STRUCTURE THEOREMS FOR A SPECIAL CLASS OF BANACH ALGEBRAS

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Introduction. In this paper we prove analogues of two of the Wedderburn structure theorems for a special class of Banach algebras (we prefer the term "Banach algebra" to the more common term "normed ring"). Our theorems assert, first, that any algebra of the special kind we shall consider is a direct sum of simple ones (simple = no closed 2-sided ideals) and, secondly, that every simple algebra of this type is a full matrix algebra—where by a full matrix algebra we mean the set of all finite or infinite (countable or uncountable) matrices for which the sum of the squares of the absolute values of the elements converges (the matrix elements being complex numbers).

We use the term Banach algebra, or simply B-algebra, for a set which is an algebra over the complex numbers (though without any assumptions about a basis, finite or otherwise) whose underlying linear space has a norm with respect to which it is a Banach space, and which satisfies the condition $\|xy\| \leqq \|x\| \, \|y\|$. These assumptions are the same as those of Gelfand [III](1) except that we assume neither commutativity nor the existence of a unit. The special algebras which we consider (we call them H^*-algebras) satisfy the additional conditions: (1) the underlying Banach space is a Hilbert space (of arbitrary dimension), (2) each element x has an "adjoint" x^* in a certain rather strong sense.

The part played in our theory by the fact that the underlying Banach space is a Hilbert space is two-fold. It makes direct sum decompositions easier through the possibility of taking orthogonal complements, and it opens up the possibility of using the spectral theory of operators on Hilbert space. Actually we do not use the spectral theory but instead use simplified forms of some of the technique used in that theory. It is here that the adjoint elements in our algebra are essential for their existence makes it possible, through the spectral theory technique, to construct idempotents.

Our consideration of these H^*-algebras arose from a consideration of the L_2-algebra of a compact group. Segal [VIII] has defined the group algebra of a locally compact group to be the space L_1 of integrable functions (that is, complex-valued functions, integrable with respect to the Haar measure of the group) with convolution for multiplication. In the case of a compact group the space L_2 (of complex-valued functions whose square is integrable with

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(1) Numbers in brackets refer to the bibliography at the end of this paper.

respect to the Haar measure of the group, and again with convolution for multiplication) also forms an algebra and this algebra is an H^*-algebra; our original concern was with theorems about this particular kind of H^*-algebra. (We should mention here that our concern with these L_2-algebras arose from reading Segal's paper [IX] in manuscript form, and from many conversations in which he pointed out various features of group algebras of topological groups; hence our indebtedness to Segal is great.) This L_2-algebra has the significant property that all the transformations in its regular representation are completely continuous, and this property makes all the simple constituents of this algebra finite-dimensional. Hence H^*-algebras (as we shall see) are more general than such L_2-algebras. On the other hand, the group algebra of a non-compact locally compact group (that is, the L_1-algebra—unfortunately L_2 does not form an algebra in this case) is not an H^*-algebra but it is close to being an H^*-algebra. We say it is close to being an H^*-algebra because the Hilbert space L_2 is dense in it and, for groups whose right and left Haar measures coincide, there is also an adjoint defined. So a general H^*-algebra lies, in degree of complication, somewhere between the group algebra of a compact group and the group algebra of a non-compact locally compact group. We hope that our theorems about H^*-algebras may be suggestive about the structure of these group algebras.

Of the papers in our bibliography the one closest to this paper is probably that of Gelfand and Neumark [IV]. Although both their assumptions and conclusions differ from ours it may be helpful to point out the nature of the differences. The assumptions differ in that we assume the underlying Banach space to be a Hilbert space, and that they assume the existence of a unit. Since in some vague sense there are more B-algebras with a unit than for which the underlying Banach space is a Hilbert space our theory is more special than theirs. On the other hand our theory would essentially become extinct if we were to assume a unit since our results show that an H^*-algebra contains a unit if and only if it is finite-dimensional. The main conclusion of [IV] is that B-algebras of their kind are algebras of operators on Hilbert space and it is an achievement to find the Hilbert space since it was not there at the beginning. In our case, though, it is there at the beginning and the regular representation of an H^*-algebra is a representation in terms of operators on Hilbert space (this representation is faithful except in trivial exceptional cases which we describe). Since a full matrix algebra (in our sense) is a very special type of algebra of operators on Hilbert space our conclusion that an H^*-algebra is a direct sum of full matrix algebras can be thought of as a concrete characterization of the particular type of algebra of operators with which we deal. This characterization, incidentally, can also be described in terms of another representation, and a full matrix algebra described as an algebra of all operators of Hilbert-Schmidt type on a function space L_2. In this description, however, the norm of an element in the algebra will not coincide with the norm of the corresponding operator.

In §1 we give our basic definitions and in §2 we eliminate from further consideration a trivial kind of special situation. In §3 we prove theorems about the existence of idempotents and in §4 we use these idempotents to obtain our desired structure theorems. In §5 we consider the L_2-algebra of a compact group.

We are indebted to S. Eilenberg for many important suggestions which have introduced great simplifications in a number of our proofs, and to R.M. Thrall for a number of helpful discussions.

1. Definitions.

Definition 1.1. A *B-algebra* (or *Banach algebra*) is a set A which contains more than two elements, is a ring in the sense of algebra [X, p. 35], and satisfies the following further conditions:

1. A is an algebra in the following sense: for each complex number λ and $x \in A$ there is defined an element of A, denoted by λx, subject to the following conditions (λ and μ being any complex numbers, x and y being any elements in A):

a. $\lambda(x+y)=\lambda x+\lambda y$,

b. $(\lambda+\mu)x=\lambda x+\mu x$,

c. $1x=x$,

d. $(\lambda\mu)x=\lambda(\mu x)$,

e. $(\lambda x)(\mu y)=(\lambda\mu)xy$.

2. The underlying linear space of A is a (complex) Banach space with the norm of x ($x \in A$) denoted by $\|x\|$.

3. The inequality $\|xy\| \leqq \|x\| \, \|y\|$ holds for all x, y in A.

The condition 3 on the norm is essentially nothing more than continuity of the product xy. If we were to assume only continuity then a variation of a theorem of Banach [II, p. 67] could be used to prove that there is a constant M such that $\|xy\| \leqq M\|x\| \, \|y\|$. Then changing the norm to a new norm $\|x\|'$ defined by $\|x\|' = M\|x\|$ we would have a norm satisfying 3, and this change of norm would not affect any of our results.

If E and F are any two sets in a B-algebra A then we shall use the notation EF for the set of all elements of the form xy, with $x \in E$ and $y \in F$, and the notation E^n for the set of all products of n elements from E; we shall use the similar notations yE, Ey (for $y \in A$), and so on. We denote the set consisting of the zero element alone by (0), and we use the symbol $[y| \cdots]$ for the set of all y in A which satisfy the condition $\cdots$.

Definition 1.2. An *H^*-algebra* is a B-algebra which satisfies the following further conditions:

1. The underlying Banach space of A is a Hilbert space (of arbitrary dimension).

2. For each $x \in A$ there is an element in A, denoted by x^* and called an *adjoint* of x, such that for all y, z in A we have both $(xy, z)=(y, x^*z)$ and

$(yx, z) = (y, xz^*)$ (here, as throughout this paper, the symbol (x, y) stands for the Hilbert space inner product of x and y).

Condition 2 in this definition says that for each x there is an x^* such that the operators (on A) defined by $y \rightarrow x^*y$ and $y \rightarrow yx^*$ are adjoints (in the ordinary sense of linear transformations on Hilbert space) of the operators $y \rightarrow xy$ and $y \rightarrow yx$, respectively. This means that both regular representations of A are closed under the operation of taking adjoints.

That the adjoint x^* of x need not be unique is shown by the following example: consider any Hilbert space and make it into an algebra by defining the product of each pair of elements to be 0. It is trivial that this is an H^*-algebra in which every element is an adjoint of every element. Theorem 2.2 below shows that every H^*-algebra can be split into two parts, one of which will be of this trivial kind, and in the other of which adjoints will be unique.

The following facts about adjoints in an H^*-algebra are obvious: (1) if x^* is an adjoint of x then x is an adjoint of x^*, (2) if x^* and y^* are adjoints of x and y respectively then $\bar{\lambda}x^* + \bar{\mu}y^*$ is an adjoint of $\lambda x + \mu y$, and y^*x^* is an adjoint of xy, (3) every element of the form xx^* or x^*x is self-adjoint, that is, is an adjoint of itself.

If E is a subset of an H^*-algebra we shall denote by E^* the set of all adjoints of all elements in E, and we call E^* the *adjoint* of E. If $E = E^*$ we call E *self-adjoint*.

Next we consider some examples of H^*-algebras. Our first example is a matrix algebra; since we shall, in general, have little concern about whether our matrices are finite or infinite, countable or uncountable, we formulate the example in a way that allows all possibilities.

Example 1. Let J be an arbitrary set of elements and consider the space of those complex-valued functions $a(i, j)$ defined on $J \times J$ which satisfy the condition $\sum_{ij} |a(i, j)|^2 < \infty$. We make this set into an H^*-algebra by the following definitions: if $a = a(i, j)$, $b = b(i, j)$, and λ is any complex number then

$$
\begin{aligned}
(a + b)(i, j) &= a(i, j) + b(i, j), \\
(ab)(i, j) &= \sum_k a(i, k)b(k, j), \\
(\lambda a)(i, j) &= \lambda a(i, j), \\
(a, b) &= \alpha \sum_{ij} a(i, j)\bar{b}(i, j) \qquad (\alpha \geqq 1), \\
(a^*)(i, j) &= \bar{a}(j, i)(^2).
\end{aligned}
$$

It is easy to verify that with these definitions this set becomes an H^*-algebra. If n is the cardinal number of J then this algebra is called the *full matrix H^*-algebra of order n*, or sometimes simply a *full matrix algebra*.

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(2) We use $\bar{a}(j, i)$ for the complex conjugate of $a(j, i)$.

The α in the preceding definition is any constant greater than or equal to 1. Clearly for any such α we will have an H^*-algebra and since we wish to prove that every simple H^*-algebra is a full matrix algebra we have to allow the possibility of an arbitrary α in this definition.

Example 2. *The set of all complex-valued functions $K(s, t)$ of two real variables which belong to L_2 on the unit square, with the following definitions*:

$$\begin{aligned}(K_1 + K_2)(s, t) &= K_1(s, t) + K_2(s, t),\\ (K_1K_2)(s, t) &= \int K_1(s, p)K_2(p, t)dp,\\ (\lambda K)(s, t) &= \lambda K(s, t),\\ (K_1, K_2) &= \int\int K_1(s, t)\,\overline{K}_2(s, t)dsdt,\\ (K^*)(s, t) &= \overline{K}(t, s).\end{aligned}$$

Examples 1 and 2 are special cases of the general situation where one considers functions of two variables, defining the product and the norm in terms of integration with respect to some measure, but in example 1 (the full matrix algebra) each point has measure $\alpha \geqq 1$, while in example 2 the measure is Lebesgue measure. In fact, these examples are still more alike, for the algebra of example 2 is easily seen, through a consideration of Fourier expansions, to be isomorphic to the full matrix algebra of order $\aleph_0$.

Example 3. *The set of all sequences* (*of any fixed cardinal number*) (a_j) *for which* $\sum_j |a_j|^2 < \infty$, *with the definitions*:

$$\begin{aligned}(a_j) + (b_j) &= (a_j + b_j),\\ (a_j)(b_j) &= (a_jb_j),\\ \lambda(a_j) &= (\lambda a_j),\\ ((a_j), (b_j)) &= \textstyle\sum_j a_j\bar{b}_j,\\ (a_j)^* &= (\bar{a}_j).\end{aligned}$$

This example is really nothing but the subalgebra of all diagonal elements from a full matrix algebra.

Example 4. The *L_2-algebra of a compact group*. Let G be a compact topological group and consider $L_2(G)$ ($L_2(G)$ is the space of complex-valued functions of integrable square with respect to the Haar measure of G) with the definitions ($f(\sigma)$, $g(\sigma)$ being functions in $L_2(G)$):

$$\begin{aligned}(f + g)(\sigma) &= f(\sigma) + g(\sigma),\\ (fg)(\sigma) &= \int f(\sigma\tau^{-1})g(\tau)d\tau,\end{aligned}$$

$$(\lambda f)(\sigma) = \lambda f(\sigma),$$

$$(f, g) = \int f(\sigma)\bar{g}(\sigma)d\sigma,$$

$$(f^*)(\sigma) = \bar{f}(\sigma^{-1}).$$

This example is isomorphic to a subalgebra of a type of H^*-algebra given in example 2. The mapping which takes each $f(\sigma)$ into the two-variable function $F(\sigma, \tau)$ defined by $F(\sigma, \tau) = f(\sigma\tau^{-1})$ can easily be seen to be an isomorphism of this algebra into that counterpart of example 2 which is formed on $L_2(G \times G)$ instead of L_2 of the unit square.

We conclude this section with a few more definitions. All ideals that we shall consider will be closed so we include this property in the definition of an ideal.

DEFINITION 1.3. A *left, right,* or *2-sided ideal* in a B-algebra is a left, right, or 2-sided ideal in the sense of algebra which has the additional property of being closed in the topology given by the norm.

We consider (0) and A to be ideals, and use the term *proper ideal* for all others. If E is any set in A then we refer to the smallest (left, right, or 2-sided) ideal containing E as the (left, right, or 2-sided) ideal *generated* by E.

In finite-dimensional algebra two elements, or sets, are sometimes called orthogonal if their algebraic product is the zero element. In dealing with H^*-algebras we also have the concept of orthogonality in terms of the inner product of the underlying Hilbert space. We mention here that whenever we refer to two elements or sets in an H^*-algebra as orthogonal we shall always mean it in terms of the inner product of the Hilbert space.

If E is a set in an H^*-algebra A then we denote the orthogonal complement of E (that is, the set of all elements which are orthogonal to every element in E) by E^P. A trivial proof shows that the orthogonal complement of a left, right, or 2-sided ideal is again the same kind of ideal. We shall say that a subset of A (or a collection of subsets of A) *spans* A if the smallest closed linear subspace of A which contains the subset (or collection of subsets) is A.

DEFINITION 1.4. Let A be an H^*-algebra and $\{A_\alpha\}$ a family of subalgebras. A is the *direct sum* of the subalgebras A_α if they are mutually orthogonal and span A. We indicate this relation by writing $A = \sum_\alpha A_\alpha$.

From Hilbert space theory we know that if $A = \sum_\alpha A_\alpha$ then each element of A has a unique expansion in terms of components which are in the A_α. It is easy to see that the following conditions are equivalent (in case the subalgebras A_α are closed):

1. Each A_α is a 2-sided ideal in A.
2. For every $x_\alpha \in A_\alpha$ and $x_\beta \in A_\beta$, with $\alpha \neq \beta$, we have $x_\alpha x_\beta = 0$.
3. For every x and y in A (with expansions $x = \sum_\alpha x_\alpha$, $y = \sum_\alpha y_\alpha$) we have $xy = \sum_\alpha x_\alpha y_\alpha$.

2. **Proper H^*-algebras.** Now we define the notion of a proper H^*-algebra. This is an H^*-algebra which contains no elements that annihilate the whole algebra and, as we shall show, is equivalent to the non-existence of nilpotent ideals. We shall show that any H^*-algebra may be decomposed into a proper algebra and an algebra whose square is (0). For this reason (and because we shall later prove that a proper H^*-algebra is a direct sum of minimal ideals) it might seem reasonable to call such an algebra "semi-simple" and to call the component whose square is (0) the "radical" of the H^*-algebra. Because of the trivial nature of these aspects of our theory and because we feel they are not at all suggestive as to what semi-simplicity should be for a general Banach algebra, we shall not use this classical terminology.

LEMMA 2.1. *If x is an element in an H^*-algebra A then $xA=(0)$ is equivalent to $Ax=(0)$.*

Proof. Let y, z be any two elements in A and let x^*, y^*, z^* be any adjoints of x, y, z respectively. Because $xy=0$ we have

$$0 = (xy, z) = (x, zy^*) = (z^*x, y).$$

Since y was arbitrary we conclude that $z^*x=0$ and since z was arbitrary we conclude that $Ax=(0)$.

DEFINITION 2.1. An H^*-algebra is *proper* if it satisfies the following two equivalent conditions:

1. The only x in A such that $xA=(0)$ is $x=0$.
2. The only x in A such that $Ax=(0)$ is $x=0$.

THEOREM 2.1. *An H^*-algebra is proper if and only if every element has a unique adjoint.*

Proof. First suppose A is a proper H^*-algebra, and let x_1^* and x_2^* be adjoints of an element x. Then we have, for all y, z,

$$(xy, z) = (y, x_1^*z) = (y, x_2^*z),$$

hence

$$(y, [x_1^* - x_2^*]z) = 0$$

for all y, z. This implies that $[x_1^*-x_2^*]\,z=0$ for all z, hence $x_1^*-x_2^*=0$, that is, $x_1^*=x_2^*$.

Now suppose A is not proper, so there exists an element $x_0\in A$ such that $x_0\neq 0$ and $x_0A=Ax_0=(0)$. Then if x is any element and x^* any adjoint of x it is trivial that x^*+x_0 is also an adjoint of x.

LEMMA 2.2. *If x is an element in a proper H^*-algebra then $x\neq 0$ implies $xx^*\neq 0$, $x^*x\neq 0$ and $x^*\neq 0$.*

Proof. If $x^*x=0$ we have, for all $y\in A$, $\|xy\|^2=(x^*xy, y)=0$, that is,

$xA = (0)$, hence $x = 0$. Thus $x \neq 0$ implies $x^*x \neq 0$ and this implies $x^* \neq 0$. In the same way we see that $xx^* \neq 0$.

LEMMA 2.3. *If x is a self-adjoint element in a proper H^*-algebra then $x \neq 0$ implies $x^n \neq 0$ for every positive integer n.*

Proof. Lemma 2.2 implies that $x^2 \neq 0$ and, by repeated application, that $x^m \neq 0$ for m any power of 2. Hence $x^n \neq 0$ for all n.

LEMMA 2.4. *If R is a right ideal in a proper H^*-algebra A and $x \in A$ then $xA \subset R$ implies $x \in R$. (The corresponding lemma for left ideals also holds.)*

Proof. Write $x = x_1 + x_2$ where $x_1 \in R$, $x_2 \in R^P$; then $xz = x_1z + x_2z$. We have, for all $z \in A$, $x_2z \in R^P$ and $x_2z = xz - x_1z \in R$, hence $x_2z = 0$. Thus $xz = x_1z$, $(x - x_1)z = 0$, for all z, so $x = x_1 \in R$.

LEMMA 2.5. *Every 2-sided ideal in a proper H^*-algebra is self-adjoint.*

Proof. Let I be the 2-sided ideal in the H^*-algebra A. If $x \in I$ and $y \in I^P$ then $xy = 0$, hence for all $z \in A$ we have $(xy, z) = 0$. Thus $(y, x^*z) = 0$ for all $y \in I^P$ and all $z \in A$, so x^*z is orthogonal to I^P for all $z \in A$. Hence $x^*z \in I$ for all $z \in A$, which, by the preceding lemma, implies that $x^* \in I$.

LEMMA 2.6. *If R is a right ideal in a proper H^*-algebra then the right ideal generated by R^n is R. (The corresponding lemma for left ideals also holds.)*

Proof. Let R_0 be the right ideal generated by R^n and let $x \in R \cap R_0^P$. Since $R_0 \subset R$ it will be sufficient to show that $x = 0$. Consider the element $x(x^*x)^n$; this obviously belongs both to R^n and R_0^P, hence $x(x^*x)^n = 0$. If $x \neq 0$ then $x^*x \neq 0$ by Lemma 2.2 and then $(x^*x)^{n+1} \neq 0$ by Lemma 2.3; since $(x^*x)^{n+1} \neq 0$ implies $x(x^*x)^n \neq 0$ we conclude that $x = 0$.

Now we prove that we can always decompose an H^*-algebra into a proper algebra and an algebra whose square is (0). The component whose square is (0) will be just the set of annihilators of the original algebra.

DEFINITION 2.2. If A is an H^*-algebra then the *trivial ideal* is the set A_0 defined by

$$A_0 = [y \mid Ay = (0)] = [y \mid yA = (0)].$$

Lemma 2.1 shows that the two sets involved in this definition are equal. It is clear that A_0 is a self-adjoint 2-sided ideal, and that $A_0^2 = (0)$.

THEOREM 2.2. *Every H^*-algebra is the direct sum of its trivial ideal and another self-adjoint 2-sided ideal which is proper.*

Proof. Let A be the H^*-algebra, A_0 its trivial ideal, and $A_1 = A_0^P$. Then A_1 is a self-adjoint 2-sided ideal and $A = A_0 + A_1$. It only remains to be shown that A_1 is proper. Suppose $x_1 \in A_1$ and $x_1A_1 = (0)$; we also have $x_1A_0 = (0)$

(because this holds for all $x \in A$) and since $A = A_0 + A_1$ this implies $x_1 A = (0)$. Thus $x_1 \in A_0$; since $x_1 \in A_1 \cap A_0$ we conclude that $x_1 = 0$.

It is now easy to see that an H^*-algebra is proper if and only if it contains no nilpotent 2-sided ideals. Lemma 2.6 shows that if A is proper then it can contain no nilpotent ideals. On the other hand, if A contains no nilpotent 2-sided ideals then in the decomposition of Theorem 2.2 the trivial ideal must be (0), so A is equal to the other component, which is proper.

The next two lemmas are needed for the proof of Theorem 2.3. In these lemmas we denote by E the set of all elements of the form $x_1y_1 + \cdots + x_ny_n$.

LEMMA 2.7. *If A is proper then E is dense in A.*

Proof. E is obviously a linear subspace of A with the property that if $z \in E$ and x is any element of A then zx and xz both belong to E; it is then clear that $\overline{E}$ has the same properties, that is, $\overline{E}$ is a 2-sided ideal. Since $A^2 \subset E \subset \overline{E}$ we conclude from Lemma 2.6 (applied with $R = A$ and $n = 2$) that $\overline{E} = A$, that is, E is dense in A.

LEMMA 2.8. *If A is proper then for any $x \in A$ and $y \in E$ we have $(x, y) = (y^*, x^*)$.*

Proof. First we prove this for $y \in A^2$, that is, for $y = uz$. We have

$$(x, y) = (x, uz) = (u^*x, z) = (u^*, zx^*) = (z^*u^*, x^*) = ((uz)^*, x^*) = (y^*, x^*).$$

The lemma now follows from the linearity of the adjoint and inner product.

THEOREM 2.3. *If A is a proper H^*-algebra then $\|x\| = \|x^*\|$ and consequently the transformation $x \rightarrow x^*$ is continuous.*

Proof. If $y \in E$ then we know by Lemma 2.8 that $\|y\| = \|y^*\|$. Now let x be any element in A and let $\{x_n\}$ be a sequence of elements from E which converges to x (Lemma 2.7 shows the existence of such a sequence). Then $x_n - x_m \in E$ and hence $\|x_n^* - x_m^*\| = \|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. Hence the sequence $\{x_n^*\}$ is convergent to an element x'. We see that $x' = x^*$ because

$$(xy, z) = \lim_n (x_n y, z) = \lim_n (y, x_n^* z) = (y, x'z)$$

and similarly $(yx, z) = (y, zx')$. Hence the sequence $\{x_n^*\}$ converges to x^*. Since $x_n \rightarrow x$, $x_n^* \rightarrow x^*$ and $\|x_n\| = \|x_n^*\|$ (this last by Lemma 2.8) it follows that $\|x\| = \|x^*\|$.

3. **Existence of idempotents.** In this section we prove that every proper H^*-algebra contains a maximal family of primitive self-adjoint idempotents. Then, in the following section, we shall use this family in essentially the same way as in the finite-dimensional case to obtain our desired structure theorems.

In obtaining this maximal family of idempotents our first job is to prove the existence of any self-adjoint idempotents, or even any idempotents at all, and in doing this we have to use other methods than are used for finite-dimensional algebras. This is because our assumptions do not yield in any simple manner the existence of minimal ideals(3). In the finite-dimensional case the existence of idempotents is obtained by taking a minimal left (or right) ideal and proving (easily) that if it is not nilpotent it must contain an idempotent. The greater difficulty we have with this point can best be emphasized by considering example 2 of §1. In this algebra of functions $K(s, t)$ consider, for each real number λ $(0 \leqq \lambda \leqq 1)$, the subalgebra L_λ of all functions $K(s, t)$ which vanish whenever $t > \lambda$. It is easily seen that each L_λ is a left ideal; this shows that an H^*-algebra can contain a continuously decreasing family of left ideals whose intersection is (0) and whose union is the full algebra. Hence we cannot hope to find minimal ideals by an arbitrary continued subdivision process.

We mention (but shall not use) the trivial fact that for H^*-algebras the existence of minimal ideals is equivalent to the existence of maximal ideals, because the orthogonal complement of a maximal (minimal) ideal is minimal (maximal).

Once we know that every H^*-algebra contains a self-adjoint idempotent it is relatively easy to obtain a maximal family of primitive idempotents. We obtain such a family by showing first that any self-adjoint idempotent can be decomposed into a finite number of primitive idempotents and then using Zorn's lemma (or transfinite induction) to obtain a maximal family.

Our procedure for proving the existence of an idempotent is a simplification of the procedure used by F. Riesz [VII] in a proof of the spectral resolution theorem for self-adjoint operators on Hilbert space. The analogy of our situation with that in the spectral theorem is clear for in the spectral theorem one starts with a self-adjoint operator and tries to find certain idempotent operators, while we start with a self-adjoint element in an H^*-algebra and try to find an idempotent in the algebra. In the Riesz proof of the spectral theorem polynomial functions of the given self-adjoint operator are first considered and then, in terms of monotone convergence, the notion of a function of an operator is extended to more general functions. The functions considered in this way are then general enough so that the corresponding operators include the idempotent operators needed in the spectral theorem. In our case the whole procedure is simpler because our main object (at this stage) is to find a single self-adjoint idempotent—we do not try to find a whole family of them and we do not try to relate it (at this stage, again) to given elements in the algebra. For that reason we do not need to consider the general functional calculus that is used in the spectral theory but instead can choose a particular sequence of polynomials (the sequence λ^n) and show that the corresponding

(3) N. Jacobson has pointed out to me that the essential result of this section is the semi-simplicity, in a certain sense, of H^*-algebras, and that certain general theorems of his (now in the process of publication) could be used to good advantage in deriving the results of this section.

elements of the algebra (the elements x^n—with x properly chosen, of course) converge to a self-adjoint idempotent. We do this however through a consideration of the monotone properties of the sequences λ^n and x^n—à la Riesz. That we can obtain an idempotent through consideration of a single (self-adjoint) element x and limits of polynomials in it is to be expected since if every H^*-algebra is to contain an idempotent then the subalgebra generated by x (which will again be an H^*-algebra, provided x is self-adjoint) must contain an idempotent.

We need the following lemma which is well known as a theorem about linear transformations on Hilbert space. For the sake of completeness we include a proof.

LEMMA 3.1. *If x is any self-adjoint element in an H^*-algebra then*

$$\sup_{\|y\|=1} |(xy, y)| = \sup_{\|y\|=1} \|xy\|.$$

Proof (4). By the Schwarz inequality we have (for all x and y) $|(xy, y)| \leqq \|xy\| \, \|y\|$ which shows immediately that

$$\sup_{\|y\|=1} |(xy, y)| \leqq \sup_{\|y\|=1} \|xy\|.$$

Now we prove the opposite inequality. We note first that if y is such that $xy=0$ then $(xy, y)=\|xy\|=0$ so we only need consider y's for which $xy\neq 0$. For any y and z we have

$$\begin{aligned} 4(xy, z) &= (x[y+z], [y+z]) - (x[y-z], [y-z]) \\ &\leqq \{\sup |(xu, u)|\}\{\|y+z\|^2 + \|y-z\|^2\} \\ &= 2\{\sup |(xu, u)|\}\{\|y\|^2 + \|z\|^2\} \end{aligned}$$

(where the $\sup|(xu, u)|$ is taken over all u for which $\|u\|=1$). Taking $z=xy/\|xy\|$ we then have, for all y with $\|y\|=1$ and $xy\neq 0$,

$$\|xy\| \leqq \sup |(xu, u)|$$

which proves the lemma.

DEFINITION 3.1. An *idempotent* in an H^*-algebra is an element e such that $e^2=e\neq 0$. If an idempotent is self-adjoint we call it an *sa-idempotent.*

THEOREM 3.1. *Every proper H^*-algebra contains an sa-idempotent.*

Proof. We first prove the following fact which will be used in the proof (this fact is also well known as a theorem about linear transformations on Hilbert space):

(α) If y is a self-adjoint element which satisfies the conditions:

(i) $0 \leqq (yu, u) \leqq (u, u)$ for all u,

(4) Taken from F. Riesz [VII].

(ii) $\sup (yu, u)=1$ (this sup being taken over all u with $\|u\|=1$), then y^2 also satisfies conditions (i) and (ii). (Actually (i) implies that y is self-adjoint.) Because y is self-adjoint we have $(y^2u, u)=(yu, yu)\geqq 0$ for all u. Now we prove that y^2 satisfies (ii), and this obviously implies that y^2 satisfies the second inequality in condition (i). By the preceding lemma we have

$$\sup_{\|u\|=1} \|yu\| = \sup_{\|u\|=1} (yu, u) = 1.$$

Hence

$$\sup (y^2u, u) = \sup (yu, yu) = \sup \|yu\|^2 = 1$$

(these sups being taken over all u with $\|u\|=1$). This proves (α).

Now we choose an element x with the following properties:

a. $x=y^2$ for some self-adjoint y,

b. $0\leqq (xu, u) \leqq (u, u)$ for all u,

c. $\sup_{\|u\|=1} (xu, u)=1$.

It is clear that a and c together imply b. To prove we can find such an x we first choose any $z\neq 0$ and consider the element z^*z. By Lemma 2.2 we know that $z^*z\neq 0$ and we also know not only that z^*z is self-adjoint but that $(z^*zu, u)=(zu, zu)\geqq 0$ for all u. Multiplying z^*z by a suitable positive number it is clear that we obtain a self-adjoint element y such that $(yu, u)\geqq 0$ and $\sup (yu, u)=1$ (this sup being taken over all u with $\|u\|=1$). We define x by $x=y^2$ and will now prove that this x has the required properties. By definition it satisfies a, and since y has properties b and c it follows from (α) that x also has properties b and c.

Now we prove that if x is any element with properties a, b, and c, then the sequence x^n converges to an *sa*-idempotent, as $n\to\infty$. Here we follow Riesz [VII] very closely. We show first that

(β) For each u the sequence (x^nu, u) is a decreasing sequence of non-negative numbers.

The numbers (x^nu, u) are non-negative because $(x^nu, u)=(y^{2n}u, u)$ $=(y^nu, y^nu)\geqq 0$. The sequence is decreasing because

$$(x^{n+1}u, u) - (x^nu, u) = (xy^nu, y^nu) - (y^nu, y^nu) \leqq 0$$

(by c), that is, $(x^{n+1}u, u)\leqq (x^nu, u)$.

Now we use (β) to prove that the sequence x^n is convergent; first we prove that for every u the sequence x^nu is convergent. By (β) we have, for $m\leqq n$,

$$(x^{2m}u, u) \geqq (x^{m+n}u, u) \geqq (x^{2n}u, u)$$

so as $m, n\to\infty$ these three terms tend to the same limit. Because of

$$\begin{aligned}\|x^mu - x^nu\| &= (x^mu - x^nu, x^mu - x^nu)\\ &= (x^{2m}u, u) - 2(x^{m+n}u, u) + (x^{2n}u, u)\end{aligned}$$

we see that $\|x^mu-x^nu\|\to 0$ as $m, n\to\infty$, thus showing that the sequence x^nu

converges for every u. Taking $u=x$ it follows that the sequence x^n itself converges.

Denote the limit of the sequence x^n by e; then e is self-adjoint because each x^n is self-adjoint. Because $x^{2n}\to e^2$ we have $e^2=e$ and it only remains to be shown that $e\neq 0$. By repeated application of (α) we see that (for m a power of 2) sup $(x^m u, u)=1$, hence (by Lemma 3.1) sup $\|x^m u\|=1$ (these sups being taken for u such that $\|u\|=1$). Since $\|x^m u\| \leqq \|x^m\| \, \|u\| = \|x^m\|$ we have $\|x^m\| \geqq 1$. This implies $e\neq 0$.

Definition 3.2. Two idempotents, e and f, are *doubly orthogonal* if $(e,f)=0$ and $ef=fe=0$.

Definition 3.3. An idempotent is *primitive* if it can not be expressed as the sum of two doubly orthogonal idempotents.

Definition 3.4. A *minimal* left (right, or 2-sided) ideal is an ideal not equal to (0) which contains no left (right, or 2-sided) ideal other than itself and (0).

Our aim now is to prove the existence of a maximal family of doubly orthogonal primitive *sa*-idempotents. For this purpose we need the following lemmas about decompositions of ideals and decomposition of idempotents.

Lemma 3.2. *Let A be a proper H^*-algebra, e an idempotent, and R the right ideal defined by $R=eA$. If R is the direct sum of a finite number of right ideals,*

$$R = R_1 + R_2 + \cdots + R_n,$$

and if we write

$$e = e_1 + e_2 + \cdots + e_n \qquad (e_i \in R_i)$$

then the e_i are doubly orthogonal idempotents and $R_i=e_iA$. If e is an sa-idempotent then each e_i is an sa-idempotent. (The corresponding lemma for left ideals also holds.)

Proof. Because R is the direct sum of the R_i we know that $(e_i, e_j)=0$ for $i\neq j$. We have

$$\text{(a)} \qquad e_i = ee_i = e_1e_i + \cdots + e_ne_i$$

and because the R_j are right ideals we know that $e_je_i\in R_j$. Because of this and because the R_j are orthogonal we must have (since the e_je_i add up to an element in R_i) $e_je_i=0$ for $i\neq j$. It is then clear from (a) that $e_i=(e_i)^2$. Thus we have proved that the e_i are doubly orthogonal idempotents. It follows trivially that $R_i=e_iA$.

Now suppose that e is an *sa*-idempotent. We have

$$e_1 + \cdots + e_n = e = e^2 = (e_1 + \cdots + e_n)e = e_1e + \cdots + e_ne$$

and because $e_1+\cdots+e_n$ and $e_1e+\cdots+e_ne$ are both expansions of e in terms of components in the R_i we conclude (from the uniqueness of such an expansion) that $e_i=e_ie$. Using this and (a) we know that $e_i=e_ie=ee_i$, and taking adjoints we have $e_i^* = ee_i^* = e_i^* e$. Because the R_i are orthogonal we have $(e_i y, e_j z) = 0$ for all y, z (provided $i \neq j$), hence $(e_j^* e_i y, z) = 0$ for all y, z, and hence $e_j^* e_i = 0$ for $i \neq j$. Using this and the fact that $e_i^* = e_i^* e$ we find $e_i^* = e_i^*(e_1 + \cdots + e_n) = e_i^* e_i$. Taking adjoints we have $e_i = e_i^* e_i$, hence $e_i = e_i^*$.

Lemma 3.3. *Let A be a proper H^*-algebra, e an sa-idempotent, and R the right ideal defined by $R = eA$. If e can be expressed as a finite sum of doubly orthogonal sa-idempotents,*

$$e = e_1 + \cdots + e_n$$

and if we define R_i by $R_i = e_i A$ then R is the direct sum of the right ideals R_i. (The corresponding lemma for left ideals also holds.)

Proof. The only thing we need prove is that the R_i are orthogonal, but this is clear because $e_i e_j = 0$ implies $(e_i e_j y, z) = 0$ for all y, z, and hence $(e_i y, e_j z) = 0$ for all y, z.

Lemma 3.4. *If R is a right ideal (in a proper H^*-algebra A) of the form $R = eA$, where e is an idempotent, then R is minimal if and only if e is primitive. (The corresponding lemma for left ideals also holds.)*

Proof. If R is not minimal then R can be split into two orthogonal proper right ideals, $R = R_1 + R_2$. Then by Lemma 3.2 it follows that there exist doubly orthogonal idempotents e_1 and e_2 such that $e = e_1 + e_2$.

If, on the other hand, e is not primitive we write $e = e_1 + e_2$ where e_1 and e_2 are doubly orthogonal idempotents. Then we define $R_1 = e_1 A$ and we shall show that $R_1 \subset R$ but $R_1 \neq (0)$ and $R_1 \neq R$. We have $ee_1 = (e_1 + e_2)e_1 = (e_1)^2 = e_1$ which shows that $R_1 \subset R$, and that $e_2 \in R$. Since $e_1 \in R_1$ we see that $R_1 \neq (0)$ and because $e_1 e_2 = 0$ we see that e_2 is not in R_1, hence $R_1 \neq R$.

Theorem 3.2. *If e is an sa-idempotent in an H^*-algebra then e is the sum of a finite number of doubly orthogonal primitive sa-idempotents.*

Proof. Consider the right ideal $R = eA$. If R is minimal then (by the preceding lemma) e itself is primitive. If not we find a proper right ideal R_1 which is properly included in R, and write $R = R_1 + R_1^p$. We continue this process, at each stage splitting each summand (which is not minimal) into orthogonal right ideals. Then at each stage we have a decomposition of R into orthogonal right ideals, $R = R_1 + \cdots + R_n$. For each such decomposition we have, by Lemma 3.2, a decomposition of e into doubly orthogonal *sa*-idempotents, $e = e_1 + \cdots + e_n$. Since in any B-algebra an idempotent must have norm greater than or equal to 1 (because of $\|e\| = \|e^2\| \leq \|e\|^2$) we have

$$\|e\|^2 = \|e_1\|^2 + \cdots + \|e_n\|^2 \geqq n.$$

This shows that our subdivision process on the right ideals must end at some stage. Hence we have R the direct sum of a finite number of minimal right ideals and e the sum of a finite number of doubly orthogonal primitive *sa*-idempotents.

THEOREM 3.3. *Every proper H^*-algebra contains a (nonvacuous) maximal family of doubly orthogonal primitive sa-idempotents.*

Proof. We know, by Theorem 3.1 and Theorem 3.2, that every proper H^*-algebra contains a primitive *sa*-idempotent. A simple application of Zorn's lemma (or transfinite induction) then shows the existence of a maximal family of doubly orthogonal primitive *sa*-idempotents.

4. **Structure theorems.** In this section we use the idempotents obtained in the last section to prove our structure theorems. We operate with these idempotents in essentially the same way as is done in the finite-dimensional case. We use the known fact that the only B-algebra which is a field is the algebra of complex numbers.

The following theorem says that every proper H^*-algebra is a direct sum of minimal left ideals and of minimal right ideals, but also says slightly more because it shows that the same idempotents can be used in both the left and right decomposition.

THEOREM 4.1. *Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive sa-idempotents in a proper H^*-algebra A. Then*

$$A = \sum_i e_iA = \sum_i Ae_i,$$

that is, A is the direct sum of the minimal left ideals Ae_i and A is a direct sum of the minimal right ideals e_iA.

Proof. Let $A_1 = \sum_i e_iA$ and $A_2 = A_1^P$; we shall show that $A = \sum_i e_iA$ by showing that $A_2 = (0)$. Suppose that A_2 contains an element $y \neq 0$ and we shall obtain a contradiction. If $y \in A_2$ then $x = yy^*$ also belongs to A_2 (because A_2 is a right ideal), is not equal to 0 (by Lemma 2.2) and is self-adjoint.

Now we consider that (closed) subalgebra A' of A generated by x and we use this subalgebra to prove that A_2 contains an *sa*-idempotent. We know that $x \in A_2$ and we find that every x^n is in A_2 because, for any y in A_1, we have $(y, x^n) = (yx^{n-1}, x) = 0$ (using the fact that A_1 is a right ideal). It follows that $A' \subset A_2$. Since A' is again an H^*-algebra it must contain an *sa*-idempotent; hence we know that A_2 contains an *sa*-idempotent, that we shall denote by f.

Applying Theorem 3.2 to f we can decompose it into a finite number of doubly orthogonal primitive *sa*-idempotents, $f = f_1 + \cdots + f_n$, and then applying Lemma 3.3 we see that the f_i are in A_2. Thus we have found at least one primitive *sa*-idempotent f_1 in A_2.

To obtain the desired contradiction we only need prove that f_1 is doubly orthogonal to every e_i. Since $f_1 \in A_2$ we have $(e_iy, f_1z) = 0$ for all y, z. Taking $y = e_i$ and $z = f_1$ we see that $(e_i, f_1) = 0$. Rewriting this in the form $(f_1e_iy, z) = (y, e_if_1z) = 0$ (for all y, z) we conclude that $f_1e_i = e_if_1 = 0$.

DEFINITION 4.1. An H^*-algebra is *simple* if it contains no proper 2-sided ideals.

THEOREM 4.2. *Every proper H^*-algebra A is a direct sum of simple H^*-algebras, each of which is a minimal 2-sided ideal in A.*

Proof. We first express A as a direct sum of minimal right ideals $R_i = e_iA$. Then for each R_i we consider the 2-sided ideal $I(R_i)$ generated by R_i, and we denote by $\mathfrak{J}$ the family of all these 2-sided ideals. We shall show that the members of $\mathfrak{J}$ are the simple algebras demanded by the theorem.

First we prove that $I(R_i)$ is minimal, by contradiction. If this is false then we can split $I(R_i)$ into two orthogonal 2-sided proper ideals, $I(R_i) = I_1 + I_2$. Since R_i is minimal we must have $I_k \cap R_i = (0)$ or R_i, and since $I(R_i)$ is the ideal generated by R_i we must then have $I_k \cap R_i = (0)$ (for $k = 1, 2$). Now let r be any element of R_i and write $r = y + z$, with $y \in I_1$ and $z \in I_2$. Then $r = e_i r = e_i y + e_i z$. Since $e_i y$, $e_i z$ belong to R_i they must be 0, hence $r = 0$. This shows that $R_i = (0)$, which is a contradiction.

Next we show that A is the direct sum of the ideals in I. To do this it is sufficient to show that each $I(R_i)$ is itself a direct sum of right ideals chosen from the family $\{R_k\}$. To show this latter it is clearly sufficient to show that if $R_j \cap I(R_i) \neq (0)$ then $R_j \subset I(R_i)$. But this is trivially true, by the minimality of the R_i.

To complete the proof we need to show that each $I(R_i)$ is itself a simple H^*-algebra. Since a 2-sided ideal is self-adjoint, $I(R_i)$ is an H^*-algebra. Now we show it is simple, by showing that any 2-sided ideal J in the algebra $I(R_i)$ is a 2-sided ideal in A. For if $y \in J$ and $x \in A$ then $x = x_1 + x_2$, with $x_1 \in I(R_i)$, $x_2 \in I(R_i)$, and then $xy = (x_1 + x_2)y = x_2 y \in J$, and similarly $yx \in J$. (Here we are using the fact that if I_1 and I_2 are orthogonal 2-sided ideals then any product xy, with $x \in I_1$ and $y \in I_2$, is 0.)

Gelfand [III] has proved that the only field which is a Banach algebra is the field of complex numbers. We shall want the following analogue, which differs in that "field" is replaced by "division algebra," "Banach algebra" is replaced by "H^*-algebra," and the adjoint is asserted to be the ordinary complex conjugate. These differences are quite trivial but nevertheless we give a complete proof. Since we have not so far considered whether our algebras had a unit we have made no restriction in this lemma to make the unit of the complex numbers have norm 1, and this accounts for the arbitrary constant, α, that appears. If we were to insist that the unit of the complex numbers have norm 1 some later statements would become more complicated.

LEMMA 4.1. *If A is an H^*-algebra which is a division algebra then A is the algebra of complex numbers with complex conjugate for adjoint. The norm need not be the ordinary absolute value but the norm of the unit may be any number $\alpha \geqq 1$; then the norm of any complex number λ will be $|\lambda|\alpha$ (where $|\lambda|$ = absolute value of λ).*

Proof. Let e be the unit of A and x any element of A. Applying Gelfand's theorem ([III, p. 8]—that the only field which is a Banach algebra is the field of complex numbers) to the closed subfield of A generated by e and x we conclude that $x=\lambda e$, where λ is a complex number. Hence every x in A is of this form, so A is the algebra of complex numbers. Because $(\lambda e)^*=\bar{\lambda}e^*=\bar{\lambda}e$ we see that the adjoint is the ordinary complex conjugate, and we also have, if $\alpha=\|e\|$, that $\|\lambda e\|=|\lambda|\,\|e\|=|\lambda|\alpha$.

In the following proof we follow Albert [I, p. 29] in outline but with certain modifications which are necessary both because our algebras are infinite-dimensional, and because we must make sure that our adjoint operation and norm turn out to be the desired ones.

THEOREM 4.3. *Every simple H^*-algebra is a full matrix H^*-algebra.*

Proof. Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive *sa*-idempotents. By Theorem 4.1 we know that $A=\sum_i e_iA=\sum_i Ae_i$. We define subsets, A_{ij}, of A by $A_{ij}=e_iAe_j$, and we break the first part of the proof into a number of parts, concerned with these sets A_{ij}.

(i) Each A_{ii} is a division algebra.

If a, $b\in A_{ii}$ and $a\neq 0$ we show first that there is a $y\in A$ such that $ay=b$. Consider the right ideal aA. We have $aA\neq(0)$ because $aA\ni ae_i=a\neq 0$, and we have $aA\subset e_iA$. From the minimality of e_iA we conclude that $aA=e_iA$, and hence there exists a y in A such that $ay=b$.

Now we prove that if $a\neq 0$ and a, $b\in A_{ii}$ then there exists a $z\in A_{ii}$ such that $az=b$. Consider the $y\in A$ for which $ay=b$ and define $z=e_iye_i$; then $az=a(e_iye_i)=(ae_i)ye_i=aye_i=be_i=b$.

In the same way we find a solution to the equation $za=b$. Hence A_{ii} is a division algebra.

(ii) Every element in A_{ii} is of the form λe_i for some complex number λ.

A_{ii} is obviously closed in A and not equal to 0; because the e_i are self-adjoint A_{ii} is itself an H^*-algebra. Since it is a division algebra it must be just the algebra of complex numbers with the idempotent e_i for the unit, that is, every element is of the form λe_i.

(iii) Each A_{ij} is not equal to (0).

Suppose that $A_{ij}=(0)$, that is, $e_iAe_j=(0)$. This would mean that the right annihilator of the right ideal e_iA contains $e_j\neq 0$ and hence that that right annihilator is not equal to (0). Since this right annihilator is a 2-sided ideal and A is simple this implies that $e_iAA=(0)$. But this is a contradiction since $(e_i)^3\neq 0$.

(iv) $A_{ij}A_{jk}=A_{ik}$; $A_{ij}A_{kl}=(0)$ if $j\neq k$.

It is trivial that $A_{ij}A_{kl}=(0)$ for $j\neq k$, since $e_je_k=0$ if $j\neq k$. We have $A_{ij}A_{jk}=(e_iAe_j)(e_jAe_k)=(e_iA)(e_jAe_k)=(0)$ then, as in the proof of (iii) we would have $e_iAA=(0)$, which would be a contradiction. We have $e_iAe_jA=e_iA$

since e_iAe_jA is a right ideal, not equal to (0), included in e_iA. Hence $e_iAe_jAe_k = e_iAe_k$, that is, $A_{ij}A_{jk} = A_{ik}$.

(v) Each A_{ij} is a 1-dimensional linear subspace of A.

We see, as above, that there is a $z \in A$ such that $ze_jA = e_iA$; then $ze_jAe_j = e_iAe_j$. Thus we have a linear transformation, $y \to zy$, of e_jAe_j onto e_iAe_j. Since e_jAe_j is 1-dimensional and we already know that $A_{ij} \neq (0)$, this implies that e_iAe_j is 1-dimensional.

(vi) $A_{ij}^* = A_{ji}$.

Since the e_i are self-adjoint we have $(e_iye_j)^* = e_jy^*e_i$, which obviously implies (vi).

(vii) We can find elements $f_{1j} \in A_{1j}$ and $f_{j1} \in A_{j1}$, for $j \neq 1$, such that $f_{1j}f_{j1} = e_1$.

Let f_{1j}, g_{j1} be nonzero elements in A_{1j}, A_{j1} respectively ($j \neq 1$). Since A_{1j} and A_{j1} are 1-dimensional and $A_{1j}A_{j1} = A_{11} \neq (0)$ we must have $f_{1j}g_{j1} \neq 0$. Then $f_{1j}g_{j1} = \lambda e_1$ ($\lambda \neq 0$) for some complex number λ. Defining $f_{j1} = (1/\lambda)g_{j1}$ we have the required elements f_{1j} and f_{j1}.

Now we shall prove the existence of a set of "matrix units" e_{ij} with the following properties

$$\begin{aligned} e_{ij} &\in A_{ij}, & e_{ii} &= e_i, \\ e_{ij}e_{jk} &= e_{ik}, & e_{ij}e_{kl} &= 0 \text{ if } j \neq k, \\ e_{ij}^* &= e_{ji}, & (e_{ij}, e_{kl}) &= 0 \text{ unless } i = k \text{ and } j = l, \\ \|e_{ij}\| &= \|e_{kl}\| \text{ for all } i, j, k, l. \end{aligned}$$

We define $e_{ii} = e_i$. Next we define the e_{1j} and e_{j1}. Consider the elements f_{1j} and f_{j1} obtained in (vii) (for $j \neq 1$). Then $f_{1j}^* = \lambda f_{j1}$ for some complex number λ. We have

$$(f_{1j}, f_{1j}) = (f_{1j}, e_1 f_{1j}) = (f_{1j}f_{1j}^*, e_1) = \lambda(f_{1j}f_{j1}, e_1) = \lambda(e_1, e_1)$$

which shows, since both (f_{1j}, f_{1j}) and (e_1, e_1) are positive, that λ is real and positive. Now we define e_{1j} and e_{j1} by

$$e_{1j} = \lambda^{-1/2} f_{1j}, \qquad e_{j1} = e_{1j}^*.$$

Then we have

$$e_{1j}e_{j1} = \lambda^{-1/2} f_{1j}(\lambda^{-1/2} f_{j1})^* = \lambda^{-1} f_{1j} f_{1j}^* = \lambda^{-1} f_{1j} \lambda f_{j1} = e_{11}.$$

Thus far we have defined e_{ii}, e_{i1}, and e_{1i} for all i. Now we define e_{ij} for all i, j by

$$e_{ij} = e_{i1}e_{1j}.$$

Since the e_{ii} given by this definition (by taking $i = j$) is obviously idempotent, and in A_{ii}, and since we know that every element in A_{ii} is of the form λe_i, it is clear that this coincides with our old definition of $e_{ii}=e_i$, and it is also clear that $e_{ij}\in A_{ij}$, $e_{ij}e_{jk}=e_{ik}$, $e_{ij}e_{kl}=0$ for $j\neq k$. We also have $e_{ij}^*=e_{ji}$ because $e_{ij}^*=(e_{i1}e_{1j})^*=e_{1j}^*e_{i1}^*=e_{j1}e_{1i}=e_{ji}$.

Because the right ideals $R_i=e_iA$ are orthogonal, and so are the left ideals $L_i=Ae_i$, it is clear that $(e_{ij}, e_{kl})=0$ unless $i=k$ and $j=l$. We also note that all the e_{ij} have the same norm because

$$\begin{aligned} \|e_{ij}\|^2 &= (e_{ij}, e_{ij}) \\ &= (e_{ik}e_{kj}, e_{ij}) \\ &= (e_{ik}, e_{ij}e_{kj}^*) \\ &= (e_{ik}, e_{ij}e_{jk}) \\ &= (e_{ik}, e_{ik}) \\ &= \|e_{ik}\|^2, \end{aligned} \qquad \begin{aligned} \|e_{ij}\|^2 &= (e_{ij}, e_{ij}) \\ &= (e_{ij}, e_{ik}e_{kj}) \\ &= (e_{ik}^*e_{ij}, e_{kj}) \\ &= (e_{ki}e_{ij}, e_{kj}) \\ &= (e_{kj}, e_{kj}) \\ &= \|e_{kj}\|^2. \end{aligned}$$

Now we are ready to complete the proof of Theorem 4.3. We define the number α by $\alpha=\|e_{ij}\|^2$ (here using the fact that all the e_{ij} have the same norm). Since the A_{ij} span A (this comes from Theorem 3.2) and are 1-dimensional we see that the e_{ij} span A. Hence each $y\in A$ can be expressed uniquely in the form

$$y = \sum_{ij} a_{ij}e_{ij}$$

where the a_{ij} are complex numbers. Thus we make each $y\in A$ correspond to a matrix (a_{ij}) of complex numbers, and it is clear that distinct elements of A correspond to distinct matrices. Because the e_{ij} are orthogonal and all have the same norm $\alpha^{1/2}$ it is clear that the matrices we obtain in this way are precisely all matrices (a_{ij}) for which $\sum_{ij}|a_{ij}|^2<\infty$, and that $\|y\|^2=\alpha\sum_{ij}|a_{ij}|^2$. Since our adjoint operation is conjugate linear and continuous it is clear that if y corresponds to (a_{ij}) then y^* corresponds to $(\bar{a}_{ji})$. And finally, because the e_{ij} multiply like matrix units it is clear that multiplication in A corresponds to matrix multiplication. Thus Theorem 4.3 is proved.

When we want to refer to the algebra of complex numbers as an H^*-algebra as in Lemma 4.1 we shall call it the "complex number H^*-algebra."

COROLLARY 4.1. *Every proper abelian H^*-algebra is the direct sum of complex number fields.*

Proof. This is an immediate consequence of Theorem 4.2 and Theorem 4.3 since the only abelian full matrix algebra is the algebra of complex numbers.

This corollary says that the only abelian proper H^*-algebras are sequence algebras of the type of example 3, §1, but where the sequences considered are those sequences (a_j) such that $\sum_j|a_j|^2\alpha_j<\infty$ (with the α_j a fixed sequence of numbers greater than or equal to 1) and with the norm of the sequence (a_j) defined to be $\sum_j|a_j|^2\alpha_j$.

5. **The L_2-algebra of a compact group.** We have mentioned previously that Segal [VIII] has defined the group algebra of a locally compact group to be the space L_1 (of complex-valued functions which are integrable with respect to the Haar measure of the group) with convolution for multiplication; we shall call this B-algebra the L_1-algebra of the group. We have also mentioned that for a compact group the space L_2 is also an algebra (with convolution for multiplication) and is not only a B-algebra but a proper H^*-algebra (see §1). In this section we consider the structure of this kind of an H^*-algebra, and show that it is a direct sum of finite-dimensional full matrix algebras. This result was obtained after reading Segal's paper in manuscript form and should only be considered as a variation of his theorem on the structure of the L_1-algebra of a compact group. Nevertheless, because of the relative ease with which we can determine the structure of this L_2-algebra and also the simpler form that the structure theorem takes for the L_2-algebra we shall consider it independently of Segal's results.

As in [IX] we shall call a Banach algebra A *completely continuous* if for every $x \in A$ the operator T defined by $Ty = xy$ is completely continuous. Then the L_2-algebra is completely continuous; this is proved in [IX] for the L_1-algebra and is essentially proved for the L_2-algebra in every proof of the Peter-Weyl theorem [XI] hence we shall not give a proof of it here. We shall state the theorems in this section for a completely continuous proper H^*-algebra and, from the remarks just made, this will include the case of the L_2-algebra of a compact group.

We shall base the theorems of this section as little as possible on the theorems of previous sections and make use of the spectral theorem for completely continuous operators instead. We do this for two reasons. In the first place it is possible to give here a proof which parallels the proof of the Peter-Weyl theorem but which is in purely algebraic terms, and this we consider interesting. In the second place we feel that the proof given here, because of its lack of emphasis on idempotents and its use of the spectral theorem, is possibly more suggestive of techniques for use in general group algebras. We shall however make use of Theorem 4.4 for a finite-dimensional proper H^*-algebra. We could, if we preferred, refer to the Wedderburn theorem instead of using this theorem, but then we would have to prove that the full matrix algebra given by the Wedderburn theorem would have the same norm and *-operation as those demanded in our definition of a full matrix H^*-algebra. Since this last point involves a little trouble and since Theorem 4.4 in the finite-dimensional case is even more elementary than the Wedderburn theorem (because in the finite-dimensional case it is trivial to find the necessary idempotents) we have decided to assume this result.

We remark here that the structure theorems for the L_1 and L_2 algebras bear the same relation to these algebras that the Peter-Weyl theorem bears to the family of representations of the group. In the case of the L_2-algebra, when the structure theorem is translated back to the representation theory it includes both the Peter-Weyl theorem and the orthogonality relations.

Now we recall the spectral theorem for completely continuous self-adjoint operators on Hilbert space. It asserts that if T is such an operator then there exists a countable sequence $\{\lambda_i\}$ of real numbers, all not equal to 0, and a countable sequence $\{M_i\}$ of mutually orthogonal closed linear subspaces with the following properties: (1) the sequence $\{\lambda_i\}$ is bounded and 0 is its only limit point, (2) all elements y of M_i satisfy the equation $Ty=\lambda_i y$, (3) each M_i has finite dimension, (4) the subspaces M_i together with the closed linear subspace M_0 of all y for which $Ty=0$ span the Hilbert space, (5) an operator on the Hilbert space commutes with T if and only if it commutes with the projection operator of each M_i.

The following theorem contains our application of the spectral theorem.

THEOREM 5.1. *Let A be a completely continuous proper H^*-algebra and let x be any self-adjoint element in A. Then A can be expressed as a direct sum of right ideals,*

$$A = R_0 + R_1 + \cdots + R_n + \cdots$$

where each R_i, for $i\neq 0$, is finite-dimensional. If $x\neq 0$ then there is at least one $i\neq 0$ for which $R_i\neq(0)$. Each $y\in R_0$ satisfies the equation $xy=0$ and each $y\in R_i$ satisfies the equation $xy=\lambda_i y$ ($\lambda_i\neq 0$).

Proof. We apply the spectral theorem to the completely continuous self-adjoint operator T defined by $Ty=xy$. We obtain a sequence of numbers and a sequence of closed linear subspaces of A, $\{R_i\}$ (we now call them R_i instead of M_i), with the properties mentioned above. If $R_i=(0)$ for all $i\neq 0$ then we have $xA=(0)$, and hence $x=0$. Thus if $x\neq 0$ there must be at least one $i\neq 0$ for which $R_i\neq(0)$.

It only remains to show that the R_i are right ideals. Since every operator T_z defined by $T_z y=yz$ commutes with T it follows from the spectral theorem that every T_z commutes with that projection operator which projects A onto R_i. This means exactly that if $y\in R_i$ and z is any element of A then $yz\in R_i$, that is, R_i is a right ideal.

Since we want to show that a completely continuous H^*-algebra is a direct sum of finite-dimensional full matrix algebras we need a way of finding 2-sided ideals in A, and also of knowing that they are finite-dimensional. The following theorem is helpful for this purpose because it shows how to use finite-dimensional right ideals (whose existence we already have, by the previous theorem) to obtain finite-dimensional 2-sided ideals. It will then be easy to break these finite dimensional 2-sided ideals down into finite-dimensional minimal 2-sided ideals. It is interesting to note here that, because of our special Hilbert space conditions on our algebra, the orthogonal complement of the 2-sided ideal I plays the role that would otherwise be played by the quotient algebra A/I. In fact it is easy to see that A/I is again an H^*-algebra and as such is isomorphic to I^P.

THEOREM 5.2. *If R is a right ideal in a proper H^*-algebra and if $I=[y \mid Ry=(0)]$ then I and I^P are 2-sided ideals. If R is finite-dimensional then so is I^P. If $R\neq(0)$ then $I^P\neq(0)$.*

Proof. Obviously I, and hence I^P, is a 2-sided ideal. Now we consider the mapping which assigns to each $z\in I^P$ the linear transformation of R into R defined by $y\rightarrow yz$ $(y\in R)$. This mapping is obviously linear and since the space of linear transformations of R into itself is finite-dimensional it will be sufficient, in proving I^P finite-dimensional, to prove that distinct elements of I^P go into distinct linear transformations on R. If z_1, z_2 belong to I^P and go into the same linear transformation then we have $yz_1=yz_2$ for all $y\in R$. This means that $R(z_1-z_2)=(0)$, hence $z_1-z_2\in I$. Since also $z_1-z_2\in I^P$ we conclude that $z_1-z_2=0$, $z_1=z_2$. Thus I^P is finite-dimensional.

If $R\neq(0)$ then we must have $I^P\neq(0)$ since otherwise we should have $I=A$ and $RA=(0)$; this latter contradicts our assumption that A is proper.

LEMMA 5.1. *A finite-dimensional proper H^*-algebra is a direct sum of minimal 2-sided ideals.*

Proof. Choose a minimal 2-sided ideal I_1 and write $A=I_1+I_1^P$. If $I_1^P=(0)$ or is minimal then the theorem is proved. If not we choose a minimal 2-sided ideal I_2 in I_1^P and write $A=I_1+I_2+(I_1+I_2)^P$. Continuing this process a finite number of times we arrive at the desired decomposition.

THEOREM 5.3. *A completely continuous H^*-algebra A is a direct sum of full matrix H^*-algebras, each of which is a 2-sided ideal in A.*

Proof. Consider the family F_1 of all minimal finite-dimensional 2-sided ideals in A (for the moment this family may be vacuous). Clearly any two members of this family either coincide or have only 0 in common. Since a finite-dimensional simple proper H^*-algebra is a full matrix algebra our theorem will be proved if we can show that the ideals belonging to F_1 span A.

Now consider the family F_2 of all finite-dimensional 2-sided ideals in A (for the moment this family may be vacuous). Since every finite-dimensional proper H^*-algebra is a direct sum of minimal 2-sided ideals it will be sufficient to prove that the ideals belonging to the family F_2 span A. To prove this it is sufficient to prove that for any $y\neq0$ in A there is a finite-dimensional 2-sided ideal to which y is not orthogonal.

Now let y be any nonzero element in A and consider the self-adjoint x defined by $x=yy^*$. By Lemma 2.2 and Lemma 2.3 we know that $x^2y\neq0$. Consider the decomposition of A into right ideals obtained by applying Theorem 5.1 with this x,

$$A = R_0 + R_1 + \cdots + R_n + \cdots .$$

We define the 2-sided ideals I_i by

$$I_i = [y \mid R_i y = (0)]$$

and we know of their orthogonal complements I_i^P that for $i \neq 0$ they are finite-dimensional and that at least one I_i^P, with $i \neq 0$, not equal to (0). Hence it will be sufficient to prove that y can not be orthogonal to all the I_i^P ($i \neq 0$), or equivalently, that y can not belong to all the ideals I_i ($i \neq 0$).

Let the decomposition of x into components in the R_i be

$$x = x_0 + x_1 + \cdots + x_n + \cdots .$$

Then if y were in all the I_i ($i \neq 0$) we would have

$$x^2 y = x(x_0 + x_1 + \cdots + x_n + \cdots) y = x x_0 y = 0.$$

This contradicts the fact that $x^2 y \neq 0$. Hence y can not belong to all the I_i ($i \neq 0$) and the theorem is proved.

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