LOO-KENG HUA AND SZU-HOA MIN

1. Introduction. It is the purpose of the paper to establish the following result⁽¹⁾:

For a prime $p \equiv 17 \pmod{24}$, there is no E.A. in $R(p^{1/2})$ except possibly for $p = 17, 41, 89, 113$, and 137 .

It is known that for $p = 17, 41$ the field is Euclidean. The only doubtful cases are $p = 89, 113$, and 137 .

The method used is a refinement of the previous paper, by introducing Euler's summation formula and a new estimation of character sum.

The numbering of theorems and lemmas continues from I, and the notations of I are preserved, for example p denotes a prime, $[\xi]$ denotes the integral part of ξ and $q_1 < q_2 < q_3$ denote the least three primes which are quadratic non-residues, mod p .

2. Lemmas involving primes.

LEMMA 17. For $11 \leq x \leq 10^6$,

$$li\ x - li\ x^{1/2} < \pi(x) < li\ x$$

(Rosser, Theorems 1 and 3).

LEMMA 18. For $x \geq 400$, $\vartheta(x) > (1 - 0.139)x$.

Proof. For $x < 10^6$ use Theorem 7⁽²⁾ of Rosser, and for $x > 10^6$, use (10)⁽³⁾ of Rosser.

LEMMA 19. For $401 \leq a \leq b \leq a^2$,

$$\sum_{a \leq q \leq b} \frac{1}{q^2} < \frac{1.2142}{a \log a}$$

where q runs over primes.

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⁽¹⁾ Editor's footnote. The proof given here is entirely different from that given in L. Rédei, *Zur Frage des Euklidischen Algorithmus in quadratischen Zahlkörpern*, Math. Ann. vol. 118 (1942) pp. 588–608. In that paper Rédei proved that the Euclidean algorithm does not exist in quadratic fields $R(\mu^{1/2})$ if $\mu > 41$ has the form $24n + 17$. The paper is unknown to the author and arrived only recently in this country.

⁽²⁾ For $0 < x \leq 10^6$, $x - 2.78x^{1/2} < \vartheta(x)$.

⁽³⁾ For $x > e^{13.8}$, $\vartheta(x) > (1 - 0.0393)x$.

Proof. By Lemma 1 (of I) and Lemma 18,

$$\begin{aligned}
 \sum_{a \leq q \leq b} \frac{1}{q^2} &\leq \sum_{a \leq n \leq b} \frac{\vartheta(n) - \vartheta(n-1)}{n^2 \log n} \\
 &= \sum_{a \leq n \leq b} \vartheta(n) \left(\frac{1}{n^2 \log n} - \frac{1}{(n+1)^2 \log(n+1)} \right) \\
 &\quad + \frac{\vartheta(b)}{(b+1)^2 \log(b+1)} - \frac{\vartheta(a-1)}{a^2 \log a} \\
 &< 1.0376 \left(\sum_{a \leq n \leq b} n \left(\frac{1}{n^2 \log n} - \frac{1}{(n+1)^2 \log(n+1)} \right) \right. \\
 &\quad \left. + \frac{b}{(b+1)^2 \log(b+1)} \right) - \frac{(1-0.139)(a-1)}{a^2 \log a} \\
 &= 1.0376 \sum_{a \leq n \leq b} \frac{1}{n^2 \log n} + \frac{0.1766(a-1)}{a^2 \log a} \\
 &< \frac{1.0376}{\log a} \left(\frac{1}{a} - \frac{1}{b} + \frac{1}{a^2} \right) + \frac{0.1766}{a \log a} \\
 &< \frac{1.2142}{a \log a}.
 \end{aligned}$$

3. Some results concerning trigonometrical sums.

LEMMA 20. If A is an integer not less than 1, $0 < y \leq \pi/2A$, and $|x| \leq 1$, we have

$$\left| \sum_{n=1}^A (-1)^n \frac{\sin xyn}{\sin yn} \right| \leq 1.$$

Proof. Without loss of generality we assume $x > 0$. The function $\sin xz/\sin z$ increases monotonically for $0 < z \leq \pi/2$. In fact

$$\frac{d}{dz} \frac{\sin xz}{\sin z} = \frac{x \sin z \cos xz - \cos z \sin xz}{\sin^2 z} \geq 0,$$

since the numerator of the expression vanishes at $z=0$ and its derivative is $(1-x^2) \sin z \sin xz \geq 0$ for $0 < z \leq \pi/2$.

Thus the sum is an alternating series with increasing absolute values of terms. Since $0 < x \leq 1$ and $0 < yn \leq \pi/2$, we have

$$0 < \sin xyn/\sin yn \leq 1;$$

the lemma follows immediately.

LEMMA 21. Let A be an integer not less than 1 and m be a positive integer not greater than $p/2A$ and let

$$S_m = S_m(A) = 2 \sum_{x=1}^{[(p-1)/2m]} \left(\frac{\sin \pi(A+1)mx/p}{\sin \pi mx/p} \right)^2.$$

Then

$$|S_m - p(A+1)/m + (A+1)^2| < 3A + 1$$

and

$$S_1 = p(A+1) - (A+1)^2.$$

Proof. For the sake of brevity, we write

$$e(x) = e^{2\pi ix},$$

and we use $\Re x$ to denote the real part of x .

We have

$$\begin{aligned} S_m &= 2 \sum_{x=1}^{[(p-1)/2m]} \sum_{a=0}^A \sum_{n=-a}^a e(mnx/p) \\ &= 2 \left[\frac{p-1}{2m} \right] (A+1) + 4 \sum_{a=1}^A \sum_{n=1}^a \Re \left(\frac{2e((mn/p)([(p-1)/2m] + 1/2)) - e(mn/2p) + e(-mn/2p)}{2(e(mn/2p) - e(-mn/2p))} \right) \\ &= 2 \left[\frac{p-1}{2m} \right] (A+1) - A(A+1) \\ &\quad + 4 \sum_{a=1}^A \sum_{n=1}^a \Re \frac{e((mn/p)([(p-1)/2m] + 1/2))}{e(mn/2p) - e(-mn/2p)} \\ &= 2 \left[\frac{p-1}{2m} \right] (A+1) - A(A+1) \\ &\quad + 2 \sum_{a=1}^A \sum_{n=1}^a \frac{\sin([(p-1)/2m] + 1/2)2\pi mn/p}{\sin \pi mn/p} \\ &= 2 \left[\frac{p-1}{2m} \right] (A+1) - A(A+1) \\ &\quad + 2 \sum_{a=1}^A \sum_{n=1}^a (-1)^n \frac{\sin([(p-1)/2m] - p/2m + 1/2)2\pi mn/p}{\sin \pi mn/p} \\ &= 2 \left[\frac{p-1}{2m} \right] (A+1) - A(A+1) + 2A\theta, \end{aligned}$$

where $|\theta| \leq 1$, by Lemma 20, since $m \leq p/2A \leq p/2a$ and

$$-1 + 1/2m \leq [(p-1)/2m] - (p-1)/2m \leq 0.$$

Since

$$-2(A+1) \leq 2[(p-1)/2m](A+1) - p(A+1)/m \leq -(A+1)/m < 0$$

we have

$$-4A - 2 < S_m - p(A+1)/m + A(A+1) < 2A.$$

Hence the inequality of the lemma follows.

In particular, when $m=1$, we have $\theta=0$. Hence the equality follows.

LEMMA 22. For $\alpha \geq 0$,

$$\sum_{n=1}^{\infty} \frac{\sin^2 \pi n \alpha}{n^2} = \frac{\pi^2}{2} (\{\alpha\} - \{\alpha\}^2),$$

where $\{\alpha\} = \alpha - [\alpha]$.

(See, for example, Whittaker-Watson, *Modern Analysis*, 4th ed., Example 3, p. 163.)

LEMMA 23. Let $A \geq 0$, $m > 0$, then

$$-\frac{4p}{\pi^2 m} - \frac{8}{\pi^2} < S_m - \frac{p^2}{m^2} \left(\left\{ \frac{(A+1)m}{p} \right\} - \left\{ \frac{(A+1)m}{p} \right\}^2 \right) < \frac{5p}{6m}.$$

Proof. (1) Using $\sin x \leq x$ for $0 \leq x \leq \pi/2$, we have

$$\begin{aligned} S_m &> 2 \sum_{x=1}^{[(p-1)/2m]} \frac{\sin^2 \pi(A+1)mx/p}{(\pi mx/p)^2} \\ &> \frac{2p^2}{\pi^2 m^2} \left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} - \sum_{x=[(p-1)/2m]+1}^{\infty} \frac{1}{x^2} \right) \\ &> \frac{2p^2}{\pi^2 m^2} \left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} - \frac{2m}{p} - \frac{4m^2}{p^2} \right), \end{aligned}$$

since $[(p-1)/2m]+1 > (p-1)/2m - (2m-1)/2m+1 = p/2m$ and

$$\sum_{x=[(p-1)/2m]+1}^{\infty} \frac{1}{x^2} < \int_{p/2m}^{\infty} \frac{dx}{x^2} + \frac{4m^2}{p^2} = \frac{2m}{p} + \frac{4m^2}{p^2}.$$

By Lemma 22,

$$S_m > \frac{p^2}{m^2} \left(\left\{ \frac{(A+1)m}{p} \right\} - \left\{ \frac{(A+1)m}{p} \right\}^2 \right) - \frac{4p}{\pi^2 m} - \frac{8}{\pi^2}.$$

(2) For $0 \leq x \leq 0.44$, we have

$$(1-x)^{-2} \leq 1+5x$$

and for $0 < x \leq \pi/2$, we have

$$\sin x > x - x^3/6.$$

Therefore, for $1 < x \leq (p-1)/2m$,

$$\frac{1}{(\sin \pi mx/p)^2} < \frac{1}{(\pi^2 m^2 x^2/p^2)(1-\pi^2 m^2 x^2/6p^2)^2} < \frac{p^2}{\pi^2 m^2 x^2} \left(1 + \frac{5\pi^2 m^2 x^2}{6p^2}\right)$$

since

$$\pi^2 m^2 x^2/6p^2 < \pi^2/24 < 0.44.$$

Hence, by Lemma 22,

$$\begin{aligned} S_m &< 2 \sum_{x=1}^{[(p-1)/2m]} \frac{p^2}{\pi^2 m^2} \frac{\sin^2 \pi(A+1)mx/p}{x^2} \left(1 + \frac{5\pi^2 m^2 x^2}{6p^2}\right) \\ &< \frac{2p^2}{\pi^2 m^2} \left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} + \sum_{x=1}^{[(p-1)/2m]} \frac{5\pi^2 m^2}{6p^2} \right) \\ &< \frac{p^2}{m^2} \left(\left\{ \frac{(A+1)m}{p} \right\} - \left\{ \frac{(A+1)m}{p} \right\}^2 \right) + \frac{5p}{6m}. \end{aligned}$$

4. Result concerning character sums.

LEMMA 24. *Let $p > 10^6$ and let*

$$N = N(A) = 2 \sum_{x=1}^{(p-1)/2'} \left(\frac{\sin \pi(A+1)x/p}{\sin \pi x/p} \right)^2,$$

where the prime denotes that the sum runs over quadratic non-residues mod p . If $q_1=3$, $q_2 > p^{1/2}/37.5$, $q_3 \geq (p/5)^{1/2}$, then, for $1 \leq A < 10p^{1/2}$, we have

$$N > 20p(A+1)/81 - 0.05314pp^{1/2}.$$

Proof. We have

$$N > (S_3 - S_9) + (S_{27} - S_{81}) - S_{3q_2} - \sum_{(p/5)^{1/2} \leq q \leq (p-1)/6} S_{3q},$$

where q runs over the primes, since all positive integers less than $3(p/5)^{1/2}$ which are not divisible by q_2 but are exactly divisible by either 3 or $3^3=27$ are quadratic non-residues, mod p .

Since $2A \times 27 < 540p^{1/2} < p$, we have, by Lemma 21,

$$\begin{aligned} (1) \quad S_3 - S_9 + S_{27} &> (1/3 - 1/9 + 1/27)p(A+1) - (A+1)^2 - 9A - 3 \\ &= (7/27)p(A+1) - (A+1)^2 - 9A - 3. \end{aligned}$$

Since $81(A+1) < 810(p^{1/2}+1) < p$, we have, by Lemma 23,

$$(2) \quad S_{81} < p(A+1)/81 - (A+1)^2 + 5p/486.$$

By the same lemma, we have

$$(3) \quad S_{3q_2} < \frac{p^2}{9q_2^2} \times \frac{1}{4} + \frac{5p}{18q_2} = \frac{p^2}{36q_2^2} + \frac{5p}{18q_2} < 0.03909pp^{1/2},$$

since $\{x\} - \{x\}^2 \leq 1/4$ for all x . Similarly,

$$\begin{aligned} \sum_{(p/5)^{1/2} \leq q \leq (p-1)/6} S_{3q} &< \sum_{(p/5)^{1/2} \leq q \leq (p-1)/6} \left(\frac{p^2}{36q^2} + \frac{5p}{18q} \right) \\ &< \frac{p^2}{36} \frac{1.2142}{(p/5)^{1/2} \log (p/5)^{1/2}} + \frac{5p}{18} \left(\log \frac{(5p)^{1/2}}{6} + \left(\frac{5}{p} \right)^{1/2} \right), \end{aligned}$$

by Lemma 19. Hence

$$\begin{aligned} \sum_{(p/5)^{1/2} \leq q \leq (p-1)/6} S_{3q} &< \frac{1.2142(5)^{1/2}}{36 \times 6.103} pp^{1/2} + \frac{5pp^{1/2}}{18000} (5.9208 + 0.0023) \\ (4) \quad &= \frac{2.71508}{219.708} pp^{1/2} + \frac{5 \times 5.9231}{18000} pp^{1/2} \\ &= (0.01236 + 0.00165) pp^{1/2} \\ &= 0.01401 pp^{1/2}. \end{aligned}$$

By (1), (2), (3) and (4), we have

$$\begin{aligned} N &> 20p(A+1)/81 - 9A - 3 - 5p/486 - (0.03909 + 0.01401)pp^{1/2} \\ &> 20p(A+1)/81 - 9A - 3 - 0.00002pp^{1/2} - 0.0531)pp^{1/2} \\ &> 20p(A+1)/81 - 0.05314pp^{1/2}. \end{aligned}$$

LEMMA 25. Let $p \equiv 1 \pmod{4}$ and

$$S = \sum_{a=1}^A \sum_{n=1, 3 \nmid n}^a \left(\frac{n}{p} \right).$$

Then under the conditions of Lemma 24, we have, for $p^{1/2} < A < 10p^{1/2}$,

$$S < 41Ap^{1/2}/81 + 0.21256p.$$

Proof. We have

$$\begin{aligned} S &= \sum_{a=1}^A \sum_{n=1}^a \left(\frac{n}{p} \right) + \sum_{a=1}^A \sum_{n=1}^{[a/3]} \left(\frac{n}{p} \right) \\ &\leq \sum_{a=1}^A \sum_{n=1}^a \left(\frac{n}{p} \right) + 2 \sum_{a=1}^{[A/3-1]} \sum_{n=1}^a \left(\frac{n}{p} \right) + 3 \left| \sum_{n=1}^{[A/3]} \left(\frac{n}{p} \right) \right|. \end{aligned}$$

As in the proof of Lemma 5 of I, we have

$$p^{1/2} \sum_{a=1}^A \sum_{n=1}^a \left(\frac{n}{p} \right) = \sum_{x=1}^{(p-1)/2} \left(\frac{x}{p} \right) \left(\frac{\sin \pi(A+1)x/p}{\sin \pi x/p} \right)^2 \\ = S_1(A)/2 - N(A).$$

Therefore, by Lemmas 21 and 24,

$$S \leq \frac{1}{2p^{1/2}} S_1(A) - \frac{1}{p^{1/2}} N(A) \\ + \frac{3}{2p^{1/2}} S_1 \left(\left[\frac{A}{3} - 1 \right] \right) - \frac{3}{p^{1/2}} N \left(\left[\frac{A}{3} - 1 \right] \right) + A \\ < \frac{1}{2p^{1/2}} \left(p(A+1) - (A+1)^2 - \frac{40}{81} p(A+1) + 0.10628pp^{1/2} \right) \\ + \frac{3}{2p^{1/2}} \left(p \left[\frac{A}{3} \right] - \left[\frac{A}{3} \right]^2 - \frac{40}{81} p \left[\frac{A}{3} \right] + 0.10628pp^{1/2} \right) + A \\ < \frac{1}{2p^{1/2}} \left(\frac{41}{81} p(A+1) - (A+1)^2 + 0.10628pp^{1/2} \right) \\ + \frac{3}{2p^{1/2}} \left(\frac{41}{81} p \frac{A}{3} + 0.10628pp^{1/2} \right) \\ < \frac{41}{81} Ap^{1/2} + 0.21256p.$$

LEMMA 26. Let $p \equiv 1 \pmod{4}$, then under the conditions of Lemma 24, we have

$$q_3 < 3p^{1/2}.$$

Proof. We have, for $A < q_3$,

$$\sum_{a=1}^A \sum_{n=1, 3 \nmid n}^a \left(\frac{n}{p} \right) > \sum_{a=1}^A \sum_{n=1, 3 \nmid n}^a 1 - \sum_{a=1}^A \sum_{n=1, q_2 \mid n}^a 1 + \sum_{a=1}^A \sum_{n=1, 3q_2 \mid n}^a 1 \\ > A(A+1)/3 - (1/3q_2)A(A+1) - 2A.$$

By Lemma 7 of I, we have $q_3 < 7.5p^{1/2} < 10p^{1/2}$, therefore for $A = q_3 - 1$ we have, by Lemma 25 (if $A < p^{1/2}$, the result is trivial),

$$(A(A+1)/3)(1 - 1/q_2) - 2A < 41Ap^{1/2}/81 + 0.21256p, \\ (A(A+1)/3)(1 - 0.0375) - 2A < 0.5062Ap^{1/2} + 0.21256p, \\ 0.3208A^2 - 1.6792A < 0.5062Ap^{1/2} + 0.21256p, \\ 0.3208A^2 - 0.5079Ap^{1/2} - 0.21256p < 0.$$

Hence

$$A < (1/0.6416)(0.5079 + (0.5079^2 + 4 \times 0.3208 \times 0.21256)^{1/2})p^{1/2},$$

which implies

$$q_3 < 3p^{1/2}.$$

5. Lemmas concerning sums.

LEMMA 27. *We have, for $q > 0$,*

$$\frac{(A+2)(A-q)}{2q} < \sum_{a=1}^A \left[\frac{a}{q} \right] < \frac{1}{2q} \left(A - \frac{(q-2)}{2} \right)^2.$$

Proof. We have

$$\begin{aligned} \sum_{a=1}^A \left[\frac{a}{q} \right] &= \sum_{q \leq a < 2q} + \sum_{2q \leq a < 3q} + \cdots + \sum_{(A/q-1)q \leq a < [A/q]q} \left[\frac{A}{q} - 1 \right] \\ &\quad + \sum_{[A/q]q \leq a \leq A} \left[\frac{A}{q} \right] \\ &= q \left(1 + 2 + \cdots + \left[\frac{A}{q} - 1 \right] \right) + \left(A - \left[\frac{A}{q} \right]q + 1 \right) \left[\frac{A}{q} \right] \\ &= \frac{1}{2} q \left[\frac{A}{q} \right] \left(\left[\frac{A}{q} \right] - 1 \right) + \left(A - \left[\frac{A}{q} \right]q + 1 \right) \left[\frac{A}{q} \right] \\ &= \frac{1}{2} \left[\frac{A}{q} \right] \left(2A + 2 - \left[\frac{A}{q} \right]q - q \right) \\ &= \frac{1}{2q} (A - \vartheta q)(A - q + \vartheta q + 2) \\ &= \frac{1}{2q} \left(\left(A - \frac{q}{2} + 1 \right)^2 - \left(\vartheta q + 1 - \frac{q}{2} \right)^2 \right), \quad \vartheta = \frac{A}{q} - \left[\frac{A}{q} \right], \end{aligned}$$

which lies between $(A+2)(A-q)/2q$ and $(A-q/2+1)^2/2q$.

LEMMA 28 (EULER'S SUMMATION FORMULA). *Let $b_1(x) = x - [x] - 1/2$ and*

$$b_2(x) = \frac{1}{12} + \int_0^x b_1(x) dx.$$

Let $b > a$ and let $g(x)$ with its first two derivatives be continuous for $a \leq x < b$. Then

$$\begin{aligned} \sum_{a \leq m < b} g(m) &= \int_a^b g(x) dx + g(b)b_1(-b) - g(a)b_1(-a) \\ &\quad + g'(b)b_2(-b) - g'(a)b_2(-a) - \int_a^b g''(x)b_2(-x) dx. \end{aligned}$$

LEMMA 29. For $a - \lambda q > 3q \geq 9$,

$$\sum_{\nu=1}^{[(a-\lambda q)/3q]} li \frac{a}{3\nu + \lambda} < \frac{a}{3} \log \frac{\log(a/(3 + \lambda))}{\log q} + \frac{(3\lambda + 10.7322)a}{12(3 + \lambda)^3 \log(a/(3 + \lambda))} \\ - \frac{3q^2 b_2(-(a - \lambda q)/3q)}{a \log q} + \frac{a}{3q} li q \\ - \left(\frac{1}{2} + \frac{\lambda}{3} \right) li \frac{a}{3 + \lambda} + li q b_1 \left(-\frac{a - \lambda q}{3q} \right) + R,$$

where $R = li q$ or 0 according as $3q$ does or does not divide $a - \lambda q$.

Proof. Let $g(x) = li a/(3x + \lambda)$. Then

$$g'(x) = - \frac{3a}{(3x + \lambda)^2 \log(a/(3x + \lambda))}, \\ g''(x) = \frac{9a(2 \log(a/(3x + \lambda)) - 1)}{(3x + \lambda)^3 \log^2(a/(3x + \lambda))},$$

and for $1 \leq 3x + \lambda \leq (1/e)a$, $g''(x)$ is monotonically decreasing, since the function $(2u - 1)/u^2$ is monotonically decreasing for $u > 1$.

By Lemma 28,

$$\sum_{\nu=1}^{[(a-\lambda q)/3q]} li \frac{a}{3\nu + \lambda} = \int_1^{(a-\lambda q)/3q} li \frac{a}{3t + \lambda} dt + \frac{1}{2} li \frac{a}{3 + \lambda} \\ + li q b_1 \left(-\frac{a - \lambda q}{3q} \right) + \frac{a}{4(3 + \lambda)^2 \log(a/(3 + \lambda))} \\ - \frac{3ab_2(-(a - \lambda q)/3q)}{(a/q)^2 \log q} \\ - \int_1^{(a-\lambda q)/3q} \frac{9a(2 \log(a/(3x + \lambda)) - 1)b_2(-x)}{(3x + \lambda)^3 \log^2(a/(3x + \lambda))} dx + R \\ = \frac{a}{3} \log \frac{\log(a/(3 + \lambda))}{\log q} + \frac{a}{3q} li q - \frac{1}{3} (3 + \lambda) li \frac{a}{3 + \lambda} \\ + \frac{1}{2} li \frac{a}{3 + \lambda} + li q b_1 \left(-\frac{a - \lambda q}{3q} \right) \\ + \frac{a}{4(3 + \lambda)^2 \log(3/(3 + \lambda))} - \frac{3q^2 b_2(-(a - \lambda q)/3q)}{a \log q} \\ + \frac{3^{1/2} a \theta}{12(3 + \lambda)^3 \log(a/(3 + \lambda))} + R,$$

where $\theta \leq 1$, since

$$\begin{aligned}
& \int_1^{(a-\lambda q)/3q} li \frac{a}{3t+\lambda} dt \\
&= \frac{1}{3} \left((3t+\lambda) li \frac{a}{3t+\lambda} \right)_1^{(a-\lambda q)/3q} \\
&\quad - \frac{a}{3} \int_1^{(a-\lambda q)/3q} \frac{d(a/(3t+\lambda))/dt}{a/(3t+\lambda) \log(a/(3t+\lambda))} dt \\
&= \frac{1}{3} \left((3t+\lambda) li \frac{a}{3t+\lambda} \right)_1^{(a-\lambda q)/3q} - \frac{a}{3} \left(\log \log \frac{a}{3t+\lambda} \right)_1^{(a-\lambda q)/3q} \\
&= \frac{a}{3q} li q - \frac{1}{3} (3+\lambda) li \frac{a}{3+\lambda} + \frac{a}{3} \log \frac{\log(a/(3+\lambda))}{\log q}
\end{aligned}$$

and

$$\left| \int_0^x b_2(-x) dx \right| \leq \frac{3^{1/2}}{216} \quad (4).$$

The lemma follows immediately.

LEMMA 30. *Under the conditions of Lemma 29, and the condition $\lambda > 0$, we have*

$$\begin{aligned}
& \sum_{a=\lambda q}^A \left[\sum_{\nu=0}^{[(a-\lambda q)/3q]} li \frac{a}{3\nu+\lambda} < \frac{1}{6} A(A+2) \log \log \frac{A}{3+\lambda} \right. \\
& \quad - \frac{1}{6} (A^2 + A + (3+\lambda)q) \log \log q + (A+1) li \frac{A}{\lambda} - \frac{(2\lambda+3)A}{6} li \frac{A}{3+\lambda} \\
& \quad + \frac{li q}{24q} (4A^2 + 8A + (2\lambda-3)^2 q^2 + 4) + \frac{3\lambda + 10.7322}{12(\lambda+3)^3} \frac{A}{\log A/(3+\lambda)} \\
& \quad + \frac{q^2}{8 \log q} \left(\log \frac{A}{(3+\lambda)q} + \frac{1}{(3+\lambda)q} \right) - \lambda \int_q^{A/\lambda} \frac{xdx}{\log x} \\
& \quad \left. + \frac{2\lambda(3+\lambda)^2 + 3\lambda + 10.7322}{12(3+\lambda)} \int_q^{A/(3+\lambda)} \frac{xdx}{\log x} \right].
\end{aligned}$$

Proof. By Lemma 29, we have

$$\sum_{a=\lambda q}^A \left[\sum_{\nu=0}^{[(a-\lambda q)/3q]} li \frac{a}{3\nu+\lambda} \right] = \sum_{a=\lambda q}^A li \frac{a}{\lambda} + \sum_{a=(3+\lambda)q}^A \left[\sum_{\nu=1}^{[(a-\lambda q)/3q]} li \frac{a}{3\nu+\lambda} \right]$$

(4) For $0 < x \leq 1$, $\int_0^x b_2(-x) dx = \int_0^x ((x^2-x)/2 + 1/12) dx = (x/12)(2x^2-3x+1)$, which takes its extrema at $x = (3 \pm 3^{1/2})/6$. For $x=1$, the integral equals zero. Since $b_2(x-1) = b_2(x)$, we have the required formula.

$$\begin{aligned}
&< \int_{\lambda q}^A li \frac{x}{\lambda} dx + li \frac{A}{\lambda} + \sum_{a=(3+\lambda)q}^A \frac{a}{3} \log \frac{\log(a/(3+\lambda))}{\log q} \\
&+ \frac{3\lambda + 10.7322}{12(3+\lambda)^3} \left(\int_{(3+\lambda)q}^A \frac{xdx}{\log(x/(3+\lambda))} + \frac{A}{\log(A/(3+\lambda))} \right) \\
&- \sum_{a=(3+\lambda)q}^A \frac{3q^2}{a \log q} b_2 \left(-\frac{a-\lambda q}{3q} \right) \\
&+ \frac{1}{6q} (A + \lambda q + 3q)(A - \lambda q - 3q + 1) li q \\
&- \left(\frac{1}{2} + \frac{\lambda}{3} \right) \int_{(3+\lambda)q}^A li \frac{x}{3+\lambda} dx \\
&+ li q \sum_{a=(3+\lambda)q}^A \left(b_1 \left(-\frac{a-\lambda q}{3q} \right) + R' \right)
\end{aligned}$$

where $R'=1$ or 0 according as $3q$ does or does not divide $a-\lambda q$.

It is plain that

$$\begin{aligned}
\int_{\lambda q}^A li \frac{x}{\lambda} dx &= A li \frac{A}{\lambda} - \lambda q li q - \int_{\lambda/q}^A \frac{xdx}{\lambda \log x/\lambda}, \\
\sum_{a=(3+\lambda)q}^A \frac{a}{3} \log \log \frac{a}{3+\lambda} &< \frac{1}{3} \int_{(3+\lambda)q}^A x \log \log \frac{x}{3+\lambda} dx + \frac{A}{3} \log \log \frac{A}{3+\lambda} \\
&= \frac{1}{6} \left(A^2 \log \log \frac{A}{3+\lambda} - (3+\lambda)^2 q^2 \log \log q \right) \\
&+ \frac{A}{3} \log \log \frac{A}{3+\lambda} - \frac{1}{6} \int_{(3+\lambda)q}^A \frac{xdx}{\log x/(3+\lambda)} \\
&- \sum_{a=(3+\lambda)q}^A \frac{b_2(-(a-\lambda q)/3q)}{a} < \frac{1}{24} \sum_{a=(3+\lambda)q}^A \frac{1}{a} \\
&< \frac{1}{24} \left(\log \frac{A}{(3+\lambda)q} + \frac{1}{(3+\lambda)q} \right),
\end{aligned}$$

since $b_2(x) \geq -1/24$,

$$\int_{(3+\lambda)q}^A li \frac{x}{3+\lambda} dx = A li \frac{A}{3+\lambda} - (3+\lambda)q li q - \int_{(3+\lambda)q}^A \frac{xdx}{(3+\lambda) \log x/(3+\lambda)}$$

and

$$\sum_{a=(3+\lambda)q}^A \left(b_1 \left(-\frac{a-\lambda q}{3q} \right) + R' \right) = \sum_{n=0}^{A-(3+\lambda)q} \left(b_1 \left(-\frac{n}{3q} \right) + R'' \right) = S,$$

say, where $R'' = 1$ or 0 according as $3q \mid n$ or $3q \nmid n$. Clearly,

$$\begin{aligned} S &= \sum_{n=0}^{A-(3+\lambda)q} \left(-\frac{n}{3q} - \left[-\frac{n}{3q} \right] - \frac{1}{2} + R'' \right) \\ &= \sum_{n=0}^{A-(3+\lambda)q} \left(-\frac{n}{3q} + \left[\frac{n}{3q} \right] + \frac{1}{2} \right), \end{aligned}$$

since $[-n/3q] = -[n/3q] + R'' - 1$. By Lemma 27,

$$\begin{aligned} S &< -\frac{1}{6q} (A - (3 + \lambda)q)(A - (3 + \lambda)q + 1) \\ &\quad + \frac{1}{6q} \left(A - (3 + \lambda)q - \frac{3q - 2}{2} \right)^2 + \frac{1}{2} (A - (3 + \lambda)q + 1) \\ &= \frac{1}{6q} (A - \lambda q) + \frac{(3q - 2)^2}{24q}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{a=\lambda q}^A \left[\sum_{r=0}^{[(a-\lambda q)/3q]} li \frac{a}{3r + \lambda} \right] &< A li \frac{A}{\lambda} - \lambda q li q - \int_{\lambda q}^A \frac{x dx}{\lambda \log x / \lambda} + li \frac{A}{\lambda} \\ &\quad - \frac{1}{6} (A + \lambda q + 3q)(A - \lambda q - 3q + 1) \log \log q \\ &\quad + \frac{1}{6} \left(A^2 \log \log \frac{A}{3 + \lambda} - (3 + \lambda)^2 q^2 \log \log q \right) + \frac{A}{3} \log \log \frac{A}{3 + \lambda} \\ &\quad - \frac{1}{6} \int_{(3+\lambda)q}^A \frac{x dx}{\log x / (3 + \lambda)} \\ &\quad + \frac{3\lambda + 10.7322}{12(3 + \lambda)^3} \left(\int_{(3+\lambda)q}^A \frac{x dx}{\log x / (3 + \lambda)} + \frac{A}{\log A / (3 + \lambda)} \right) \\ &\quad + \frac{q^2}{8 \log q} \left(\log \frac{A}{(3 + \lambda)q} + \frac{1}{(3 + \lambda)q} \right) \\ &\quad + \frac{1}{6q} (A + \lambda q + 3q)(A - \lambda q - 3q + 1) li q \\ &\quad - \left(\frac{1}{2} + \frac{\lambda}{3} \right) \left(A li \frac{A}{3 + \lambda} \right) - (3 + \lambda)q li q \\ &\quad - \int_{(3+\lambda)q}^A \frac{x dx}{(3 + \lambda) \log x / (3 + \lambda)} + li q \left(\frac{A - \lambda q}{6q} + \frac{(3q - 2)^2}{24q} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{6} A(A+2) \log \log \frac{A}{3+\lambda} - \frac{1}{6} (A^2 + A + (3+\lambda)q) \log \log q + (A+1) li \frac{A}{\lambda} \\
 &\quad - \frac{2\lambda+3}{6} A li \frac{A}{3+\lambda} + \frac{li q}{24q} (4A^2 + 8A + (2\lambda-3)^2 q^2 + 4) \\
 &\quad + \frac{3\lambda+10.7322}{12(3+\lambda)^3} \frac{A}{\log(A/(3+\lambda))} + \frac{q^2}{8 \log q} \left(\log \frac{A}{(3+\lambda)q} + \frac{1}{(3+\lambda)q} \right) \\
 &\quad - \int_{\lambda q}^A \frac{x dx}{\lambda \log x/\lambda} + \frac{2\lambda(3+\lambda)^2 + 3\lambda + 10.7322}{12(3+\lambda)} \int_q^{A/(3+\lambda)} \frac{x dx}{\log x}.
 \end{aligned}$$

LEMMA 31. *Under the conditions of Lemma 29.*

$$\begin{aligned}
 &\sum_{a=q}^A \left(\sum_{\nu=0}^{[(a-q)/3q]} li \frac{a}{3\nu+1} + \sum_{\nu=0}^{[(a-2q)/3q]} li \frac{a}{3\nu+2} \right) \\
 &< \frac{1}{6} A(A+2) \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \frac{1}{6} (2A^2 + 2A + 9q) \log \log q \\
 &\quad + \frac{li q}{12q} (4A^2 + 8A + q^2 + 4) + \frac{0.03A}{\log A/5} \\
 &\quad + \frac{q^2}{8 \log q} \left(\log \frac{A^2}{20q^2} + \frac{q}{20q} \right) + A^2 \left(\int_{1/2}^1 \frac{(1-x)dx}{\log Ax} + \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} \right. \\
 &\quad \left. + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} + 0.1016 \int_{q/A}^{1/5} \frac{x dx}{\log Ax} \right) + li A + li \frac{A}{2}.
 \end{aligned}$$

Proof. By Lemma 30, the sum in question is less than

$$\begin{aligned}
 &\frac{1}{6} A(A+2) \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \frac{1}{6} (2A^2 + 2A + 9q) \log \log q \\
 &\quad + (A+1) li A + (A+1) li \frac{A}{2} - \frac{5}{6} A li \frac{A}{4} - \frac{7}{6} A li \frac{A}{5} \\
 &\quad + \frac{li q}{12q} (4A^2 + 8A + q^2 + 4) + \frac{0.03A}{\log(A/5)} \\
 &\quad + \frac{q^2}{8 \log q} \left(\log \frac{A^2}{20q^2} + \frac{9}{20q} \right) - \int_q^A \frac{x dx}{\log x} - \int_{2q}^A \frac{x dx}{2 \log x/2} \\
 &\quad + \frac{45.7322}{48} \int_q^{A/4} \frac{x dx}{\log x} + \frac{116.7322}{60} \int_q^{A/5} \frac{x dx}{\log x}.
 \end{aligned}$$

Since

$$A \operatorname{li} A + A \operatorname{li} \frac{A}{2} - \frac{5}{6} A \operatorname{li} \frac{A}{4} - \frac{7}{6} A \operatorname{li} \frac{A}{5} \\ = A \int_{A/2}^A \frac{dx}{\log x} + 2A \int_{A/4}^{A/2} \frac{dx}{\log x} + \frac{7}{6} A \int_{A/5}^{A/4} \frac{dx}{\log x}$$

and

$$- \int_q^A \frac{xdx}{\log x} - \int_{2q}^A \frac{xdx}{2 \log x/2} + \frac{45.7322}{48} \int_q^{A/4} \frac{xdx}{\log x} + \frac{116.7322}{60} \int_q^{A/5} \frac{xdx}{\log x} \\ = - \int_{A/2}^A \frac{xdx}{\log x} - 3 \int_{A/4}^{A/2} \frac{xdx}{\log x} - 2.0472 \int_{A/5}^{A/4} \frac{xdx}{\log x} \\ - 0.1016 \int_q^{A/5} \frac{xdx}{\log x},$$

we have the lemma by replacing x in the integrals by Ax .

6. The growth of smaller quadratic non-residues.

LEMMA 32. Let $p \equiv 1 \pmod{4}$, $p > 10^6$, $q_1 = 3$. Then $5q_2q_3 < p$.

Proof. Suppose on the contrary that $5q_2q_3 \geq p$. Then

$$(1) \quad q_3 > (p/5)^{1/2}.$$

By Lemma 7 of I, we have $q_3 \leq 4p^{1/2}/(1-1/3)(1-1/5) = 7.5p^{1/2}$.

If $q_2 < p^{1/2}/37.5$, the lemma is immediate. If $q_2 \geq p^{1/2}/37.5$, by Lemma 26,

$$(2) \quad q_3 < 3p^{1/2}.$$

The lemma follows for $q_2 \leq p^{1/2}/15$. Then we may assume that

$$(3) \quad q_2 > p^{1/2}/15.$$

By Lemma 10 of I, we have

$$\frac{1}{2} \sum_{a=1}^A \sum_{n=1,3,5}^a \left(1 - \left(\frac{n}{p} \right) \right) \leq \sum_{a=q_3}^A \left(\sum_{q_3 \leq q \leq a} \left[\frac{a}{q} \right] - \sum_{q_3 \leq q \leq a/3} \left[\frac{a}{3q} \right] \right) \\ + \sum_{a=q_2}^A \left(\left[\frac{a}{q_2} \right] - \left[\frac{a}{3q_2} \right] \right) \\ = \sum_{a=q_3}^A \left(\sum_{v=1}^{[a/q_3]} \pi \left(\frac{a}{v} \right) - \left[\frac{a}{q_3} \right] (\pi(q_3) - 1) \right) \\ - \sum_{v=1}^{[a/3q_3]} \pi \left(\frac{a}{3v} \right) + \left[\frac{a}{3q_3} \right] (\pi(q_3) - 1) \\ + \sum_{a=q_2}^A \left(\left[\frac{a}{q_2} \right] - \left[\frac{a}{3q_2} \right] \right).$$

Denoting the first side of the inequality by S_0 , we have

$$(4) \quad S_0 < \sum_{a=q_2}^A \sum_{\substack{[a/q_3] \\ p=1, 2, 3}} \pi\left(\frac{a}{p}\right) - \sum_{a=q_2}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right) (\pi(q_3) - 1) \\ + \sum_{a=q_2}^A \left(\left[\frac{a}{q_2} \right] - \left[\frac{a}{3q_2} \right] \right) = S' - S'' + S''', \text{ say.}$$

(1) Suppose that $p > 10^8$. Let us take $A = [10p^{1/2}]$. By Lemma 25,

$$(5) \quad S_0 > (A^2/3 - 41Ap^{1/2}/81 - 0.21256p)/2 \\ > (33.32 - 5.1 - 0.22)p/2 = 14p$$

since

$$A^2/3 > (100p - 20p^{1/2})/3 = 33.32p + 4p/300 - 20p^{1/2}/3 > 33.32p.$$

By (3),

$$(6) \quad S''' < \sum_{a=q_2}^A \left(\frac{a}{q_2} - \frac{a}{3q_2} + 1 \right) < \frac{A(A+1)}{3q_2} + A \\ < 50(10p^{1/2} + 1) + 10p^{1/2} = 510p^{1/2} + 50 \\ < 0.06p.$$

By Lemma 2 of I, we have, since $q_3 > (p/5)^{1/2} \geq 10^4/5^{1/2} > 51^2$,

$$S'' > (0.9607 \operatorname{li} q_3 - 2.7) \sum_{a=q_3}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right).$$

By Lemma 27 and (2),

$$\sum_{a=q_3}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right) \\ > \frac{1}{2q_3} \left((A+2)(A-q_3) - \frac{1}{3} \left(A+1 - \frac{3q_3}{2} \right)^2 \right) \\ = \frac{1}{2q_3} \left(\frac{2}{3} A^2 + \frac{4}{3} A - \frac{3}{4} q_3^2 - q_3 - \frac{1}{3} \right) \\ > \frac{p}{q_3} \left(\frac{100}{3} - \frac{1}{3 \times 10^8} - \frac{3^3}{8} - \frac{3}{2 \times 10^4} - \frac{1}{6 \times 10^8} \right) \\ > \frac{29.957}{q_3} p.$$

Then

$$(7) \quad S'' > (29.957/q_3)(0.9607 \operatorname{li} q_3 - 2.7) > (28.779 \operatorname{li} q_3/q_3 - 0.03)p.$$

Finally, by Lemma 2 of I and Lemma 31,

$$\begin{aligned}
S' &< 1.0376 \sum_{a=q_3}^A \sum_{\substack{v=1,3 \\ v \nmid a}}^{a/q_3} (li \, a/v + 1.85) \\
&< 1.0376 \left\{ \frac{1}{6} A(A+2) \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) \right. \\
&\quad - \frac{1}{6} (2A^2 + 2A + 9q_3) \log \log q_3 + \frac{li \, q_3}{12q_3} (4A^2 + 8A + q_3^2 + 4) \\
&\quad + \frac{0.03A}{\log A/5} + \frac{q_3^2}{8 \log q_3} \left(\log \frac{A^2}{20q_3^2} + \frac{9}{20q_3} \right) + A^2 \left(\int_{1/2}^1 \frac{(1-x)dx}{\log Ax} \right. \\
&\quad + \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472)dx}{\log Ax} \\
&\quad + 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} \left. \right) + li \, A + li \, \frac{A}{2} \Big\} \\
&\quad + 1.91956 \sum_{a=q_3}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{6} A(A+2) \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \frac{1}{6} (2A^2 + 2A + 9q_3) \log \log q_3 \\
&< \frac{1.0001}{6} A^2 \left(\log \frac{\log (5/2) p^{1/2} \log 2 p^{1/2}}{(\log p^{1/2} - (\log 5)/2)^2} + \frac{2 \log \log (p/5)^{1/2}}{[10 p^{1/2}]} \right) \\
&< \frac{1.0001}{6} A^2 \left(\log \frac{(4 \log 10 + \log 2.5)(4 \log 10 + \log 2)}{(4 \log 10 - (\log 5)/2)^2} + 0.0002 \right) \\
&< \frac{1.0001}{6} A^2 \left(\log \frac{10.267 \times 9.9035}{8.40562^2} + 0.0002 \right) \\
&= \frac{1.0001}{6} A^2 (\log 1.42 + 0.0002) \\
&< \frac{0.351}{6} A^2 < 5.85p.
\end{aligned}$$

$$\begin{aligned}
\frac{li \, q_3}{12q_3} (4A^2 + 8A + q_3^2 + 4) &< \frac{\log q_3}{q_3} p \left(\frac{100}{3} + \frac{2 \times 10}{3 \times 10^4} + \frac{3^2}{12} + \frac{1}{3 \times 10^8} \right) \\
&< 34.0841 li \, q_3 p / q_3,
\end{aligned}$$

$$\frac{0.03A}{\log A/5} < \frac{0.3p^{1/2}}{\log (2p^{1/2} - 1)} = \frac{0.3p}{10^4 \log (2 \times 10^4 - 1)} < 0.00001p,$$

$$\begin{aligned} \frac{q_3^2}{8 \log q_3} \left(\log \frac{A^2}{20q_3^2} + \frac{9}{20q_3} \right) &< \frac{9p}{8 \log (3p^{1/2})} \left(\log \frac{10^2 \times 5}{20} + \frac{9(5)^{1/2}}{20 \times 10^4} \right) \\ &< \frac{9}{8 \times 10.30895} (3.21889 + 0.0002)p \\ &< 0.36p \end{aligned}$$

(since $(p/5)^{1/2} < q_3 < 3p^{1/2}$).

$$\begin{aligned} \int_{1/2}^1 \frac{(1-x)dx}{\log Ax} &< \frac{1}{\log A/2} \int_{1/2}^1 (1-x)dx = \frac{1}{8 \times 10.81977} < 0.0116, \\ \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} &< \frac{7}{32 \log A/4} < \frac{7}{32 \times 10.126635} < 0.0217, \\ \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} &< \frac{1}{\log A/5} \left(\frac{1.1667}{20} - \frac{9 \times 2.0472}{800} \right) < 0.00357, \\ 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} &< \frac{0.00204}{\log q_3} < 0.0001, \\ li A + li A/2 < 2 li A < 2 \times \frac{10p^{1/2}}{\log 10p^{1/2} - 2} \quad (5) &< \frac{2p}{10^3(\log 10^5 - 2)} \\ &< 0.0003p \end{aligned}$$

and

$$\sum_{a=q_3}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right) \leq \frac{A(A+1)}{3q_3} + A < 0.0085p.$$

We have

$$\begin{aligned} S' &< 1.0376p \{ 5.85 + 34.0841 li q_3/q_3 + 0.00001p + 0.36 \\ &\quad + 10^2(0.0116 + 0.0217 + 0.0036 + 0.0001) + 0.0003 \} \\ (8) \quad &+ 1.91956 \times 0.0085p \\ &< p(1.0376 \times 10 + 0.02 + 1.0376 \times 34.084) li q_3/q_3 \\ &< p(10.40 + 35.366 li q_3/q_3). \end{aligned}$$

By (4), (5), (6), (7), and (8), we have $14p < p(10.40 + 35.366 li q_3/q_3) - (28.779 li q_3/q_3 - 0.03)p + 0.06p$, that is,

$$\begin{aligned} 14 &< 10.40 + 0.03 + 0.06 + 6.59 li q_3/q_3 \\ &< 10.49 + 6.59 \times 1/(\log (10^4/5^{1/2}) - 2) < 12, \end{aligned}$$

which is absurd.

(6) For $e^4 \leq x$, $li x < x/(\log x - 2)$. In fact, the inequality is true for $x = e^4$, and $d li x/dx \leq d(x/(\log x - 2))/dx$.

(2) Suppose $10^6 < p < 10^8$. We take $A = [6p^{1/2}]$. By Lemma 25,

$$(5') \quad \begin{aligned} S_0 &> (A^2/3 - 41A p^{1/2}/81 - 0.21256p)/2 \\ &> (12 - 0.004 - 3.0371 - 0.21256)p/2 > 4.3731p. \end{aligned}$$

By Lemma 27

$$\begin{aligned} S''' &< \frac{1}{2q_2} \left(\left(A - \frac{q_2}{2} + 1 \right)^2 - \frac{(A+2)(A-3q_2)}{3} \right) \\ &= \frac{1}{2q_2} \left(\frac{2}{3} A^2 + \frac{4}{3} A + \frac{(q_2+2)^2}{4} \right) \\ &= \frac{1}{q_2} \left(\frac{1}{3} A^2 + \frac{2}{3} A + \frac{(q_2+2)^2}{8} \right) < \frac{p}{q_2} \left(12 + \frac{4}{10^3} + \frac{(q_2+2)^2}{8p} \right) \\ &< 5q_3 \left(12.004 + \frac{(p+10q_3)^2}{200pq_3^2} \right), \end{aligned}$$

for $q_2 > p/5q_3$, and the function is decreasing for $q_2 < q_3 < 3p^{1/2}$ (by (2)). Since

$$\begin{aligned} \frac{(p+10q_3)^2}{200pq_3^2} &= \frac{1}{200p} \left(\frac{p}{q_3} + 10 \right)^2 < \frac{1}{200} \left(5^{1/2} + \frac{10}{p^{1/2}} \right)^2 < \frac{1}{200} \left(5^{1/2} + \frac{1}{100} \right)^2 \\ &< 0.0253, \end{aligned}$$

$$(6') \quad S''' < 60.1465q_3.$$

Since $11 < (p/5)^{1/2} < q_3 < 3p^{1/2} < 10^6$, we have, by Lemma 18, $S'' > (li \, q_3 - li \, q_3^{1/2} - 1) \sum_{a=q_3}^A ([a/q_3] - [a/3q_3])$.

By Lemma 27 and (2), we have as in (1)

$$\begin{aligned} \sum_{a=q_3}^A \left(\left[\frac{a}{q_3} \right] - \left[\frac{a}{3q_3} \right] \right) &> \frac{1}{2q_3} \left(\frac{2}{3} A^2 + \frac{4}{3} A - \frac{3}{4} q_3^2 - q_3 - \frac{1}{3} \right) \\ &> \frac{p}{2q_3} \left(\frac{2}{3} \cdot 6^2 - \frac{8}{p^{1/2}} + \frac{4}{3} \frac{6}{p^{1/2}} - \frac{3}{4} \frac{q_3^2}{p} - \frac{q_3}{p^{1/2}} - \frac{1}{p} \right) \\ &> \frac{p}{2q_3} \left(24 - \frac{3}{4} \frac{q_3^2}{p} - \frac{3}{p^{1/2}} - \frac{1}{p} \right) \\ &> \frac{p}{2q_3} \left(23.9969 - \frac{3}{4} \frac{q_3^2}{p} \right) > \frac{p}{q_3} \left(11.9984 - \frac{3}{8} \frac{q_3^2}{p} \right), \\ \frac{1}{q_3} &< \left(\frac{5}{p} \right)^{1/2} < \frac{p^{1/2}}{10^3} = 0.002237, \\ \frac{li \, q_3^{1/2}}{q_3} &< \frac{li \, (10^3/5^{1/2})^{1/2}}{10^3/5^{1/2}} < \frac{li \, 21.176}{447.21} < \frac{10.75}{447.21} < 0.02405. \end{aligned}$$

Therefore

$$(7'') \quad S'' > p(li\,q_3/q_3 - 0.0263)(11.9984 - 3q_3^2/8p).$$

Finally, by Lemmas 17 and 31, we have

$$\begin{aligned} S' &= \sum_{a=q_3}^A \sum_{r=1,3,4}^{a/q_3} \pi\left(\frac{a}{r}\right) < \sum_{a=q_3}^A \sum_{r=1,3,4}^{a/q_3} li\,\frac{a}{r} \\ &< \frac{1}{6} A(A+2) \log\left(\log\frac{A}{4} \log\frac{A}{5}\right) - \frac{1}{6} (2A^2 + 2A + 9q_3) \log\log q_3 \\ &\quad + \frac{li\,q_3}{12q_3} (4A^2 + 8A + q_3^2 + 4) + \frac{0.03A}{\log A/5} + \frac{q_3^2}{8 \log q_3} \left(\log\frac{A^2}{20q_3^2} + \frac{9}{20q_3}\right) \\ &\quad + A^2 \left(\int_{1/2}^1 \frac{(1-x)dx}{\log Ax} + \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} \right. \\ &\quad \left. + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} + 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} \right) + li\,A + li\,\frac{A}{2} \\ &< \frac{A^2}{3} \left\{ \frac{1}{2} \left(1 + \frac{2}{A}\right) \log\left(\log\frac{A}{4} \log\frac{A}{5}\right) - \log\log q_3 + \frac{6\,li\,A}{A^2} \right. \\ &\quad + \frac{li\,q_3}{q_3} \left(1 + \frac{2}{A} + \frac{q_3^2}{4A^2} + \frac{1}{A^2}\right) + \frac{0.09}{A \log A/5} \\ &\quad + \frac{3q_3^2}{8A^2 \log q_3} \left(\log\frac{A^2}{20q_3^2} + \frac{9}{20q_3}\right) + 3 \left(\int_{1/2}^1 \frac{(1-x)dx}{\log Ax} \right. \\ &\quad \left. + \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} \right. \\ &\quad \left. + 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} \right) \Big\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{A} \log\left(\log\frac{A}{4} \log\frac{A}{5}\right) &< \frac{1}{5999} \log(\log 1500 \log 1200) \\ &< \frac{1}{5999} \log(7.31323 \times 7.09008) < 0.0007, \\ \frac{6\,li\,A}{A^2} &< \frac{6}{5999(\log 5999 - 2)} < 0.00016, \\ \frac{0.09}{A \log A/5} &< \frac{0.09}{5999} < 0.00002, \end{aligned}$$

$$\begin{aligned} \frac{3p}{8A^2 \log q_3} \left(\log \frac{A^2}{20q_3^2} + \frac{9}{20q_3} \right) &< \frac{3}{286 \log 10^3/5^{1/2}} \left(\log \frac{36}{4} + \frac{9(5)^{1/2}}{20 \times 10^3} \right) < 0.0038, \\ \int_{1/2}^1 \frac{(1-x)dx}{\log Ax} &< \frac{1}{\log A/2} \int_{1/2}^1 (1-x)dx < \frac{1}{8.006} \times \frac{1}{8} < 0.0157, \\ \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} &< \frac{1}{\log A/4} \int_{1/4}^{1/2} (2-3x)dx < 0.03, \\ \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} &< \frac{1}{\log A/5} \int_{1/5}^{1/4} (1.1667 - 2.0472x)dx < 0.005, \\ 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} &< 0.1016 \times \frac{0.002}{\log q_3} < \frac{0.1016 \times 0.002}{6.103} < 0.0001, \end{aligned}$$

and the total sum of the last four integrals is less than 0.0508, we have

$$\begin{aligned} (8') \quad S' &< 12p \left\{ \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \log \log q_3 \right. \\ &\quad + \frac{li q_3}{q_3} \left(1 + \frac{2}{5.999p^{1/2}} + \frac{q_3^2}{143.952p} + 0.00001 \right) \\ &\quad \left. + 0.0038 \frac{q_3^2}{p} + 0.15328 \right\} \end{aligned}$$

(for $0.0007 + 0.00016 + 0.00002 + 3 \times 0.0508 = 0.15328$).

By (4), (5'), (6'), (7') and (8'), we have

$$\begin{aligned} 4.3731 &< 12 \left\{ \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \log \log q_3 \right. \\ &\quad + \frac{li q_3}{q_3} \left(1 + \frac{2}{5.999p^{1/2}} + \frac{q_3^2}{143.952p} + 0.00001 \right) \\ &\quad \left. + 0.0038 \frac{q_3^2}{p} + 0.15328 \right\} \\ &\quad - (li q_3/q_3 - 0.0263)(11.9984 - 3q_3^2/8p) + 60.1465q_3/p, \\ \frac{4.3731}{12} &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \log \log q_3 \\ &\quad + \frac{li q_3}{q_3} (0.00014 + 0.00034 + 0.03825q_3^2/p + 0.00001) \\ &\quad + 0.0263 \left(1 - \frac{q_3^2}{32p} \right) + 0.0038q_3^2/p + \frac{60.1465}{12} \frac{q_3}{p} + 0.15328. \end{aligned}$$

Since

$$\frac{\text{li } q_3}{q_3} < \frac{1}{\log q_3 - 2} < \frac{1}{4.103} < 0.24373,$$

we have

$$\begin{aligned} 0.3644 &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \log \log q_3 \\ &\quad + 0.24373(0.00049 + 0.03825q_3^2/p) \\ &\quad + 0.0263(1 - q_3^2/32p) + 0.0038q_3^2/p \\ &\quad + 5.013q_3/p + 0.15328, \end{aligned}$$

that is,

$$\begin{aligned} 0.3644 &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) - \log \log q_3 + 0.0123q_3^2/p \\ &\quad + 5.013q_3/p + 0.1797. \end{aligned}$$

Then

$$\begin{aligned} (9) \quad \log \log q_3 &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) + 0.0123q_3^2/p \\ &\quad + 5.013q_3/p - 0.1847. \end{aligned}$$

Since $(p/5)^{1/2} < q_3 < 3p^{1/2}$, we have

$$\begin{aligned} \log \log q_3 &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) + 0.0123 \times 9 \\ &\quad + \frac{5.013 \times 3}{10^3} - 0.1847 \\ &< \frac{1}{2} \log \left(\log \frac{A}{4} \right)^2, \end{aligned}$$

that is,

$$(10) \quad q_3 < (3/2)p^{1/2}.$$

For $p < 10^8$, we have by (10),

$$\begin{aligned} \frac{d}{dq_3} \left(\log \log q_3 - \frac{0.0123}{p} q_3^2 - \frac{5.013q_3}{p} \right) \\ = \frac{1}{q_3 \log q_3} - \frac{0.0246q_3}{p} - \frac{5.013}{p} \\ > \frac{2}{3p^{1/2} \log ((3/2) \times 10^4)} - \frac{0.0246}{p^{1/2}} \frac{3}{2} - \frac{5.013}{p} > 0. \end{aligned}$$

Therefore (9) gives

$$\begin{aligned} \log \log \left(\frac{p}{5} \right)^{1/2} &< \frac{1}{2} \log \left(\log \frac{3}{2} p^{1/2} \log \frac{6}{5} p^{1/2} \right) \\ &+ \frac{0.0123}{p} \left(\left(\frac{p}{5} \right)^{1/2} \right)^2 + \frac{5.013}{p} \left(\frac{p}{5} \right)^{1/2} - 0.1847 \\ &< \frac{1}{2} \log \left(\log \frac{3}{2} p^{1/2} \log \frac{6}{5} p^{1/2} \right) \\ &+ 0.00246 + \frac{5.013}{10^3(5)^{1/2}} - 0.1847 \\ &< \frac{1}{2} \log \left(\log \frac{3}{2} p^{1/2} \log \frac{6}{5} p^{1/2} \right) - 0.1799 \end{aligned}$$

or

$$\begin{aligned} 0.35 &< 0.1799 \times 2 < \log \frac{\log (3p^{1/2}/2) \log (6p^{1/2}/5)}{(\log (p/5)^{1/2})^2} \\ &< \log \frac{\log (3/2 \times 10^3) \log (6/5 \times 10^3)}{(\log 10^3/5^{1/2})^2} \\ &< \log 1.393 < 0.332, \end{aligned}$$

which is absurd.

Thus the lemma is established.

7. Proof of the theorem for E. A.

THEOREM 4. *For a prime $p \equiv 17 \pmod{24}$, there is no E.A. in $R(p^{1/2})$ except possibly for $p = 17, 41, 89, 113$ and 137 .*

Proof. (1) Suppose $q_2 = 5$. Since

$$40q_3 < 7.5 \times 40p^{1/2} = 300p^{1/2} < p$$

if $p > 300^2 = 90000$, the theorem is true for $p > 90000$. For $p < 90000$, it is verified directly that the condition

$$40q_3 < p$$

holds except for $p = 17, 113, 233$ and 257 . But

$$233 = 2.5 \cdot q_3 + 41.3 \quad (q_3 = 11),$$

$$257 = 5q_3 + 74.3 \quad (q_3 = 7),$$

by Lemma 15 (of I) the field $R(p^{1/2})$ has no E.A. except possibly for $p = 17, 113$ and 137 .

(2) Suppose $q_2 \neq 5$. For $p > 10^6$, the theorem follows from Lemma 32 and Lemma 16 (of I). For $p \leq 10^6$, it is verified directly that the condition

$$5q_2q_3 < p$$

is satisfied except for $p = 41, 89$ and 271 . But

$$271 = 3q_3 + q_2 \cdot 34 \qquad (q_2 = 7, q_3 = 11),$$

by Lemma 14 (of I) we have the theorem.

NATIONAL TSING HUA UNIVERSITY,
KUNMING, YUNNAN, CHINA.