APPLICATION OF ANALYTIC FUNCTIONS TO TWO-DIMENSIONAL BIHARMONIC ANALYSIS

BY
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1. **Introduction.** As is well known, the repeated Laplace equation

$$\nabla^4 H = 0, \qquad \nabla^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2 \tag{1.1}$$

occurs in several branches of applied mathematics. One of its applications is in elasticity in connection with *bending of plates*. Here (1.1) is satisfied by the normal deflection of the middle plane of a bent, thin, uniform, initially plane plate, in regions *free* from *normal load*(1). If proper displacements and slopes are applied to the boundary, then (1.1) has to be solved subject to the boundary conditions

$$H = \alpha, \qquad \partial H/\partial n = \beta \tag{1.2}$$

where $\partial H/\partial n$ is the normal derivative, and α, β are prescribed functions over the boundary. For a plate which is *loaded* with a *normal pressure* p, the normal deflection satisfies the nonhomogeneous equation

$$\nabla^4 H = p/B, \qquad B = E\delta^3/12(1-\sigma^2) \tag{1.3}$$

where B is the "bending stiffness" of the plate, E is Young's modulus, δ the plate thickness, and σ Poisson's ratio. In solving (1.3) subject, say, to boundary conditions of "built-in edges"

$$H = 0, \qquad \partial H/\partial n = 0, \tag{1.4}$$

if a particular solution H_0, not necessarily fulfilling (1.4), can be found, then the problem may be reduced to finding a "complementary solution" satisfying (1.1) and proper boundary conditions of the form (1.2). In particular, for uniform pressure p, solutions of (1.3) are given by

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(1) See for instance Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge University Press, 1927, chap. 22, §§313, 314; or else Geiger and Scheel, *Handbuch der Physik*, vol. 6, 1st ed., Geckeler, chap. 3, part 7. These references will be cited in the following as "Love," "*Handbuch*," respectively.

In the derivation of (1.1) and (1.3) below, assumptions for the normal deflection analogous to that of the simple beam theory are made, namely that the deflection is largely due to the curvatures induced by the moments about the x, y axes, existing in the plate material, and that the shearing stresses X_z, Y_z, while in static equilibrium with the normal loads and the bending moments, produce negligible deflections.

$$(1.5) \qquad H = [(p/B)x^4/4!, \ (p/B)r^4/64], \qquad r^2 = x^2 + y^2,$$

and the reduction of the loaded clamped edge problem to (1.1), (1.2) may always be carried out. If known tractions (shears and moments) are applied over the boundary, the boundary conditions are more complicated and involve second and third order derivatives(2).

Another application of (1.1) in elasticity is in connection with *stretching of thin plates*. To each such state of stress in a plate there corresponds a solution H of this equation, known as "Airy's function," unique except for an additive first degree polynomial $P=Ax+By+C$, and such that the components of stress are given in terms of H by means of

$$(1.6) \qquad X_x = \partial^2 H/\partial y^2, \qquad X_y = -\partial^2 H/\partial x \partial y, \qquad Y_y = \partial^2 H/\partial x^2;$$

here X_x, X_y, Y_y are the average values of the respective stress components throughout the plate thickness. If given tractions are applied over the boundary, equations (1.6) yield boundary conditions involving second-order derivatives; however, by integration of the traction across any curve C from P_1 to P_2 one is led to

$$(1.7) \qquad \partial H/\partial x\Big]_{P_1}^{P_2} = -Y, \qquad \partial H/\partial y\Big]_{P_1}^{P_2} = X,$$

where the bracket is defined by

$$(1.8) \qquad \phi\Big]_{P_1}^{P_2} = \phi(P_2) - \phi(P_1),$$

and X, Y are the resultant traction components across C(3). From (1.7) one may derive once more boundary conditions of the form (1.2).

The displacement components u, v in the x, y-directions under plane stress may also be determined in terms of H, and as follows. Let

$$(1.9) \qquad s = \nabla^2 H.$$

By virtue of (1.1), $\nabla^2 s = 0$, that is, s is harmonic. There exists consequently a conjugate harmonic function t, such that $s+it$ is an analytic function of $x+iy$. Let

$$(1.10) \qquad S + iT = \int (s + it)d(x + iy)$$

(2) The boundary conditions now become (see Love, *Handbuch*, loc. cit.) $M_B = -B[\nabla^2 w + (1-\sigma)\partial^2 w/\partial n^2]$, $S + \partial M_T/\partial s = -B[\partial(\nabla^2 w)/\partial n + (1-\sigma)(\partial/\partial s)(\partial^2 w/\partial n \partial t)]$, where M_B = applied bending moment over boundary, M_T = applied twisting moment over boundary, S = applied shear moment over boundary, and where n, t are rectangular co-ordinates along the tangent to the boundary and the outer normal at the point in question oriented the same way as the positive x and y directions, and s is the arc length along the bounding curve.

(3) If C is traversed from P_1 toward P_2, equations (1.7) hold with X, Y representing the components of traction exerted by the region to the right of C upon the region to its left.

be the integral of this analytic function, then

$$(1.11) \qquad \begin{aligned}(u, v) = (1/2G)\big[(S, T)/(1 + \sigma) - (\partial/\partial x, \partial/\partial y)H\big] \\ + \text{ a rigid displacement;}\end{aligned}$$

here u, v are the average values of u, v throughout the plate thickness and $G=E/2(1+\sigma)$ is the rigidity modulus. For the proof of these relations the reader must be referred to Love(4) or the *Handbuch*(5). Their derivation is based upon the assumptions that normal stress component Z_z vanishes everywhere, while the shears X_z, Y_z vanish over the faces of the plate. The latter conditions are satisfied if the faces are free from traction, the former holds very nearly if the thickness is not large. The resulting state of stress is known as "generalized plane stress."

Airy's function also applies in case of "plane strain" in which u, v are independent of z, while the normal displacement w vanishes or at least $\partial w/\partial z$ is everywhere constant. The only modification required in applying (1.6)–(1.11) is that $1/(1+\sigma)$ in (1.11) be replaced by $1-\sigma$(6).

A further application of (1.1) occurs in the slow flow of viscous incompressible fluids. Here (1.1) is satisfied by the flow function (sometimes referred to as flux function or stream function) in terms of which the velocity components u, v are given by (7)

$$(1.12) \qquad u = -\,\partial H/\partial y, \qquad v = \partial H/\partial x.$$

Corresponding to prescribed motion along the boundaries, one is led to assign values of $\partial H/\partial x$, $\partial H/\partial y$.

It is clear that the boundary conditions (1.2) are essentially equivalent to the boundary condition.

$$(1.13) \qquad \partial H/\partial x = \gamma, \qquad \partial H/\partial y = \delta,$$

where γ, δ are proper functions along the boundary. The special case of (1.13) where γ, δ are constants is of great physical interest. For bent plates this corresponds to a *rigid displacement* of the boundary; where the deflection is caused by displacing different parts of the boundary relative to each other, each part being displaced as a rigid body, the right-hand constants in (1.13) need not have the same constant values for each part. In the case of Airy's function the conditions (1.13) with constant γ, δ correspond to a boundary *free from stress*. For the motion of a viscous fluid, constant γ, δ in (1.13) correspond to a boundary moving as a rigid body and possessing translational motion. In all these cases (of constant γ, δ), by subtracting a proper first

(4) Love, loc. cit., pp. 204–208.

(5) Geiger and Scheel, loc. cit., vol. 6, pp. 109–113.

(6) Love, loc. cit., pp. 204–208. *Handbuch*, vol. 6, pp. 109–113.

(7) H. Lamb, *Hydrodynamics*, 5th ed., §342, p. 580. By equating the body forces to zero in equation (3) of p. 580, the biharmonic equation results for the flow function.

degree polynomial P from H, (1.13) may be reduced to

$$\partial H/\partial x = 0, \qquad \partial H/\partial y = 0 \tag{1.14}$$

and hence also to (1.4). When (1.13) applies over a composite boundary but with possibly different constants γ, δ over each point, the polynomials P which reduce (1.13) to (1.14) will in general differ for each part of the boundary.

In a sense (1.14) is the analogue of the condition of constancy of harmonic functions, frequently encountered in potential theory.

2. **Solutions of (1.1) in terms of harmonic and analytic functions.** It may be shown by repeated application of Green's theorem that solutions of (1.1) in a real domain in the (x, y)-plane are *analytic* in x and y(8). Change variables from x, y to the complex variables

$$z = x + iy, \qquad \bar{z} = x - iy, \qquad i^2 = -1, \tag{2.1}$$

whence

$$x = (z + \bar{z})/2 = R(z), \qquad y = (z - \bar{z})/2i = I(z), \tag{2.2}$$

R, I denoting respectively "the real part of," "the imaginary part of" (more precisely "the coefficient of i in the imaginary part of"). It is found from the formal rules for transforming derivatives that

$$\partial/\partial x = \partial/\partial z + \partial/\partial \bar{z}, \qquad \partial/\partial y = i(\partial/\partial z - \partial/\partial \bar{z}) \tag{2.3}$$

whence the Laplacian becomes

$$\nabla^2 = 4\partial^2/\partial z \partial \bar{z} \tag{2.4}$$

while the repeated Laplace equation (1.1) takes the form

$$\partial^4 H/\partial z^2 \partial \bar{z}^2 = 0. \tag{2.5}$$

Now H is originally considered for real x and y only, hence only for values of z, $\bar{z}$ that are conjugate imaginaries of each other; however, the analyticity of H allows one to extend H to complex x, y close to the real values, and hence to values of z, $\bar{z}$ which vary independently. One may therefore consider the partials $\partial/\partial z$, $\partial/\partial \bar{z}$ as bona fide partial derivatives, and show that (1.1) and (2.5) apply for all z, $\bar{z}$.

Integration of (2.5) yields

$$2H = f(z) + f_1(\bar{z}) + \bar{z}g(z) + zg_1(\bar{z}), \tag{2.6}$$

where f, f_1, g, g_1 are arbitrary analytic functions of their respective complex arguments. From (2.6) follows on applying (2.4)

(8) See, for instance, the author's paper: *Green's formulas for analytic functions*, Ann. of Math. vol. 39 (1938) pp. 46–48.

$$\nabla^2 H = 2[g'(z) + g_1'(\bar{z})]. \tag{2.7}$$

If it is assumed that

$$g_1(\bar{z}) = \overline{g(z)}, \qquad f_1(\bar{z}) = \overline{f(z)}, \tag{2.8}$$

then (2.6) takes on the form

$$2H = f(z) + \overline{f(z)} + \bar{z}g(z) + z\overline{g(z)} \tag{2.9}$$

or simply

$$H = R[f(z) + \bar{z}g(z)]; \tag{2.10}$$

at the same time (2.7) becomes

$$\nabla^2 H = 2[g'(z) + \overline{g'(z)}] = 4R[g'(z)]. \tag{2.11}$$

It will be shown that (2.6) and hence (2.10), (2.11) may be assumed to apply without loss of generality in the *real case*, that is, when H is real for real x, y.

Indeed, considering the real case, subtract the real quantity

$$2[g'(z) + \overline{g'(z)}]$$

from $\nabla^2 H$ as given by (2.7); the result, namely

$$g_1'(\bar{z}) - \overline{g'(z)}, \tag{2.12}$$

is real. However, (2.12) is also analytic in $\bar{z}$. Now an analytic function of a complex variable, which is real for all values of the latter, must reduce to a (real) constant. Absorbing this constant into g' one is led to the vanishing of (2.12), whence

$$g_1'(\bar{z}) = \overline{g'(z)}, \tag{2.13}$$

and integrating and absorbing the constant of integration into g, one obtains the first relation (2.8). It now follows from (2.6) that

$$f'(z) + f_1(\bar{z})$$

is real. From this the second relation (2.8) is established in a similar fashion by subtracting the real quantity

$$2[f'(z) + \overline{f'(z)}]$$

and integrating.

Unless the contrary is explicitly stated, the real case will be assumed throughout the following.

We shall adopt the notation

$$f(z) = F(x, y) + iF'(x, y), \quad F = R[f(z)], \quad F' = I[f(z)], \tag{2.14}$$

and similarly for $g(z)$, G, G'.

The following alternative forms may be used in place of (2.9), (2.10) for representing the general (real) solution of (1.1)([9]):

$$H = R[f(z) + z\bar{z}g(z)] = R(f + r^2 g) = F + r^2 G, \tag{2.15}$$

$$H = R[f(z) + (z + \bar{z})g(z)/2] = R(f + xg) = F + xG, \tag{2.16}$$

$$H = R[f(z) + (z - \bar{z})g(z)/2i] = R(f + yg) = F + yG, \tag{2.17}$$

$$\begin{aligned} H &= R[f(z) + (\bar{z}/z)g(z)] = R[f(z) + e^{-2i\theta}g(z)] \\ &= F + \cos 2\theta G + \sin 2\theta G'; \end{aligned} \tag{2.18}$$

here r, θ are polar co-ordinates in the $[z=(x+iy)]$-plane. Obviously for the same function H, the analytic functions f, g, and the harmonic functions F, G are not the same in the various representations; their mutual relations are readily found, however. Thus

$$g\,|_{(2.10)} = zg\,|_{(2.15)} = (g/2)\,|_{(2.16)} = (ig/2)|_{(2.17)} = (g/z)\,|_{(2.18)}, \tag{2.19}$$

$$f\,|_{(2.10)} = f\,|_{(2.15)} = f + zg/2\,|_{(2.16)} = f + zg/2i\,|_{(2.17)} = f\,|_{(2.18)}, \tag{2.20}$$

where the notation is self-explanatory.

We shall now express various boundary conditions for formulas of §1 in terms of z, $\bar{z}$, using the form (2.9) for H.

The boundary conditions (1.13) specifying $\partial H/\partial x$, $\partial H/\partial y$ yield

$$\partial H/\partial x - i\partial H/\partial y = 2\partial H/\partial z = \gamma - i\delta \tag{2.21}$$

and (using the form (2.9) for H)

$$2\partial H/\partial z = f'(z) + \bar{z}g'(z) + \overline{g(z)} = (\gamma - i\delta). \tag{2.22}$$

Conversely, (2.22) or its "conjugate equation"

$$2\partial H/\partial \bar{z} = \overline{f'(z)} + \overline{zg'(z)} + g(z) = (\gamma + i\delta) \tag{2.23}$$

is equivalent to (1.13)([10]). The boundary conditions (1.14) therefore reduce to

$$\begin{aligned} 2\partial H/\partial \bar{z} &= \overline{f'(z)} + \overline{zg'(z)} + g(z) = 0, \\ 2\partial H/\partial z &= f'(z) + \bar{z}g'(z) + \overline{g(z)} = 0. \end{aligned} \tag{2.24}$$

Equations (1.7) may be replaced by

$$\begin{aligned} X + iY &= \partial H/\partial y - i\partial H/\partial x]_{P_1}^{P_2} = -2i\partial H/\partial \bar{z}]_{P_1}^{P_2} \\ &= (-i)[\overline{f'(z)} + g(z) + \overline{zg'(z)}]_{P_1}^{P_2} \end{aligned} \tag{2.25}$$

or by its conjugate equation

([9]) The form (2.9) is probably due to Goursat; the forms given on the right hand of (2.15), (2.16) are due to Almansi, Annali di Mathematica (3) vol. 2 (1899) §3.

([10]) Since the real case is understood, γ and δ are real. In the contrary case (2.6) is used in place of (2.9) and $\partial H/\partial \bar{z}$ is equated to $(\bar{\gamma}+i\bar{\delta})/2$.

$$X - iY = 2i\partial H/\partial z\big]_{P_1}^{P_2} = i\big[f'(z) + \overline{g(z)} + \bar{z}g'(z)\big]_{P_1}^{P_2}. \tag{2.26}$$

Using the form (2.9), (2.10) for H and recalling (2.11), equations (1.9), (1.10), (1.11) for plane stress can be transformed respectively into

$$s = \nabla^2 H = 4R[g'(z)], \tag{2.27}$$

$$S + iT = 4g(z), \tag{2.28}$$

$$\begin{aligned} 2G[u - iv] &= 4\overline{g(z)}/(1 + \sigma) - 2\partial H/\partial z \\ &= 4\overline{g(z)}/(1 + \sigma) - [f'(z) + \bar{z}g'(z) + \overline{g(z)}] \\ &= \bar{g}(3 - \sigma)/(1 + \sigma) - f' - \bar{z}g', \end{aligned} \tag{2.29}$$

while for the case of plane strain

$$2G[u - iv] = (3 - 4\sigma)\bar{g} - f' - \bar{z}g' \tag{2.30}$$

takes the place of (2.29). These expressions are to be credited to N. Muschelisvili(11); they were also obtained by the author independently in 1931(12). Boundary conditions corresponding to given displacements at the boundary now lead to prescribed values of

$$\nu\bar{g} - f' - \bar{z}g' = 2G(u - iv) \tag{2.31}$$

over the boundary, where

$$\begin{aligned} \nu &= (3 - \sigma)/(1 + \sigma) \quad \text{for plane stress,} \\ \nu &= (3 - 4\sigma) \qquad\qquad \text{for plane strain.} \end{aligned} \tag{2.32}$$

3. **The boundary conditions $H = \partial H/\partial n = 0$ over a straight line.** We prove the following theorem:

Let H be biharmonic and let it satisfy the boundary conditions

$$H = \partial H/\partial x = 0 \quad along \quad x = 0. \tag{3.1}$$

Represent H in the form

$$H = R[f(z) + xg(z)]. \tag{2.16}$$

Then the following two functional equations hold:

$$f(z) + \bar{f}(-z) = 0, \tag{3.2}$$

$$k(z) + \bar{k}(-z) = 0, \qquad k(z) = g(z) + f'(z). \tag{3.3}$$

Conversely, if (2.16), (3.2), (3.3) *hold, then* (3.1) *is satisfied.*

The notation in (3.2), (3.3) is explained as follows: As stated in §2, the

(11) Zeitschrift für angewandte mathematik und mechanik vol. 13 (1933) pp. 264–265.

(12) See the author's paper entitled *Thermal stresses in cylindrical pipes*, Philosophical Magazine (7) vol. 24 (1937) p. 209.

conjugate of an analytic function of z, say of $f(z)$, is analytic in $\bar{z}$; denote it by $\bar{f}(\bar{z})$:

$$\overline{f(z)} = \bar{f}(\bar{z}). \tag{3.4}$$

The latter function $\bar{f}$ may, of course, be considered for the argument z as well as for $-z$, resulting in $\bar{f}(z)$, $\bar{f}(-z)$ respectively. More explicitly, if for any constant a

$$f(z) = \sum f_n(z - a)^n \tag{3.5}$$

is the expansion of $f(z)$ near $z=a$ in integral powers of $z-a$, then

$$\overline{f(z)} = \sum \bar{f}_n(\bar{z} - \bar{a})^n$$

and $\bar{f}(z)$ is given by

$$\bar{f}(z) = \sum \bar{f}_n(z - \bar{a})^n \tag{3.6}$$

and its analytic continuations. The function $\bar{f}$ will be said to be "conjugate to" the function f. It is of interest to note that if $f(z)$ is broken up into its real and imaginary parts as in (2.14), then

$$\overline{f(z)} = \bar{f}(\bar{z}) = F(x, y) - iF(x, y), \tag{3.7}$$

$$\bar{f}(z) = F(x, -y) - iF(x, -y), \tag{3.8}$$

$$\bar{f}(-z) = F(-x, y) - iF(-x, y), \tag{3.9}$$

so that while $\bar{f}(z)$ involves reflection in the axis of reals along with change of sign of i, $\bar{f}(-z)$ involves reflection in the imaginary axis along with a similar change of sign.

To prove (3.2) put $H=0$ for $x=0$. There results from (2.9)

$$0 = R[f(z)] = [f(z) + \overline{f(z)}]/2 \qquad \text{for } x = 0. \tag{3.10}$$

Replacing

$$\overline{f(z)}$$

by $\bar{f}(\bar{z})$ and noting from (2.2) that for $x=0$, $\bar{z}=-z$, there follows

$$f(z) + \bar{f}(-z) = 0.$$

This has been proved for pure imaginary z. However, an analytic function of z cannot vanish at an infinite number of points of the z-plane having a finite limit point in its region of analyticity without vanishing identically. Hence (3.2) holds for *all* z.

To establish (3.3) now apply $\partial/\partial x$ to (2.16) and put $\partial H/\partial x=0$ for $x=0$. Noting that R, $\partial/\partial x$ are permutable, and since $\partial f(z)/\partial x = df(z)/dz = f'(z)$ one obtains

$$0 = R[f'(z) + g(z)] = R[k(z)] \qquad \text{for } x = 0$$

whence (3.3) follows in a similar manner.

In terms of F, G, the results obtained appear even simpler. The boundary conditions (3.1) lead from (2.16) to

$$(3.11) \qquad F = 0, \qquad F_x + G = 0 \qquad \text{for } x = 0,$$

where $F_x = \partial F/\partial x$, and the functional equations (3.2), (3.3) simply amount to

$$(3.12) \qquad F^* = -F,$$

$$(3.13) \qquad K^* = -K, \qquad K = F_x + G,$$

where stars denote reflection about $x=0$, that is substitution of $-x$ for x:

$$(3.14) \qquad F^*(x, y) = F(-x, y).$$

Equations (3.12), (3.13) thus state that the harmonic functions F, K are odd about $x=0$.

Conversely, let (3.2), (3.3) hold. Putting $x=0$ in (3.2) or its equivalent (3.12) leads to the first equation (3.11); (3.3) or (3.13) similarly leads to the second equation (3.11). Hence (3.1) holds.

So far H has been assumed biharmonic and hence analytic in a region *enclosing* $x=0$. It will be shown now that if the boundary conditions (3.1) hold, then analytic continuation across $x=0$ is always possible.

Let R be a region abutting on $x=0$ along a segment S. Let H be biharmonic in R including the boundary points of R which comprise S and suppose that along S, H satisfies (3.1). It will be shown that H may always be continued analytically across S to the region R^* which is the mirror image of R in $x=0$ by means of (3.2), (3.3) or their equivalent equations (3.12), (3.13), and so continued H will be analytic in $R+R^*$ inclusive of the points of S.

Indeed from the first equation (3.11), if one recalls a familiar theorem of Schwartz, follows that F may be continued analytically to R^* by means of negative reflection in $x=0$, that is, into the function $F(x, y) = -F^*(x, y) = -F(-x, y)$; in other words, by means of (3.12). Similarly from the second equation (3.11) follows that $K = F_x + G$ may be similarly continued analytically across S into R^* by means of negative reflection. Hence $G = K - F_x$, too, is analytically continuable into R^*. Similar conclusions apply to the analytic functions f, g, k of which F, G, K form the real parts.

If H satisfies (3.1) along two different intervals S_1, S_2 of $x=0$ (for instance, to each side of an isolated singular point P lying on $x=0$), then the branch of H obtained by analytic continuation into R^* across S_1 need not, of course, be the same as the branch obtained by continuation across S_2.

If H satisfies (3.1) and if P is isolated singular point of H lying in R, then H is also singular at P^*, the mirror image of P in $x=0$; otherwise analytic continuation from R^* into R would make P a point at which H is analytic. Moreover, the nature of the singularity at P^* is completely determined by the singularity at P. Indeed, let H_1, H_2 be two biharmonic functions, both satisfying (3.1) and possessing the same singularity at P so that H_1-H_2 is analytic at P. Then H_1, H_2 will possess the same singularity at P^*, because H_1-H_2 must also be analytic at P^*. The precise aspect of the singularity at P^* corresponding to specific singularities at P will be considered below and in the next section.

A simple yet general method of obtaining harmonic functions F which are odd in x is by putting

$$F = F_1 - F_1^* = F_1(x, y) - F_1(-x, y), \tag{3.15}$$

where F_1 is an arbitrary harmonic function. The corresponding solution of (3.2) is given by

$$f(z) = f_1(z) - \bar{f}_1(-z) \tag{3.16}$$

where f_1 is an arbitrary analytic function. This will now be applied to obtain biharmonic functions H satisfying (3.1) and having prescribed singularities in $x>0$. This problem is encountered in the following section in the determination of proper Green's function in $x>0$.

Let

$$H_1 = R[f_1(z) + xg_1(z)] = F_1 + xG_1 \tag{3.17}$$

be given, and suppose that f_1, g_1 possess proper singularities in $x>0$ but suppose that they are analytic in $x\leq 0$; the problem is to find

$$H_2 = R[f_2(z) + xg_2(z)] = F_2 + xG_2 \tag{3.18}$$

where f_2, g_2 are analytic in $x>0$ and such that

$$H = H_1 + H_2 = R[f(z) + xg(z)] = F + xG, \quad f = f_1 + f_2, \quad g = g_1 + g_2, \tag{3.19}$$

satisfies (3.1). As shown above, (3.1) may be replaced by (3.2), (3.3) or by (3.11).

Applying (3.16) toward the solution of (3.2) we put

$$f_2(z) = -\bar{f}_1(-z). \tag{3.20}$$

The function $k(z)$ occurring in (3.3) now becomes([13])

$$k(z) = f_1'(z) + \bar{f}_1'(-z) + g_1(z) + g_2(z) \tag{3.21}$$

and g_2 must be chosen in it so that (3.3) is satisfied. Note that if this were done again by means of (3.16) by identifying the sum of the first three terms

([13]) The term $\bar{f}_1'(-z)$ is to be interpreted by first differentiating the function $\bar{f}_1(z)$, then putting $-z$ in place of z and not as the result of first putting $-z$ for the argument of $\bar{f}_1$, then differentiating. A similar notation is used in (3.22), (3.23).

on the right-hand side of (3.21) with $f_1(z)$, then the resulting g_2 would admit singularities in $x>0$, introduced by the reflection of the function $\bar{f}_1'(-z)$. To avoid this we put

$$g_2(z) = -2\bar{f}_1'(-z) - \bar{g}_1(-z) \tag{3.22}$$

and are thus led to

$$k(z) = f_1'(z) - \bar{f}_1'(-z) + g_1(z) - \bar{g}_1(-z). \tag{3.23}$$

It will be observed that both the first two terms on the right-hand side of (3.23) as well as the last two terms agree in form with the right-hand side of (3.16); hence (3.3) is satisfied. Summarizing, the equations (3.20), (3.23) furnish a solution of the problem in question. In terms of the harmonics F_1, G_1, the result may be stated as follows:

$$F_2 = -F_1^*, \tag{3.24}$$

$$G_2 = -[2F_{1x} + G_1]^*, \tag{3.25}$$

where, as in (3.14), a star indicates reflection in $x=0$.

Denote the linear functional operation leading from H_1 to H_2 by L:

$$H_2 = L(H_1), \tag{3.26}$$

this operation being given more explicitly by

$$\begin{aligned} L(H_1) &= R\{-\bar{f}_1(-z) - x[2\bar{f}_1'(-z) + \bar{g}_1(-z)]\} \\ &= (-F_1 + 2xF_{1x} + xG_1)^* \end{aligned} \tag{3.27}$$

provided that H_1 is given by (3.17). An alternative form for L is given by

$$L(H_1) = (-H_1 + 2xH_{1x} - x^2\nabla^2 H_1)^*; \tag{3.28}$$

this has the advantage of operating on H_1 as a whole (rather than on its component parts F_1, G_1). To prove this write (3.27) as follows:

$$L(H_1) = [-(F_1 + xG_1) + 2x(F_{1x} + G_1)]^* = [-H_1 + 2x(F_{1x} + G_1)]^* \tag{3.29}$$

and utilize the relations

$$H_{1x} = F_{1x} + G_1 + xG_{1x}, \qquad \nabla^2 H_1 = 2G_{1x}$$

which follow from (3.17) by differentiation.

Suppose it turns out that

$$H_2 \equiv L(H_1) = H_1. \tag{3.30}$$

Then $H = H_1 + H_2 = 2H_1$ and H_1 satisfies (3.1) directly. Conversely, if H_1 satisfies (3.1), then (3.30) holds. To prove this, applying (3.12), (3.13) to F_1, G_1, there results

$$F_1^* = -F_1 \tag{3.31}$$

$$(F_{1x} + G_1)^* = -(F_{1x} + G_1). \tag{3.32}$$

Differentiation of (3.31) yields

$$(F_{1x})^* = F_{1x} \tag{3.33}$$

and hence from (3.32)

$$(G_1)^* = -(2F_{1x} + G_1). \tag{3.34}$$

Multiplying (3.34) by x and subtracting from (3.31)

$$F_1^* - xG_1^* = (F_1 + xG_1)^* = (F_{1x} + 2xF_{1x} + xG_1)$$

whence applying the star operation,

$$(F_1 + xG_1) = (F_{1x} + 2xF_{1x} + xG_1)^*; \tag{3.35}$$

a glance at (3.27) reveals that (3.25) is none other than (3.30). It has thus been shown that (3.30) is both a necessary and a sufficient condition for the function H_1 to satisfy the boundary conditions (3.1).

In common with ordinary reflection L has the property that its square is the identity:

$$L^2(H) = H \tag{3.36}$$

where H is an arbitrary biharmonic function. Indeed, as previously shown, for an arbitrary biharmonic H,

$$H + L(H) \tag{3.37}$$

is biharmonic and satisfies (3.1). Therefore this function must also satisfy (3.30). Hence

$$L[H + L(H)] = H + L(H)$$

whence (3.36) follows.

4. Green's function for the half-plane. We now apply the results of §3 to the determination of the Green's function Γ for the problem of a bent plate for a half plane, corresponding to the boundary conditions (1.4). Corresponding to the differential equation (1.1) and the boundary conditions (1.4), the Green's function for a region R is defined by the following: In R, Γ satisfies (1.1) with the exception of one point, the pole of Γ; at the boundary, Γ satisfies (1.4); at the pole Γ is singular after the manner of

$$H_1 = r_1^2 \ln r_1, \tag{4.1}$$

where r_1 is the distance from the pole, that is, $\Gamma = H_1 +$ an analytic biharmonic function.

The physical significance of the Green's function lies in the fact that it represents the deflection of the plate corresponding to a normal concentrated point load at the pole[14], while the boundary is "built-in" or clamped. To show that Γ is the proper analogue of the familiar Green's function γ of potential theory, corresponding, say, to the boundary condition of vanishing potential, recall that γ for a region R is harmonic everywhere inside R with the exception of a pole, vanishes at the boundary of R, and becomes infinite at the pole after the manner of $\ln r_1$. Physically γ represents the potential due to a concentrated charge at the pole, while the boundary is held at zero potential. A well known formula of Green expresses the solution of the Dirichlet problem for a region R at any point P in terms of its boundary values and γ. By repeated application of Green's theorem one proves similarly that the solution of (1.1) over a region R satisfying the boundary conditions (1.2) is given by

$$H(P) = (1/8\pi)\int_B [\alpha(s)\partial(\nabla^2\Gamma)/\partial n - \beta(s)\nabla^2\Gamma]ds \tag{4.2}$$

where the pole of Γ is at P and the integrand is evaluated over the boundary B, over which the integration is carried out[15].

We shall choose the half plane $x>0$ and put the pole at $z=b>0$ on the real positive axis. The singularity of Γ in $x>0$ at $z=b$ is given by (4.1) which is biharmonic except at $z=b$. Since

$$\begin{aligned} H_1 = r_1^2 \ln r_1 = |z-b|^2 \ln |z-b| &= R[(\bar{z}-b)(z-b)\ln(b-z) \\ &= R[\bar{z}_1 z_1 \ln z_1], \end{aligned} \tag{4.3}$$

where $z_1=z-b$, there results upon expressing H_1 in the form (2.9)

$$g_1(z) = 2(z-b)\ln(b-z), \qquad f_1(z) = (b^2-z^2)\ln(b-z). \tag{4.4}$$

The determination of H_2 may now be carried out by means of (3.20) and (3.22). It turns out to be less laborious, however, to adopt the alternative procedure of using (3.26), (3.28). From

$$H_{1x} = (2r_1 \ln r_1 + r_1)\partial r_1/\partial x = (2\ln r_1 + 1)(x-b), \tag{4.5}$$

$$\nabla^2 H_1 = 4R[\partial^2(z_1\bar{z}_1 \ln z_1)/\partial z_1 \partial\bar{z}_1] = 4R(\ln z_1 + 1) = 4(\ln r_1 + 1), \tag{4.6}$$

there results

$$H_2 = (-r_1^2 \ln r_1 - 4xb\ln r_1 - 2xb - 2x^2)^*. \tag{4.7}$$

The last term may be omitted without affecting the values of either H_2 or H_{2x} at $x=0$. Hence we put

[14] By using (1.3) for a distributed load, then concentrating the load at a point, it can be shown that (4.1) represents the deflection in an infinite plate due to a load $L=B/8\pi$.

[15] More exactly, one proves in this way that if a biharmonic function satisfying (1.2) exists, it must be given by (4.2). We avoid the question of existence theorems of these solutions here. They are similar to the existence theorems of the Dirichlet problem and no doubt may be treated by similar methods (say by variational methods and by means of integral equations).

$$H_2 = (-r_1^2 \ln r_1 - 4xb \ln r_1 - 2xb)^* = -r_2^2 \ln r_2 + 4xb \ln r_2 + 2xb, \tag{4.8}$$

whence

$$\Gamma = H_1 + H_2 = r_1^2 \ln r_1 - r_2^2 \ln r_2 + 4xb \ln r_2 + 2xb. \tag{4.9}$$

Note that since $r_2^2 = r_1^2 + 4xb$, (4.9) may be replaced by

$$\Gamma = r_1^2(\ln r_1 - \ln r_2) + 2xb. \tag{4.10}$$

From (4.8), (4.5), (4.6) follows

$$H_2 = [-H_1 - 2bH_{1x} - b^2\nabla^2 H_1 + 2b^2]^*. \tag{4.11}$$

Now $-H_1^*$ is the negative image of H_1 obtained by replacing x by $-x$ and changing signs, and might be described as the deflection due to a force $-L$ placed at $z=-b$, where L is the applied force due to H_1 at $z=b$. The singularity of H_{1x} is also at $z=-b$ and might be described as due to a concentrated *moment* (about the y-axis) placed at $z=-b$; this singularity may further be visualized as the limit as δ approaches zero of forces L/δ, $-L/\delta$ placed at $z=-b$ and $z=-b+\delta$. The singularities of $\nabla^2 H_1$ are of even higher order. $\nabla^2 H_1$ may be visualized as the limiting deflection due to a force $-4L/\delta^2$ at $(x=-b,\ y=0)$ and four forces each equal to L/δ^2 and placed at $(x, y) = (b \pm \delta, \pm \delta)$, as δ approaches zero, or again, $\nabla^2 \overline{H}_1$ may be described as the deflection due to a proper bending moment applied around the boundary of a circular hole $r_2=$constant.

It is clear that biharmonic functions satisfying (3.1) and possessing point singularities at $z=b$ corresponding to various x, y-derivatives of $r_1^2 \ln r_1$ may be obtained by a similar method. However, they may also be obtained by proper differentiation of Γ with respect to either b or y. Since $\partial r_1/\partial b = -\partial r_1/\partial x$, $\partial r_2/\partial b = \partial r_2/\partial x$, it follows that the b-derivatives may be expressed in terms of x-derivatives. In particular, if H is singular at $z=b$ after the manner of

$$H_1 = \partial(r_1^2 \ln r_1)/\partial y \tag{4.12}$$

then (4.11) can still be applied for finding H_2 except for the omission of the constant term $2b^2$. However, if

$$H_1 = \partial(r_1^2 \ln r_1)/\partial x = -\partial(r_1^2 \ln r_1)/\partial b, \tag{4.13}$$

one obtains by differentiating (4.10) with respect to b a function H_2 which is not derivable from (4.11) without even further modifications. The rule (4.11) by no means replaces the general rule (3.28).

From (4.10) follows

$$\nabla^2\Gamma\big|_{x=0} = 8b^2/(b^2+y^2), \qquad \frac{\partial}{\partial x}\nabla^2\Gamma\big|_{x=0} = -16b^3/(b^2+y^2)^2$$

and hence (4.2) becomes ([16])

$$H(x, y) = (1/\pi) \int_{-\infty}^{+\infty} \{\alpha(y')2x^3[x^2 + (y - y')^2]^{-2} - \beta(y')x^2[x^2 + (y - y')^2]^{-1}\} dy'. \tag{4.14}$$

Since the region $x>0$ is infinite, proper restrictions must be postulated for the behavior of H and its derivative at infinity in order that (4.2) apply without an added contribution arising from the boundary at infinity.

Green's functions for other regions but for the same boundary conditions (1.4) will be considered below.

5. **Other problems for the half-plane.** In case of Airy's function the boundary conditions (1.4) apply not to fixed but to *completely free* boundaries. The Green's function Γ of the preceding section is still of interest on account of its use in (4.2) for obtaining biharmonic functions satisfying (1.2). However, as Airy's functions, H_1, Γ given by (4.1), (4.10) suffer from the difficulty that the displacements u, v associated with them are multiple-valued. We proceed to construct for the infinite plane, and for the half-plane with a free or a fixed edge, the Airy's functions corresponding to the stress induced by a concentrated point force (acting, of course, in a direction *parallel* to the plate).

We shall adopt the form (2.10) for H rather than (2.16) in order to be able to use the displacement equation (2.31). Put

$$g(z) = \ln z. \tag{5.1}$$

There results from (2.31)

$$2G(u - iv) = \nu \log \bar{z} - \bar{z}/z - f'. \tag{5.2}$$

Multiple-valued displacements will be avoided by choosing

$$f' = -\nu \log z \tag{5.3}$$

whereupon (5.2) yields

$$2G(u - iv) = 2\nu \log r - e^{-2i\theta}. \tag{5.4}$$

To examine the singularity of the resulting stress distribution at $z=0$ apply (2.25). There results

$$-\nu \ln \bar{z} + z/\bar{z} + \ln z\Big]_{P_1}^{P_2} = iX - Y. \tag{5.5}$$

Upon going once around the origin counterclockwise the left-hand member of (2.26) increases by $(1+\nu)2\pi i$. Hence there is a traction exerted upon the region inside any circle $|z| = r$ by the region outside it of amount $(1+\nu)2\pi$ in the direction of positive x. The Airy's function H in question therefore

([16]) See L. F. Richardson, Proceedings of the Physical Society of London vol. 23 (1911) p. 78.

represents the stress due to a concentrated point force $-(1+\nu)2\pi$. With a slight modification of f' we put

$$f = -\nu z \ln z, \qquad g = \ln z$$

and obtain

$$\begin{aligned} H &= [L/2\pi(\nu+1)]R[\nu z \ln z - \bar{z} \ln z] \\ &= (L/2\pi)[(\nu-1)x \ln r/(\nu+1) - y\theta] \end{aligned} \tag{5.6}$$

as the Airy's function corresponding to an applied load L in the direction of positive x (see Fig. 1a). The corresponding displacements are given by (5.4) and agree with the expressions given by Love[17].

By proper differentiation and superposition one obtains from (5.6)

$$H = [L/2\pi(\nu+1)]R[(\nu-1)\ln z - \bar{z}/z], \tag{5.7}$$

$$H = (L/\pi)R(i \ln z) = -(L/2\pi)\theta, \tag{5.8}$$

$$H = (L/\pi)(\nu-1)/(\nu+1)R(\ln z) = (L/\pi)(\nu-1)/(\nu+1)\ln r \tag{5.9}$$

as the Airy's functions corresponding to applied singular loads shown schematically in Figs. 1b, 1c, 1d. In every case $F\delta = L$ and δ is allowed to ap-

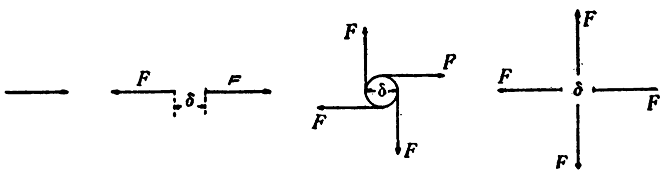

FIG. 1a. FIG. 1b. FIG. 1c. FIG. 1d.

proach zero. The singularity of Airy's function (5.8) may be described as a "concentrated moment" at the origin, while (5.9) is said to be due to a "center of pressure" or "expansion nucleus" at the origin; both states of stress (5.8), (5.9) are radially symmetric about $r=0$.

Consider next Airy's functions for the half-plane $x>0$ with a concentrated load at some internal point $z=b>0$, when the boundary $x=0$ is either completely free (no applied traction) or held rigidly (no displacement allowed). First let the force be in the x-direction and at $z=b>0$. Adapting (5.6) we put

$$H_1 = R[\nu(z-b)\ln(b-z) - (\bar{z}-b)\ln(b-z)] \tag{5.10}$$

and let $H = H_1 + H_2$ where H_2 is free from singularities in $x>0$. If $x=0$ is a free boundary, the boundary conditions (3.1) may be used and we may apply the results of §3, say by means of (3.26), (3.28). As a variation, however, we shall indicate a different procedure; the latter, while considerably longer, has

(17) Loc. cit., p. 208.

the advantage of being applicable with little modification to the boundary condition of no displacement.

Continuing with the form (2.10) for H as well as for H_1, H_2 and recalling (2.26) it will be noted that the boundary condition

$$(5.11)\qquad 2\partial H/\partial z = f'(z) + \bar{z}g'(z) + \overline{g(z)} = \text{const.}$$

applies along a free edge; the right-hand constant may be equated to zero by adding a linear term in z, $\bar{z}$ to H if necessary; this we suppose to be the case:

$$(5.12)\qquad f'(z) + zg'(z) + \overline{g(z)} = 0.$$

Suppose now that (5.12) holds for $x=0$, then replacing $\overline{g(z)}$ by $\bar{g}(\bar{z})$ and $\bar{z}$ by $-z$ and substituting $f=f_1+f_2$, $g=g_1+g_2$ we obtain

$$(5.13)\qquad f_1'(z) + f_2'(z) + \bar{g}_1(-z) + \bar{g}_2(-z) - zg_1'(z) - zg_2'(z) = 0.$$

From §3 it follows that if f_1, g_1 are singular at P in $x>0$ but not at P^*, the reflection of P in $x=0$ (this is the case with f_1, g_1 of (5.10) with P at $z=b>0$), then f_2, g_2 will be singular at P^*. Equating separately to zero the parts of the left-hand member of (5.13) which are singular at P and at P^* respectively, we obtain

$$(5.14)\qquad f_1'(z) + \bar{g}_2(-z) - zg_1'(z) = 0,\quad f_2'(z) + \bar{g}_1(-z) - zg_2'(z) = 0.$$

If f_1, g_1 are given, the first equation (5.14) determines g_2; the second one then yields f_2.

Similarly (2.3) yields as the condition of zero displacement

$$(5.15)\qquad \nu\overline{g(z)} - \bar{z}g'(z) - f'(z) = \text{const.}$$

and the constant may again be chosen as zero. Similar manipulation of (5.15) leads to

$$(5.16)\qquad \begin{aligned} \nu\bar{g}_2(-z) + zg_1'(z) - f_1'(z) &= 0,\\ \nu\bar{g}_1(-z) + zg_2'(z) - f_2'(z) &= 0. \end{aligned}$$

To obtain the Airy's function sought, we merely put in (5.14), (5.16)

$$(5.17)\qquad f_1 = \left[\nu(z-b) + b\right]\ln(b-z),\qquad g_1 = b\ln(b-z)$$

and determine g_2, f_2.

We now turn to the determination of Airy's function for the half plane $x>0$, corresponding to given tractions applied over $x=0$. This problem may be reduced to the boundary conditions (1.2) and may be solved by means of (4.14). The following direct treatment, however, is also of interest.

Returning to the proof of §3 it will be noted that either (3.2) or (3.12) is equivalent to the boundary condition $H=0\big|_{x=0}$. Equations (1.6), (1.7) show that this corresponds to vanishing normal tractions X_x over $x=0$, that is, to purely shearing stress applied over $x=0$. Similarly either (3.3) or (3.13) is equivalent to the boundary condition $\partial H/\partial x|_{x=0}$, corresponding to vanishing shearing tractions X_y and purely normal applied stresses over $x=0$.

As a very simple solution of (3.2), (3.12) we choose

$$f = 0, \qquad H = R(xg) = xG. \tag{5.18}$$

There results from (1.7), (1.6) along the boundary $x=0$

$$-Y = R(g) = G(0, y), \tag{5.19}$$

$$X_y = -\partial Y/\partial y = -\partial^2 H/\partial x \partial y = R[ig'(z)] = -I[g'(z)] = G_y(0, y). \tag{5.20}$$

A similar simple solution of (3.3), (3.13) is obtained by letting $k=0$, hence by choosing g, f, H in accordance with

$$g = -f', \qquad H = R(f - xf'), \tag{5.21}$$

resulting in vanishing $\partial H/\partial x$, X_y at $x=0$. There is obtained along the boundary $x=0$ from (1.6), (1.7)

$$X = \partial H/\partial y| = R[\partial f(z)/\partial y]| = R[if'(z)]| = -I[f'(z)], \tag{5.22}$$

$$X_x = \partial X/\partial y = \partial^2 H/\partial y^2 = R(\partial^2 f/\partial y^2) = -R[f''(z)]. \tag{5.23}$$

By properly choosing f, g above, many interesting stress distributions result. As examples of special interest we consider first the special case of (5.18),

$$f = 0, \qquad g = i \ln z. \tag{5.24}$$

This leads to

$$H = x\theta \tag{5.25}$$

and to the boundary resultant traction components

$$X = 0, \qquad -Y = \begin{cases} \pi/2 & \text{for} \quad x = 0,\ y > 0, \\ -\pi/2 & \text{for} \quad x = 0,\ y < 0, \end{cases} \tag{5.26}$$

corresponding to a shearing force π in the y-direction applied at the origin along the otherwise free boundary $x=0$. The similar case of a normal concentrated force at the origin is obtained by choosing in (5.21)

$$\ln z = f', \tag{5.27}$$

resulting essentially in

$$H = R(z \ln z - x \ln z) = -y\theta. \tag{5.28}$$

$$Y = 0, \qquad -X = I(\ln z) = \begin{cases} \pi/2 & \text{for} \quad x = 0,\ y > 0, \\ \pi/2 & \text{for} \quad x = 0,\ y < 0, \end{cases} \tag{5.29}$$

corresponding to an applied force π in the normal or x-direction along $x=0$.

By proper superposition of the two solutions just found, any traction over $x=0$ may be obtained.

Further examples of interest are obtained by choosing $e^{-\lambda z}$ for g or f in (5.18), (5.21), where λ is a real positive constant. There result a sinusoidal shearing and normal tractions over $x=0$. The corresponding Airy's functions are also obtainable directly by seeking solutions of (1.1) of the product form:

$$u = Y(y)X(x). \tag{5.30}$$

Such solutions are found to be

$$u = e^{\pm\lambda i y}X(x) \tag{5.31}$$

where X satisfies the differential equation

$$X^{(4)} - 2\lambda^2 X'' + X = 0 \tag{5.32}$$

and therefore consists of a linear combination of

$$e^{\lambda x},\ xe^{\lambda x},\ e^{-\lambda x},\ xe^{-\lambda x}. \tag{5.33}$$

For real λ in the half plane $x>0$ only the last two terms are used to avoid infinite stresses for $x=+\infty$. By choosing X so that $X'(0)=0$, there result a sinusoidal normal stress and zero shearing stress; similarly, $X''(0)=0$ yields a sinusoidal shearing traction and vanishing normal traction.

Arbitrary tractions over $x=0$ may now be found by superposing these solutions, say by means of proper harmonic analysis, for instance, by means of Fourier integrals.

6. **Circular boundary.** Consider first a circular boundary $r=a$ along which the boundary conditions (1.4) hold so that

$$H = 0, \qquad \partial H/\partial r = 0 \qquad \text{for } r = a. \tag{6.1}$$

We shall adopt for H the form (2.15) slightly modified, however, by replacing r^2 by r^2-a^2:

$$H = F + (r^2 - a^2)G = R[f + (r^2 - a^2)g]. \tag{6.2}$$

The factor (r^2-a^2) vanishes along the circular boundary, hence the form (6.2) is a close analogue of (2.16) that proved so convenient for the rectilinear boundary $x=0$.

Applying (6.1) along the boundary there results

$$F = 0, \qquad \partial F/\partial r + 2aG = 0 \qquad \text{for } r = a. \tag{6.3}$$

The analogy to the rectilinear boundary can now actually be put to use by introducing the conformal transformation

$$z = ae^{w}, \quad w = u + iv; \qquad r = ae^{u}, \quad \theta = v, \tag{6.4}$$

whereupon $r=a$ goes into $u=0$ and (6.3) becomes

(6.5) $$F = 0, \qquad \partial F/\partial u + 2a^2 G = 0 \qquad \text{for } u = 0.$$

These conditions differ from (3.11) only in that x, y, G are replaced respectively by u, v, $2a^2G$. Hence the solution of §3 may be adapted to the present problem. In particular it follows that if (6.1) applies along a segment S of the circle $r=a$, that F, G, H may be continued analytically from a region R to one side of the segment S to the region R^* which is the inverse image of R in the circle $r=a$. If P is an isolated singular point of H to one side of $r=a$, other than $r=0$, then P^*, the reflection (or inverse image) of P in $r=a$, is a singular point of (the analytic continuation of) H.

Adapting equations (3.2), (3.3) to the present case we state: A necessary and sufficient condition that H satisfy (6.1) along a circle $r=a$ or any segment thereof is that H be represented in the form (6.2) where the functions f, g satisfy the functional equations

(6.6) $$f(z) + \bar{f}(a^2/z) = 0,$$

(6.7) $$k(z) + \bar{k}(a^2/z) = 0, \qquad k(z) = 2a^2g(z) + zf'(z).$$

These relations follow by replacing f, g, z by f, $2a^2g$, w in (3.2), (3.3), then expressing w in terms of z (equation (6.4)). If

(6.8) $$f(z) = F(r, \theta) + iF(r, \theta),$$

then the nature of $\bar{f}(a^2/z)$ is rendered clear from

(6.9) $$\bar{f}(a^2/z) = F(a^2/r, \theta) - iF(a^2/r, \theta)$$

which shows that $\bar{f}(a^2/z)$ is equal to the conjugate of the value that f takes on at the inverse image point of z, $a^2/\bar{z}$.

Equations (3.19)–(3.29) may now be adapted to obtain biharmonic functions satisfying (6.1). There results from (3.19), (3.24), (3.25)

(6.10) $$\begin{aligned} H &= F_1 + F_2 + (r^2 - a^2)(G_1 + G_2) \\ &= R\{f_1(z) + f_2(z) + (r^2 - a^2)[g_1(z) + g_2(z)]\} \end{aligned}$$

where F_1, G_1 are arbitrary harmonics and F_2, G_2 are given by

(6.11) $$F_2 = -F_1^*, \quad G_2 = -[(r/a^2)\partial F_1/\partial r + G_1]^* = -(1/r)(\partial F_1/\partial r)^* - G_1^*$$

where stars now indicate inversion in the circle $r=a$:

(6.12) $$F^*(r, \theta) = F(a^2/r, \theta).$$

In terms of f_1, f_2, g_1, g_2 in (6.10) this is equivalent to([18])

(6.13) $$f_2(z) = -\bar{f}_1(a^2/z),$$

([18]) In (6.15), $\bar{f}_1'(a^2/z)$ is to be interpreted as follows. First derive the function $\bar{f}_1(z)$ as in (3.6), then differentiate it obtaining $\bar{f}_1'(z)$ and finally substitute a^2/z in place of z. In (6.12) the term $(\partial F/\partial r)^*$ is to be interpreted by carrying out the r-differentiation, then replacing r by a^2/r.

$$(6.14) \qquad g_2(z) = -\bar{g}_1(a^2/z) - \bar{f}_1'(a^2/z)/z,$$

where f_1, g_1 are arbitrary.

To revert to the form (2.15), $f_1 - a^2 g_1$ in H_1 is replaced by f_1, resulting in [19]

$$(6.15) \qquad \begin{aligned} H_1 &= R[f_1(z) + r^2 g_1(z)], \\ -H_2 &= R\{[\bar{f}_1(a^2/z) + r^2\bar{g}_1(a^2/z) \\ &\qquad + (r^2 - a^2)z^{-1}[\bar{f}_1'(a^2/z) + a^2\bar{g}_1'(a^2/z)]\}, \\ H &= H_1 + H_2, \end{aligned}$$

where again f_1, g_1 are arbitrary.

If f_1, g_1 (in either (6.10) or (6.15) or F_1, G_1 in (6.10)) possess singularities in $r > a$ but are analytic in $r \leqq a$, then H_2 is analytic in $r > a$ and H possesses the same singularities in $r > a$ as H_1. With the possible exception of $r = 0$, the last statement also holds with the inequalities reversed: by choosing H_1 properly singular for $r < a$ but analytic in $r \geqq a$, H_2 will be free from singularities in $r \leqq a$ and H will possess the same singularities as H_1 there, except possibly for $r = 0$ where the behavior of H_2 depends upon the nature of H_1 at $z = \infty$. The singularity due to H_2 at $z = 0$, if any, must be removed without interfering with the boundary conditions (6.1).

As an elementary example, consider the well known case of a circular hole in an infinite tension member, the edge of the hole being free from stress. Let

$$(6.16) \qquad X_x = X_y = 0, \quad Y_y = 1; \qquad H_1 = x^2/2$$

be the uniform tensile stress in a uniform plate (without the hole) and its Airy's function. This describes the behavior of the stress and Airy's function at infinity even in the presence of the hole. To transform H_1 to the form (2.15) note that

$$H_1 = (z + \bar{z})^2/8 = R(z^2 + z\bar{z})/4 = R[(z^2 + r^2)/4]$$

whence

$$f_1 = z^2/4, \qquad g_1 = 1/4.$$

Applying (6.15) there results

$$H_2 = R(a^4/4z^2 - r^2/4 - a^2r^2/2z^2)$$

whence

$$(6.17) \qquad H = R(z^2/4 + a^4/4z^2 - a^2r^2/2z^2) = \cos 2\theta(r^2/4 + a^4/4r^2 - a^2/2).$$

As an example of a less elementary nature we shall obtain the Green's functions for the problem of the bent plate for the region R outside the circle

[19] To avoid confusion it is preferable to introduce first a new symbol, say ϕ, in place of $f_1 - a^2 g_1$ in (6.13), (6.14), then replace ϕ by f_1.

$r=a$, with the "built in" boundary condition (6.1) at $r=a$. Let the pole be chosen at $z=b>a$ and let H_1 be given by (4.3). Using the form (2.15) for H there results

$$
\begin{aligned}
H_1 &= R[(\bar{z}-b)(z-b)\ln(b-z)] \\
&= R[r^2(z-b)z^{-1}\ln(b-z) - b(z-b)\ln(b-z)]
\end{aligned}
\tag{6.18}
$$

whence

$$
g_1 = (z-b)z^{-1}\ln(b-z), \qquad f_1 = -b(z-b)\ln(b-z). \tag{6.19}
$$

Substitution in (6.15) leads after considerable algebraic reduction to

$$
H_2 = -r_1^2 \ln r_2 + r_1^2 \ln r/b + (a^2 - r^2)(1 - br^{-1}\cos\theta) \tag{6.20}
$$

where r_2 is the distance from a^2/b, the inverse image point of b in the circle $r=a$:

$$
r_2 = |z - a^2/b|. \tag{6.21}
$$

There results for the Green's function

$$
H = \Gamma = r_1^2[\ln r_1 + \ln(r/b) - \ln r_2] + (a^2 - r^2)(1 - br^{-1}\cos\theta). \tag{6.22}
$$

The boundary conditions (6.1) at $r=a$ may be checked directly on (6.22).

As pointed out above, equations (6.15) may lead to a singularity at $z=0$. This, indeed, is the case at present, the terms of (6.22) which are singular at $z=0$ being

$$
r_1^2 \ln(r/b) - (a^2b/r)\cos\theta.
$$

Replacing them by a linear combination of $1, r, r\cos\theta$ having the same value along $r=a$ as well as the same r-derivative there, one obtains as a further possible Green's function

$$
H = \Gamma = r_1^2[\ln r_1 - \ln r_2 + \ln(a/b)] + (a^2 - r^2)(a^2 - b^2)/2a^2. \tag{6.23}
$$

This has the advantage over (6.22) of having no singularity at $r=0$.

If the Green's function Γ for the region *inside* the circle $r=a$ is sought, b is chosen positive and less than a, and (6.23) still furnishes the proper Green's function. While for the region external to a circle either (6.22) or (6.23) can be used for Γ, for the region internal to a circle the Green's function is unique, only (6.23) will do, (6.22) being disqualified by the singularity at the center $r=0$.

If one allows a, the radius of the circle, to become infinite, (6.23) will be found to approach the Green's function (4.10) for the half-plane, once proper consideration is given to the different position of the origin in the two cases.

We give the interpretation of the singularity of (6.23) at $z=a^2/b$. The inverse image of $z=b$ in the circle $r=a$ can be carried out along similar lines to that of the Green's function for the half-plane by transforming the term $-r^2 \ln r_2$ as follows:

$$-r_1^2 \ln r_2 = -r_2^2 \ln r_2 - [(a^2 - b^2)/b][x - a^2/b] \ln r_2 - (a^2 - b^2) \ln r_2.$$

If we identify $H' = -r_2^2 \ln r_2$ with the deflection due to a concentrated force at $z=a^2/b$, equal and opposite to the initial force at $z=b$, and express the remaining terms in terms of $\partial H'/\partial x$ and $\nabla^2 H'$, we can interpret them as deflections due to a concentrated moment and a quadrupole of forces or a circularly distributed bending moment at $z=a^2/b$.

Application of (4.2) to (6.23) yields the following solution of (1.1), (1.2) for the circular region $r<a$ (see Fig. 2)

$$\begin{aligned} H_P = {} & (a/2\pi) \int_0^{2\pi} \{[\cos(\theta - \theta_1) + (b/a^2)\cos\theta_1]\alpha(\theta)/r_1\} d\theta \\ & + [(a^2 - r^2)/4\pi] \int_0^{2\pi} \beta(\theta) d\theta; \end{aligned} \tag{6.24}$$

here H_P is the value at P of the biharmonic function which satisfies the boundary conditions

$$H = \alpha(\theta), \qquad \partial H/\partial r = \beta(\theta) \tag{6.25}$$

along $r=a$.

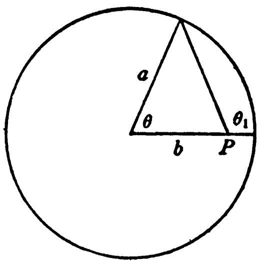

FIG. 2.

Equation (6.25) applies as well to the exterior of the circle $r=a$, provided, however, that the boundary conditions are (1.2) and not (6.25), and provided that H is properly restricted at infinity so that no additional terms in (4.2) arise from the boundary at infinity when Green's theorem is applied.

An alternative treatment of the boundary conditions (1.2), say for the inside of the circle $r=a$, is to use the form (6.2) directly. There results in place of (5.27)

$$F = \alpha, \qquad \partial F/\partial r + 2aG = \beta \qquad \text{for } r = a. \tag{6.26}$$

The determination of F leads to a Dirichlet problem. After F has been found and from the first equation the boundary value of $\partial F/\partial r$ determined, G is likewise obtained as the solution of a further Dirichlet problem.

A further well known treatment of the problem for the interior of a circle is by resolving α and β in Fourier series in θ and using power series in z for f and g.

7. **Airy's functions for a circular region.** Turning to Airy's functions for a circular disc corresponding to applied tractions over its boundary, we recall from §1 that this may be reduced to the boundary conditions (1.2); hence the results of the preceding section may be applied to the determination of this Airy's function. The following solution in terms of X, Y, the resultant tractions (from a fixed point on the boundary to any point on it), is somewhat more convenient, however.

Recalling (3.25) (and using the form (2.10) for H) one obtains over $r=a$

$$f'(z) + \bar{z}g'(z) + \overline{g(z)} = -iX - Y. \tag{7.1}$$

To treat this boundary condition it will be noted that it is in general impossible to assign boundary values of an analytic function over the boundary of a closed region. Thus no function $\gamma(z)$, analytic in $|z| < a$, need exist which takes on assigned boundary values $\gamma(\theta)$ over $|z| = a$. However, it is always possible to find two functions, $\gamma_1(z)$, $\gamma_2(z)$, each analytic in z for $|z| < a$, and such that

$$[\gamma_1(z) + \gamma_2(\bar{z})] = \gamma(\theta) \qquad \text{for } |z| = a. \tag{7.2}$$

The two functions γ_1, γ_2 are essentially unique, any two different determinations differing from each other at most by constants. The proof of these statements follows readily by starting with the Fourier series resolution of $\gamma(\theta)$:

$$\gamma(\theta) = \sum \gamma_n e^{ni\theta} \tag{7.3}$$

and putting

$$\gamma_1(z) = \sum_{n=1}^{\infty} \gamma_n (z/a)^n + C,\ \gamma_2(\bar{z}) = \sum_{n=1}^{\infty} \gamma_{-n}(\bar{z}/a)^n + C', \tag{7.4}$$
$$C + C' = \alpha_0.$$

Applying the above to the right-hand member of (7.1) let $-iX - Y$ be represented as the sum of the boundary values of $\gamma_1(z)$, $\gamma_2(\bar{z})$:

$$\gamma_1(z) + \gamma_2(\bar{z}) = -iX - Y = \gamma(\theta) \tag{7.5}$$

along the boundary $r=a$, where γ_1, γ_2 are analytic in their respective arguments in the circle $r < a$. It will be assumed that

$$I[\gamma_2'(0)] = 0. \tag{7.6}$$

This is a natural restriction which is equivalent to the vanishing of the moment about the origin of the applied tractions; the vanishing of their resultant follows automatically from the fact that X, Y, assumed single-valued on $r=a$, return to their initial values as θ increases by 2π. The boundary condition (7.1) will be written in the form

$$f'(z) + (a^2/z)[g'(z) - g'(0)] + [\bar{z}g'(0) + \bar{g}(\bar{z})] = \gamma_1(z) + \gamma_2(\bar{z}), \tag{7.7}$$

thus avoiding the singularity at $z=\infty$. To solve this equate respective functions of z, $\bar{z}$ on both sides:

$$\bar{z}g'(0) + \bar{g}(\bar{z}) = \gamma_2(\bar{z}), \tag{7.8}$$

$$f'(z) + (a^2/z)[g'(z) - g'(0)] = \gamma_1(z). \tag{7.9}$$

Equation (7.8) would involve a contradiction were it not for (7.6). When the latter is satisfied we solve (7.8) by putting

$$g(z) = \bar{\gamma}_2(z) - \overline{\gamma_2'(0)}z/2. \tag{7.10}$$

Equation (7.9) is then used to yield $f'(z)$, and by integration, $f(z)$.

As an example we shall obtain the Airy's function for a circular disk $r<a$ subject to several concentrated loads over its boundary. To this end let (7.1), (7.5) hold with

$$\gamma_1(z) = (1/2\pi)\sum_{i=1}^{n} \overline{C}_i \ln (z - z_i), \qquad \gamma_2(z) = -(1/2\pi)\sum \overline{C}_i \ln (\bar{z} - \bar{z}_i), \tag{7.11}$$

where $z_1, z_2, \cdots, z_n$ are n points lying on $z=a$ and C_i are n (possibly complex) constants satisfying the relations

$$\sum_i \overline{C}_i = 0, \qquad I\left(\sum_i C_i\bar{z}_i\right) = 0. \tag{7.12}$$

Equation (7.1) now reduces to

$$-iX - Y = (i/\pi)\sum_i \overline{C}_i\theta_i \tag{7.13}$$

where θ_i are the respective arguments or angles of $z-z_i$. It is readily shown that in view of (7.12) the right-hand member of (7.13) is constant over each arc segment $r=a$, between adjacent points z_i, thus corresponding to no applied tractions over the internal points of these arc segments. Upon crossing z_j in direction of increasing θ and without getting out of $r\leqq a$ the right-hand member of (7.13) increases by $-C_ji$; hence $X-iY$ increases by an amount C_j. Thus (7.11) corresponds to the "complex" forces $C_1, C_2, \cdots$ applied at $z_1, z_2, \cdots$, that is, to forces whose components in the x, y-directions are equal respectively to the real and imaginary parts of $C_1, C_2, \cdots$. Equations (7.12) correspond to the conditions of equilibrium for these applied forces (zero resultant and zero moment).

Applying (7.9), (7.10) to (7.11) we obtain

$$
\begin{aligned}
g(z) &= -(1/2\pi)\sum C_i[\ln(z - z_i) - z/2z_i],\\
f(z) &= (1/2\pi)\sum \ln(z - z_i)[\overline{C}_i(z - z_i) - C_i\bar{z}_i]
\end{aligned}
\tag{7.14}
$$

and hence

$$
\begin{aligned}
H &= R\{\sum \ln(z - z_i)[(z\overline{C}_i - \bar{z}C_i) - (z_i\overline{C}_i - \bar{z}_iC_i) - (z\bar{z}/2)\sum C_i/z_i\}\\
&= -(1/\pi)\sum \theta_i I[\overline{C}_i(z - z_i)] - (r^2/4\pi)\sum R(C_i/z_i).
\end{aligned}
\tag{7.15}
$$

The Airy's function corresponding to distributed tractions is easily written down by replacing the above summations by integrations.

For the region external to the circle $r=a$ a similar treatment can be given. However, X, Y need no longer be single-valued along $r=a$ since the applied tractions now may possess a nonvanishing resultant or moment, this being kept in equilibrium by a proper stress distribution at infinity. Moreover, the displacement components must be single-valued for $r>a$. The Airy's function (7.15) is disqualified on both of these accounts.

We now consider a *single* point force (F_x, F_y) applied over the boundary $r=a>0$. Corresponding to it we write the boundary condition (7.1) as

$$
f'(z) + \bar{z}g'(z) + \overline{g(z)} = 2iC(\theta_1 - \theta/2) + \text{constant}
\tag{7.16}
$$

where θ_1, θ are the arguments of $z-a$, z and the complex constant C is given by

$$
2\pi iC = -iF_x - F_y.
\tag{7.17}
$$

Indeed, over $|z|=a$, $d\theta=2d\theta_1$, and the right-hand member of (7.16) reduces to a constant except at $z=a$, and when z describes a semi-circle about $z=a$ while staying in $r>a$, θ_1 increases by π. Now note that

$$
\begin{aligned}
\theta &= I(\ln z) = (\ln z - \ln \bar{z})/2i,\\
\theta_1 &= I[(\ln)(z - a)] = [\ln(z - a) - \ln(\bar{z} - a)]/2i
\end{aligned}
\tag{7.18}
$$

and replace the right-hand member of (7.16) by

$$
C[\ln(z - a) - \ln(\bar{z} - a)] - (C/2)(\ln z - \ln \bar{z}) + K(\ln z + \ln \bar{z})
\tag{7.19}
$$

where the K-terms reduce to a constant over $|z|=a$. Replacing $\bar{z}$ by a^2/z in the left-hand member of (7.16) and equating the analytic functions of z and $\bar{z}$ on both sides one obtains first g, then f':

$$
\begin{aligned}
g(z) &= -\overline{C}\ln(z - a) + (\overline{K} + \overline{C}/2)\ln z,\\
f'(z) &= C\ln(z - a) + (K - C/2)\ln z + \overline{C}/z(z - a) - (K + \overline{C}/2)/z^2.
\end{aligned}
\tag{7.20}
$$

The displacements (u, v) are determined by (5.2). Their multiple-valuedness in $r>a$ is correctly represented by

$$(7.21)\qquad 2G(u - iv) = - (K + C/2) \ln z - \nu(K + 3C/2) \ln \bar{z} + \cdots$$

where the terms explicitly indicated are obtained by replacing $\ln(z-a)$ by $\ln z$ and the terms not explicitly shown are single-valued. The displacements are single-valued provided

$$(7.22)\qquad (K + C/2) = \nu(\overline{K} + 3\overline{C}/2);$$

by expressing K in the form K_1+iK_2, recalling (7.17) and equating real and imaginary parts one determines K.

8. **Regions with multiple boundaries.** We shall now consider briefly regions whose boundary consists of several portions each of which is represented by a different analytic equation in the coordinates x, y or $z, \bar{z}$. Simple examples of these regions are given by the infinite strip

$$(8.1)\qquad - a/2 \leqq x \leqq a/2,$$

the region between two concentric circles

$$(8.2)\qquad a \leqq r \leqq c,$$

the angular region

$$(8.3)\qquad - \alpha/2 \leqq \theta \leqq \alpha/2,$$

and so forth. While detailed expressions for the Green's functions or treatments of boundary value problems for these regions are outside the scope of this paper, the following comments are of interest from the point of view of analytic function theory.

Application of the method of alternate "reflections" in the sense explained in §§3, 4, across *each* boundary $x=-a/2$, $x=a/2$ of the strip (8.1), to the function

$$(8.4)\qquad H_0 = r_0^2 \ln r_0, \qquad r_0 = |z - b|, \qquad b \text{ real}$$

which has the singularity of the Green's function at $(x, y)=(b, 0)$ leads to the series

$$(8.5)\qquad H = H_0 + (H_{-1} + H_1) + (H_{-2} + H_2) + \cdots$$

where for any $n>0$

$$(8.6)\qquad H_{-n} = L(H_{n-1}), \qquad H_n = M(H_{1-n}),$$

and where the linear operators L, M are defined by

$$(8.7)\qquad L(H) = \{- 1 + 2[x + (a/2)]\partial/\partial x - (x + a/2)^2\nabla^2\} H \big|_{x\to(-a-x)},$$

$$(8.8)\qquad M(H) = \{- 1 + 2[x - (a/2)]\partial/\partial x - (x - a/2)^2\nabla^2\} H \big|_{x\to(a-x)}.$$

The series (8.5) gives the correct singularities of the analytic continuation of Γ—these singularities are located only at

$$z_n = (-1)^n b + na, \tag{8.9}$$

this point being singular for H_n only—and formally satisfies the conditions (1.4) along $x=a/2$, $x=-a/2$. The calculation of its successive terms gets very irksome and the question of its convergence is still an open one.

Similar remarks apply to the regions (8.2), (8.3), where for (8.2) operators L, H are replaced by reflections across the circular boundaries of (8.2) in the sense of §6, while for (8.3) the reflections are across the intersecting lines bounding (8.3).

Application of (2.24) to H expressed in the form (2.10) leads to

$$f'(z) + \overline{g(z)} + \bar{z}g'(z) = f'(z) + \bar{g}(\bar{z}) + \bar{z}g'(z) = 0 \tag{8.10}$$

along both $x=-a/2$, $x=a/2$. Now along $x=-a/2$

$$(z+\bar{z})/2 = -a/2, \qquad \bar{z} = -z-a \tag{8.11}$$

and (8.10) becomes

$$f'(z) + \bar{g}(-z-a) + (-z-a)g'(z) = 0. \tag{8.12}$$

Since this holds for all the points of $x=-a/2$, it holds for all z. Similarly along $x=a/2$

$$(z+\bar{z})/2 = a/2, \qquad \bar{z} = -z-a \tag{8.13}$$

and (8.10) yields for all z

$$f'(z) + \bar{g}(a-z) + (a-z)g'(z) = 0. \tag{8.14}$$

Elimination of $f'(z)$ from (8.12), (8.14) leads to

$$2ag'(z) + \bar{g}(a-z) - \bar{g}(-a-z) = 0. \tag{8.15}$$

The last equation applies even when H is represented in the form (2.16) since, as shown in (2.19), the functions $g(z)$ for the forms (2.10), (2.16) differ from each other only by a factor 2.

In the functional equation (8.15), the terms $\bar{g}(a-z)$, $\bar{g}(-a-z)$ involve the values of g at the reflection of z in the lines $x=a/2$, $x=-a/2$ respectively. This equation is to be compared with the simple periodicity equation which results for a harmonic function vanishing over the boundaries of strip (8.1).

Special simple solutions of (8.15) are given by

$$g(z) = e^{\lambda z} - e^{-\bar{\lambda} z} \tag{8.16}$$

provided λ is a root of

$$\sinh \lambda + \lambda a = 0, \tag{8.17}$$

and by

$$g(z) = e^{\lambda z} + e^{\bar{\lambda} z} \tag{8.18}$$

where λ is a root of

$$\sinh \lambda a - \lambda a = 0. \tag{8.19}$$

Corresponding functions $f(z)$ may now be found from (8.12) or (8.14).

Another way of arriving at essentially the same solutions is by using biharmonic functions of the product from (5.31). The boundary conditions (1.4) reduce to $X = X' = 0$ for $x = \pm a/2$. By choosing a linear combination of the functions (5.33) which is even and odd in x one is led to (8.17), (8.19) respectively.

It is planned to discuss the expansion of the Green's function in terms of these characteristic functions in a future paper.

The strip (8.1) with prescribed values of H, $\partial H/\partial x$ over the boundaries can be treated by means of the Fourier integral. As an example of the Fourier integral method see Filon's treatment of Airy's function for the case of equal and opposite forces at the opposite points of the strip boundary[20].

For the ring (8.2) a similar application of (8.10) to the two boundaries leads upon elimination of f' to the functional equation

$$(c^2 - a^2) z^{-1} g'(z) + \bar{g}(c^2/z) - \bar{g}(a^2/z) = 0. \tag{8.20}$$

Solutions of this are available of the form

$$g(z) = z^{\mu+1} + C z^{1-\bar{\mu}} \tag{8.21}$$

where μ satisfied the equation

$$(c^2 - a^2)\mu \pm [(1/c^2)^{\mu} - (1/a^2)^{\mu}] = 0. \tag{8.22}$$

An alternative way of arriving at the same solutions is by means of the product solutions of (1.1):

$$H = e^{i\mu\theta} R(r) \tag{8.23}$$

where R is a linear combination of

$$r^{-\mu},\ r^{-\mu+2},\ r^{\mu},\ r^{\mu+2}. \tag{8.24}$$

The functions (8.21), (8.23) are not single-valued over the ring (8.2). Single-valued biharmonic functions are furnished by (8.23), (8.24) with positive integer $\mu \neq 1$, and by

$$1,\ \ln r,\ r^2,\ r^2 \ln r, \tag{8.25}$$

$$e^{i\theta}[r^{-1},\ r,\ r^3,\ r \ln r] \tag{8.26}$$

(20) L. N. Filon, Philos. Trans. Roy. Soc. London Ser. A vol. 201 (1903) pp. 63–155.

for $\mu=0, 1$. By means of Fourier series the Green's function can be expressed in terms of these functions and the general solutions of the (1.1), (1.2) for the ring (8.2) obtained.

Turning to the angular wedge (8.3) we express its boundaries in the form

$$\bar{z} = e^{-i\alpha}z, \qquad \bar{z} = e^{i\alpha}z. \tag{8.27}$$

Application of (8.10) leads to

$$f'(z) + e^{-i\alpha}zg'(z) + \bar{g}(e^{-i\alpha}z) = 0, \quad f'(z) + e^{i\alpha}zg'(z) + \bar{g}(e^{i\alpha}z) = 0 \tag{8.28}$$

and hence to

$$2i \sin \alpha zg'(z) + \bar{g}(e^{i\alpha}z) - \bar{g}(e^{-i\alpha}z) = 0. \tag{8.29}$$

It can be shown that with the exception of $g\equiv 0$ any solution of this functional equation must be singular at $z=0$. Indeed, if a solution of (8.29) analytic at $z=0$ did exist, then expanding in integral powers of z:

$$g = \sum g_n z^n \tag{8.30}$$

it will be found that each term has to satisfy (8.29) separately and hence

$$g_n[2ni \sin \alpha] + \bar{g}_n[2ni \sin n\alpha] = 0. \tag{8.31}$$

This leads to $g_n=0$ even fcr $\alpha=\pi/2$.

This last conclusion is quite at variance with the analogous result for harmonic functions F where the condition $F=0$ over edges of a right angle leads upon reflection across the edges to $F=R(f)$, where f is a single-valued analytic function at the corner point.

Proper product solutions for the wedge are furnished by

$$r^n\left[\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta + \text{const.} \begin{matrix}\sin\\ \cos\end{matrix}\, \overline{n-2}\,\theta\right]. \tag{8.32}$$

9. **Inversion. General conformal mapping.** Under a conformal transformation

$$z = z(w), \qquad w = u + iv \tag{9.1}$$

solutions of (1.1) do *not* transform into solutions of the same equation in the coordinates u, v. Indeed, under (9.1), the general function (2.10) becomes

$$H = R[f(w) + \overline{z(w)}g(w)] \tag{9.2}$$

where, properly speaking, $f(w)=f[z(w)]$, $g(w)=g[z(w)]$. One recognizes the obvious difference between this and the general biharmonic function in the (u, v) coordinates:

$$H = R[f(w) + \bar{w}g(w)]. \tag{9.3}$$

Even though (9.2) still involves two analytic functions, one cannot use conformal mapping for the biharmonic problem in a manner similar to the harmonic one.

While the above statement is true of general conformal mapping, there is one outstanding difference, namely inversion in a circle. Using the form (2.15) for H:

$$H = F + r^2G \tag{9.4}$$

one obtains after inversion in $r=a$

$$H^* = F^* + (a^4/r^2)G^* \tag{9.5}$$

whence follows that

$$r^2H^* = r^2F^* + a^4G^* \tag{9.6}$$

is of the same form as the right-hand member of (9.4) and thus biharmonic. Thus while inversion does not transform solutions of (1.1) into other solutions of this equation, inversion followed by multiplication by r^2 does([21]).

Moreover, if H satisfies (1.4) along a particular curve or boundary C, then r^2H will also satisfy the same boundary conditions along C^*, the inverse image of C. For the Airy's function this corresponds to the condition of a free boundary. More generally the conditions

$$H = P, \qquad \partial H/\partial n = \partial P/\partial n, \qquad P = C_0 + C_1x + C_2y \tag{9.7}$$

may hold along any portion of C free from stress. This is transformed into

$$r^2H^* = C_0r^2 + C_1a^2x + C_2a^2y = r^2P^*, \quad \partial(r^2H^*)/\partial n = \partial(r^2P^*)/\partial n. \tag{9.8}$$

The term C_0r^2 in Airy's function corresponds to a state of constant tension or pressure. By subtracting this term a free boundary portion still results.

Both Green's functions of §§4, 6 can be obtained by proper inversion of the Green's function of a circular region with pole at the center. Likewise the Airy's function (5.25), corresponding to a concentrated lead at a point P of the rectilinear boundary of a half-plane, can be made to yield by inversion and subtraction of a constant pressure term the solution for a disk subject to two equal and opposite forces, one at P, the image of the other at the image of the point at infinity.

For general conformal transformations (9.1) the boundary condition (2.24) becomes transformed into

$$0 = 2\partial H/\partial w \tag{9.9}$$

and hence (2.9), (9.2) yield

(21) This result is due to Mitchell, Proc. London Math. Soc. vol. 34 (1902) p. 134; see also Love, p. 154.

$$0 = f'(w) + \overline{z(w)}g'(w) + z'(w)\overline{g(w)}, \tag{9.10}$$

while the boundary condition (2.22) is transformed into

$$\begin{aligned} 2\partial H/\partial w &= f'(w) + \overline{z(w)}g'(w) + z'(w)\overline{g(w)} \\ &= (\gamma - i\delta)dz/dw = (\gamma - i\delta)z'(w). \end{aligned} \tag{9.11}$$

A more thorough discussion of these boundary conditions and of the functional equations resulting therefrom will be postponed for a future occasion. For the present we close with an illustration of the use of (9.10) for the determination of the Airy's function for a tension member with an elliptical hole.

The mapping function

$$z/c = w + 1/w, \qquad c > 0, \tag{9.12}$$

maps the outside of $|w| = a$ on the outside of the ellipse of semi-axes $c(a+1/a)$, $c(a-1/a)$ in the z-plane. Equation (9.10) now takes on the form

$$f'(w) + c[(a^2/w) + (w/a^2)]g'(w) + c(1 - 1/w^2)\bar{g}(a^2/w) = 0. \tag{9.13}$$

At infinity the behavior of the Airy's function is described by (see (6.17))

$$H = x^2/2 = R(z^4/4 + z\bar{z}/4) = R(w^2/4 + \bar{z}w/4) \tag{9.14}$$

thus making

$$f'(w) = w/2, \qquad g(w) = w/4 \tag{9.15}$$

for large $|w|$. It turns out that (9.13) can be satisfied by putting

$$f'(w) = w/2 + A/w + B/w^3, \qquad g(w) = w/4 + C/w \tag{9.16}$$

where A, B, C are real constants, thus adding to f', g in (9.14) polynomials in $1/w$. Indeed, (9.13) yields upon equation of net coefficients of w, $1/w$, $1/w^3$ to zero,

$$\begin{gathered} 1/2 + c/4a^2 + Cc/a^2 = 0, \qquad A + ca^2/2 + 2Cc/a^2 = 0, \\ B - Cca^2 - a^2c/4 = 0, \end{gathered} \tag{9.17}$$

and these determine the values C, A, and B in turn.

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