GEOMETRICAL CHARACTERIZATIONS OF SOME FAMILIES OF DYNAMICAL TRAJECTORIES

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1. **Introduction.** A broad problem in differential geometry is that of characterizing, by a set of geometrical properties, the family of curves which is defined by a given system of differential equations, of a more or less special form. The problem has been studied especially by Kasner and his students, and characterizations have been obtained for various families of curves which are of geometrical or physical importance(1). However, the interesting problem of characterizing the family of trajectories of an electrified particle moving in a static magnetic field does not seem to have been considered heretofore. The present paper gives the principal results of a study of this problem.

Specifically, our chief problem is that of characterizing the five-parameter family of trajectories of a particle moving according to one or the other of the following systems of differential equations of motion:

$$\ddot{x} = \omega\dot{y} - \psi\dot{z}, \qquad \ddot{y} = \phi\dot{z} - \omega\dot{x}, \qquad \ddot{z} = \psi\dot{x} - \phi\dot{y}; \tag{1}$$

$$\begin{aligned} \frac{d}{dt}\,\frac{\dot{x}}{[1 - c^{-2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)]^{1/2}} &= \omega\dot{y} - \psi\dot{z}, \\ \frac{d}{dt}\,\frac{\dot{y}}{[1 - c^{-2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)]^{1/2}} &= \phi\dot{z} - \omega\dot{x}, \\ \frac{d}{dt}\,\frac{\dot{z}}{[1 - c^{-2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)]^{1/2}} &= \psi\dot{x} - \phi\dot{y}. \end{aligned} \tag{2}$$

Here x, y, and z are the rectangular coordinates of the particle with respect to a fixed set of axes, the dots indicate differentiation with respect to the time t, and c is a positive constant. ϕ, ψ, and ω are functions of x, y, and z. We assume that these functions are of class C^2 throughout a certain open region to which our considerations are restricted, and that the functions do not all vanish identically in any three-dimensional part of that region(2).

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(1) We shall not have much occasion to refer to the literature. However, in order to indicate the background of our problem, we list a few typical articles in the Bibliography at the end of the paper. References to the entries in the Bibliography will be made by numbers in brackets.

(2) If ϕ, ψ, and ω were all zero throughout a three-dimensional region, the family of trajectories in that region would be merely the four-parameter family of straight line segments traversing the region.

The system of equations (1) can be regarded as the system of equations of motion of an electrified particle moving, in accordance with the laws of Newtonian dynamics, in a static magnetic field. If the equations are interpreted in this way, the functions ϕ, ψ, and ω are proportional to the components of the magnetic induction parallel to the coordinate axes. The functions are also proportional to the electric charge carried by the particle, and they are inversely proportional to the mass of the particle.

Likewise, if c denotes the speed of light *in vacuo*, equations (2) can be regarded as the equations of motion of an electrified particle moving according to the laws of special relativistic dynamics in a static magnetic field.

It will be shown in §4 that the families of trajectories resulting from the systems of differential equations (1) and (2) are actually identical. For this reason, most of our attention will be focused on the simple system of equations (1) and its consequences.

The characterization of the five-parameter family of trajectories resulting from equations (1) is given in §3. In other parts of the paper we define and characterize certain other families of curves, which are related in various ways to that family.

In §2 we define and characterize certain auxiliary families of plane curves, called associated planar families.

In §4 we separate the five-parameter family of trajectories into ∞^1 four-parameter subfamilies, called natural families of trajectories, and we characterize a natural family.

Finally, in §5 we give a brief discussion of the family of trajectories of a particle moving in a fixed plane according to a system of differential equations of motion which is, in a certain sense, the plane analogue of the system of equations (1). It is only in quite special cases that this motion in a plane has a realistic physical interpretation in terms of an electrified particle in a magnetic field; and the discussion is given here mainly for the sake of its purely mathematical interest.

Before concluding this introductory section we shall subject our problem to some final adjustments, and we shall then give the system of differential equations which defines directly the family of trajectories resulting from the equations of motion (1).

It is easily seen that the system of equations (1) is invariant in form under changes of the rectangular coordinate system, provided ϕ, ψ, and ω are transformed as the components of a vector point function. In virtue of this invariance, and of what we have previously assumed about ϕ, ψ, and ω, we can assume that the coordinate axes are so oriented that ω does not vanish identically in any three-dimensional part of the region under consideration. We shall actually assume something more; namely, that the region is so restricted that ω is never zero in it. Thus we shall be studying the family of trajectories in the neighborhood of what we naturally call an ordinary point; and we shall not consider the properties of the family in the neighborhoods of points belonging to certain possible exceptional loci, of at most two dimensions.

The system of differential equations which defines the family of trajectories of a particle moving according to the equations of motion (1) is obtained by eliminating the time from the latter equations. The elimination is straightforward, and the details need not be given here. The resulting system of equations can be written in the form:

$$
\begin{aligned}
y''' &= \frac{-\omega'+\phi'z'-\omega'y'^2+\psi'y'z'}{-\omega+\phi z'-\omega y'^2+\psi y'z'}\, y'' \\
&\quad + \left[\frac{-3\omega y'+2\psi z'}{-\omega+\phi z'-\omega y'^2+\psi y'z'} + \frac{(\phi+\psi y')(\psi-\phi y'-\omega y'z'+\psi z'^2)}{(-\omega+\phi z'-\omega y'^2+\psi y'z')^2}\right] y''^2, \\
z'' &= \frac{\psi-\phi y'-\omega y'z'+\psi z'^2}{-\omega+\phi z'-\omega y'^2+\psi y'z'}\, y'',
\end{aligned} \tag{3}
$$

where the primes indicate total differentiation with respect to x.

2. **The associated planar families of curves and their characteristic properties.** Consider an arbitrary plane Π. Through each point of Π there pass ∞^2 trajectories, of the family defined by equations (3), which are tangent to the plane at the point. We project the third order differential elements belonging to these curves, at their points of tangency to Π, orthogonally upon Π. Thus we get ∞^4 differential elements of the third order in the plane. This aggregate of differential elements is defined, with reference to a coordinate system in Π, by a differential equation of the third order. This differential equation defines a certain three-parameter family of plane curves, which we call *the associated planar family of curves in the plane* Π. We are now going to characterize the associated planar families of curves(3).

Obviously, the associated planar family of curves in a plane $z=$constant is defined by the equation

$$
y''' = \frac{1}{\omega}\left[\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}\, y'\right] y'' + \left[\frac{3y'}{1+y'^2} + \frac{(\phi+\psi y')(\psi-\phi y')}{\omega^2(1+y'^2)^2}\right] y''^2, \tag{4}
$$

where the z which occurs in the functions ϕ, ψ, ω, $\partial\omega/\partial x$, and $\partial\omega/\partial y$ is given the appropriate constant value(4).

Equation (4) is a special case of the more general equation

$$
y''' = G(x, y, y')y'' + H(x, y, y')y''^2. \tag{5}
$$

Consequently, we know from Kasner's work that the family of curves defined

(3) More exactly, the associated planar families that satisfy a certain condition, which is implicit in the discussion of §§1 and 2, and which will be explained explicitly in §3.

(4) It is convenient, for the time being, to consider the coordinate system as having been transformed, so that the plane under consideration is represented by the equation $z=0$.

by equation (4) possesses the following property(5).

Property A. *If, for each of the ∞^1 curves passing through a given point O: (x, y) in a given direction, we construct the parabola which osculates the curve at O, the locus, Γ, of the foci of these parabolas is a circle passing through O.*

It should be remarked that if the lineal element (x, y, y') is such that $G(x, y, y')$ is zero, the locus Γ is a straight line passing through O. The above statement of Property A is valid for this case, provided that we regard the circle as having an infinite radius.

Conversely, it is known that if a three-parameter family of plane curves has Property A, the family is defined by some differential equation of the form (5).

For the sake of convenience in the statement and discussion of the further properties, we introduce two new systems of rectangular coordinates in the plane Π. The one, the (ξ, η) system, has its origin at the point O, and the ξ and η axes are parallel to the x and y axes, respectively. The other, the (u, v) system, is obtained from the (ξ, η) system by means of the rotation about O such that the u-axis, which is the transform of the ξ-axis, is the common tangent of the ∞^1 curves considered above. Analytically, the two coordinate systems are related by the equations

$$\xi + y'\eta = (1 + y'^2)^{1/2}u, \qquad -y'\xi + \eta = (1 + y'^2)^{1/2}v.$$

Consider a family of curves defined by an equation of the form (5). It is readily shown that the coordinates, in the (u, v) system, of the center of the circle Γ, which corresponds in the manner described to the lineal element (x, y, y'), are given by the formulae:

$$(6) \quad u_c = 3(4G)^{-1}(1 + y'^2)^{1/2}, \quad v_c = (4G)^{-1}[3y' - (1 + y'^2)H](1 + y'^2)^{1/2}.$$

In our case we have

$$u_c = \frac{3\omega(1 + y'^2)^{1/2}}{4(\omega_x + \omega_y y')}, \qquad v_c = \frac{-(\phi + \psi y')(\psi - \phi y')}{4\omega(\omega_x + \omega_y y')(1 + y'^2)^{1/2}},$$

where $\omega_x = \partial\omega/\partial x$, $\omega_y = \partial\omega/\partial y$. It follows that

$$\frac{v_c}{u_c} = \frac{\phi^2 + \psi^2}{6\omega^2} \sin 2(\theta - \alpha),$$

where $\theta = \arc\tan y'$, $\alpha = \arc\tan (\psi/\phi)$. Thus we have a second property of the associated planar family of curves, which we can state as follows:

Property B. *The circle Γ which corresponds, according to Property* A, *to the lineal element (x, y, y') is so situated that the tangent, v_c/u_c, of the angle be-*

(5) See [1].

tween the line drawn from O to the center of Γ and the lineal element either is zero for all values of y', or is proportional to the sine of twice the angle, $\theta-\alpha$, between the lineal element and a direction Δ_{II} which is fixed for the point O. In the latter case the factor of proportionality is a function of the coordinates of O.

When the lineal element (x, y, y') is such that $G=0$, the locus Γ degenerates into the straight line

$$3u + [3y' - (1 + y'^2)H]v = 0.$$

In this case we naturally interpret the line drawn from O to the center of Γ as the line through O perpendicular to the straight line. It is easily verified that the above statement of Property B, with this interpretation of the phraseology, holds in this degenerate case.

We have $v_c/u_c=0$ for all values of y' when, and only when, the point O is a zero of both of the functions ϕ and ψ.

Now suppose, conversely, that a family of curves having Property A also has Property B. By equations (6), we must have the relation

$$H = \frac{3y'}{1 + y'^2} + \frac{(\phi_1 + \psi_1 y')(\psi_1 - \phi_1 y')}{\omega_1^2(1 + y'^2)^2},$$

where ϕ_1, ψ_1, and ω_1 are functions of x and y.

The expression for u_c can be written in the form

$$u_c = (3/4)\omega(\omega_x^2 + \omega_y^2)^{-1/2} \sec(\theta - \beta),$$

where

$$\beta = \text{arc tan } (\omega_y/\omega_x).$$

Hence, we have another property of the associated planar family, which we can state in the following form.

PROPERTY C. *Either the locus Γ corresponding to the lineal element (x, y, y') degenerates into a straight line for all values of y', or there is associated with the point O a certain point P, such that if we draw a line through P perpendicular to OP, intersecting the u-axis in a point which we shall call Q, the center of the circle Γ corresponding to (x, y, y') is on the line passing through Q perpendicular to the u-axis.*

We have the first situation described in the statement of the property when, and only when, the point O is a zero of both of the functions ω_x and ω_y. In this case we consider that the associated point P does not exist.

Conversely, it is easily seen that if a family of curves having Property A also has Property C, the function $G(x, y, y')$ in the defining differential equation is a linear function of y'.

Suppose that there exist lineal elements (x, y, y') for which the corresponding loci Γ are not straight lines. Then there exists a point O for which the associated point P exists, and all points in the neighborhood of O have this property.

Let ξ_P and η_P denote the coordinates of the point P in the (ξ, η) system of coordinates. We readily find that

$$\xi_P = \frac{3}{4}\,\frac{\omega_x/\omega}{(\omega_x/\omega)^2 + (\omega_y/\omega)^2}, \qquad \eta_P = \frac{3}{4}\,\frac{\omega_y/\omega}{(\omega_x/\omega)^2 + (\omega_y/\omega)^2};$$

and therefore

$$\omega_x/\omega = \frac{3}{4}\,\frac{\xi_P}{\xi_P^2 + \eta_P^2}, \qquad \omega_y/\omega = \frac{3}{4}\,\frac{\eta_P}{\xi_P^2 + \eta_P^2}.$$

Since ω_x/ω and ω_y/ω are the partial derivatives of $\log \omega$, we have the following property.

PROPERTY D. *When the initial point O is changed, the coordinates of the associated point P, if it exists, change in the manner described by the equation*

$$\frac{\partial}{\partial y}\,\frac{\xi_P}{\xi_P^2 + \eta_P^2} = \frac{\partial}{\partial x}\,\frac{\eta_P}{\xi_P^2 + \eta_P^2}.$$

As has been said above, if a family of curves having Property A has also Property C, the function G in the defining differential equation is of the form

$$G = g_1(x, y) + g_2(x, y)y'.$$

If the functions g_1 and g_2 are not both zero at the point $O: (x, y)$, the associated point P exists, and we have the relations

$$g_1 = \frac{3}{4}\,\frac{\xi_P}{\xi_P^2 + \eta_P^2}, \qquad g_2 = \frac{3}{4}\,\frac{\eta_P}{\xi_P^2 + \eta_P^2}.$$

Hence, if a family of curves having Properties A and C also has Property D, we have the relation

$$\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x},$$

and we can write $g_1 = \omega_x/\omega$, $g_2 = \omega_y/\omega$, where ω is some function of x and y. If the point P associated with the point O does not exist, these formulae still hold, it being understood that ω_x and ω_y are then both zero at the point O.

We see, therefore, that if a three-parameter family of plane curves has Properties A, B, C, and D, it is defined by a differential equation of the form

$$y''' = \frac{1}{\omega}\left[\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}\,y'\right]y'' + \left[\frac{3y'}{1 + y'^2} + \frac{(\phi_1 + \psi_1 y')(\psi_1 - \phi_1 y')}{\omega_1^2(1 + y'^2)^2}\right]y''^2.$$

If we multiply the numerator and the denominator of the second term of the coefficient of y''^2 by $(\omega/\omega_1)^2$, this differential equation assumes the form (4). Therefore, the set of properties A, B, C, and D is completely characteristic of families of curves defined by differential equations of the form (4).

3. The characteristic properties of the family of curves defined by equations (3). We are now ready to proceed with the solution of our main problem, that is, the problem of characterizing the family of curves defined by (3).

One property of the family of curves can be inferred at once from equations (1). In fact, the differential equations of motion show that the acceleration vector $(\ddot{x}, \ddot{y}, \ddot{z})$ is perpendicular to the velocity vector $(\dot{x}, \dot{y}, \dot{z})$ and to the vector (ϕ, ψ, ω). (Of course, this statement is vacuous for points at which $\phi=\psi=\omega=0$. As we have explained previously, we assume that there are no such points in the region under consideration.) Hence we have the following property.

PROPERTY I. *The principal normals (at O) of the ∞^3 curves passing through an arbitrary point O are all perpendicular to a certain direction Δ associated with O.*

Conversely, let us consider a family of curves in three-dimensional space, such that ∞^1 curves pass through each point in each direction. The family is defined by some system of differential equations of the form

$$y''' = f_1(x, y, z, y', z', y''), \qquad z'' = f_2(x, y, z, y', z', y''). \tag{7}$$

We shall impose Property I on the family of curves, and see what effect this restriction has on the form of the differential equations.

The direction cosines of the principal normal of the typical curve at the typical point $O:(x, y, z)$ are proportional to

$$-(y'y''+z'z''), \qquad (1+z'^2)y''-y'z'z'', \qquad (1+y'^2)z''-y'z'y''.$$

If the family has Property I, we must have

$$\begin{aligned} -\lambda(x, y, z)(y'y''+z'z'') + \mu(x, y, z)\left[(1+z'^2)y''-y'z'z''\right] \\ + \nu(x, y, z)\left[(1+y'^2)z''-y'z'y''\right] = 0, \end{aligned}$$

where λ, μ, and ν are the direction cosines of the direction Δ[6]. Therefore, the function f_2 in the second of equations (7) must be

$$f_2 = \frac{\mu - \lambda y' - \nu y'z' + \mu z'^2}{-\nu + \lambda z' - \nu y'^2 + \mu y'z'}\, y''. \tag{8}$$

This function is of the same form as the right-hand member of the second of equations (3). Hence Property I is the geometrical meaning of the special form of that equation.

Since, as has been pointed out in §1, the system of equations (1) is in-

(6) In our case λ, μ, and ν are proportional to ϕ, ψ, and ω, respectively.

variant in form under changes of the rectangular coordinate system, the associated planar family of curves that we have studied in §2 (that is, the family in a plane z=constant, such that ω does not vanish in the two-dimensional part of the plane under consideration) is typical of the associated planar family of curves in any plane Π such that the component of the vector (ϕ, ψ, ω) perpendicular to Π does not vanish in the two-dimensional part of the plane under consideration. Therefore we can state the following property.

Property II. *Let Π be a plane such that the direction Δ, which is associated with a point O of the plane, according to Property* I, *is not contained in* Π *for any point O belonging to the two-dimensional part of* Π *under consideration*(7). *Then the associated planar family of curves in* Π *possesses Properties* A, B, C, D.

Properties I and II are not sufficient to characterize completely the family of curves defined by equations (3); we also need some properties which have the effect of relating the associated planar families of curves in different planes. Two simple properties of the kind, which are adequate for our purpose, will now be stated.

Property III. *The direction Δ_{Π}, referred to in the statement of Property* B, *is the orthogonal projection upon the plane* Π *of the direction Δ, referred to in the statement of Property* I; *and the factor of proportionality, referred to in the statement of Property* B, *is* $(1/6)(\tan \gamma)^2$, *where γ is the angle between Δ and the normal to the plane* Π.

In order to be able to give a concise statement of Property IV, we shall first explain some matters of notation.

As in §2, we let x and y be rectangular coordinates in the plane Π; also we set up a rectangular, (ξ, η), coordinate system in the plane, with its origin at the arbitrarily chosen point O, and with the ξ and η axes parallel to the x and y axes, respectively.

By Property C, there is associated, in general, with the point O a certain point P, the coordinates of which are ξ_P and η_P. Let P_1 denote the inverse of P with respect to the circle

$$\xi^2 + \eta^2 = 3/4,$$

and let ξ_{P_1} and η_{P_1} denote the coordinates of P_1.

Then we have the following property.

Property IV. *If the point P exists, the coordinates ξ_{P_1} and η_{P_1} are the partial logarithmic derivatives, with respect to x and y, respectively, of a function which is the component perpendicular to the plane* Π *of a vector which has the direction Δ, and which, while it is dependent upon the point O, is not dependent upon the plane* Π.

(7) Henceforth we shall call such a plane an *ordinary plane.*

These properties follow at once from the details of the work by which we established Properties B and C in §2.

Now we shall consider a five-parameter family of curves having Property I (and being defined, therefore, by a system of differential equations (7), with the function f_2 given by (8)); and we shall see how, by imposition of the various properties contained in Properties II, III, and IV upon the family, the defining system of differential equations is gradually specialized into the form (3).

We shall make use of the following transformation of coordinates

$$\begin{aligned} x &= X, & y &= cY - sZ, & z &= sY + cZ, \\ X &= x, & Y &= cy + sz, & Z &= -sy + cz, \end{aligned} \tag{9}$$

where $c=\cos\epsilon$, $s=\sin\epsilon$, ϵ being an arbitrary angle. Under this transformation of coordinates the functions λ, μ, and ν are transformed as follows:

$$\begin{aligned} \lambda &= \Lambda, & \mu &= cM - sN, & \nu &= sM + cN, \\ \Lambda &= \lambda, & M &= c\mu + s\nu, & N &= -s\mu + c\nu, \end{aligned} \tag{9'}$$

since they are the components of a unit vector. Here, of course, Λ, M, and N are to be interpreted as functions of X, Y, and Z.

In terms of the new variables, the system of equations (7), with f_2 given by (8), has the form

$$\begin{aligned} cY''' - sZ''' &= f_1(X, cY - sZ, sY + cZ, cY' - sZ', sY' + cZ', cY'' - sZ''), \\ Z'' &= \frac{M - \Lambda Y' - NY'Z' + MZ'^2}{-N + \Lambda Z' - NY'^2 + MY'Z'} Y'', \end{aligned} \tag{10}$$

where the primes indicate differentiation with respect to X.

For the sake of brevity, let us write the second of equations (10) in the form

$$Z'' = K(X, Y, Z, Y', Z')Y''.$$

Then we have

$$Z''' = (K_X + K_Y Y' + K_Z Z')Y'' + (K_{Y'} + KK_{Z'})Y''^2 + KY''',$$

where the subscripts indicate partial differentiation in the usual way. Consequently, the first of equations (10) takes the form

$$\begin{aligned} (c - sK)Y''' = {} & f_1(X, cY - sZ, sY + cZ, cY' - sZ', sY' + cZ', (c - sK)Y'') \\ & + s(K_X + K_Y Y' + K_Z Z')Y'' + s(K_{Y'} + KK_{Z'})Y''^2. \end{aligned}$$

We shall consider the associated planar family of curves in a plane $Z=$constant, assuming that N does not vanish in the two-dimensional part of the plane under consideration. This family is defined by the equation

$$(11)\qquad \begin{aligned}(c - sK_0)Y''' = {}& f_1(X, cY - sZ, sY + cZ, cY', sY', (c - sK_0)Y'') \\ & + s(K_X + K_Y Y')_0 Y'' + s(K_{Y'} + KK_{Z'})_0 Y''^2,\end{aligned}$$

where the subscripts 0 indicate that the expressions to which they are appended are to be evaluated for $Z' = 0$.

Now we impose the condition that the family of curves defined by (11) shall have Property A. Under this condition the function $f_1(X, \cdots, (c - sK_0)Y'')$ must be of the form

$$\begin{aligned} & g(X, cY - sZ, sY + cZ, cY', sY')(c - sK_0)Y'' \\ & \qquad + h(X, cY - sZ, sY + cZ, cY', sZ')(c - sK_0)^2 Y''^2;\end{aligned}$$

and hence the function $f_1(x, y, z, y', z', y'')$ must be of the form

$$(12)\qquad f_1(x, y, z, y', z', y'') = g(x, y, z, y', z')y'' + h(x, y, z, y', z')y''^2.$$

(In fact, we can always write

$$f_1 = gy'' + hy''^2 + l(x, y, z, y', z'/y', y'');$$

and the function l must vanish when $Z' = -sy' + cz' = 0$. However, c and s are arbitrary, subject to the one restriction $c^2 + s^2 = 1$. Consequently, l must vanish for all values of the argument z'/y'. Hence, l must be identically zero.)

Now equation (11) can be written in the form

$$(13)\qquad \begin{aligned} Y''' = {}& \left[g(X, cY - sZ, sY + cZ, cY', sY') + \frac{s}{c - sK_0}(K_X + K_Y Y')_0\right] Y'' \\ & + \left[(c - sK_0)h(X, cY - sZ, sY + cZ, cY', sY') \right. \\ & \left. + \frac{s}{c - sK_0}(K_{Y'} + KK_{Z'})_0\right] Y''^2. \end{aligned}$$

We next impose the condition that the family of curves defined by equations (7), with the functions f_1 and f_2 given by equations (12) and (8), respectively, shall be such that every associated planar family of curves in an ordinary plane shall have Property B, and such that the five-parameter family of curves shall have Property III.

It follows from this condition and equation (13), without any difficulty, that the function

$$h(X, cY - sZ, sY + cZ, cY', sY')$$

is given by the equation

$$(14)\qquad h = \frac{1}{c - sK_0}\left[\frac{3Y'}{1 + Y'^2} + \frac{(\Lambda + MY')(M - \Lambda Y')}{N^2(1 + Y'^2)^2}\right] - \frac{s(K_{Y'} + KK_{Z'})_0}{(c - sK_0)^2}.$$

In the formula (14) we now transform the variables back to the (x, y, z) system of coordinates, by means of equations (9) and (9'). Here we make use of the relations

$$cY' = y', \qquad sY' = z',$$

which follow from equations (9), and from the equation $Z' = 0$ which was used in deriving equation (11). After some tedious, but entirely straightforward, algebraic reduction, we obtain the result

$$\begin{aligned} h(x, y, z, y', z') &= \frac{2\mu z' - 3\nu y'}{-\nu + \lambda z' - \nu y'^2 + \mu y' z'} \\ &\quad + \frac{(\lambda + \mu y')(\mu - \lambda y' - \nu y' z' + \mu z'^2)}{(-\nu + \lambda z' - \nu y'^2 + \mu y' z')^2}. \end{aligned} \tag{15}$$

The function h given by (15) is of the same form as the coefficient of y''^2 in the first of equations (3). Thus, we have determined the geometrical significance of the special form of that coefficient.

We now consider a five-parameter family of curves such that every associated planar family of curves in an ordinary plane possesses Properties A and B, and such that the five-parameter family possesses Properties I and III. We impose the condition that every associated planar family in an ordinary plane shall have Properties C and D, and that the five-parameter family shall have Property IV.

Its results from equation (13), and from the condition that we are imposing, that the function $g(X, cY-sZ, sY+cZ, cY', sY')$ is of the form

$$g = \frac{1}{\Omega}\left[\frac{\partial\Omega}{\partial X} + \frac{\partial\Omega}{\partial Y} Y'\right] - \frac{s}{c - sK_0}(K_X + K_Y Y')_0. \tag{16}$$

By Property IV, Ω is the component perpendicular to the plane $Z = \text{constant}$ of a vector which has the direction cosines Λ, M, N.

In the formula (16) we now transform the variables back to the (x, y, z) system of coordinates, in the manner that has been described above in connection with the derivation of equation (15). After some lengthy, but elementary, reduction, we obtain the result

$$g = \frac{-\omega' + \phi' z' - \omega' y'^2 + \psi' y' z'}{-\omega + \phi z' - \omega y'^2 + \psi y' z'}. \tag{17}$$

Here, of course, ϕ, ψ, and ω are the components, in the (x, y, z) system of coordinates, of the vector referred to in the statement of Property IV.

In (8) and (15) we can obviously replace λ, μ, and ν by ϕ, ψ, and ω, respectively. Hence, we have shown that if a five-parameter family of curves possesses Properties I, II, III, and IV, it is defined by some system of differential equations of the form (3). Thus, those properties are, together, completely characteristic of families of curves defined by systems of equations of the form (3).

The concept of the associated planar families of curves is somewhat recondite, and it would certainly be desirable to have a characterization of the family of curves defined by equations (3) which did not depend upon that concept. So far we have no complete characterization of that kind. However, we are able to state another property, not involving that concept, which, together with Property I, goes part way toward characterizing the family of curves.

Let us consider the ∞^1 curves, belonging to the family defined by equations (3), which pass through a given point $O:(x, y, z)$ in a given direction (y', z'). By Property I, all of these curves have the same tangent, principal normal, and binormal at the point O. The different curves have different osculating spheres at O.

Let η and ζ be rectangular coordinates in the common normal plane at O, η and ζ being measured from O in the directions of the principal normal and the binormal, respectively.

A straightforward, but somewhat lengthy, calculation leads to the result that the locus of the centers of the ∞^1 osculating spheres is represented by the equation

$$(A\eta - \zeta)(B\eta + C) + (D\eta + E)\eta = 0, \tag{18}$$

where

$$\begin{aligned}
A &= (1 + y'^2 + z'^2)^{-1/2}[(1 + y'^2)k - y'z'],\\
B &= (k_x + k_y y' + k_z z')[(1 + y'^2 + z'^2)(1 + k^2) - (y' + kz')^2]^{1/2},\\
C &= (1 + y'^2 + z'^2)^{3/2}(k_{y'} + kk_{z'}),\\
D &= (1 + y'^2 + z'^2)^{-1/2}[(1 + y'^2 + z'^2)(1 + k^2) - (y' + kz')^2]^{3/2}g,\\
E &= [(1 + y'^2 + z'^2)(1 + k^2) - (y' + kz')^2][(1 + y'^2 + z'^2)h - 3(y' + kz')].
\end{aligned}$$

Here g, h, and k are the functions (of x, y, z, y', and z') occurring in the equations

$$y''' = gy'' + hy''^2, \qquad z'' = ky'', \tag{19}$$

defining the family of curves. *The result is not dependent upon the special forms of these functions which we have in the case of equations* (3). Of course, the subscripts in the symbols k_x, k_y, k_z, $k_{y'}$, and $k_{z'}$ indicate partial differentiation in the usual way.

Equation (18) represents a hyperbola, which passes through the point O, and has one of its asymptotes parallel to the ζ axis. We find that the distance, η_0, of the point O from that particular asymptote is given by the formula:

$$(20)\quad \eta_0 = -\frac{(1+y'^2+z'^2)^{3/2}(k_{y'}+kk_{z'})}{[(1+y'^2+z'^2)(1+k^2)-(y'+kz')^2]^{1/2}(k_x+k_yy'+k_zz')}.$$

It is worth remarking that since the direction of the common tangent to the ∞^1 curves is determined by y' and z', and since the direction of the common binormal is determined by y', z', and k, the complicated right-hand member of equation (20) actually admits of a geometrical interpretation.

For the hyperbola to reduce to a pair of straight lines it is necessary that $C=0$, or that $BE=CD$. It is readily verified that neither of these conditions is satisfied, in general, when equations (19) are specialized into equations (3). (The first condition requires that k satisfy the equation $k_{y'}+kk_{z'}=0$; the second condition requires that a special relation exists between g, h and k[8].)

We now see that any family of curves defined by a system of equations of the form (19), and, in particular, the family of curves defined by the equations (3), possesses the following property:

PROPERTY V. *The locus of the centers of the osculating spheres (at O) of the ∞^1 curves which pass through a fixed point O in a fixed direction is a hyperbola, which:*

(i) *passes through O;*

(ii) *has one of its asymptotes parallel to the common binormal of the ∞^1 curves at O;*

(iii) *is such that the distance, η_0, of O from that asymptote is given by the formula* (20).

Conversely, it is readily found that if a family of curves defined by a system of equations of the form

$$y''' = f(x, y, z, y', z', y''), \qquad z'' = k(x, y, z, y', z')y''$$

(and therefore having the property that all of the curves passing through a given point in a given direction have the same osculating plane at the point) has Property V, the function f must be of the form

$$f = g(x, y, z, y', z')y'' + h(x, y, z, y', z')y''^2.$$

Thus Properties I and V, together, characterize the five-parameter family of curves to the extent of limiting the defining system of differential equations to the form:

$$y''' = gy'' + hy''^2, \qquad z'' = \frac{\psi - \phi y' - \omega y'z' + \psi z'^2}{-\omega + \phi z' - \omega y'^2 + \psi y'z'}\, y'',$$

(8) Kasner and others have studied families of curves defined by systems of equations of the form (19), where the functions k happened to satisfy the partial differential equation $k_{y'}+kk_{z'}=0$. (See, for instance, [2 and 10].) In any such case, of course, the locus of the centers of the osculating spheres is a straight line.

where ϕ, ψ, and ω are functions of x, y, and z, and g and h are functions of x, y, z, y', and z'.

4. **Natural families of trajectories.** The differential equations of motion (1) possess the integral

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2E_c. \tag{21}$$

Here E_c is a constant of integration, which is, aside from a constant multiplier, the kinetic energy of the particle. The subscript c is intended to distinguish this constant, in the classical or Newtonian theory, from an analogous constant E_r in the relativistic theory.

Consequently, the five-parameter family of curves defined by equations (3) breaks up into ∞^1 four-parameter subfamilies, the different subfamilies corresponding to different values of E_c. We shall call each of these subfamilies a natural family of trajectories.

In virtue of equation (21), the second and third of equations (1) are jointly equivalent to the following pair of equations

$$\begin{aligned} y'' &= (2E_c)^{-1/2}(1 + y'^2 + z'^2)^{1/2}(-\omega + \phi z' - \omega y'^2 + \psi y'z'), \\ z'' &= (2E_c)^{-1/2}(1 + y'^2 + z'^2)^{1/2}(\psi - \phi y' - \omega y'z' + \psi z'^2), \end{aligned} \tag{22}$$

where the primes denote differentiation with respect to x. This system of differential equations defines the natural family of trajectories corresponding to the energy constant E_c.

The relativistic equations of motion (2) likewise possess an integral such as (21). However, in this case it is advantageous to write the integral in the form

$$c^2[1 - c^{-2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)]^{-1/2} = E_r. \tag{23}$$

Thus the five-parameter family of trajectories of a particle moving according to the equations (2) also breaks up into ∞^1 four-parameter natural families of trajectories.

It follows from equations (2) and (23) that the natural family of trajectories corresponding to the energy constant E_r is defined by the following system of differential equations:

$$\begin{aligned} y'' &= c(E_r^2 - c^4)^{-1/2}(1 + y'^2 + z'^2)^{1/2}(-\omega + \phi z' - \omega y'^2 + \psi y'z'), \\ z'' &= c(E_r^2 - c^4)^{-1/2}(1 + y'^2 + z'^2)^{1/2}(\psi - \phi y' - \omega y'z' + \psi z'^2). \end{aligned} \tag{24}$$

The equation $2E_c = (E_r^2 - c^4)/c^2$ establishes a one-to-one correspondence between the significant values of E_c and E_r ($E_c > 0$, $E_r > c^2$)(9). *Comparing the systems of equations* (22) *and* (24), *we see at once that the natural families of trajectories in the Newtonian and relativistic cases, for corresponding values*

(9) It is to be noted that corresponding values of E_c and E_r correspond to different values of the speed of the particle.

of the energy constants E_c and E_r, are identical.

From the last result we immediately infer the following: *The five-parameter family of trajectories of a particle moving according to equations* (1) *is identical with the family of trajectories of a particle moving according to equations* (2). Hence Properties I, II, III, and IV are also characteristic of the family of trajectories resulting from the equations of motion (2).

Of course, the identity of the families of trajectories in the Newtonian and relativistic cases results from the fact that the forces exerted by the magnetic field on the particle do no work, so that the speed of the particle, and consequently also its mass in the relativistic case, remain constant. With other types of forces, there may be great differences between the corresponding families of trajectories in the Newtonian and relativistic cases([10]).

Now we proceed to obtain a set of properties characterizing a natural family of trajectories, say the family defined by equations (22).

One property follows immediately from Property I of §3. We can state the property as follows.

PROPERTY $\mathrm{I_n}$. *The principal normals (at O) of the ∞^2 curves passing through an arbitrary point O are all perpendicular to a certain direction Δ associated with O.*

In a natural family of trajectories one curve passes through each point in each direction. Such a family of curves is defined by some system of differential equations of the form

$$y'' = f_1(x, y, z, y', z'), \qquad z'' = f_2(x, y, z, y', z').$$

When we impose the requirement that the family of curves shall possess Property $\mathrm{I_n}$, we find at once that the system of differential equations specializes into the form:

$$\begin{aligned} y'' &= f(x, y, z, y', z')(-\nu + \lambda z' - \nu y'^2 + \mu y'z'), \\ z'' &= f(x, y, z, y', z')(\mu - \lambda y' - \nu y'z' + \mu z'^2). \end{aligned} \tag{25}$$

Here, as before, λ, μ, and ν denote the direction cosines of the direction Δ.

Let us consider a trajectory which passes through the point $O:(x, y, z)$ in a direction making an angle θ with the direction Δ associated with O. We find that the radius of curvature, ρ, of the trajectory at O is given by the formula

$$\rho^{-2} = (2E_c)^{-1}(\phi^2 + \psi^2 + \omega^2)(\sin \theta)^2.$$

Hence we have the following property.

([10]) Compare, for instance, the results obtained by Kasner [1] for a certain problem in the Newtonian case with the results obtained by the present author [6] for the corresponding problem in the relativistic case.

PROPERTY II_n. *All trajectories (belonging to the natural family corresponding to E_c) which pass through a point O: (x, y, z) in directions making a fixed angle θ with the direction Δ associated with O have the same curvature at O. This curvature is proportional to* $\sin\theta$, *the factor of proportionality depending upon O.*

Conversely, suppose that a four-parameter family of curves having Property I_n also has Property II_n.

The family is defined by some system of differential equations of the form (25). Consequently, the radius of curvature of the curve passing through the point O: (x, y, z) in a direction making an angle θ with the direction Δ associated with O is given by the formula

$$\frac{1}{\rho^2} = \frac{[f(x, y, z, y', z')]^2}{1 + y'^2 + z'^2} (\sin\theta)^2.$$

In virtue of Property II_n, the function $f(x, y, z, y', z')$ must be of the form

$$f(x, y, z, y', z') = g(x, y, z)(1 + y'^2 + z'^2)^{1/2}.$$

Therefore, the defining system of differential equations (25) reduces to the form

$$\begin{aligned} y'' &= g(x, y, z)(1 + y'^2 + z'^2)^{1/2}(-\nu + \lambda z' - \nu y'^2 + \mu y'z'), \\ z'' &= g(x, y, z)(1 + y'^2 + z'^2)^{1/2}(\mu - \lambda y' - \nu y'z' + \mu z'^2). \end{aligned} \tag{26}$$

The system of equations (26) is of the same form as (22). Therefore, it follows that Properties I_n and II_n are together completely characteristic of natural families of trajectories.

5. **The case of motion in a plane.** So far we have been considering a particle moving in three-dimensional space. Now it is natural to ask what corresponds to the foregoing results in the case in which the particle moves in a fixed plane. For sundry physical reasons, it is only under quite special conditions that an electrified particle can move in a fixed plane in a static magnetic field. However, if we ignore questions of physical realizability, we can set up a system of differential equations of motion, analogous to the system (1), for motion in a plane; and we can proceed to study the resulting family of trajectories, simply for the sake of its mathematical interest. In this concluding section we give a brief account of the results obtained in this way.

As the plane analogue of the system of equations (1), we take the following:

$$\ddot{x} = \omega(x, y)\dot{y}, \qquad \ddot{y} = -\omega(x, y)\dot{x}. \tag{27}$$

Here x and y are the rectangular coordinates of the particle, and $\omega(x, y)$ is a function of class C^2 which we assume does not vanish in the region under consideration.

Eliminating the time from equations (27), we obtain the following equation defining the three-parameter family of trajectories of the particle:

$$y''' = \frac{1}{\omega}(\omega_x + \omega_y y')y'' + \frac{3y'}{1 + y'^2}y''^2. \tag{28}$$

We shall first characterize the family of curves defined by this equation.

We see at once that one characteristic property is just Property A, stated in §2. For the sake of uniformity in the notation, we shall refer to this property, in the remainder of this discussion, as Property $\mathrm{I_p}$.

Referring to equations (6), we see that in the present case we have $v_c = 0$. Therefore, we have the following property.

PROPERTY $\mathrm{II_p}$. *The circle* Γ, *which corresponds in the manner described in the statement of Property* $\mathrm{I_p}$ *to the lineal element* (x, y, y'), *has its center on the common tangent at O of the* ∞^1 *curves considered.*

Conversely, if a family of curves having Property $\mathrm{I_p}$ also has Property $\mathrm{II_p}$, we have $v_c = 0$ and, therefore, the function $H(x, y, y')$ in the defining differential equation is $3y'(1+y'^2)^{-1}$.

The last two of the set of characteristic properties are essentially the same as Properties C and D stated in §2. In the present connection we shall designate these Property $\mathrm{III_p}$ and Property $\mathrm{IV_p}$, respectively. In virtue of Property $\mathrm{II_p}$, Property $\mathrm{III_p}$ can be stated more directly as follows:

PROPERTY $\mathrm{III_p}$. *Associated with the point O there is a certain point P, such that the center of the circle* Γ *lies on the line passing through P perpendicular to OP.*

If a family of curves having Property $\mathrm{I_p}$ also has Properties $\mathrm{III_p}$ and $\mathrm{IV_p}$, the function $G(x, y, y')$ in the defining differential equation has the form

$$G(x, y, y') = \frac{1}{\omega}\left[\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}y'\right],$$

where ω is some function of x and y. Hence the set of four properties we have obtained is completely characteristic of a family of curves defined by a differential equation of the form (28).

The differential equations of motion (27) possess an integral which can be written in the form

$$\dot{x}^2 + \dot{y}^2 = 2E_c,$$

where E_c is a constant of integration. Consequently, the three-parameter family of curves defined by equation (28) consists of ∞^1 two-parameter subfamilies, each particular one of which corresponds to a particular value of the energy constant E_c. As before, we call each of these subfamilies a natural family of trajectories.

It is easily seen that the natural family of trajectories corresponding to the constant E_c is defined by the differential equation

$$y''(1 + y'^2)^{-3/2} = - (2E_c)^{-1/2}\omega(x, y).$$

Hence, a natural family of trajectories is completely characterized by the following single property:

All of the ∞^1 *curves passing through a given point have the same curvature at that point.*

So far our considerations have been based upon the Newtonian equations of motion (27). As would be expected in the light of the discussion given in §4, we obtain the same sets of characteristic properties, both for the total three-parameter family of trajectories and for the two-parameter natural families of trajectories, if we start with the relativistic equations of motion

$$\frac{d}{dt}\frac{\dot{x}}{[1 - c^{-2}(\dot{x}^2 + \dot{y}^2)]^{1/2}} = \omega(x, y)\dot{y},$$

$$\frac{d}{dt}\frac{\dot{y}}{[1 - c^{-2}(\dot{x}^2 + \dot{y}^2)]^{1/2}} = - \omega(x, y)\dot{x}.$$

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