ORTHONORMAL SETS OF PERIODIC FUNCTIONS OF THE TYPE $\{f(nx)\}$

BY

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1. **Introduction.** This paper considers some new problems in the characterization of functions or classes of functions by orthogonality relations. In contradistinction to the usual problems treated in this connection, those taken up here are *nonlinear*. Other alternative formulations of this investigation are possible. Thus, as one instance, our work may be considered a chapter in the theory of special systems of quadratic equations in an infinite number of variables. It is expected that the methods used and the results obtained will direct attention to a wide variety of allied significant questions.

Consider an odd(1) function $f(x) \in L_2(-\pi, \pi)$ which is periodic of period 2π and satisfies the conditions

$$\text{I} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(nx)f(mx)dx = \delta_{nm}, \qquad n, m = 1, 2, \cdots .$$

A natural conjecture is that $f(x)$ must be $\sin kx$, $k = \pm 1, \pm 2, \cdots$. The special case $\{f(nx)\}$ complete proves entirely misleading and in the absence of further restrictions the conjecture is false. There are in fact an infinite number of solutions, not of the form conjectured, but they are hardly obvious. Our main interest is the study of such functions.

We list our principal conventions. We write $[f(x), g(x)] = \pi^{-1}\int_{-\pi}^{\pi} f(x)g(x)dx$ and the norm, $\|f\| = [f(x), f(x)]^{1/2}$ (this is $\pi^{-1/2}$ times the usual norm). The space $L_2(-\pi, \pi)$ consists of the measurable *odd* functions with $\|f\| < \infty$. Unless otherwise stated, all functions of the real variable x are understood to be in $L_2(-\pi, \pi)$. The class K consists of functions $f(x)$ which are odd, of period 2π, and satisfy I. The more extensive class for which I is satisfied when $\inf(n, m) \leq N$ is denoted by K_N. In view of the Riesz-Fischer theorem [2, pp. 10, 23](2), $f(x) \in K$ implies $f(x) \sim \sum_{i=1}^{\infty} a_i \sin ix$, $\{a_i\} \in l_2$. We shall write a for $\{a_n\}$. The terms "norm" and "completeness" are used in connection with the spaces l_2 or $L_2(-\pi, \pi)$ only. The subclass $K' \subset K$ consists of

Presented to the Society, November 27, 1943, under the title *Orthonormal sequences*; received by the editors May 4, 1944.

(1) The requirement that $f(x)$ be odd *is not essential*. Indeed all our results are valid for functions whose mean value is 0 on $-\pi \leq x \leq \pi$. Thus $f(x) \sim \sum a_n e^{inx}/2$ where $a_n = a_{-n}$, $a_0 = 0$. However, with complex coefficients the correspondent of $f(x) = \sin kx$ is $f(x) = \alpha_k \cos kx - \beta_k \sin kx$ with $|a_k| = |\alpha_k + i\beta_k| = 1$. It seems preferable to gain uniqueness by requiring either the α_k's or β_k's in the expansion of $f(x)$ to vanish.

(2) Numbers in brackets refer to the Bibliography.

functions for which $a \in l_1$. In order to avoid cumbersome repetition we write $a \in K$, K', or K_N to indicate the fact that the corresponding $f(x)$ belongs to these classes. Moreover, we shall often refer to a or $f(x) \in K$, K' as a solution of our problem. We have as the equivalent of I,

$$\text{II} \qquad \sum_{k=1}^{\infty} a_{mk}a_{nk} = \delta_{mn}, \qquad (m, n) = 1,$$

where δ_{mn} is the Kronecker delta and (m, n), as is customary, denotes the greatest common divisor of m and n. We shall use the convention $a_t = 0$, t nonintegral. Thus we can write II as

$$\sum_{k=1}^{\infty} a_k a_{mk/n} = \delta_{mn}, \qquad (m, n) = 1.$$

Most of the results obtained fall naturally into two main categories. Accordingly, with certain important exceptions the first ten sections are primarily concerned with the specialization $a \in l_1$, and the remaining sections with $a \in l_2$. A brief *partial* summary of some of the main conclusions follows:

The numbers refer here to sections. In (2) it is shown that there are an infinite number of nonvanishing a_n's if at least two of them are nonzero. If $f(x) \in K_1$ and $\{f(nx)\}$ is complete then $f(x) = \pm \sin x$. If $f(x) \in K$ and $\{f(nx)\}$ is not complete the addition of no finite collection of functions can complete the system. In (3) we develop an important criterion involving the Dirichlet series associated with $f(x)$. Explicit solutions of our problem are given in (4). In (5) it is shown that no nontrivial linear combination of two functions in K' can be a solution though, for certain special types, a linear combination of 3 solutions may be a solution. A formulation of our problem as a nonlinear integral equation furnishes the content of (6). In (7) it is shown that the hypothesis $f(x) \in K_{2N}$ precludes the possibility of representing $\sin x, \cdots, \sin 2Nx$ even with the addition of N new functions to $\{f(nx)\}$. Examples are given to show that if $g_r(mx)$, $r = 1, \cdots, N$; $m = 1, 2, \cdots$, are adjoined then $\{f(nx)\}$ may be completed. Section (8) indicates some curious identities for a_n's, corresponding to certain solutions in K'. In (9) it is shown that in a natural sense the most general transformations leaving the class K' invariant are generated by a special type of rational function. The most general solutions in K' involving powers of m^{-z} alone are restricted types of rational function of m^{-z}. Section (10) is perhaps of special interest and considers certain additional restrictions under which a solution in K' is unique. It is shown in (11) that K is norm closed in l_2 and that any finite number of a_n's may be chosen in an essentially arbitrary manner. In (12) a more general criterion for a solution, involving the associated Dirichlet expansion, is presented. On the basis of this criterion an example of a solution in K but not in K' is given. The last section involves, in part, considerations not special to our problem and shows in particular that if $a_n \neq 0$ for two values of n then $\sum a_n w^n$ has a singularity at $w=1$.

2. **Preliminary results.** In this section we present some results useful for the purpose of proper orientation.

THEOREM 2.1. *If* $a \in K'$ *then* $|\sum a_n| = 1$.

Since $\{a_n\}$ is absolutely convergent, so also is $(\sum a_n)^2 = \sum_{m,n} a_n a_m$, and we may rearrange as a sum of series each corresponding to one in II.

THEOREM 2.2. *If* $a \in K$ *then either there is just one or else an infinite number of nonzero terms.*

Let a_M be the first and a_N the last nonzero term in a. Let $(M, N) = D$ and define $M' = M/D$, $N' = N/D$ so that $(M', N') = 1$. Then from II using $m = M'$, $n = N'$ we have $a_M a_N = \delta_{M'N'}$. Since $a_M a_N \neq 0$ it follows that $M' = N' = 1$ and $M = N$.

THEOREM 2.3. *If* $a \in K$ *and* $0 < |a_p| < 1$, *p a prime, then* $a_{kp} \neq 0$ *for an infinite number of values of k.*

Suppose to the contrary that $a_{kp} = 0$ for every $k > N$ and $a_{pN} \neq 0$. Let a_M be the first nonzero term in a. Define $M' = M/(M, Np)$ and $N' = Np/(M, Np)$. Since $(M', N') = 1$ we derive from II, $a_M\, a_{Np} = \delta_{M'N'}$ whence $M = Np$. Since $M \leqq p$ we must have $M = p$. Thus a_p is the only nonvanishing term in a; but then $|a_p| < 1$ and $\|a\| = 1$ are in manifest contradiction.

The following theorems indicate the great restriction imposed by a completeness, or closure, hypothesis. (The remarks in §7 are highly pertinent.)

THEOREM 2.4. *If* $f(x) \in K_1$ *and* $\{f(nx)\}$ *is complete then* $f(x) = \pm \sin x$.

This theorem is included in a formally more general statement.

THEOREM 2.5. *If* $f_1(x) \in K_1$ *and* $[f_1(x), f_j(nx)] = \delta_{1j}\delta_{1n}$ *and if* $\{f_j(nx) \mid j = 1, \cdots, N;\ n = 1, 2, \cdots\}$ *is a complete system then* $f_1(x) = \pm \sin x$.

Write $f_1(x) \sim \sum a_n \sin nx$. For arbitrary $\epsilon > 0$, $\|\sin x - \sum_{j=1}^{N} \sum_{m=1}^{M} c_{j,m} f_j(mx)\| < \epsilon$, where M, $c_{j,m}$ may depend on the choice of ϵ. Hence[3]

$$
\begin{aligned}
|a_n| &= |[f_1(x), \sin nx]| \\
&\leqq \left| \left[f_1(x), \sum_{j=1}^{N} \sum_{m=1}^{M} c_{j,m} f_j(mnx) \right] \right| \\
&\qquad + \left| \left[f_1(x), \sin nx - \sum_{j=1}^{N} \sum_{m=1}^{M} c_{j,m} f_j(mnx) \right] \right| \\
&\leqq \delta_{1j}\delta_{1mn} |c_{j,m}| + \epsilon \|f_1(x)\|.
\end{aligned}
$$

(3) Except when otherwise stated, the term "Dirichlet series" will be restricted to series of the form $\sum b_n/n^s$.

Since $\epsilon>0$ is arbitrary, it follows that $a_n=0$ for $n>1$. Moreover $\|f_1(x)\|=1$ and therefore $f_1(x)=\pm\sin x$.

The next theorem shows how sparsely distributed are the solutions of I.

THEOREM 2.6. *If* $\pm\sin x\neq f(x)\in K$ *the system* $\{f(nx)\}$ *cannot be completed by the adjunction of a finite number of new functions.*

Suppose the theorem false. Then $\{g_i(x)|i=1, \cdots, N\}$, $\{f(nx)\}$ is complete. Let Q be the orthogonal complement in Hilbert space of the closed linear manifold determined by $\{f(nx)\}$ [2]. Obviously we may consider Q to be the linear extension of $\{g_i(x)|i=1, \cdots, N\}$ where $[g_i(x), g_j(x)]=\delta_{ij}$. We write $g_i(x)\sim\sum_{k=1}^{\infty}A_i^k \sin kx, f(x)\sim\sum_{j=1}^{\infty}a_j \sin jx$.

Our assumption requires that

$$\sin kx = \sum_{i=1}^{N} c_i^k g_i(x) + \sum_{n=1}^{\infty} d_n f(nx).$$

On multiplying by $g_i(x)$ or $f(nx)$ and integrating over $-\pi\leqq x\leqq\pi$ we may verify that $c_i^k=A_i^k$, $d_n=a_{k/n}$.

Thus

$$1 = \sum_{i=1}^{N} A_i^k g_i(x) + \sum_{n=1}^{\infty} a_{k/n} f(nx). \tag{2.1}$$

Since the functions on the right side are mutually orthogonal

$$1 = \sum_{i=1}^{N} (A_i^k)^2 + \sum_{n} (a_{k/n})^2. \tag{2.2}$$

By Bessel's inequality

$$N = \frac{1}{\pi}\sum_{i=1}^{N}\int_{-\pi}^{\pi}(g_i(x))^2dx \geqq \sum_{i=1}^{N}\sum_{k=1}^{M}(A_i^k)^2,$$

for arbitrary M. Hence, in view of equation (2.2),

$$N \geqq M - \sum_{k=1}^{M}\sum_{n=1}^{\infty}(a_{k/n})^2$$

or

$$1 - N/M \leqq \sum_{k=1}^{M}\sum_{n=1}^{\infty}(a_{k/n})^2/M. \tag{2.3}$$

Let p be any positive integer. Then, for every integer m not divisible by p, we have

$$\sum_{n}(a_{m/n})^2 \leqq 1 - (a_p)^2.$$

The number of positive integers below $M+1$ which are not divisible by p is $M-[M/p]$, where $[M/p]$ denotes the greatest integer not exceeding M/p. Thus

$$\sum_{m=1}^{M} \sum_{n=1}^{\infty}{}' (a_{m/n})^2 \leqq M - \left(M - \left[\frac{M}{p}\right]\right) a_p^2. \tag{2.4}$$

Plainly $[M/p]/M \leqq 1/p$. Then by equations (2.3) and (2.4,

$$N/M \geqq a_p^2(p-1)/p.$$

Since N is fixed and M is arbitrarily large it follows that $a_p=0$ for $p>1$ or $f(x)=\pm\sin x$. For results that in some respects generalize Theorem 2.6, cf. Theorems 7.1 and 7.2.

3. **Dirichlet series formulation.** Consider now the Dirichlet series(3),

$$\phi(z) = \sum a_n n^{-z}, \tag{3.1}$$

where $\{a_n\}$ is the sequence of Fourier constants for $f(x)$. It is trivial that $a\in l_2$ or $a\in l_1$ implies absolute convergence of this Dirichlet series for $R(z)>1/2$ or $R(z)\geqq 0$ respectively. If $f(x)\in K'$, K, K_N the corresponding $\phi(z)$ will be said to belong to K', K, K_N.

THEOREM 3.1. *The relation $a\in K'$ implies and is implied by $a\in l_1$, and* $\left|\sum a_n n^{iy}\right| = 1$.

It is convenient for many purposes to replace considerations on a line by those involving the complex plane.

THEOREM 3.2. *Necessary and sufficient conditions that $\phi(z)$ be in K' are* (a) *$\phi(z)$ is meromorphic,* (b) *$\phi(z)$ admits a Dirichlet expansion converging absolutely for $R(z)\geqq 0$, and* (c) III: $\phi(z)\phi(-z)=1$.

The essentials of these theorems are brought in relief by making the preliminary stronger assumption that the absolute convergence abscissa in equation (3.1) is $R(z)=-\epsilon$, $\epsilon>0$. This is to say $\sum a_n n^{-z}$ and $\sum a_n n^{z}$ both converge absolutely in the strip $|R(z)|<\epsilon$. In this strip we have, after permissible rearrangement of terms,

$$\phi(z)\phi(-z) = \frac{1}{2}\sum_{(m,n)=1}\left(\left(\frac{m}{n}\right)^z + \left(\frac{n}{m}\right)^z\right)\sum_{k=1}^{\infty} a_{km}a_{kn}. \tag{3.2}$$

If $f(x)\in K'$ then, in view of II,

$$\phi(z)\phi(-z) = 1. \tag{III}$$

This functional equation valid in the domain $|R(z)|<\epsilon$ may be combined with equation 3.2 to define $\phi(z)$ throughout the finite plane. Since possible zeros to the right of the imaginary axis are of finite multiplicity, it follows that in the finite plane $\phi(z)$ can have polar singularities only. That is to say $\phi(z)$ is meromorphic. We now proceed to the proof of Theorem 3.1.

Since a_n is real, the conjugate of $\sum a_n n^{iv}$ is $\sum a_n n^{-iv}$. We recall that the definition of the class K' involves $a \in l_1$. Hence $\phi(iy)\phi(-iy)$ has the representation given in equation 3.2 with iy replacing z. Therefore

$$1 = \phi(iy)\phi(-iy) = \phi(iy)\overline{\phi(iy)} = \left| \sum a_n n^{iv} \right|^2.$$

For the reverse implication we must show that II is implied by III when $a \in l_1$. A little reflection will show that our purpose will clearly be accomplished if we show that the condition $\sum B_r r^{iv} = 0$ where r ranges over all positive rational numbers and $\sum |B_r| < \infty$ guarantees $B_r \equiv 0$. This is essentially well known [3, chap. 2]. Indeed if $\psi(t) = \sum_{r<t} B_r$ then $\psi(t)$ is of bounded variation and

$$0 = \sum B_r r^{iv} = \int_{-\infty}^{\infty} e^{itv} d\psi(t).$$

Then, either from the uniqueness of the integral representation or the fact that

$$L_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum B_r r^{iv} \right|^2 dy = \sum (\psi(r+) - \psi(r-))^2 = 0,$$

we infer $B_r \equiv 0$.

In order to establish Theorem 3.2 we require a preliminary result.

LEMMA 3.1. *If $a \in K'$ then $\phi(z)$ defined by equation* 3.1 *can be continued to the whole finite plane by $\phi(-z) = 1/\phi(z)$ and is then a meromorphic function with poles at the negatives of the zeros.*

Since $a \in l_1$, the Dirichlet series for $\phi(z)$ converges absolutely uniformly for $R(z) \geqq 0$ and hence defines a function holomorphic for $R(z) > 0$ and continuous on $R(z) = 0$. In view of Theorem 3.1, $\phi(z)$ has no zeros on the imaginary axis. Let D_1 be the half plane $R(z) > 0$ from which the obviously isolated zero points have been removed by nonoverlapping circles in $R(z) > 0$. Let D_2 be the reflection of D_1 in $z = 0$. Define $\phi_1(z)$ in D_2 by $\phi_1(-z) = 1/\phi(z)$, $z \in D_1$. Plainly $\phi_1(z)$ is analytic throughout D_2 since the derivative exists at $-z_1 \in D_2$ and is equal to

$$-\phi'(z)/\phi(z)^2 \big|_{z=z_1}.$$

On approaching $z = iy_0$, through values in D_2 we get $L_{x\to 0+, y\to y_0} \phi_1(-x+iy) = L_{x\to 0+, y\to y_0} 1/\phi(x-iy) = 1/\phi(-iy_0)$. The limit through D_1 values is $\phi(iy_0)$ which is identical with $1/\phi(-iy_0)$ in view of Theorem 3.1. According to a well known extension of the Schwartz reflection principle [4, p. 157] it follows that $\phi_1(z)$ is the analytic continuation of $\phi(z)$ and hence that $\phi(z)$ is analytic in the open part of the union of the closures of D_1 and D_2. On allowing the radii of the circles, cutting out the zeros in D_1, to approach 0 the assertion concerning the position of the poles is easily verified.

Evidently Theorem 3.1 and Lemma 3.1 together imply the necessity conditions in Theorem 3.2 (with the understanding that when $\phi(z_0)=0$, III is to be interpreted as $L_{z\to z_0}\phi(z)\phi(-z)=1$). On the other hand hypotheses (b) and III yield the sufficiency condition in Theorem 3.1. Our proof is therefore complete.

4. **Some examples in** K'. The replacement of I or II by III constitutes a notable simplification. Thus for solutions in K' we seek a meromorphic function $\phi(z)$ whose zeros have positive real parts, and whose poles are at the reflected points through the origin. Furthermore, $\phi(z)$ is to satisfy III and admit an absolutely convergent Dirichlet series for $R(z)\geqq 0$. A basic example of such a function is given by

$$\phi(z) = (m^{-z} - A)/(1 - Am^{-z}) \tag{E}$$

where A is real with $|A|<1$ and m is an integer not less than 2. A function of this type as well as the corresponding sequence a and function $f(x)$ will be called an *elementary solution*. The expansion for $\phi(z)$ is given by

$$\phi(z) = -A + (A^{-1} - A)\sum_{n=1}^{\infty} (Am^{-z})^n, \qquad |A| < 1.$$

Thus $a_k=0$ for k not a power of m; $a_1=-A$ and for $k=m^n$

$$a_k = (1 - A^2)A^{n-1}, \qquad n = 1, 2, \cdots. \tag{4.1}$$

Solutions of the form m^{-z} with $m=1, 2, \cdots$ are referred to as *unit solutions.*

THEOREM 4.1. *If* $\phi_j(z)\in K'$, $j=1, \cdots, N$, *then* $\phi(z)=\prod_1^N\phi_j(z)\in K'$.

The proof is trivial since the Dirichlet series for $\phi_j(z)$, $j=1, \cdots, N$, and hence for $\phi(z)$ also, converge absolutely for $R(z)\geqq 0$ and III is obviously satisfied.

Remark. It is natural to consider infinite product of solutions $\phi_j(z)\in K'$. Solutions in K may actually be generated in this way under certain further restrictions. If for instance the functions $\phi_j(z)$ are elementary solutions any infinite product involves an infinite number of zeros (or zeros of arbitrarily high multiplicity) in a bounded domain, and hence the presence of essential singularities in the finite plane. Accordingly such new classes of solutions *cannot* be members of K'.

Functions in K' may sometimes be factored into products of functions similar to the elementary solutions except that the constants A need not be real. For instance, the product

$$\prod_{j=1}^{N} (p^{-z} + \alpha\omega_j)/(1 + \alpha\omega_j p^{-z})$$

is a solution, where $\omega_1, \cdots, \omega_N$ are the Nth roots of unity, α is real with $|\alpha|<1$ and p is an integer not less than 1. In fact, this product is itself the elementary solution obtained from (E) by setting $m=p^N$, $A=-\alpha^N$.

Formally, the function $e^{Q(z)}$ where $Q(z)$ is odd satisfies III, but this observation is not of as much value as might be expected since, for one thing, the requirement of expansibility in a Dirichlet series remains, but cf. §§11, 12, 13. The same remark applies to the expression $\phi(z)=F(-z)/F(z)$. However, the last form is of direct application. Thus suppose $a^\sigma \in K'$ with $\sum_{n=1}^\infty |a_n{}^\sigma| = M_\sigma$. Write $\phi_\sigma(z)$ for the corresponding function in K' and $\{a_\sigma\}$ for a set of N real constants.

THEOREM 4.2. *If* $\sum_1^N|\alpha_\sigma M_\sigma|<1$ *then* $\phi(z) = ((1+\sum_1^N\alpha_\sigma\phi_\sigma(-z))/(1+\sum_1^N\alpha_\sigma\phi_\sigma(z)))\prod_1^N\phi_\rho(z)$ *belongs to* K'.

For $z=0$, $|\sum_1^N\alpha_\sigma\phi_\sigma(0)| \leqq \sum_1^N|\alpha_\sigma|<1$ since $M_\sigma \geqq 1$ (Theorem 2.1). By continuity $|\sum_1^N\alpha_\sigma\phi_\sigma(z)|<1$ if $|z|<\delta$ for some $\delta>0$. Hence

$$\left(1+\sum_1^N \alpha_\sigma\phi_\sigma(z)\right)^{-1} = \sum_{n=0}^\infty\left(-\sum_{\sigma=1}^N \alpha_\sigma\phi_\sigma(z)\right)^n.$$

Now, for $R(z)\geqq 0$

$$\sum_{n=0}^\infty\left|-\sum_{\sigma=1}^N \alpha_\sigma\phi_\sigma(z)\right|^n \leqq \sum_{n=0}^\infty\left(\sum_{\sigma=1}^N|\alpha_\sigma|\sum_{m=1}^\infty|a_m{}^\sigma|\right)^n$$
$$\leqq \sum_{n=0}^\infty\left(\sum_{\sigma=1}^N|\alpha_\sigma M_\sigma|\right)^n<\infty.$$

Therefore $\sum_{n=0}^\infty(-\sum_{\sigma=1}^N\sum_{m=1}^\infty \alpha_\sigma a_m{}^\sigma/m^z)^n$ is absolutely convergent as a multiple series in n, σ, and m at least in the half circle D_δ: $R(z)\geqq 0$, $|z|<\delta$. Accordingly the terms may be rearranged. Now

$$\prod_{\rho=1}^N\phi_\rho(z)\left(1+\sum_1^N\alpha_\sigma\phi_\sigma(-z)\right) = \prod_{\rho=1}^N\phi_\rho(z)+\sum_1^N\alpha_\sigma P_\sigma(z),$$

where $P_\sigma(z)$ is $\prod_{\rho=1}^N\phi_\rho(z)/\phi_\sigma(z)$. Hence the numerator in the expression for $\phi(z)$ is a sum of absolutely convergent Dirichlet series for $R(z)\geqq 0$. It follows finally that $\phi(z)$ is expansible in a Dirichlet series, absolutely convergent for $z\in D_\delta$ and then for $R(z)\geqq 0$ also. Plainly $\phi(z)$ satisfies III. Our proof is complete.

An interesting class of solutions is obtained by replacing $\phi_\rho(z)$, $\rho=1,\cdots,N$, in the expression for $\phi(z)$ in Theorem 4.2 by the unit solutions $n_\rho{}^{-z}$. It will be noted that the factor $\prod_1^N\phi_\rho(z)$ may in this case be replaced by M^{-z} where M is the least common multiple of $\{n_\rho\}$. The solutions obtained in this way and all *finite products* of such solutions are called *quasi elementary solutions*. Every elementary solution is of course a quasi elementary solution. It is not true, however, that every quasi elementary solution is the product of elementary solutions as is shown by the simple example $N=2$, α_1, $\alpha_2\neq 0$, $n_1=2$, $n_2=3$.

5. **Combinations of solutions.** If $f(x)$ and $\phi(z)$ are corresponding functions in K, we can write $f(x)=T\phi(z)$. We can define a sort of composition operation, denoted by a star, for $f(x)$ analogous to that occurring in other fields, by writing $f_1(x)^*f_2(x)=T\phi_1(z)\phi_2(z)$. (The transformations T, T^{-1} can be given an explicit representation by using for instance the integral kernel $E(x, z) =\sum_{k=1}^{\infty}k^{-z}\sin kx$ at least for certain *restricted* cases. Thus with c denoting the absolute convergence abscissa of the Dirichlet series for $\phi(z)$,

$$f(x) = L_{u\to\infty}\frac{1}{2ui}\int_{a-iu}^{a+iu}\phi(-z)E(x, z)dz, \qquad c < -a, \qquad a > 1,$$

$$\phi(z) = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)E(x, z)dx, \qquad R(z) > 1.)$$

THEOREM 5.1. *If $a^{(\sigma)}\in K'$, $\sigma=1, 2$, and $\phi_1(z)$, $\phi_2(z)$ and $f_1(x)$, $f_2(x)$ are the associated functions, then*

$$K' \supset f_1(x)^*f_2(x) \sim \sum_{n=1}^{\infty} a_n^{(1)}f_2(nx) \sim \sum_{n=1}^{\infty} a_n^{(2)}f_1(nx).$$

Consider $\sum_{n=1}^{\infty}a_n^{(1)}f_2(nx)$. Since $\{f_\sigma(nx)\}$ is orthonormal for $\sigma=1$ and $\sigma=2$ an elementary application of the Riesz-Fischer Theorem shows that the series represents a function, $f(x)$, in $L_2(-\pi, \pi)$. Since $L_{N\to\infty}\sum_{n=N}^{\infty}(a_n^{(1)})^2=0$ and $[\sin nx, f(mx)]=0$ for $m>n$ it follows easily that

$$d_m = [f(x), \sin mx] = \sum_{r=1}^{m} a_r^{(1)}a_{m/r}^{(2)}.$$

Moreover $\{d_m\}\in l_1$, for $\sum_1^{\infty}\sum_1^{\infty}a_n^{(1)}a_m^{(2)}$ is obviously absolutely convergent. It may be verified that, at least for $R(z)\geqq 0$, $\phi_1(z)\phi_2(z)=\sum d_m/m^z$. Whence $f(x)=T\phi_1(z)\phi_2(z)\subset K'$.

THEOREM 5.2. *If $\phi_1(z)$, $\phi_2(z)\in K'$ and $|\phi_1(z)/\phi_2(z)|\neq 1$ then no proper linear combination is a solution in K'.*

Let $\psi(z)=\alpha\phi_1(z)+\beta\phi_2(z)$, $\alpha\beta\neq 0$. There is no loss of generality in assuming $\phi_i(0)=1$, $i=1, 2$. For a solution $\alpha+\beta=\pm 1$. Plainly then, $\phi_1(z)\phi_2(-z) +\phi_1(-z)\phi_2(z)=2$. That is to say $(\phi_1(z)/\phi_2(z))^2-2(\phi_1(z)/\phi_2(z))+1=0$ or $\phi_1(z)/\phi_2(z)=1$ in contradiction with our hypothesis.

We proceed to show that with three different functions in K' it is sometimes possible to determine a distinct linear combination also in K'. Thus let

$$\phi(z) = \sum_{j=1}^{3} \alpha_j\phi_j(z).$$

It follows at once on proper choice of signs of $\pm\phi_j(0)$ that $\sum_1^3\alpha_j=1$ and, in view of Theorem 5.2, $\alpha_j\neq 0, j=1, 2, 3$. Let $\xi(z)=\phi_1(z)/\phi_2(z)$, $\eta(z)=\phi_2(z)/\phi_3(z)$ and $\xi(z)\eta(z)=\phi_1(z)/\phi_3(z)$. Write also $A_j=\alpha_j^{-1}$. We observe that not only $\phi(z)$, but $\eta(z)$ and $\xi(z)$ also, formally satisfy III and the condition that the functions take on the value 1 for $z=0$. Thus we obtain the quadratic equation for $\eta(z)$

$$\begin{aligned}A_3(\xi(z)+\xi(z)^{-1}-2)+A_1(\eta(z)+\eta(z)^{-1}-2)& \\ +A_2(\xi(z)\eta(z)+(\xi(z)\eta(z))^{-1}-2)&=0,\end{aligned} \tag{5.1}$$

with discriminant

$$(\xi(z)-1)^2(A_3^2(\xi(z)+1)^2-4(A_1+A_3)(A_2+A_3)\xi(z)).$$

If we require that $\eta(z)$ be a rational function of $\xi(z)$ we obtain a simple result. Indeed, it is easy to show by consideration of the discriminant of equation 5.1, that, after possible relabelling of indices, we must have $\alpha_1=1$ and $\alpha_2+\alpha_3=0$. Then

$$\eta(z)=\frac{\alpha_3\xi(z)-1}{\xi(z)(\alpha_3-\xi(z))}\cdot$$

This leads to

$$\phi_3(z)=\frac{\alpha_3\phi_2(z)-\phi_1(z)}{\alpha_3\phi_1(z)-\phi_2(z)}\cdot\phi_1(z),$$

and finally to

$$\phi(z)=\phi_1(z)+\alpha_3(\phi_3(z)-\phi_2(z))=\phi_2(z)\phi_3(z)/\phi_1(z).$$

The relation III is satisfied formally. In order to satisfy conditions (a) and (b) of Theorem 3.2 it is sufficient to require that $\xi(z)$ be a solution and that $|\alpha_3|<1$. We state a partial summary of our conclusions in the following theorem.

THEOREM 5.3. *If $\phi_j(z)$, $j=1, 2, 3$, are algebraically independent solutions in K', no linear combination involving at least two of the functions can be a solution. If $\phi_1(z)/\phi_2(z)\in K'$ and is rational in $\phi_2(z)/\phi_3(z)$ then the essentially unique linear combination $\phi_1(z)+\alpha(\phi_3(z)-\phi_2(z))=\phi_2(z)\phi_3(z)/\phi_1(z)\in K'$ where $\phi_3(z)=(\alpha\phi_2(z)-\phi_1(z))/(\alpha\phi_1(z)-\phi_2(z))\cdot\phi_1(z)$ and $|\alpha|<1$.*

6. **An integral equation.** An interesting alternative formulation of conditions I, II, III for a *restricted* situation will now be indicated. We write $h(t)=\sum_1^\infty a_n e^{-nt}$ and K^δ for the class of functions in K' such that $h'(t)=o(t^{\delta-1})$ for $t\to 0$ where $\delta>0$ and $h'(t)=dh(t)/dt$.

THEOREM 6.1. *If $\phi(z)\in K^\delta$ then $\int_0^\infty sh'(s)h'(su)ds=(1+u)^{-2}$ for $u>0$.*

We observe first that [4, problem 1, p. 314]

$$\Gamma(z)\phi(z) = \int_0^\infty h(t)t^{z-1}dt, \qquad R(z) > 0.$$

By Abel's theorem and the fact that $a \in l_1$, it is clear that $L_{t\to 0}h(t) = h(0)$ is finite. Moreover

$$|h(t)| < \sum e^{-nt} < e^{-t}/(1 - e^{-t}), \quad |h'(t)| < e^{-t}/(1 - e^{-t})^2.$$

Thus both $h(t)$ and $h'(t) = O(e^{-t})$, $t \to \infty$. Hence, it is easy to see that, at least for $R(z) > 0$,

$$(6.1) \qquad \int_0^\infty t^z h'(t)dt = L_{a\to 0, b\to\infty} h(t)t^z \Big|_a^b - z\int_0^\infty h(t)t^{z-1}dt$$
$$= -\Gamma(z+1)\phi(z), \qquad R(z) > 0.$$

Actually the left side of equation 6.1 exists as a Lebesgue integral for $R(z) > -\delta$ and the right side is analytic for $R(z) > -\epsilon$ for some choice of $\epsilon > 0$ (Theorem 3.2). Hence by analytic continuation equation (6.1) is valid for $R(z) > -\eta$, $\eta = \min(\epsilon, \delta)$.

If we replace z by $-z$ in equation (6.1) there results

$$(6.2) \qquad \int_0^\infty t^{-z}h'(t)dt = -\Gamma(1-z)\phi(-z), \qquad R(z) < \eta.$$

Hence, in view of Theorem 3.2 and a standard relation for Γ functions,

$$(6.3) \quad \frac{z\pi}{\sin \pi z} = \phi(z)\Gamma(1+z)\phi(-z)\Gamma(1-z) = \int_0^\infty h'(t)t^z dt \int_0^\infty h'(s)s^{-z}ds, \qquad |R(z)| < \eta.$$

Plainly for $|R(z)| < \eta$

$$(6.4) \qquad \left| s\int_0^\infty h'(su)u^z du \right| = \left| \int_0^\infty h'(t)(t/s)^z dt \right|$$
$$\leqq s^{-R(z)} \int_0^\infty t^{R(z)} |h'(t)| dt, \qquad s > 0.$$

Hence

$$\int_0^\infty \int_0^\infty sh'(s)h'(su)u^z du dz, \qquad |R(z)| < \eta,$$

is a Lebesgue integral obviously equal to the right side of equation (6.3). Therefore by Fubini's theorem

$$(6.5) \qquad \frac{z\pi}{\sin \pi z} = \int_0^\infty u^{z-1} \int_0^\infty ush'(s)h'(su)dsdu, \qquad |R(z)| < \eta.$$

We show now that the inner integral in equation 6.5, denoted by $F(u)$, is continuous for $u>0$. We write

$$(6.6)\quad |F(u+\Delta u)-F(u)| = \left|\int_0^{\gamma/u}+\int_{\gamma/u}^{\infty} sh'(s)\Delta(uh'(su))dsdu, \quad |R(z)|<\eta.\right.$$

Consider $u>0$ fixed and ϵ an arbitrary positive quantity. Manifestly for sufficiently small γ, say γ_0, the contribution of the integral over the range $0\leqq s\leqq \gamma_0/u$ in equation 6.6 is inferior to $\epsilon/2$, for all Δu in $0\leqq \Delta u<u/2$. Since $h'(t)\in L_1(0, \infty)$, $L_{s\to\infty}uh'(su)=0$, and $h'(t)$ is continuous for $t>0$, it follows that for $\Delta u<\rho(\gamma_0)$ the contribution of the integral over the range $\gamma_0/u\leqq s<\infty$ is inferior to $\epsilon/2$. Hence the left side of equation (6.6) is dominated by ϵ for $\Delta u<\min(u/4, \rho(\gamma_0))$, which is the result desired. We observe further that $F(u)$ is Lebesgue summable over any finite range, $0\leqq u\leqq a<\infty$.

It can be shown that $z\pi/\sin \pi z$ is the Mellin transform of $u/(1+u)^2$. Accordingly the Mellin transforms of $F(u)$ and $u/(1+u)^2$ are the same for $|R(z)|<\eta$. Since both functions are Lebesgue summable on any finite range, it is at once clear, on making an exponential transformation, that the hypotheses of Theorem (6.6) of Widder [5, p. 244] are satisfied. Combining the continuity of $F(u)$ and $u/(1+u)^2$ for $u>0$ with the assertion of the theorem quoted, we have, at least for $u>0$,

$$(6.7)\qquad \int_0^{\infty} sh'(s)h'(su)ds = (1+u)^{-2}.$$

Remark. We can write equation (6.7) in the somewhat more perspicuous form

$$\int_{-\infty}^{\infty} P(t)P(t-v)dt = e^v/(1+e^v)^2,$$

where $P(t)=e^{-t}h'(e^{-t})$.

Remark. The theorem is undoubtedly true with weaker conditions on $h(t)$ even without going to $L_{p,p>1}$ spaces. It would be of interest to determine whether equation 6.7 (or generalizations to L_p spaces) has solutions for which the associated sequence a does not satisfy II.

7. **Completeness.** Actually the completeness assertions are implicit as *special instances* of most of the results collected in this section. We show first that the hypotheses of Theorem 2.6 can be weakened significantly. Let the linear extension of $\sin x, \cdots, \sin nx$ be denoted by E_n.

THEOREM 7.1. *If* $\pm\sin x\neq f(x)\in K_2$, *then the system* $\{f(nx)\}$ *cannot be completed by the adjunction of a single function* $g(x)$.

Evidently there is no loss of generality in assuming $g(x)$ normalized and orthogonal to $\{f(nx)\}$. Since the Fourier expansion of $f(nx)$ does not contain $\sin kx$ for $k<n$ and the linear manifolds determined by $f(x)$, $f(2x)$ and by $\{f(nx)|n>2\}$ are orthogonal, it is easy to see that our hypotheses require the consideration of

$$\sin x = d_1 g(x) + D_1 f(x), \tag{7.1}$$

$$\sin 2x = d_2 g(x) + D_2 f(x) + B f(2x). \tag{7.2}$$

Suppose $g(x) \sim \sum A_n \sin nx$ and $f(x) \sim \sum a_n \sin nx$. Then by the argument used in Theorem 2.6, it is apparent that $d_i = A_i$, $D_i = a_i$, $i = 1, 2$, and $B = a_1$. Let $\psi(z) = \sum A_n/n^z$, $\phi(z) = \sum a_n/n^z$. Then the correspondent of $f(2x)$ is $2^{-z}\phi(z)$. Hence we have for $R(z) > 1/2$

$$1 = A_1\psi(z) + a_1\phi(z), \qquad 2^{-z} = A_2\psi(z) + (a_1 2^{-z} + a_2)\phi(z).$$

We observe that $a_1^2 + A_1^2 = 1$ and we need only consider the case that $a_1, A_1 \neq 0$. For $k > 1$, $a_k = -A_1 A_k / a_1$ from equation (7.1). Moreover from equation (7.2), $1 = A_2^2 + a_1^2 + a_2^2$ whence $A_2 = \pm a_1 A_1$. Hence, in case $A_2 = -a_1 A_1$,

$$\phi(z) = \begin{vmatrix} A_1 & 1 \\ A_2 & 2^{-z} \end{vmatrix} \Big/ \begin{vmatrix} A_1 & a_1 \\ A_2 & a_1 2^{-z} + a_2 \end{vmatrix} = \frac{2^{-z} + a_1}{a_1 2^{-z} + 1}.$$

That is to say, $\phi(z)$ is an elementary solution. Accordingly $f(x)$ is actually in K' and this, by Theorem 2.6, is absurd. The contradiction follows similarly for the other case.

Remark. It is interesting to note that the analysis of the theorem would have led logically to the elementary solution had the latter been unknown. In point of fact, this theorem followed the authors' discovery of the elementary solutions.

The proof above hinges on the fact that two equations in the two unknowns $\psi(z)$ and $\phi(z)$ are sufficient to determine $\phi(z)$ as an *elementary solution* and hence Theorem 2.6 applies. It might therefore be expected that, for the obvious generalization with the hypothesis $f(x) \in K_{N+1}$, there would be $N+1$ equations in $\psi_j(z)$, $j = 1, \cdots, N$, and $\phi(z)$ from which we could again infer that $\phi(z) \in K$. The theorem below requires instead the hypothesis $f(x) \in K_{2N+1}$. It should be observed however that as a partial balance for the strengthened hypothesis we are able to assert a good deal more than lack of completeness. In this connection see also the second remark below.

THEOREM 7.2. *If $\pm \sin x \neq f(x) \in K_{2N+1}$ then even with the adjunction of N new functions $\{g_i(x) \mid i = 1, \cdots, N\}$ the closed linear extension of $\{f(nx)\}$ and $\{g_i(x)\}$ does not contain E_{2N}.*

Suppose the theorem false. Just as in the previous theorem we may show that we may require $g_i(x)$, $i = 1, \cdots, N$, to be orthogonal to $\{f(nx)\}$ for all n and that in the expansion of $\sin kx$ only the first k functions in $\{f(nx)\}$ need be taken. By applying the Schmidt orthogonalization process to the linear combinations of the $g_i(x)$'s occurring in the expansion of $\sin x, \cdots, \sin 2Nx$, it may easily be established that $\{g_i(x)\}$ may be replaced by an equivalent set $\{g_i(x)'\}$ where $[g_i(x)', g_j(x)'] = \delta_{ij}$ and at most one new $g_j(x)'$ is introduced in each of the successive equations expressing $\sin kx$ in terms of $\{g_j(x)'\}$ and $\{f(nx)\}$. We drop the primes in reference to these functions and write $g_i(x) \sim \sum_k A_i^k \sin kx$. Finally we can show that we must have

$$\sin kx = \sum_{i=1}^{\min(k,N)} A_i^k g_i(x) + \sum_{n=1}^{k} a_{k/n} f(nx), \qquad k = 1, \cdots, 2N+1.$$

We assume first that $A_k^k \neq 0$ for $k = 1, \cdots, N$. (*In this case the requirement* $f(x) \in K_{N+2}$ *suffices.*) Let $\psi_i(z) = \sum_k A_i^k / k^z$, $\phi(z) = \sum a_n / n^z$. Then we have the system

$$k^{-z} = \sum_{i=1}^{\min(k,N)} A_i^k \psi_i(z) + \sum_{n=1}^{k} a_{k/n} n^{-z} \phi(z), \tag{7.3}$$

$$k = 1, \cdots, N+2, \; R(z) > 1/2.$$

Denote $\sum_{n=1}^{k} a_{k/n} n^{-z}$ by $B_k(z)$ and $B_k(z) - a_1 k^{-z}$ by $B_k'(z)$. The first $N+1$ equations yield

$$\phi(z) = \begin{vmatrix} A_1^1 & 0 & 0 & 1 \\ \vdots & \ddots & & \vdots \\ A_1^k & A_k^k & 0 & k^{-z} \\ \vdots & & \ddots & \vdots \\ \cdot & & A_N^N & N^{-z} \\ A_1^{N+1} & \cdot & A_N^{N+1} & (N+1)^{-z} \end{vmatrix} \div \begin{vmatrix} A_1^1 & 0 & 0 & B_1(z) \\ \vdots & \ddots & & \vdots \\ A_1^k & A_k^k & 0 & \vdots \\ \vdots & & \ddots & \vdots \\ \cdot & & A_N^N & B_N(z) \\ A_1^{N+1} & \cdot & A_N^{N+1} & B_{N+1}(z) \end{vmatrix} \tag{7.4}$$

$$= (\Delta(z) + \Pi(N+1)^{-z})/(a_1(\Delta(z) + \Pi(N+1)^{-z}) + Q(z)),$$

where

$$\Delta(z) = \begin{vmatrix} A_1^1 & 0 & 0 & 1 \\ \vdots & \ddots & & \vdots \\ A_1^k & A_k^k & 0 & k^{-z} \\ \vdots & & \ddots & \vdots \\ \cdot & & A_N^N & N^{-z} \\ A_1^{N+1} & \cdot & A_N^{N+1} & 0 \end{vmatrix},$$

$$Q(z) = \begin{vmatrix} A_1^1 & 0 & 0 & B_1'(z) \\ \vdots & \ddots & & \vdots \\ A_1^k & A_k^k & 0 & \vdots \\ \vdots & & \ddots & \vdots \\ \cdot & & A_N^N & B_N'(z) \\ A_1^{N+1} & \cdot & A_N^{N+1} & B_{N+1}'(z) \end{vmatrix}$$

and $\Pi = \Pi_1^N A_k^k$. Since the identical vanishing of the denominator in equation (7.4) implies the system in equation (7.3) is incompatible, it is impossible that $a_1 = 0$, $Q(z) = 0$; while if $a_1 \neq 0$, $Q(z) \equiv 0$ we infer that $\phi(z) = 1/a_1$ and hence $|a_1| = 1$, that is, $f(x) = \pm \sin x$. Hence let us suppose $Q(z) \not\equiv 0$. We show that this leads to a contradiction. If the $N+1$st equation is replaced by the $N+2$nd and the system is solved for $\phi(z)$ the result is

$$\phi(z) = (\Delta'(z) + \Pi(N+2)^{-z}) \div (a_1(\Delta'(z) + \Pi(N+2)^{-z}) + Q'(z)).$$

On equating the two expressions for $\phi(z)$ there results

$$(7.5) \qquad Q(z)(\Delta'(z) + \Pi(N+2)^{-z}) = Q'(z)(\Delta(z) + \Pi(N+1)^{-z}).$$

We observe that the finite Dirichlet expansions for $Q(z)$ and $Q'(z)$ contain no terms in n^{-z} with $2n > N+2$ and that the expansion for $\Delta(z)$ or $\Delta'(z)$ terminates with the N^{-z} term. Write

$$Q(z) = \sum b_j/j^z, \qquad Q'(z) = \sum b_j'/j^z$$

and let r and l be the largest j values for which b_j and b_j' respectively are not 0. Now equation (7.5) is an identity in view of the uniqueness of Dirichlet expansions [4, p. 309]. Hence $b_r = b_l'$ and $(r(N+2))^z = (l(N+1))^z$. Therefore $r = (N+2)k$, $l = (N+1)k$, k an integer. In view of the restriction $1 \leqq r \leqq (N+2)/2$, this is impossible. Thus $Q(z) \equiv 0$.

In the general case where a diagonal term A_i^i may vanish our procedure is the following: For convenience we shall refer to the highest subscript on a $\psi_i(z)$ in any one of the equations of the system (7.3) as the *terminal* subscript for that equation. Let M be the smallest integer such that the equations for $(M+1)^{-z}$ and $(M+2)^{-z}$ have terminal subscripts *no larger* than that for the equation for M^{-z}. Let m be the terminal subscript for the latter equation. Since $m \leqq N$ one readily observes that $M \leqq 2N-1$. From the first M equations of the set (7.3) omit each equation in which the terminal subscript is no larger than for the preceeding equation. Omit also all equations following the $M+2$nd. The resulting system of $m+2$ equations involves as unknowns $m\,\psi_i$'s and $\phi(z)$. Furthermore the matrix of the coefficients of the $\psi_i(z)$, $i = 1, \cdots, m$, in the first m equations is triangular and *no diagonal term vanishes*. Now the argument for the special case in the previous paragraphs depended only on the fact that the product of diagonal terms there denoted by Π did not vanish, and not at all on the fact that consecutive equations were chosen from the system in (7.3). Accordingly, just as before, we can conclude $a_1 = \pm 1$.

Remark. The further study of functions of the type defined in (7.5) which are expected to lead to solutions in K_{N+1}, though not in K, is foreign to our purpose in this paper.

Remark. The form of the above proof suggests that $f(x) \in K_{2N+1}$ could be replaced by $f(x) \in K_M$ and E_{2N+1} by E_M with $M < 2N+1$. Actually, the theorem stated is the best possible as is shown by the example $g_k(x) = \sin(2k-1)x$, $k = 1, 2, \cdots, N$ and $f(x) = \sin 2x$. In this case E_{2N} is actually contained in the closed linear extension of $\{f(nx)\}$ and $\{g_j(x)\}$ though E_{2N+1} is not. However, the question of the least M for which $f(x) \in K_M$ will ensure $\{f(nx), g_j(x)\}$ not complete is still open.

We now turn to the closer study of the sort of situation considered in Theorem 2.5, namely the problem of completing by adjunction of the sequences $\{f_j(mx) \mid j = 2, \cdots, N; m = 1, 2, \cdots\}$.

THEOREM 7.3. *If $f_j(x) \in K'$, $j = 1, 2$, then the system $[f_1(mx), f_2(nx)] = 0$ is incompatible.*

Indeed let $\phi_j(z) = \sum_n a_n^{(j)}/n^z$ correspond to $f_j(x)$, $j = 1, 2$. Since a^1 and $a^2 \in l_1$ it is permissible to interchange orders of summation in the product series for $\phi_1(iy)\phi_2(-iy)$ and then the hypotheses of the theorem require

$$\phi_1(iy)\phi_2(-iy) = \sum_{(r,s)=1} \sum_k a_{kr}^{(1)} a_{ks}^{(2)} \left(\left(\frac{r}{s}\right)^{iy} + \left(\frac{r}{s}\right)^{-iy} \right) = 0.$$

Hence one of $\phi_1(iy)$, $\phi_2(-iy)$, say $\phi_1(iy)$, has zeros in contradiction with Theorem 3.1.

The requirement that $f_j(x) \in K'$, $j = 1, 2$, may be waived as in the following theorem.

THEOREM 7.4. *If $a^{(1)} \in l_1$, $\{a_n^{(2)} n^\epsilon\} \in l_1$, $\epsilon > 0$, then the system $[f_1(mx), f_2(nx)] = 0$ implies either $a^{(1)}$ or $a^{(2)}$ is a null sequence.*

As usual we write $\phi_j(z) = \sum a_n^{(j)}/n^z$, $j = 1, 2$, though here $\phi_j(z)$ is generally *not* a solution. We can easily establish that $\phi_1(iy)\phi_2(-iy) = 0$ and we infer from this that there are two alternatives: (a) $\phi_1(iy) = 0$ on a dense set of points and hence by continuity $\phi_1(iy)$ vanishes identically, or (b) $\phi_2(-iy)$ vanishes for a set of points with a nonvacuous derived set. In the first case $a_n^{(1)} = 0$ for all n (cf. for instance the sufficiency argument for Theorem 3.1). The second alternative implies $a_n^{(2)} = 0$ for all n since $\sum a_n^{(2)}/n^z$ is analytic for $R(z) > -\epsilon$ [4, p. 88].

We now exhibit some examples of completing $\{f_1(nx)\}$, where $f_1(x) \in K$, by the adjunction of a finite number of sequences $\{f_j(mx) \mid j = 2, \cdots, N; m = 1, 2, \cdots; f_j(x) \in K\}$. Of course $f_1(x)$ $(\neq \pm \sin x)$ cannot be orthogonal to all $f_j(mx)$ (Theorem 2.5). Thus let $f_1(x) = \sin rx$, r a positive integer. Then

$$f_2(x) = -A \sin x + (A^{-1} - A) \sum_1^{\infty} A^n \sin r^n x, \quad 0 < |A| < 1 \tag{7.6}$$

(the correspondent of $\phi(z) = (r^{-z} - A)/(1 - Ar^{-z})$), yields the complete system $\{f_j(nx) | j = 1, 2\}$. The verification is immediate for

$$\sin x = A^{-1}(-f_2(x) + (A^{-1} - A) \sum_{n=1}^{\infty} A^n f_1(r^{n-1}x)),$$

with an obviously convergent right-hand side.

The next example indicates that $\sin x$ may be expressed in terms of a *finite* sum of terms in $f_j(nx)$. For this purpose we need merely specialize Theorem 5.3. We choose $\phi_1(z) = 1$, $\phi_2(z) = r^{-z}$, $|\alpha| > 1$ and

$$\phi_3(z) = \frac{\alpha r^{-z} + 1}{\alpha + r^{-z}},$$

so that

$$\phi(z) = r^{-z}\left(\frac{\alpha r^{-z} + 1}{\alpha + r^{-z}}\right).$$

Then, on writing $f_1(x)$ for the function corresponding to $\phi(z)$ we obtain

$$\sin x = f_1(x) + \alpha(f_2(x) - f_3(x)).$$

Hence $\{f_j(nx) | j = 1, 2, 3\}$ is complete.

Some restriction such as $f(x) \in K$ is necessary for the validity of a theorem of the type of Theorem 2.5. For instance, if $f(x) = \sin x - b \sin 2x$, $|b| < 1$, then $\{f(nx)\}$ is complete for

$$L_{N\to\infty} \sum_1^N b^n f(2^n x) = \sin x. \tag{7.7}$$

On the other hand for $|b| \geqq 1$ it is obvious that $\|\sin x - \sum_1^N b^n f(2^n x)\|$ does not approach 0. Moreover on combining the relations $\inf \|f(Nx) - \sum_1^{N-1} c_i f(ix)\| \geqq |b|$ and $|b| \geqq 1$ it is easy to verify that $\|\sin x - \sum_1^M c_n^{(M)} f(nx)\|$ cannot be made arbitrarily small for any choice of M and $\{c_n^{(M)}\}$. Accordingly completeness for this type of $f(x)$ is essentially a matter of convergence in $L_2(-\pi, \pi)$ of the formal series on the left side of equation 7.7.

For a general periodic function, $f(x) \backsim \sum b_n \sin nx$, $b_1 \neq 0$, it can be shown that there is a unique choice of constants $\{d_n\}$ such that $\sin x - \sum_1^N d_n f(nx)$ is orthogonal to $\sin x, \cdots, \sin Nx$ and that a sufficient condition for completeness of $f(nx)$ is that $\sum_1^\infty d_n f(nx) \in L_2(-\pi, \pi)$. (In general, of course, the constants $\{d_n | n = 1, \cdots, N\}$ will not minimize $\|\sin x - \sum_1^N c_n f(nx)\|$.) We observe parenthetically that for the elementary solution in equation (7.6),

$$\left\| \sin x - \sum_1^N d_n f(nx) \right\| \geqq C |A|^{-N},$$

for some $C>0$ (and $|A|<1$), in keeping, of course, with the assertion of Theorem 2.4. In this connection it is worth while to note the following simple result.

THEOREM 7.5. *If* $\sin mx$ *has an expansion in terms of* $\{f(nx)\}$ *with* $[f(x), \sin x]\neq 0$ *then* $\{f(nx)\}$ *is complete.*

By the expansion $\sin mx \backsim \sum b_n f(nx)$ we mean $L_{N\to\infty}\|\sin mx - \sum_1^N b_n f(nx)\| =0$. As a consequence, $L_{n\to\infty} b_n = 0$. Indeed

$$\begin{aligned}\|f(x)\|\,\|b_N\| &= \|b_N f(Nx)\| \\ &\leqq \left\|\sin x - \sum_1^{N-1} b_n f(nx)\right\| + \left\|\sin x - \sum_1^{N} b_n f(nx)\right\| \\ &\leqq 2\epsilon\end{aligned}$$

for N sufficiently large. Then $m^{-z}=\sum b_n n^{-z}\phi(z)$, where $\phi(z)$ is the Dirichlet series associated with $f(x)$, for $R(z)$ sufficiently large. Thus $\sum_1^\infty b_n n^{-z} = m^{-z}/\phi(z)$. Since $1/\phi(z)$ has a convergent Dirichlet expansion for $R(z)$ sufficiently large [10], it follows from the uniqueness of Dirichlet expansions that the coefficients of corresponding terms on both sides of the equation must be alike. Hence $b_n=0$ unless n is a multiple of m. Accordingly the expansion of $\sin mx$ is $\sum_{n=1}^\infty b_{nm} f(nmx)$ and then $\sin x \backsim \sum b_{mn} f(nx)$. This is a consequence of the elementary observation that for $F(x)$ periodic of period 2π and k a positive integer

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}(\sin kx - F(kx))^2 dx &= \frac{1}{k\pi}\int_{-k\pi}^{k\pi}(\sin x - F(x))^2 dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin x - F(x))^2 dx,\end{aligned}$$

whence

$$\|\sin kx - F(kx)\| = \|\sin x - F(x)\|.$$

8. **Some identities.** Under sufficient convergence restrictions on $\{a_n\}$ we can derive an infinite number of striking identities.

THEOREM 8.1. *If* $a\in K'$ *and* $\{a_n n^\epsilon\}\in l_1$, $\epsilon>0$, *then with* $\sum a_n = 1$

$$\sum a_n(\log n)^2 = \left(\sum a_n \log n\right)^2;$$

$$\sum a_n (\log n)^4/4! = \sum \frac{a_n(\log n)^3}{3!}\sum a_n \log n - \frac{\left(\sum a_n \log n\right)^4}{8}.$$

The derivation is immediate. Since $\phi(z)$ is regular at $z=0$ (Theorem 3.2), we have with $\phi(z)=\sum b_n z^n$

$$1 = \phi(z)\phi(-z) = \sum z^k \sum_{n+m=k}(-1)^m b_n b_m,$$

or

$$\delta_{0k} = \sum_{n+m=k} (-1)^m b_n b_m. \tag{8.1}$$

Now

$$b_N = \frac{d^N}{dz^N}\phi(0)/N! = \sum a_n(\log n)^N/N!. \tag{8.2}$$

From equation (8.1) we get $b_0=1$. Moreover it is plain that for k odd, $\sum_{m+n=k}(-1)^m b_n b_m$ is identically 0 since one of m, n is even and the other is odd, so that the terms $(-1)^m b_n b_m$ and $(-1)^n b_n b_m$ balance off in pairs. However, for k even we get nontrivial identities. For $k=2$ we have $2b_0b_2-b_1^2=0$ or $b_2=b_1^2/2$. For $k=4$ we get $b_4=b_1b_3-b_1^4/8$. The identities in the statement of the theorem are now a consequence of equation (8.2). If these two identities alone were required, the hypothesis $\{a_n(\log n)^4\}\in l_1$ would be sufficient as is easily verified by a simple application of Abel's transformation for series.

9. **Transformations.** A *transformation* of $\phi(z)\in K'$ is understood in this section to be a function of $w=\phi(z)$. The transformations mapping K' *into* itself are of simple type as the next theorem shows.

THEOREM 9.1. *If the transform $F(\phi(z))\in K'$ for every $\phi(z)\in K'$ then $F(w)$ is a rational function of the form $\pm\prod_1^N(w-b_j)/(1-b_jw)$, $0\leqq|b_j|<1$, subject to the condition that $F(w)$ is real for real w values.*

We observe that $F(w)$ is single-valued. Suppose this were not true for $w=w_0$. Choose $\phi(z)\in K'$ such that $w_0=\phi(z_0)$. Then $F(\phi(z_0))$ would be multiple-valued in contradiction with the single-valuedness of solutions in K'. If $F(w_0)$ is finite then $F(w)$ is regular at w_0. Let $\phi(z)\in K'$ satisfy $\phi(z_0)=w_0$ and $\phi'(z_0)\neq 0$. Obviously such a $\phi(z)$ can be found; in fact the solution 2^{-z} serves for $w_0\neq 0$ and some elementary solution for $w_0=0$. Then $w=\phi(z)$ defines a homeomorphism between a neighborhood of w_0, say $N(w_0)$, and a neighborhood of z_0, say $N(z_0)$. Then

$$\frac{\Delta F(w)}{\Delta w} = \frac{\Delta F(\phi(z))}{\Delta z}\Big/\frac{\Delta\phi(z)}{\Delta z}$$

where $w=\phi(z)$. In the neighborhoods mentioned, $\Delta w\to 0$ implies $\Delta z\to 0$ and hence

$$\left.\frac{dF(w)}{dw}\right|_{w_0} = \left.\frac{dF(\phi(z))}{dz}\Big/\frac{d\phi(z)}{dz}\right|_{z_0},$$

that is to say $F(w)$ is analytic at $w=w_0$ and obviously single-valued and analytic at all points in $N(w_0)$ for which $F(w)$ is finite.

The correspondence $w=2^{-z}$ defines a 1-1 map of the strip $0\leqq I(z)<2\pi/\log 2$ in the z-plane on the w-plane cut along the positive real axis with $z=\infty$ corresponding to $w=0$. Since $F(2^{-z})$ is meromorphic it follows that the set of singular z values in the strip is isolated and hence the singular w values can have only 0 or ∞ as limit points. If we use

$$w = \frac{1 - 2^{1-z}}{2 - 2^{-z}}$$

it is clear that $w=0$ is not a limit point. In short, the finite singular points for $F(w)$ are isolated. Now the singularities can be only poles or essential singularities since $F(w)$ is single-valued. Plainly $F(w)F(1/w)=1$ for all nonsingular w values. Hence if there is an essential singularity at w_0 there must be one at $1/w_0$. If $w_0 \neq 0$ then using $w=2^{-z}$ with w_0, w_0^{-1} corresponding to z_0 and $-z_0$ respectively, $0 \leqq I(z_0) < 2\pi/\log 2$, it would follow that a function in K' would have singularities on the axis of imaginaries or the right half-plane, which is absurd. If $w_0=0$ let

$$w = \frac{1 - 2^{1-z}}{2 - 2^{-z}}$$

and we can derive the same sort of contradiction. Accordingly we have shown that the only singularities are poles. Moreover by another appeal to the map defined by a unit function or an elementary solution it is readily shown that there are no poles for $|w| \leqq 1$. Accordingly $F(w)$ is a rational function of the form

$$k \prod_1^M (w - b_j) \Big/ \prod_1^N (wa_j - 1)$$

where $0 \leqq |a_j| < 1$. The condition $F(w)F(1/w)=1$ implies $k^2=1$, $N=M$, $a_j=b_j$. Finally since $w=2^{-z}$ maps $I(z)=0$ into $I(w)=0$ and functions in K' are real on the real axis it follows that $F(w)$ is real for real w. Thus $F(w)$ is of the type indicated in the statement of the theorem. On the other hand it is plain from the remarks in §4 that when $w=\phi(z) \in K'$ a solution in K' is defined by an $F(w)$ of the form given.

THEOREM 9.2. *The most general solution of class K' whose Dirichlet series has powers of m^{-z} alone is a finite product of elementary solutions involving m^{-z} only.*

Let $\psi(m^{-z})$ have the expansion $\sum c_n/n^z$. Now m^{-z} goes to 0 as $R(z) \to 0$ uniformly in $0 \leqq I(z) \leqq 2\pi/\log m$ and $\psi(m^{-z})$ goes to c_1. Moreover m^{-z} and $\psi(m^{-z})$ are periodic of period $2\pi/\log m$. Since $\psi(m^{-z}) \in K'$ there are no singularities in the half-plane $R(z) \geqq -\epsilon$ for some $\epsilon > 0$. Hence $\psi(w)$, $w=m^{-z}$, is analytic throughout $D=\{w \mid |w| < m^\epsilon\}$. Any singular point w_0 must correspond to a pole. Indeed obviously $|w_0| \geqq m^\epsilon > 1$. If there is a nonpolar singularity at w_0 then there is a singularity at $1/w_0$ in contradiction with the analyticity of $\psi(w)$ throughout D. Hence $\psi(w)$ is rational. It is easy to verify that $\psi(w)$ is of the form of $F(w)$ in the previous theorem and that $\psi(m^{-s})$ has the representation asserted in the present theorem.

Remark. If the restriction to class K' is weakened too much the theorem is no longer true. (For instance, cf. §12 for the case of the class K''.)

10. **Uniqueness**(4). This section is perhaps of special interest and is concerned with conditions added to I, II, III under which only unit solutions are possible. We have already obtained some results of this character. (Cf. Theorems 2.4 and 2.5.) In the theorems of this section N is understood to be a positive integer.

THEOREM 10.1. *The conditions* $\{a_n \log n/C\}$ *and* $a\in K'$ *are inconsistent unless* $C=\log N$ *and then* $a_n=\pm\delta_{nN}$.

We have $\phi'(iy)=-\sum a_n \log n n^{-iy}$ and $\phi'(-iy)=\sum a_n \log n n^{iy}$, where $\phi'(\pm iy)=\partial\phi(\pm iy)/\partial iy$, since termwise differentiation is valid in view of the uniform convergence of the resulting series. Then if we use our earlier methods (cf. §3) it follows that $\phi'(iy)\phi'(-iy)=-C^2$. In view of III

$$\phi'(-iy)/\phi(-iy) = -C^2\phi(iy)/\phi'(iy).$$

Also

$$0 = (\phi(iy)\phi(-iy))' = \phi(iy)\phi'(-iy) + \phi'(iy)\phi(-iy).$$

Hence

$$C = \pm\phi'(iy)/\phi(iy).$$

Thus $\phi(iy)=Ae^{\pm iyC}$. This requires that $C=\log N$, $N>1$, and $|A|=1$. The admissible solutions are then $a_n=\pm\delta_{nN}$.

When this theorem is stated in terms of $f(x)$ it acquires a formidable appearance. Thus let

$$G(x) = \pi\left[(\log\Gamma(x/2\pi)(\sin x/2)^{1/2}) + \frac{x}{2\pi}(\gamma+\log 2\pi) - \frac{1}{2}(\gamma+\log 2\pi)/2\right]$$

where γ is the Eulerian constant. For the theorem below it is convenient to replace the condition $f(x)$ is odd by the restriction that $f(x)$ is *even* and has mean value zero. Thus $\{a_n\}$ is now the sequence of coefficients in the Fourier cosine series expansion of $f(x)$ and $L_2(-\pi, \pi)$ no longer requires a function to be odd.

THEOREM 10.2. *If* $f'(x)\in L_2(-\pi,\pi)$ *and* $[F(nx), F(mx)]=C^2\delta_{nm}$, *where* $F(x)=\pi^{-1}\int_0^{2\pi}G(t)f'(x-t)dt$, *then a necessary and sufficient condition for* $f(x)$ *to be a solution of* I *is that* $C=\log N$ *and then* $f(x)=\pm\cos Nx$.

If $h_j(t)\in L_2(-\pi,\pi)$, $j=1, 2$, then

(4) It is understood that a sign difference is admitted. True uniqueness can be obtained by requiring the first nonzero coefficient in a to be positive.

$$H(x) = \frac{1}{\pi}\int_0^{2\pi} h_1(t)h_2(x-t)dt \in L_2(-\pi, \pi).$$

Furthermore if both $h_1(t)$ and $h_2(t)$ are even functions or odd functions, then $H(x)$ is even and $H(x)\sim\sum\alpha_n\beta_n \cos nx$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the Fourier coefficients for $h_1(t)$ and $h_2(t)$ respectively. We now observe [6, p. 15] that $f'(t)\sim\sum_1^\infty na_n \sin nt$. Moreover [7],

$$G(t) = \sum (\log n/n) \sin nt.$$

Taking $h_1(t)$ as $G(t)$, $h_2(t)$ as $f'(t)$ and $H(x)$ as $F(x)$ we easily verify that all conditions are met to give $F(x)\sim\sum_1^\infty a_n \log n \cos nx$. Furthermore [8], $\sum|a_n \log n| < \infty$. Theorem 10.1 now justifies our assertions.

Remark. The first condition may be replaced by any restrictions sufficient to guarantee the validity of termwise differentiation of the Fourier series for $f(x)$ and the finiteness of $\sum|a_n \log n|$.

THEOREM 10.3. *Both $f(x)$ and $f'(x)/C$ cannot belong to K' unless $C=N$ and then $f(x)=\pm\sin Nx$.*

This theorem is in a certain sense somewhat similar to some results of Stone's (cf. his type 4(a) [9]). However, his interest and method of proof is totally different from ours.

It is plain that the Dirichlet series for $\phi(z)$ converges absolutely for $R(z)\geqq -1$ and that the sequence $\{a_n n\}$ is associated with $\phi(z-1)$. Arguments similar to those in §3 gain the conclusion $\phi(z-1)\phi(-(z+1))=C^2$. Moreover $\phi(z+1)\phi(-z-1)=1$ in view of III. Thus $\phi(z-1)=C^2\phi(z+1)$. Accordingly $\phi(z)=C^2\phi(z+2)$. Then

$$\sum \frac{a_n}{n^z}\left(\frac{1}{n^2}-\frac{1}{C^2}\right) = 0.$$

However, this relation implies

$$a_n\left(\frac{1}{n^2}-\frac{1}{C^2}\right) = 0$$

because of the uniqueness of Dirichlet expansions. Hence either $a_n=0$ for all n or $C=\pm N$ and then $a_n=\pm\delta_{Nn}$.

THEOREM 10.4. *If $f(x)\in K'$ and is entire with $a_1\neq 0$ then $f(x)=\pm\sin x$.*

Evidently $a_n=o(n^{-l})$ for every l [6, p. 35]. Therefore $\sum a_n n^{-z}=\phi(z)$ converges for all z. From III it follows that

$$L_{R(z)\to\infty}\phi(-z) = 1/a_1, \qquad L_{R(z)\to\infty}\phi(z) = a_1.$$

Accordingly $|a_1^{-1}-\sum a_n n^z|<\epsilon$ for $R(z)>R_\epsilon$. Therefore $|\sum a_n n^{2m+1}|\leqq M$ for all m. Since termwise differentiation of the sine series for $f(x)$ is obviously permissible here, $|f^{(m)}(0)| < M$. Hence $|f(\tau)| < Me^{|\tau|}$ where $\tau = x+it$. Since $f(x)$ is periodic and

$$|f^{(l)}(x)| \leqq M \sum |x|^{m-l}/(m-l)!$$

it is plain that $|f^{(l)}(x)| \leqq Me^{\pi}$. Write $g_r(x) = \sin rx$ or $\cos rx$ accordingly as r is even or odd. Then by continued partial integration

$$(10.1) \qquad |a_n| \leqq 1/\pi n^r \int_{-\pi}^{\pi} |f^r(x)| \, |g_r(nx)| \, dx \leqq 2Me^{\pi}/n^r.$$

Since the left side of equation (10.1) is independent of r it follows that $a_n = 0$, $n > 1$. Hence $a_1 = \pm 1$.

THEOREM 10.5. *If $f(x) \in K'$ is entire with a_N the first nonvanishing coefficient in a then $f(x) = \pm \sin Nx$.*

The proof is entirely similar to that for the previous theorem.

THEOREM 10.6. *If $\phi(z)$ has no zeros and is entire of order m and satisfies* III, *and if $\phi(z)$ has a convergent Dirichlet expansion, then $\phi(z)$ is a unit solution.*

We invoke the Hadamard factorization theorem [4, p. 250] in order to write

$$\phi(z) = \phi(0)e^{p(z)},$$

where $p(z)$ is a polynomial of degree m. Since $\phi(z)$ has a convergent Dirichlet expansion, $|\phi(z)|$ must be uniformly bounded for some right half-plane. It is easy to see this is impossible unless $m \leqq 1$. Hence $\phi(z) = \phi(0)e^{b+Cz}$ and thus, in view of III, $\phi(z) = \pm n^{-z}$.

The next two theorems are in the same spirit, but are not so near the surface.

THEOREM 10.7. *If $\phi(z)$ satisfies* III *and admits a Dirichlet expansion with $a_1 \neq 0$ converging for $R(z) > -\epsilon$, $\epsilon > 0$, and $\phi(z)$ and $1/\phi(z)$ are entire, then $\phi(z) = \pm 1$.*

Since $\phi(z)$ has no zeros and $a_1 \neq 0$ it follows from a theorem of Landau's [10, p. 90] that $1/\phi(z)$ has a Dirichlet expansion converging for $R(z) > -\epsilon$. Accordingly $\phi(-z) = \sum a_n n^z = \sum b_n n^{-z}$ in the strip $|R(z)| < \epsilon$. Let $\alpha_1(t)$ be the step function with jumps of magnitude a_n at $t = \log 1/n$ and $\alpha_2(t)$ the step function with jumps b_n at $\log n$. Without changing the notation we may assume normalization of these functions so that $\alpha_i(0) = 0$ and

$$\alpha_i(t) = \frac{\alpha_i(t+0) + \alpha_i(t-0)}{2}, \qquad i = 1, 2.$$

Of course $\alpha_i(t)$, $i = 1, 2$, is of bounded variation in any finite interval. The equality of the two expressions for $\phi(-z)$ implies

$$\int_{-\infty}^{\infty} e^{-zt}d\alpha_1(t) = \int_{-\infty}^{\infty} e^{-zt}d\alpha_2(t),$$

for $|R(z)|<\epsilon$. According to a theorem of Widder's [5, Theorem 6a, p. 243] we can conclude $\alpha_1(t)=\alpha_2(t)$. Since $\alpha_1(t)$ is constant for $t>0$ and $\alpha_2(t)$ is constant for $t<0$ it follows that $a_n=b_n=0$, $n>1$, and $a_1=b_1=\phi(0)=\pm 1$.

THEOREM 10.8. *If $\phi(z)$ satisfies* III *and admits a Dirichlet expansion $\sum_{n=N}^{\infty}a_n/n^z$, $N>1$, converging for $R(z)>-\epsilon$, $\epsilon>0$, and $\phi(z)$ and $1/\phi(z)$ are entire, then $\phi(z)=\pm N^{-z}$.*

We can show [10] now that

$$N^{-z}\phi(-z) = \sum_{n=N}^{\infty} a_n\left(\frac{n}{N}\right)^z = \sum b_n l_n^{-z}$$

for $|R(z)|<\epsilon$ (where l_n takes the values in order of size of $\prod_{\nu=1}^{h}(N+k_\nu)/N$, $h\geqq 0$, and k_ν is a positive integer). The argument used in the preceding proof may be carried over with obvious modifications to cover these generalized Dirichlet series.

Remark. There is no clear reason for supposing the hypotheses of either of the two previous theorems quarantee $a\in l_1$. Hence the functions may not be in K'. On the other hand if $\phi(z)\in K'$ the writers do not know whether the convergence abscissa is always to the left of $R(z)=0$. These seem interesting topics for further investigation. Very likely the requirement $a\in K'$ together with the assumptions $\phi(z)$ and $1/\phi(z)$ are entire may yield the conclusions in the theorem.

Remark. The reader will easily verify that compact proofs of Theorems 10.4 and 10.5 are possible by making use of Theorems 10.7 and 10.8.

11. **Closure of solutions.** This section contains some results of basic interest for our problem.

THEOREM 11.1. *The set of solutions in K is closed in l_2.*

Let $\{a^\sigma | a^\sigma\in K\}$ be a sequence of solutions of II converging to a in the norm sense. Then, of course, $\|a\|=1$. For arbitrary $\epsilon>0$ choose N so that $\sum_N^\infty a_n{}^2<\epsilon^2$ and a^σ so that $\|a^\sigma-a\|<\epsilon$. Then a simple application of the triangle inequality yields $\sum_N^\infty (a_n{}^\sigma)^2<4\epsilon^2$. Let $(r, s)=1$ with $s<r$. Define m by $ms\leqq N<(m+1)s$. Then

$$\left|\sum_{n=1}^{\infty} a_{nr}a_{ns}\right| \leqq \left|\sum_{1}^{m} (a_{nr}-a_{nr}{}^\sigma)a_{ns}\right| + \left|\sum_{1}^{m} (a_{ns}-a_{ns}{}^\sigma)a_{nr}{}^\sigma\right| + \left|\sum_{1}^{m} a_{nr}{}^\sigma a_{ns}{}^\sigma\right| + \left|\sum_{m+1}^{\infty} a_{nr}a_{ns}\right|.$$

We observe that

$$\sum_{1}^{m} a_{nr}{}^{\sigma} a_{ns}{}^{\sigma} = \sum_{1}^{\infty} a_{nr}{}^{\sigma} a_{ns}{}^{\sigma} - \sum_{m+1}^{\infty} a_{nr}{}^{\sigma} a_{ns}{}^{\sigma} = - \sum_{m+1}^{\infty} a_{nr}{}^{\sigma} a_{ns}{}^{\sigma}.$$

On combining the relations above there results

$$\left| \sum_{1}^{\infty} a_{nr} a_{ns} \right| \leqq \|a - a^{\sigma}\| \, \|a\| + \|a - a^{\sigma}\| \, \|a^{\sigma}\| + \sum_{1}^{N} ((a_n{}^{\sigma})^2 + (a_n)^2) \leqq 2\epsilon + 8\epsilon^2.$$

Hence a satisfies II and the proof is complete.

Since

$$L_{A \to 1} \frac{A - 2^{-z}}{1 - A2^{-z}} \phi(z) = \phi(z)$$

it is clear that the solutions are not isolated points in l_2. The next two theorems dispel any notion that a solution is seriously restricted as regards the choice of any finite number of coefficients.

THEOREM 11.2. *For any sequence* $\{b_n | n=1, \cdots, N\}$ *with* $\sum_1^N |b_n| < 1$, *there is a quasi-elementary solution* $\phi(z)$, *whose associated a sequence has* $a_n = b_n$, $n=1, \cdots, N$.

THEOREM 11.3. *For any finite sequence* $\{b_n | n=1, \cdots, N\}$ *with* $b_1 \neq 0$ *there is a finite product of elementary solutions whose associated a sequence satisfies* $a_i = \lambda b_i$, $i=1, \cdots, N$, $\lambda \neq 0$.

Consider the quasi-elementary solution

$$\phi(z) = \frac{1 + M^z \sum_1^N b_n n^{-z}}{1 + M^{-z} \sum_1^N b_n n^z} M^{-z},$$

where M is any multiple of the L.C.M. of the n's associated with the nonzero members of $\{b_n | n=1, \cdots, N\}$ *which also satisfies* $M > N^2$. It may easily be verified that $\phi(z)$ has the property asserted in Theorem 11.2.

To demonstrate Theorem 11.3 we write

$$\psi_j(z) = \prod_{1}^{j} \phi_n(z)$$

with

$$\phi_n(z) = \frac{(n+1)^{-z} - A_n}{1 - A_n(n+1)^{-z}}.$$

As usual let a^j denote the sequence corresponding to $\psi_j(z)$. In view of equation (4.1) we have $a_1{}^1 = -A_1$, $a_2{}^1 = 1 - (A_1)^2$. Then the requirement $a_2{}^1 : a_1{}^1 = b_2 : b_1$ leads to $b_1(A_1)^2 - b_2 A_1 - b_1 = 0$. Since the product of the roots is -1 there is a unique A_1 satisfying $-1 \leqq A_1 < 1$, $0 < |A_1|$. Write $\lambda_1 = -A_1/b_1$. Thus

$$a_k^1 = \lambda_1 b_k, \qquad k = 1, 2. \tag{11.1}$$

Suppose $A_1, \cdots, A_{m-1}$ have been determined so that

$$a_k^{m-1} = \lambda_{m-1} b_k, \qquad k = 1, \cdots, m. \tag{11.2}$$

Now $\psi_m(z) = \phi_m(z)\psi_{m-1}(z)$. Hence on comparing coefficients of $1, 2^{-z}, \cdots, m^{-z}$ we obtain

$$a_k^m = -A_m a_k^{m-1}, \qquad k = 1, \cdots, m, \tag{11.3}$$

$$a_{m+1}^m = -A_m a_{m+1}^{m-1} + (1 - (A_m)^2) a_1^{m-1}. \tag{11.4}$$

Hence equations (11.2) and (11.3) imply

$$a_k^m = \lambda_m b_k, \qquad k = 1, \cdots, m, \tag{11.5}$$

where $\lambda_m = -A_m \lambda_{m-1}$. Thus (11.4) becomes

$$b_1 \lambda_{m-1} A_m^2 + (a_{m+1}^{m-1} - \lambda_{m-1} b_{m+1}) A_m - b_1 \lambda_{m-1} = 0.$$

Since $b_1 \lambda_{m-1} = (-1)^{m-1} \prod_1^{m-1} A_j \neq 0$ there is a unique solution for A_m under the restriction $-1 \leqq A_m < 1$, $0 < |A_m|$. We have finally

$$a_k^m = \lambda_m b_k, \qquad k = 1, \cdots, m + 1.$$

In view of equation (11.1) and the induction from equation (11.2) to equation (11.3) the proof is complete and the function $\psi_N(z)$ satisfies the requirements of the theorem.

12. **Functions in** K''. We now consider a different aspect of the question of determining solutions in K than that developed in the preceding section. In particular, we formulate a criterion more general than those in Theorems 3.1 or 3.2. We need a preliminary result. Let X and Y represent the set of points on the x and y axes respectively. We shall refer to an exceptional set denoted by $\{y_\sigma\} \subset Y$ which is at most denumerable and may be vacuous. We assume $0 < \rho \leqq \inf\{|y_\sigma - y_{\sigma'}| \mid \sigma \neq \sigma'\}$. Let δ_σ be a closed interval of length l independent of σ in Y containing y_σ as its midpoint. Write $Y_l = \bigcup \delta_\sigma$ and Y_l' for the complement of Y_l in Y.

THEOREM 12.1. *Let $\gamma(x, y) = \sum B_r(x) r^{iy}$ where r runs through all positive rational numbers and* (a) *the series for $\gamma(x, y)$ converges absolutely for each $x > 0$;* (b) $|\gamma(x, y)| < M$ *for $x > 0$;* (c) *for each choice of $l > 0$ the series converges uniformly to* 1 *in $y \in Y_l'$ as $x \to 0+$. Then $L_{x\to 0+} B_r(x) = 1, 0$ according as $r = 1$ or $r \neq 1$.*

Let $\gamma'(x, y) = \gamma(x, y) - 1 = \sum B_r'(x) r^{iy}$ where $B_1'(x) = B_1(x) - 1$ and $B_r'(x) = B_r(x)$, $r \neq 1$. Define $v(\lambda; x) = \sum_{\log r < \lambda} B_r'(x)$. Plainly $v(\lambda; x)$ is of bounded variation for each $x > 0$ by (a). Hence

$$\psi(x, y) = \int_{-\infty}^{\infty} e^{i\lambda y} dv(\lambda; x)$$

and [3, chap. 2]

$$\sum (B_r'(x))^2 = L_{t\to\infty} \frac{1}{2t} \int_T |\psi(x, y)|^2 dy \tag{12.1}$$

where $T = \{y \mid -t \leqq y \leqq t\}$. We may write $T = T' + T''$ where T' is the set intersection of Y_l' and T and T'' that of Y_l and T. Choose l to satisfy $lM^2/\rho < \epsilon$. By (c) we have $|\psi(x, y)|^2 < \epsilon$, $y \in Y_l'$ and $0 < x < \eta$. Hence if we examine the contribution of the ranges T' and T'' to the integral on the right side of equation (12.1) it is apparent that

$$\sum (B_r'(x))^2 \leqq \epsilon + Ml^2/\rho \leqq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary we infer that $L_{x\to 0+} B_r'(x) = 0$ which is tantamount to the conclusion sought.

We can now state the generalized criterion referred to above.

THEOREM 12.2. *Suppose the Dirichlet expansion of $\phi(z)$ has real coefficients and the absolute convergence abscissa $x = 0$;* (a) *$\phi(z)$ is uniformly bounded for* $R(z) > 0$; (b) $L_{x\to 0+}|\phi(z)| = 1$ *uniformly in $y \in Y_l'$ for each $l > 0$. Then $\phi(z) \in K$.*

Evidently $\phi(\bar{z}) = \bar{\phi}(z)$. For $R(z) > 0$ we have, by hypothesis (a), that

$$|\phi(z)|^2 = \phi(z)\phi(\bar{z}) = \frac{1}{2} \sum_{(l,s)=1} \sum_{k=1}^{\infty} a_{kl} a_{ks} (k^2 ls)^{-x} ((l/s)^{iy} + (l/s)^{-iy}).$$

Hence

$$|\phi(z)|^2 = \sum A_r(x) r^{iy}$$

where r sums over all positive rationals. Here

$$A_r(x) = \sum_{k=1}^{\infty} a_{kl} a_{ks} (k^2 ls)^{-x}, \qquad r = l/s, \ (l, s) = 1.$$

All conditions of Theorem 12.1 are met. Accordingly

$$L_{x\to 0+} \sum_{k=1}^{\infty} a_{kl} a_{ks} (k^2 ls)^{-x} = \delta_{ls}, \qquad (l, s) = 1. \tag{12.2}$$

Evidently this implies that $\sum (a_k)^2$ converges and that the value of the sum is 1. Indeed since $\sum (a_k k^{-x})^2$ is monotone it is plain that $\sum (a_n)^2 \geqq 1$. On the other hand suppose $\sum_1^N (a_k)^2 > H > 1$ for some N. Choose x so that $N^{-2x} = 2/(H+1)$. Then, in contradiction with equation (12.2),

$$\sum_1^{\infty} (a_k k^{-x})^2 \geqq \sum_1^{N} (a_k k^{-x})^2 > \frac{2H}{H+1} > 1.$$

The usual application of Schwarz' inequality then guarantees the absolute convergence of $\sum a_{kl}a_{ks}$. Accordingly the Dirichlet series $\sum a_{kl}a_{ks}(k^2ls)^{-z}$ converges for $z=0$ and hence [4, p. 291]

$$\sum a_{kl}a_{ks} = L_{x\to 0+}\sum a_{kl}a_{ks}(k^2ls)^{-x} = 0$$

for l, s not both 1, subject to $(l, s)=1$. The class of solutions satisfying the conditions of this theorem is denoted by K''.

13. **Examples in** K''. In this section we exhibit solutions of our problem which are not of class K'.

Consider then

$$\phi(z) = e^{-\tanh(z\log 2/2)} = e^{-1+2(2^{-z}/(1+2^{-z}))}.$$

Straightforward computation gives

$$\left|\phi(z)\right|^2 = e^{-2(1-2(1+2^x\cos(y\log 2))/(1+2^{2x}+2^{x+1}\cos(y\log 2))}.$$

Plainly $\phi(z)$ is uniformly bounded for $R(z)>0$. Moreover, except for

$$y_\sigma = (2\sigma+1)\pi i/\log 2, \qquad \sigma = 0, \pm 1, \pm 2, \cdots,$$

$L_{x\to 0+}\left|\phi(z)\right| =1$ and the convergence is uniform for $y\in Y_l'$, $l>0$. ($L_{x\to 0+}\left|\phi(x+iy_\sigma)\right| =0$.) Write $t=2^{-z}$. Then

$$\phi(z(t)) = e^{-2t/(1+t)}e^{-1}.$$

This function admits a Maclaurin series expansion converging absolutely for $|t|<1$. On replacing t^n by 2^{-nz} it is then obvious that $\phi(z)$ has an absolutely convergent Dirichlet expansion involving powers of 2^{-z} alone (compare Theorem 9.2) for $R(z)>0$. Accordingly $\phi(z)$ satisfies the conditions of Theorem 12.2 and we infer that the associated $a\in K$. In the same way we may show $\exp(-\sinh\tanh(z\log 2/2))$, and so on, generates a solution.

If we start with $\phi(z)=\exp(-\coth(z\log 2/2))$ the analysis is essentially the same except that $y_\sigma=2\sigma\pi/\log 2$ so that $z=0$ is singular and again $\phi(z)$ is the generating function of a sequence in K. It is worth while to point out that the associated a is not in l_1. Thus, if we write t for 2^{-z} the resulting Maclaurin series is $\sum a_m t^n$, $m=2^n$. From the form of $\phi(z)$ it is evident that $\phi(-\log t/\log 2)\to 0$ as t approaches 1 along the real axis. If $a\in l_1$ then by Abel's Theorem it follows that $\sum a_i=0$. However, we should conclude from Theorem 2.1 that $(\sum a_i)^2=1$. Hence $a\notin l_1$. The result follows also from Theorem 3.2.

Remark. A striking feature of the examples given above is the fact that they are *not* representable as the limits of finite products of elementary solutions. Indeed such products would have zeros to the right of the imaginary axis.

Remark. A limit to the extent that the conditions in Theorem 12.2 can be weakened is indicated by the function $\phi(z)=\exp(\coth(z\log 2/2))$. For this function all conditions of that theorem save (a) are met, nevertheless $\phi(z)$ does not generate a solution in K. Indeed it may easily be verified that the

Dirichlet expansion in powers of 2^{-z} of $\phi(z)$ has all its nonzero coefficients greater than those for $\exp(-\coth(z\log 2/2))$ and hence the sum of the squares cannot be 1.

14. **Singularities of associated power series.** In this section we shall be concerned with some results valid for a much wider range of Dirichlet series than those occurring in the problems of this paper. Let S be the class of Dirichlet series $\lambda(z)=\sum b_n/n^z$ satisfying (a) $\{b_n\}\in l_2$, (b) if $\lambda(z)$ has a zero it is not entire. When we replace S by K'' or K' the more general results below become assertions regarding our main problem. Thus Theorem 14.1 implies that except for unit solutions $\sum a_n w^n$ has a singularity at $w=1$, and so on.

THEOREM 14.1. *If $\lambda(z)\in S$ and more than one b_n is not 0 then $\sum b_n w^n$ has a singularity at $w=1$.*

That there is a singularity on $|w|=1$ if $\lambda(z)\in K'$ is *not a unit solution* is apparent. Indeed if the convergence radius exceeds 1, then the limit inferior of $|a_n|^{-1/n}>1$ or $\alpha^n>|a_n|$, $0\leqq\alpha<1$, for n sufficiently large. Hence $\sum a_n/n^z$ converges for all values of z. It is easy to see that Theorem 10.8 bars such a possibility. Among other things Theorem 14.1 asserts that the singularity is actually at $w=1$.

Write $h(t)=\sum b_n e^{-nt}$. We may assume convergence for $t\geqq 0$. Otherwise since $b_n\to 0$, there would be a singularity at $t=0$ [4, paragraph 7.31], and the assertion of the theorem would be granted. We have

$$\lambda(z)=\frac{1}{\Gamma(z)}\int_0^\infty h(t)t^{z-1}dt. \tag{14.1}$$

If the theorem is untrue then $h(\tau)$, $\tau=t+is$, may be continued analytically throughout a neighborhood of $\tau=0$, say for $|\tau|<\delta$. In this case cut the τ plane along the positive real axis and consider the Hankel contour C_ρ running along the upper bank of the cut from ∞ to $t=\rho$ then counter clockwise around $\tau=0$ to $t=\rho$ on the lower bank and thence to ∞. For $\rho<\delta$ we write formally

$$Q(z)=\frac{1}{2\pi i}\int_{C_\rho}(-\tau)^{z-1}h(\tau)d\tau=\frac{1}{2\pi i}\int_{\infty+}^{\rho+}+\int_{|\tau|=\rho}+\int_{\rho-}^{\infty-}(-\tau)^{z-1}h(\tau)d\tau, \tag{14.2}$$

where the $+$ and $-$ refer to the upper and lower banks respectively, with the usual convention that $\log(-\tau)=\log|\tau|-i\pi$ on the upper and $\log|-\tau|=\log(\tau)+i\pi$ on the lower bank of the cut. Let $z=re^{i\omega}$ be a non-integer, with $|\omega|<\pi/2$. Then the integral around the circle is inferior in absolute value to

$$\frac{e^{2\pi r}}{2\pi}\rho^{r\cos\omega}\int_{-\pi}^{\pi}|h(\rho e^{i\theta})|\,d\theta.$$

Since $h(\tau)$ is analytic for $|\tau|<\delta$ the term just written vanishes with ρ. Hence

$$Q(z) = L_{\rho\to 0}\frac{1}{2\pi i}\left(e^{i\pi(z-1)} - e^{-i\pi(z-1)}\right)\int_\rho^\infty h(t)t^{z-1}dt \tag{14.3}$$
$$= \Gamma(z)\sin\pi(z-1)\lambda(z)/\pi,$$

in view of equation (14.1). Since $\Gamma(1-z)\Gamma(z)=\pi/\sin\pi z$ there results

$$\lambda(z) = -1/2\pi i\Gamma(1-z)\int_{C_\rho}(-\tau)^{z-1}h(\tau)d\tau, \qquad \rho<\delta,\ R(z)\geqq\epsilon>0. \tag{14.4}$$

The integral on the right side of equation (14.4) *is analytic throughout any bounded z domain, D.* Indeed it is evident that (a) $h(\tau)(-\tau)^{z-1}$ is continuous in τ and z for τ on C_ρ and z in a bounded domain D, and that (b) τ^{z-1}, τ fixed, is analytic in D. Write $C_\rho=C^m+C^{-m}$ where C^m is the part of C_ρ included in the closed circle of radius m about $\tau=0$. Then

$$\left|\int_{C^{-m}}h(\tau)(-\tau)^{z-1}d\tau\right| \leqq \int_{C^{-m}}\left|h(\tau)e^{\tau/2}\right|\left|e^{-\tau/2}(-\tau)^{z-1}\right|d\tau. \tag{14.5}$$

For $m>\rho$ the last integral is inferior to

$$2\left[\int_m^\infty |h(t)|^2e^tdt\int_m^\infty e^{-t}t^{2(R(z)-1)}dt\right]^{1/2}. \tag{14.6}$$

Manifestly

$$|h(t)|^2 \leqq M\left|\sum_1^\infty e^{-nt}\right|^2 = Me^{-2t}/(1-e^{-t})^2.$$

Hence the product of integrals on the right side of equation 14.6 goes to 0 uniformly in $z\in D$ when $m\to\infty$. Thus (c)

$$\int_{C^{-m}}h(\tau)(-\tau)^{z-1}d\tau$$

goes to 0 uniformly in $z\in D$ when $m\to\infty$. The properties (a), (b), (c) are sufficient to justify the assertion in italics [4, p. 100].

Accordingly the right side of equation 14.4 provides the continuation of $\lambda(z)$ to the entire finite z plane. The sole possible singularities in the finite plane are the simple poles of $\Gamma(1-z)$ at $z=n$, $n=1, 2, \cdots$. However, since $\{b_n\}\in l_2$, $\sum b_nn^{-z}$ converges absolutely for $R(z)>1/2$ and these poles do not occur. Hence $\lambda(z)$ must be an entire function whose order may easily be found from the representation in equation (14.4).

Consider the Hankel loop integral in equation 14.4 for *large* $|z|$. The following crude bounds are sufficient for our purpose. We have for *fixed* ρ, $0<\rho<\min(\delta, 1)$,

$$\left|\int_{\infty}^{\rho}(-t)^{z-1}h(t)dt\right| \leqq \int_{1}^{\infty} t^{|z|-1}\,|\,h(t)\,|\,dt + \int_{\rho}^{1} t^{-|z|-1}\,|\,h(t)\,|\,dt$$
$$\leqq M\left(\int_{1}^{\infty} t^{|z|-1}e^{-t}/(1-e^{-t})dt + \int_{\rho}^{1} t^{-|z|-1}e^{-t}/(1-e^{-t})dt\right.$$
$$\leqq M_1\Gamma(|z|) + M_2\rho^{-|z|}.$$

Since

$$\Gamma(|z|) < M_3|z|^{|z|-1/2}e^{-|z|} < M_4e^{|z|^2}$$

it is clear that

$$\left|\int_{\rho}^{\infty}(-t)^{z-1}h(t)dt\right| < M_5e^{|z|^2}.$$

We have for the integral about the circle of radius ρ

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\rho e^{i(\theta+\pi)})^z h(\rho e^{i\theta})d\theta\right| < M_6\rho^{-|x|}e^{2\pi|y|} < M_7e^{|z|^2}.$$

If $-\pi+\epsilon<\arg(1-z)<\pi-\epsilon$,

$$|\Gamma(1-z)| < M_9e^{|z|^2}.$$

Since $\{b_n\}\in l_2$ it is plain that $|\lambda(z)|<M_{10}$ for $R(z)>1$. Accordingly, in estimating the order of $\lambda(z)$ we may restrict attention to the complement of a sector including the positive real axis; that is to say, we need consider $-\pi+\epsilon<\arg(1-z)<\pi-\epsilon$ alone. Combining the inequalities obtained above, we have

$$|\lambda(z)| < M_{11}e^{2|z|^2}$$

or $\lambda(z)$ is of order 2 at most. Since $\lambda(z)\in S$ and is entire, it can have no zeros in the finite plane. The Hadamard factorization theorem yields

$$\lambda(z) = \lambda(0)e^{cz+dz^2}.$$

Since $\lambda(z)$ has a Dirichlet expansion with finite convergence abscissa, it is obvious that $d=0$. Hence $\lambda(z)=b_n/n^z$ for some n. This is a contradiction with our hypothesis. Our proof is complete.

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