A STUDY OF THE PROJECTIVE DIFFERENTIAL GEOMETRY OF SURFACES BY MEANS OF A MODIFIED TENSOR ANALYSIS

BY

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1. **Introduction.** The present paper is a study of the projective differential geometry of surfaces in ordinary space by means of a system of linear homogeneous differential equations of the first order. The use of a tensor notation with intrinsic differentiation (a generalized covariant differentiation introduced in §2) enables us to express general results with great formal simplicity.

Simple forms of the analytic conditions for fixity of points, lines, planes and algebraic surfaces (obtained in §§2 and 3) make possible a remarkably compact formulation of the theory of envelopes. Applications of a particular aspect of this theory yield the characteristics of significant families of planes, thereby providing (i) basic elements for the geometric characterizations of the differential invariants introduced in §§6 and 10, and (ii) means of determining (in §12) dual systems of hypergeodesics.

The conditions that a given net of an arbitrarily selected surface S be a conjugate net or the asymptotic net, respectively, are expressed in new ways (in §4) by means of simple relations among the local coordinates of a generic point of S and their first and second intrinsic derivatives. Each of these conditions assumes its simplest form if the given net is the parametric net.

§5 is devoted to a study of the effects on the coefficients of the differential equations of the most general transformations of independent and dependent variables which leave the surface unchanged. These effects, expressed by *the law of transformation of the connection*, are first interpreted in terms of local point coordinates of a generic point of the surface and then in terms of local plane coordinates of a general tangent plane to the surface. These interpretations lead to the introduction and geometric characterization (in §6) of a new quadratic differential invariant Ω of a general pair of analytic surfaces S, S'.

An invariant quadratic differential form which plays a basic role in a former paper by the author(1) will be shown in the present paper to be a projective generalization of the Euclidean second fundamental form of S. We shall denote this form by ϕ_2 and its discriminant by d. It is shown (in §7) that the form ϕ_1 defined by the relation

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(1) P. O. Bell [1, (2.6)]. (Numbers in brackets refer to the bibliography at the end of the paper.)

$$\phi_1 = \Omega d/h,$$

in which we denote by h the discriminant of Ω, becomes the Euclidean first fundamental form when the surface S' is selected to be the locus of the center of mean curvature of S. The form ϕ_1 for the general pair of surfaces S, S' will therefore be called the *projective first fundamental form of S relative to S'*. We define the *projective normal curvature κ_n of S relative to S', for a given direction at a point x*, by the ratio of the projective fundamental forms

$$\kappa_n = \phi_2/\phi_1.$$

The sum and product of the extremal values of κ_n at x are projective invariants which we denote by K_m and K and which we call the *projective mean curvature* and the *projective total curvature*, respectively, *of S relative to S'*. The class of R-associate surfaces of S is geometrically characterized and the invariants K_m and K of S relative to a general member of this class are then determined.

In §8 a geometric characterization for ϕ_2 is obtained and a geometric relation among the forms $\phi_2^{1/2}$, ϕ_2 and the form for the projective linear element is established. In §9 a system of projective geodesics of S is defined in association with a general R-conjugate congruence. A projective theorem concerning the cusp-axis of this system is proved, and a metric analogue of this theorem is established.

Differential invariants of a general tetrad of surfaces are defined and geometrically characterized (in §10), special cases of which are shown (in §11) to be the classical *projective linear element* of a surface, the *elementary forms* of Bompiani, and Fubini's *quadratic normal form*. A principle of duality is outlined (in §12) and used to characterize *dual systems* of hypergeodesics of which *union-curves* of a congruence Γ', the ρ- and σ-*tangeodesics*, and the *duals* of these are special cases. Certain properties of the ρ- and σ-tangeodesics serve to characterize the *first canonical pencil*.

2. **The fundamental differential equations.** Consider in ordinary projective space four analytic surfaces S_i, $i=0, 1, 2, 3$, whose corresponding generic points x_i are linearly independent. The projective homogeneous coordinates of x_i form a square matrix of rank and order four whose elements are analytic functions $x_i^{(p)}$ of two independent variables u^1, u^2. The general coordinates of any point in space may consequently be expressed as a linear combination of the corresponding coordinates of the points x_i. It follows that a set of functions $\Gamma^h{}_{i\alpha}$ of u^1, u^2, which we call the connection of the surfaces S_i, can be uniquely determined such that the functions $x_i^{(p)}$ are solutions of the system of differential equations

$$\frac{\partial x_i}{\partial u^\alpha} - \Gamma^h{}_{i\alpha} x_h = 0, \qquad i = 0, 1, 2, 3; \alpha = 1, 2, \tag{2.1}$$

in which h denotes a dummy index. Throughout the present paper, except when otherwise specified, Latin indices have the range 0, 1, 2, 3 whereas Greek indices have the range 1, 2; repeated indices in upper and lower positions of adjoining symbols indicate summing over their respective ranges. The most general sets of solutions of (2.1) are the sets of coordinates of the points $\bar{x}_i$ which correspond to the points x_i by a general projective transformation

$$\bar{x}_i{}^p = c_j{}^p x_i{}^j.$$

Thus *the equations* (2.1) *determine the projective differential properties common to all projective transforms of the tetrad of surfaces* S_i.

For the sake of typographical simplicity, where no confusion can arise we shall denote the partial derivative of a quantity with respect to u^α by the symbol for the quantity with the subscript α adjoined. Thus

$$x_{i\alpha} \equiv \frac{\partial x_i}{\partial u^\alpha}, \qquad \Gamma^j{}_{i\alpha\beta} \equiv \frac{\partial \Gamma^j{}_{i\alpha}}{\partial u^\beta}, \qquad x^i{}_{\alpha\beta} \equiv \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}.$$

Plückerian line coordinates $\omega_{ij}{}^{pq}$ of the line ω_{ij} joining the points x_i, x_j are defined by the formula

$$\omega_{ij}{}^{pq} = x_i{}^p x_j{}^q - x_j{}^p x_i{}^q.$$

By making use of (2.1) in an obvious manner we find that the coordinates $\omega_{ij}{}^{pq}$ are solutions of the system of equations

$$\omega_{ij\alpha} - \omega_{ih}\Gamma^h{}_{j\alpha} - \omega_{hj}\Gamma^h{}_{i\alpha} = 0. \tag{2.2}$$

Let $|x|$ denote the determinant whose elements are the functions $x_i{}^p$ and let $\xi^j{}_i$ denote the normalized cofactor of $x_j{}^i$ in $|x|$, defined by the relations

$$\xi^r{}_i x_h{}^i = \delta^r{}_h, \tag{2.3}$$

in which the right member represents the Kronecker deltas. The functions $\xi^r{}_i$, $i=0, 1, 2, 3$, form a set of general homogeneous plane coordinates of the plane determined by the three points x_h wherein $h \neq r$. Performing partial differentiation with respect to u^α of (2.3) and making use of (2.1) yields

$$x_h{}^i \xi^r{}_{i\alpha} + \Gamma^r{}_{h\alpha} = 0.$$

Forming the inner product of the left member of this equation with $\xi^h{}_k$ we find that the plane coordinates $\xi^r{}_i$ are solutions of the system of equations

$$\xi^r{}_\alpha + \xi^h \Gamma^r{}_{h\alpha} = 0. \tag{2.4}$$

The left members of (2.1), (2.2), and (2.4) will be called the intrinsic derivatives of x_i, ω_{ij}, and ξ^r, respectively, and will be denoted simply by $x_{i,\alpha}$, $\omega_{ij,\alpha}$, and $\xi^r{}_{,\alpha}$. We define, similarly, the intrinsic derivatives of aggregates a^{ij}, a_{ij}, $a_{i_1 i_2 \cdots i_p}$, $a^{i_1 i_2 \cdots i_p}$ by the respective equations

$$a^{ij}{}_{,\alpha} = a^{ij}{}_{\alpha} + a^{ih}\Gamma^{j}{}_{h\alpha} + a^{hj}\Gamma^{i}{}_{h\alpha},$$

$$a_{ij,\alpha} = a_{ij\alpha} - a_{ih}\Gamma^{h}{}_{j\alpha} - a_{hj}\Gamma^{h}{}_{i\alpha},$$

$$a_{i_1i_2\cdots i_p,\alpha} = a_{i_1i_2\cdots i_p\alpha} - \sum_{r=1}^{p} a_{i_1i_2\cdots i_{r-1}hi_{r+1}\cdots i_p}\Gamma^{h}{}_{i_r\alpha}, \tag{2.5}$$

$$a^{i_1i_2\cdots i_p}{}_{,\alpha} = a^{i_1i_2\cdots i_p}{}_{\alpha} + \sum_{r=1}^{p} a^{i_1i_2\cdots i_{r-1}hi_{r+1}\cdots i_p}\Gamma^{i_r}{}_{h\alpha}.$$

It is evident from the forms of these equations that *intrinsic differentiation of sums and of outer and inner products of aggregates of the types here considered obeys the same rules as ordinary differentiation.*

If we put $u^\alpha = u^\alpha(t)$, $\alpha = 1, 2$, a point x whose general coordinates are functions of u^1, u^2 describes a curve C as t varies. The aggregates

$$a_{ij\cdots l,\alpha}\frac{du^\alpha}{dt}, \qquad a^{ij\cdots l}{}_{,\alpha}\frac{du^\alpha}{dt}$$

will be called the *intrinsic derivatives along the curve* C of the corresponding aggregates $a_{ij\cdots l}$, $a^{ij\cdots l}$. Clearly these aggregates reduce to the ordinary intrinsic derivatives

$$a_{ij\cdots l,\alpha}, \qquad a^{ij\cdots l}{}_{,\alpha}$$

when the curve C is the parametric u^α curve.

The integrability conditions of the system (2.1) may be obtained by demanding that the order of differentiation of x_i be immaterial, that is

$$x_{i\alpha\beta} = x_{i\beta\alpha}.$$

Making use of (2.1) to express the members of this equation as linear combinations of x_k we obtain

$$\Gamma^{k}{}_{i\alpha,\beta}x_k = \Gamma^{k}{}_{i\beta,\alpha}x_k.$$

Since the points x_k are linearly independent, we obtain the following form for the conditions of integrability

$$\Gamma^{k}{}_{i\alpha,\beta} = \Gamma^{k}{}_{i\beta,\alpha},$$

wherein intrinsic differentiation is with respect to the index k. It may be easily demonstrated that these integrability conditions, which insure the existence of integral surfaces of a system of equations (2.1), may also be geometrically interpreted as follows. Let S_i^α denote a general transversal surface of the congruence of u^α tangents to S_i, generated by the point ρ^α whose general homogeneous coordinates are given by

$$\rho^\alpha = x_{i\alpha} + a^\alpha x_i,$$

wherein a^α is a function of u^1, u^2. In correspondence with an arbitrarily selected surface S_i^α there exist infinitely many surfaces S_i^β with the property that the u^β-tangent to S_i^α at ρ^α intersects the u^α-tangent to S_i^β at ρ^β if, and only if, the integrability conditions are satisfied. The points ρ^β are defined by the above equation wherein a^β satisfies the condition

$$a^\beta{}_\alpha = a^\alpha{}_\beta, \qquad \alpha \neq \beta.$$

3. **Conditions for fixity of points, lines and algebraic surfaces.** The linearly independent points x_i, $i=0, 1, 2, 3$, serve as vertices of a moving reference frame for local point coordinates. If we define the general coordinates of the unit point by the vector relation

$$U = x_0 + x_1 + x_2 + x_3,$$

a point X whose general coordinates are given by

$$X = x^h x_h$$

has the functions x^h as its local coordinates.

A point X is fixed as t varies if, and only if, the general coordinates of X satisfy a relation of the form

$$\frac{dX}{dt} = \lambda X,$$

wherein λ is a function of t. If we substitute in this relation the right member of (3.1) and make use of (2.1), we obtain, on equating coefficients of x_k, the relations

$$x^k{}_{,\alpha} \frac{du^\alpha}{dt} = \lambda x^k,$$

where λ is a function of t, which are *necessary and sufficient that the point X be fixed as t varies.*

Local line coordinates of the line joining two points X, Y are defined by

$$\omega^{ij} = x^i y^j - x^j y^i, \qquad i < j,$$

in which x^i, y^i are the local coordinates of X, Y. We prove the following theorem.

THEOREM 3.1. *Necessary and sufficient conditions that the line ω be fixed as t varies are that the local line coordinates ω^{ij} satisfy a system of equations of the form*

$$\omega^{ij}{}_{,\alpha} \frac{du^\alpha}{dt} = \eta \omega^{ij}, \tag{3.1}$$

in which η is an arbitrary function of t.

The line $\dot{\omega}$ is fixed as t varies if, and only if, the general coordinates X, Y satisfy a system of equations of the form

$$\frac{dX}{dt} = a_1 X + a_2 Y, \qquad \frac{dY}{dt} = b_1 X + b_2 Y,$$

in which the coefficients are functions of t. If we substitute for X and Y the respective expressions $x^h x_h$ and $y^h x_h$ in these equations and make use of (2.1) we find the *systems of equations*

$$x^h{}_{,\alpha}\frac{du^\alpha}{dt} = a_1 x^h + a_2 y^h, \qquad y^h{}_{,\alpha}\frac{du^\alpha}{dt} = b_1 x^h + b_2 y^h, \tag{3.2}$$

which are necessary and sufficient that the line ω joining the points X, Y be fixed as t varies.

If we differentiate the functions ω^{ij} intrinsically we find

$$\omega^{ij}{}_{,\alpha}\frac{du^\alpha}{dt} = (x^i y^j{}_{,\alpha} + y^j x^i{}_{,\alpha} - x^j y^i{}_{,\alpha} - y^i x^j{}_{,\alpha})\frac{du^\alpha}{dt}\cdot$$

Making use of equations (3.2) in the right member of this equation we obtain (3.1) in which $\eta = a_1 + b_2$. Hence equations (3.1) are necessary conditions. To prove them sufficient let the (i, j)-line coordinate of the line joining a general pair of points z, w be denoted by (z, w). Thus

$$(z, w) = z^i w^j - w^i z^j.$$

Equations (3.1) may consequently be written in the determinantal form

$$\{(x, y_{,\alpha}) + (x_{,\alpha}, y)\}\frac{du^\alpha}{dt} + \eta(y, x) = 0.$$

Combining the first and third determinants we have

$$\left(x, y_{,\alpha}\frac{du^\alpha}{dt} - \eta y\right) = (y, x_{,\alpha})\frac{du^\alpha}{dt}\cdot$$

Therefore $x_{,\alpha}\, du^\alpha/dt$ and $y_{,\alpha}\, du^\alpha/dt$ must be defined by equations of the form (3.1). This completes the proof.

Similar arguments may be applied to prove the following theorem.

THEOREM 3.2. *A point X and a line ω are fixed as u^1, u^2 vary independently if and only if their local coordinates satisfy respective systems of equations*

$$x^i{}_{,\alpha} = \lambda_\alpha x^i,$$
$$\omega^{ij}{}_{,\alpha} = \eta_\alpha \omega^{ij}, \qquad i, j = 0, 1, 2, 3;\ \alpha = 1, 2,$$

in which λ and η are arbitrary solutions of the equations

$$\theta_{\alpha\beta} = \theta_{\beta\alpha}.$$

It may be easily verified that these equations may be reduced to *normal forms*

$$\bar{x}^i{}_{,\alpha} = 0, \qquad \bar{\omega}^{ij}{}_{,\alpha} = 0$$

by making the substitutions

$$x^i = r\bar{x}^i, \qquad \omega^{ij} = s\bar{\omega}^{ij}$$

where r, s satisfy the respective relations

$$\log r = \lambda, \qquad \log s = \eta.$$

If the equation of a system of surfaces is expressed in terms of local point coordinates, the usual theory of envelopes may be applied to the system, providing the derivatives of local point coordinates are calculated by means of the conditions of fixity of a point(2). An equation of the form

$$f(x^0, x^1, x^2, x^3, u^1, u^2) = 0$$

which is homogeneous in the variables x^i represents a two-parameter family of surfaces. The characteristic points of the family corresponding to the parameter values u^1, u^2 have local coordinates which are solutions of the three equations

$$f = 0, \qquad f_\alpha - x^h\Gamma^i{}_{h\alpha}\frac{\partial f}{\partial x^i} = 0.$$

The locus of the characteristic points is the envelope of the family.

If we put $u^\alpha = u^\alpha(t)$, $\alpha = 1, 2$, the equation $f = 0$ represents a one-parameter family of surfaces. The equations of the characteristics are found to be

$$f = 0, \qquad \left(f_\alpha - x^h\Gamma^i{}_{h\alpha}\frac{\partial f}{\partial x^i}\right)\frac{du^\alpha}{dt} = 0.$$

The focal points of the characteristics are defined by these equations together with a third equation obtained by differentiating the second equation and making use of the fixed point conditions.

It follows that the edge of regression of a one-parameter family of planes $\xi_i x^i = 0$, in which the local coordinates ξ_i are functions of $u^\alpha(t)$, is the locus of focal points whose coordinates satisfy the equations

$$(3.3) \quad x^i\xi_i = 0, \quad x^i\xi_{i,\alpha}\frac{du^\alpha}{dt} = 0, \quad x^i\left(\xi_{i,\alpha}\frac{d^2u^\alpha}{dt^2} + \xi_{i,\alpha,\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt}\right) = 0.$$

The first two of these equations define the characteristics.

Let $a_{i_1 \ldots i_p}$ denote an aggregate of functions of $u^\alpha(t)$. The characteristics of the one-parameter family of algebraic surfaces

(2) E. P. Lane [1, p. 207].

$$a_{i_1\cdots i_p}x^{i_1\cdots i_p} = 0, \qquad i_1, i_2, \cdots, i_p = 0, 1, 2, 3,$$

are readily found to be defined by the equations

$$a_{i_1\cdots i_p}x^{i_1}\cdots x^{i_p} = 0, \quad a_{i_1\cdots i_p,\alpha}\frac{du^\alpha}{dt}x^{i_1}\cdots x^{i_p} = 0.$$

In virtue of the form of the second of these equations we have

THEOREM 3.3. *The algebraic surface whose equation in local point coordinates is*

$$a_{i_1\cdots i_p}x^{i_1}\cdots x^{i_p} = 0$$

is fixed as t varies if, and only if,

$$a_{i_1\cdots i_p,\alpha}\frac{du^\alpha}{dt} = \mu a_{i_1\cdots i_p}$$

in which μ is a function of t.

COROLLARY. *The surface $a_{i_1\cdots i_p}x^{i_1}\cdots x^{i_p}=0$ is fixed as u^1, u^2 vary independently if, and only if,*

$$a_{i_1\cdots i_p,\alpha} = \mu_\alpha a_{i_1\cdots i_p},$$

where μ_α are functions of u^1, u^2.

Necessary and sufficient conditions that a plane whose local coordinates are ξ_i be fixed as u^1, u^2 vary independently are, consequently,

$$\xi_{i,\alpha} = \mu_\alpha\xi_i$$

where μ_α are functions of t.

4. **Conjugate nets of a general surface.** The general coordinates of a generic point X of a general analytic surface S are given by a vector relation of the form

$$X = x^i x_i.$$

The analytic condition that two directions du^2/du^1, $\delta u^2/\delta u^1$ be conjugate with respect to S at X is that they satisfy the determinantal equation

$$\left(X, \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}, X_{\alpha\beta}du^\alpha\delta u^\beta\right) = 0.$$

In view of the vector relations

$$dX = x_i x^i{}_{,\alpha}du^\alpha, \qquad \delta X = x_i x^i{}_{,\beta}\delta u^\beta, \qquad d\delta X = x_i x^i{}_{,\beta,\alpha}\delta u^\beta du^\alpha,$$

which follow from (2.1), this determinantal equation may be expressed in terms of local coordinates of X as follows

$$(x^i,\ x^i_{,1},\ x^i_{,2},\ x^i_{,\beta,\alpha}\delta u^\beta du^\alpha) = 0.$$

Since an asymptotic direction of S at X is a self-conjugate direction, *the differential equation of the net of asymptotic curves of S is*

$$(x^i,\ x^i_{,1},\ x^i_{,2},\ x^i_{,\beta,\alpha}du^\alpha du^\beta) = 0.$$

It follows that the parametric net of S is (a) a conjugate net if, and only if,

$$(x^i,\ x^i_{,1},\ x^i_{,2},\ x^i_{,1,2}) = 0,$$

(b) the asymptotic net of S if, and only if

$$(x^i,\ x^i_{,1},\ x^i_{,2},\ x^i_{,\alpha,\alpha}) = 0, \qquad \alpha = 1,\ 2.$$

Consequent to these relations we have the following theorems.

THEOREM 4.1. *Necessary and sufficient conditions that the parametric net of S be a conjugate net are that the local coordinates x^i of X be solutions of a system of differential equations of the form*

$$x^i_{,1,2} + bx^i + a^\alpha x^i_{,\alpha} = 0,$$

in which b, a^α are functions of u^1, u^2.

THEOREM 4.2. *Necessary and sufficient conditions that the parametric net of S be the asymptotic net of S are that the coordinates x^i satisfy a system of equations of the form*

$$x^i_{,\beta,\beta} + c_\beta x^i + \overset{\alpha}{d_\beta} x^i_{,\alpha} = 0.$$

If S is the surface S_0 of the reference tetrad, the curvilinear differential equations defining a conjugate net of S_0 are found to be

$$(\Gamma^i{}_{01},\ \Gamma^i{}_{02},\ \Gamma^i{}_{0\alpha,\beta}du^\alpha\delta u^\beta) = 0, \qquad du^2/du^1 = \lambda(u^1,\ u^2), \qquad i = 1,\ 2,\ 3.$$

The differential equation of the asymptotic net of S_0 is, therefore,

$$(\Gamma^i{}_{01},\ \Gamma^i{}_{02},\ \Gamma^i{}_{0\alpha,\beta}du^\alpha du^\beta) = 0.$$

If we assume that the tangent plane to S_0 at x_0 is the plane $x^3=0$, we have

$$x_{0\alpha} = \Gamma^h{}_{0\alpha}x_h, \qquad \Gamma^3{}_{0\alpha} = 0, \qquad \alpha = 1,\ 2.$$

It follows that, in this case, *a conjugate net of S_0 is defined by a pair of equations of the form*

$$\Gamma^j{}_{0\alpha}\Gamma^3{}_{j\beta}du^\alpha\delta u^\beta = 0, \qquad du^2/du^1 = \lambda(u^1,\ u^2),$$

and *the asymptotic net of S_0 is defined by the equation*

$$\Gamma^j{}_{0\alpha}\Gamma^3{}_{j\beta}du^\alpha du^\beta = 0.$$

Furthermore, if $\Gamma^j{}_{02}\Gamma^3{}_{j1} = \Gamma^j{}_{01}\Gamma^3{}_{j2} = 0$, the parametric curves of S_0 form a conjugate net. Similarly, if $\Gamma^j_{01}\Gamma^3_{j1}=\Gamma^j_{02}\Gamma^3_{j2}=0$, the parametric curves of S_0 form the asymptotic net of S_0.

The condition that the directions du^2/du^1, $\delta u^2/\delta u^1$ be conjugate to S_0 at x_0 assumes its simplest form

$$\Gamma^3_{\alpha\beta}du^\alpha\delta u^\beta = 0$$

if the vertices x_α, $\alpha=1, 2$, lie on the corresponding tangents to the u^α curves of S_0 at x_0, for in this case we have $\Gamma^\beta_{0\alpha}=0$, $\alpha\neq\beta$, α, $\beta=1, 2$. If we define differential forms ϕ^j_i by the relations

$$\phi^j_i = \Gamma^j_{i\beta}\delta u^\beta,$$

the direction du^2/du^1 conjugate to $\delta u^2/\delta u^1$ is given by the relation

$$du^2/du^1 = -\phi^3_1/\phi^3_2.$$

5. **Transformation of the connection.** A general analytic surface S' is undisturbed by the general transformations: (i) of *independent variables*

$$u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2), \qquad J(u, \bar{u}) \neq 0; \tag{5.1}$$

(ii) of *dependent variables* of the form

$$\bar{x}_i = \bar{\lambda}_i{}^h x_h, \tag{5.2}$$

in which $\bar{\lambda}_i{}^h$ are functions of $\bar{u}^1$, $\bar{u}^2$; (iii) of *proportionality factor*

$$X = \bar{\lambda}\overline{X} \tag{5.3}$$

in which X and $\overline{X}$ denote general homogeneous projective coordinates of a generic point of S' defined by the vector forms

$$X = x^i x_i, \qquad \overline{X} = \bar{x}^i\bar{x}_i,$$

and $\bar{\lambda}$ is a function of $\bar{u}^1$, $\bar{u}^2$.

In view of the linear independency of the points $\bar{x}_h$ the set of functions $\overline{\Gamma}^h_{i\alpha}$ exists for which

$$\frac{\partial \bar{x}_i}{\partial \bar{u}^\alpha} = \overline{\Gamma}^h_{i\alpha}\bar{x}_h.$$

Equating the right member of this equation to the derivative of the right member of (5.2) and making use of (2.1) yields

$$\overline{\Gamma}^h_{i\alpha}\bar{\lambda}_h{}^j x_j = \bar{\lambda}_i{}^h\Gamma^j_{h\beta}x_j\frac{\partial u^\beta}{\partial \bar{u}^\alpha} + \frac{\partial \bar{\lambda}_i{}^j}{\partial \bar{u}^\alpha}x_j. \tag{5.4}$$

From (5.2) and (5.3) it follows that

$$x^h = \bar{x}^i\bar{\lambda}_i{}^h\bar{\lambda}, \tag{5.5}$$

and therefore

$$\frac{\partial x^h}{\partial \bar{x}^i} = \bar{\lambda}\lambda_i{}^h. \tag{5.6}$$

On equating corresponding coefficients of x_j in (5.4), multiplying by $\bar{\lambda}$ and making use of (5.5), we obtain *the law of transformation of the connection* $\Gamma^j{}_{i\alpha}$ *under the most general transformation which leaves the surface* S' *undisturbed*(8)

$$\Gamma^h{}_{i\alpha}\frac{\partial x^j}{\partial \bar{x}^h} = \Gamma^j{}_{h\beta}\frac{\partial u^\beta}{\partial \bar{u}^\alpha}\frac{\partial x^h}{\partial \bar{x}^i} + \frac{\partial^2 x^j}{\partial \bar{u}^\alpha \partial \bar{x}^i} - \frac{\partial(\log \bar{\lambda})}{\partial \bar{u}^\alpha}\frac{\partial x^j}{\partial \bar{x}^i}. \tag{5.7}$$

Writing (5.5) in the form

$$x^j = \bar{x}^i\frac{\partial x^j}{\partial \bar{x}^i},$$

we find, on differentiating with respect to u^α,

$$\bar{x}^i\frac{\partial^2 x^j}{\partial \bar{u}^\alpha \partial \bar{x}^i} = \frac{\partial x^j}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^\alpha} - \frac{\partial \bar{x}^i}{\partial \bar{u}^\alpha}\frac{\partial x^j}{\partial \bar{x}^i}.$$

On substituting the right member of this relation in the equation obtained by forming the inner products of the members of (5.7) with $\bar{x}^i$, we obtain the *law of transformation of intrinsic derivatives of local point coordinates*

$$x^j{}_{,\beta} = \bar{x}^i{}_{,\alpha}\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\frac{\partial x^j}{\partial \bar{x}^i} + x^j\frac{\partial(\log \bar{\lambda})}{\partial u^\beta}, \tag{5.8}$$

in which $x^j{}_{,\beta}$ and $\bar{x}^i{}_{,\alpha}$ denote the intrinsic derivatives of x^j and $\bar{x}^i$ with respect to u^β and $\bar{u}^\alpha$, respectively.

Let ξ^i and $\bar{\xi}^i$ denote the faces of the tetrahedra (x_0, x_1, x_2, x_3) and $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$, respectively, which are opposite the corresponding vertices x_i and $\bar{x}_i$. General homogeneous projective plane coordinates of these faces may be defined by the functions $\xi^i{}_h$ and $\bar{\xi}^i{}_h$, $h=0, 1, 2, 3$, which are solutions of the systems of equations

$$\xi^i{}_h x_j{}^h = \bar{\xi}^i{}_h \bar{x}_j{}^h = \delta^i{}_j. \tag{5.9}$$

According as the old or new local reference tetrahedron is employed, the general homogeneous projective plane coordinates of a generic tangent plane of a surface S are given by the vector forms

$$\pi = \xi_i\xi^i, \qquad \bar{\pi} = \bar{\xi}_i\bar{\xi}^i \tag{5.10}$$

in which the local coordinates ξ_i and $\bar{\xi}_i$ are functions of u^1, u^2 and $\bar{u}^1$, $\bar{u}^2$, re-

(8) For $i, j=1, 2$ this becomes the law of transformation of an affine connection if we impose the conditions that $\partial x^h/\partial \bar{x}^i=0$, $h=0, 3$, $\partial x^j/\partial \bar{x}^i=\lambda\partial u^j/\partial \bar{u}^i$, in which λ is a constant.

spectively. In order that the forms of (5.10) represent a common plane they must be related by a *transformation of proportionality factor*

$$\pi = \bar{\mu}\bar{\pi}. \tag{5.11}$$

In view of equations (5.9) the general transformation (5.2) may be expressed in terms of general homogeneous projective plane coordinates as

$$\bar{\xi}^i = \mu^i{}_h\xi^h, \tag{5.12}$$

in which the coefficients $\bar{\mu}^i{}_h$ are the functions of $\bar{u}^1$, $\bar{u}^2$ defined by the equations

$$\bar{\mu}^i{}_h\bar{\lambda}_j{}^h = \delta^i{}_j.$$

On substituting the right member of (5.12) for $\bar{\xi}^i$ in (5.11) and equating corresponding coefficients of ξ_h, we obtain the following relations connecting local plane coordinates

$$\xi_h = \bar{\mu}\bar{\mu}^i{}_h\bar{\xi}_i. \tag{5.13}$$

In virtue of these relations and (5.5), we find that

$$\frac{\partial \xi_h}{\partial \bar{\xi}_i}\frac{\partial x^h}{\partial \bar{x}^j} = \bar{\mu}\bar{\lambda}\delta^i{}_j. \tag{5.14}$$

If now we differentiate the members of (5.11) with respect to u^α and equate corresponding coefficients of ξ^j, we readily obtain the *law of transformation of intrinsic derivatives of local plane coordinates*

$$\xi_{j,\alpha} = \bar{\xi}_{i,\beta}\frac{\partial \bar{u}^\beta}{\partial u^\alpha}\frac{\partial \xi_j}{\partial \bar{\xi}_i} + \xi_j\frac{\partial(\log \bar{\mu})}{\partial u^\alpha}. \tag{5.15}$$

The law of transformation of the connection $\Gamma^j{}_{i\alpha}$ may be written in the form

$$\mathbf{\Gamma}^h{}_{i\alpha}\bar{\lambda}_h{}^j = \Gamma^j{}_{h\beta}\bar{\lambda}_i{}^h\frac{\partial u^\beta}{\partial \bar{u}^\alpha} + \frac{\partial \bar{\lambda}_i{}^j}{\partial \bar{u}^\alpha} - \frac{\partial(\log \bar{\lambda})}{\partial \bar{u}^\alpha}\bar{\lambda}_i{}^j.$$

Compounding the members of this equation with $\bar{\mu}_j{}^k$ yields

$$\mathbf{\Gamma}^k{}_{i\alpha} = \left(\frac{\partial \bar{\lambda}_i{}^j}{\partial u^\beta} + \bar{\lambda}_i{}^h\Gamma^j{}_{h\beta}\right)\bar{\mu}^k{}_j\frac{\partial u^\beta}{\partial \bar{u}^\alpha}.$$

If we denote by $\lambda_i{}^j$ the functions of u^1, u^2 defined by the identities

$$\lambda_i{}^j(u^1, u^2) \equiv \bar{\lambda}_i{}^j(\bar{u}^1, \bar{u}^2)$$

we have for the determination of the functions $\mathbf{\Gamma}^k{}_{i\alpha}$ the forms

$$\mathbf{\Gamma}^k{}_{i\alpha} = \bar{\lambda}_i{}^j{}_{,\beta}\bar{\mu}^k{}_j\frac{\partial u^\beta}{\partial \bar{u}^\alpha}, \tag{5.16}$$

in which intrinsic differentiation is with respect to the upper index.

6. **A differential invariant of a general pair of surfaces.** If the point X of a surface S' does not lie in the plane π of a surface S, a relation free from $\bar{\mu}$ and $\bar{\lambda}$ may be obtained by making use of relations (5.5), (5.8), (5.13) and (5.15). Such a relation is the tensor equation

$$\bar{a}_{\gamma\delta} = a_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\delta}$$

in which the quantities $a_{\alpha\beta}$ are the functions of u^1, u^2 defined by

$$a_{\alpha\beta} = (x^i\xi_j\xi_{i,\alpha}x^j{}_{,\beta} - x^i\xi_i x^j{}_{,\alpha}\xi_{j,\beta})/(x^i\xi_i)^2,$$

and the quantities $\bar{a}_{\gamma\delta}$ are the corresponding barred functions of $\bar{u}^1$, $\bar{u}^2$. The quadratic differential form Ω defined by

$$\Omega = a_{\alpha\beta}du^\alpha du^\beta \tag{6.1}$$

is, therefore, an absolute invariant of the pair of surfaces S, S'.

Let X_1 and π_1 denote the point of S' and the plane of S, respectively, which correspond to the parameter values u^1+du^1, u^2+du^2. A geometric characterization of the invariant (6.1) will now be derived.

The general point coordinates of X_1 are given by the development

$$X(u^1 + du^1, u^2 + du^2) = X + \frac{\partial X}{\partial u^\alpha} du^\alpha + (_2)$$

in which $(_2)$ denotes terms of order at least two. In view of (2.1) we may write

$$\frac{\partial X}{\partial u^\alpha} = x^i{}_{,\alpha}x_i.$$

Hence, to terms of order one, the local point coordinates of X_1 are given by

$$x^i + x^i{}_{,\alpha}du^\alpha.$$

Similarly, we find that the local plane coordinates of π_1 are, to terms of order one, given by

$$\xi_i + \xi_{i,\alpha}du^\alpha.$$

Let θ_{00}, θ_{01}, θ_{10}, θ_{11} denote the forms which are defined by the relations

$$\theta_{00} = x^i\xi_i, \qquad \theta_{01} = x^i\xi_{i,\alpha}du^\alpha, \qquad \theta_{10} = \xi_i x^i{}_{;\alpha}du^\alpha, \qquad \theta_{11} = x^i{}_{,\alpha}\xi_{i,\beta}du^\alpha du^\beta.$$

Let Y, Y_1 denote the respective points of intersection of the planes π, π_1 with the line joining X, X_1. The local coordinates of any point collinear with X, X_1 are expressible in the form

$$z^i = \mu x^i + x^i{}_{,\alpha}du^\alpha.$$

The values of μ with which the points X, X_1 are associated are ∞, 1, respectively. The point z lies in the plane π or in the plane π_1 according as μ is the root of the first or of the second of the equations

$$(\mu_1 x^i + x^i{}_{,\alpha} du^\alpha)\xi_i = 0, \qquad (\mu_2 x^i + x^i{}_{,\alpha} du^\alpha)(\xi_i + \xi_{i,\beta} du^\beta) = 0$$

The values of μ with which Y, Y_1 are associated are, therefore,

$$\mu_1 = -\theta_{10}/\theta_{00}, \qquad \mu_2 = -(\theta_{10} + \theta_{11})/(\theta_{00} + \theta_{01}),$$

respectively. The cross-ratio of the four points X, Y, X_1, Y_1 is the cross-ratio of the four corresponding values of μ

$$(\infty, \mu_1, 1, \mu_2) = (\theta_{01}\theta_{10} - \theta_{00}\theta_{11})/(\theta_{00} + \theta_{01})(\theta_{00} + \theta_{10}).$$

Hence, we find that the quadratic differential invariant (6.1) is the principal part of the cross-ratio

$$(X, Y, X_1, Y_1).$$

Since the form (6.1) is symmetric in ξ_i, x^i, the above characterization may be dualized as follows: A plane π of S and a neighboring plane π_1, corresponding to the increments du^1, du^2 (except for infinitesimals of order at least two), intersect in the characteristic of π for the direction du^2/du^1. Let π_X, π_{X_1} denote the planes which pass through the respective points X, X_1 and contain the characteristic of π. *The principal part of the cross-ratio*

$$(\pi_X, \pi, \pi_{X_1}, \pi_1)$$

is the differential invariant (6.1).

The dual characterizations just described apply to pairs of surfaces S', S generated by point X and plane π, respectively, such that X does not lie in π. Let us consider, briefly, the case of surfaces S', S for which the generic point X of S' lies in the plane π of S. For this case we have

$$\xi_i x^i = 0, \qquad (x^i \xi_{i,\alpha} + x^i{}_{,\alpha}\xi_i) du^\alpha = 0, \tag{6.2}$$

in which the second equation is satisfied identically in du^α, that is, for every direction. If we assume that the point X lies on the characteristic of the plane π for a direction du^2/du^1, the point X is fixed, except for variations involving infinitesimals of order at least two, as u^1, u^2 vary in this direction. The condition for this is that the local coordinates of X satisfy the relations

$$x^i{}_{,\alpha} du^\alpha = \lambda x^i dt, \qquad (u^\alpha = u^\alpha(t)).$$

Forming the inner products of the members of this equation with ξ_i, we find, in view of equations (6.2),

$$\theta_{00} = \theta_{10} = \theta_{01} = 0. \tag{6.3}$$

Since the sets of equations (6.2) and (6.3) are symmetric in ξ_i and x^i, we may state our results in the following theorem.

THEOREM 6.1. *If at a point X of S' the tangent line to S' in a direction du^2/du^1 lies in the plane π of S, the characteristic of π for this direction passes through the point X. This condition is fulfilled if and only if the local coordinates x^i and ξ_i of X and π satisfy the equations*

$$\theta_{00} = \theta_{10} = \theta_{01} = 0.$$

It follows that *the equations* (6.3) *are satisfied independently of direction if the plane π of S is tangent to S' at X. If the surface S enveloped by π is identical with S' and the generic point X is the contact point of π with S, these equations are identities in u^1, u^2.*

7. Projective fundamental forms of a surface and associated curvatures. Let the surface S_0 be referred to asymptotic parameters with Grove's normal coordinates(4) for the point x_0. This choice allows us to select as an edge of the local tetrahedron, with equal formal simplicity, the line through x_0 of any *R-conjugate congruence.* Let the point x_3 lie on the given R-conjugate line passing through x_0, and let the points x_1, x_2 be defined by the relations

$$x_\alpha = x_{0\alpha}, \qquad u^1 = u,\ u^2 = v,$$

so that the coordinates of the points x_i satisfy the system of linear equations

$$x_{i\alpha} = \Gamma^h{}_{i\alpha} x_h \tag{7.1}$$

whose coefficients are related to those of Grove's canonical system by the equations

$$\begin{aligned}
&\Gamma^h{}_{0\alpha} = \delta^h{}_\alpha, && \Gamma^0{}_{11} = p, && \Gamma^\alpha{}_{\alpha\alpha} = \theta_\alpha, && \Gamma^2{}_{11} = \beta,\\
&\Gamma^3{}_{\alpha\alpha} = 0, && \Gamma^0{}_{22} = q, && \theta = \log R, && \Gamma^1{}_{22} = \gamma,\\
&\Gamma^\alpha{}_{12} = 0, && \Gamma^3{}_{12} = \Gamma^3{}_{21}, && \Gamma^3{}_{3\alpha} = (\theta_\alpha - (\log \Gamma^3{}_{12})\alpha)/\Gamma^3{}_{12},\\
&\Gamma^\alpha{}_{3\alpha} = (\theta_{12} + \beta\gamma - \Gamma^0{}_{12})/\Gamma^3{}_{12}, && && \Gamma^\beta{}_{3\alpha} = (\Gamma^0{}_{\alpha\alpha} + (\Gamma^\beta{}_{\alpha\alpha})_\beta + \Gamma^\beta{}_{\alpha\alpha}\Gamma^\beta{}_{\alpha\alpha})/\Gamma^3{}_{12},
\end{aligned} \tag{7.2}$$

in which a repeated index does not indicate summation, and $\alpha \neq \beta$.

The multiplier of x_3 may be so selected that the functions $\Gamma^3{}_{3\alpha}$ vanish. This selection, which is defined by the condition $\Gamma^3{}_{12} = R$, does not restrict the location of the point x_3. We readily find that the form Ω for S_0, S_3 in which S_0, S_3 play the roles of S, S' respectively is given by the equation

$$\Omega = \pi(du^1)^2 + 2(\kappa - \Gamma^0{}_{12})du^1du^2 + \chi(du^2)^2, \tag{7.3}$$

in which $\pi = p + \beta_2 + \beta\theta_2$, $\chi = q + \gamma_1 + \gamma\theta_1$, $\kappa = \beta\gamma + \theta_{12}$.

Grove(5) has defined analytically the projective curvature tensor of the

(4) We signify by "Grove's normal coordinates" a set of solutions of Grove's canonical system of differential equations. V. G. Grove [1, p. 582].

(5) V. G. Grove [2, pp. 121–122].

surface S relative to the R-conjugate line xy and the point y, and he has given the analytic basis for the introduction of an associated metric on the surface. It is a remarkable fact, which can be easily verified, that the form Ω for S_0, S_3 is characterized by the relation

$$\Omega = Kds^2$$

in which K is *the projective curvature of S_0 relative to the R-conjugate line and the point* x_3, and ds^2 is *the associated metric of* S_0. The curves defined by $\Omega=0$ will be called *the projective minimal curves* of S_0 *relative to* S_3. We shall hereafter refer to K as the *projective total curvature of* S_0 *relative to* S_3.

Let a quadratic differential invariant of the form

$$\phi_2 = 2Rdu^1du^2 \tag{7.4}$$

serve as the projective second fundamental form of S_0 relative to the R-conjugate congruence, and let h, g and d denote the discriminants of Ω, ds^2 and ϕ_2, respectively. The projective total curvature K may be defined by the formula

$$K = d/g.$$

Since $h=K^2g$, it follows that

$$K = h/d = [(\kappa - \Gamma^0{}_{12})^2 - \pi\chi]/R^2.$$

Introducing ϕ_1 by the relation $\phi_1=\Omega/K$, we now define by the relation

$$\kappa_n = \phi_2/\phi_1$$

the *projective normal curvature* κ_n *of* S_0 *relative to* S_3 for the direction du^2/du^1 at x_0. The maximum and minimum values of κ_n at a point x_0, which are the roots of the equation

$$R\kappa^2{}_n - 2(\kappa - \Gamma^0{}_{12})\kappa_n + KR = 0,$$

will be called the *projective principal normal curvatures of* S_0 *at* x_0 *relative to* S_3. The directions of S_0 at x_0 which correspond to these curvatures will be called the *projective principal directions of* S_0 *at* x_0 *relative to* S_3. The curves of S_0 having these directions at each point x_0 will be called the *projective lines of curvature of* S_0 *relative to* S_3. Their curvilinear differential equation may be readily found to be

$$\pi(du^1)^2 - \chi(du^2)^2 = 0$$

in which $\pi=p+\beta_2+\beta\theta_2$, $\chi=q+\gamma_1+\gamma\theta_1$, $\theta=\log R$. The sum of the projective principal normal curvatures of S_0 at x_0 serves to define the *projective mean curvature* K_m *of* S_0 *at* x_0 *relative to* S_3. Hence,

$$K_m = 2(\kappa - \Gamma^0{}_{12})/R,$$

in which $\kappa=\beta\gamma+\theta_{12}$, $\theta=\log R$.

Homogeneous cartesian coordinates x^i, $i=0, 1, 2, 3$, of the point x_0 of S_0 may be obtained by adjoining to ordinary rectangular coordinates of x_0 a fourth coordinate $x^3=1$. The point z at infinity on the normal to S_0 at x_0 has homogeneous cartesian coordinates z^0, z^1, z^2, 0, the first three of which may be taken as the direction cosines of the normal to S_0 at x_0. *If the generic point x_3 of S_3 is defined by the vector relation*

$$x_3 = z + K_m x_0$$

in which K_m denotes the mean curvature of S_0, and if ϕ_2 is the second fundamental form of S_0, the associated entities ds^2, K, κ_n, K_m defined, as above, relative to S_3 are the Euclidean first fundamental form, Gaussian curvature, normal curvature for a given direction, and mean curvature, respectively, of S_0(6).

Let us now specialize in projective manners the results of this section. *The harmonic invariant of the differential forms Ω and ϕ_2 vanishes if, and only if, $\Gamma^0_{12}=\kappa$. This is a necessary and sufficient condition that the projective minimal curves of S_0 relative to S_3 form a conjugate net. The surface S_3 thus selected is,* moreover, *such that the projective mean curvature of S_0 relative to S_3 vanishes.* This surface S_3 will be called the *R-associate* of S_0. The following additional geometric characterization of the *R*-associate of S_0 may be readily verified. *The harmonic conjugate of the point x_{12} with respect to the points in which the R-conjugate line through x_0 intersects the quadric of Lie at x_0 is the point x_3 which generates the R-associate of S_0 as x_0 varies over S_0.* The conjugate net characterized above as *the projective minimal net of S_0 relative to the R-associate of S_0 is the mean conjugate net of S_0 when the R-conjugate congruence is the congruence of metric normals of S_0*(7).

Let P_∞, P_0, P_1, x_3 denote the points on the Fubini-Green projective normal to S_0 at x_0 whose general homogeneous coordinates are given by the forms

$$x_0 = x, \qquad z = [x_{12} - 2^{-1}(\theta_{12} + 2\beta\gamma)x]/\beta\gamma, \qquad z + x, \qquad z + kx,$$

respectively. The point P_1 is the intersection of the projective normal to S_0 at x_0 with the quadric of Wilczynski at x_0, and the point P_0 is the harmonic conjugate of P_1 with respect to the points in which the projective normal intersects the quadric of Lie. The point x_3 is, therefore, geometrically characterized by the cross-ratio equation

$$(P_\infty, P_0, P_1, x_3) = k.$$

A convenient way to complete the geometric characterization of x_3 is to let the function k be the projective mean curvature K_m of S_0 relative to S_3. The imposition of this condition results in the characterization of K_m by the relation

$$K_m = -\theta_{12}/\beta\gamma$$

(6) See P. O. Bell [2, pp. 567–569].

(7) P. O. Bell [2, Theorem 5].

and of the point x_3 by the vector form

$$x_3 = [x_{12} - 2^{-1}(3\theta_{12} + 2\beta\gamma)x]/\beta\gamma.$$

The form of the coordinates of x_3 reveals that P_0 and x_3 are harmonic conjugates with respect to x_0 and the point of the R-associate of S_0, where $R=\beta\gamma$, which lies on the Fubini-Green projective normal of S_0 at x_0. We shall call this point x_3 the center of mean projective curvature of S_0 at x_0 relative to the points P_0, P_1 on the projective normal to S_0 at x_0. The associated projective total curvature of S_0 and the metric of S_0 are defined by the equations

$$K = (\theta^2_{12} - 4\pi\chi)/4\beta^2\gamma^2, \qquad Kds^2 = \pi(du^1)^2 - \theta_{12}du^1du^2 - \chi(du^2)^2,$$

respectively.

8. **Projective analogues of the second fundamental form and the projective linear element of a surface.** Let us again suppose that the surface S_0 is referred to asymptotic parameters. Again, let $\Gamma^3_{12}=\Gamma^3_{21}=R$, and let the points x_α, $\alpha=1, 2$, be the points defined by $x_\alpha=x_{0\alpha}$. We need not restrict x_3 to lie on the R-conjugate line which passes through x_0, but we shall assume that the form ϕ_2 defined by

$$\phi_2 = 2Rdu^1du^2$$

is a projective invariant of S_0 and that the vertices of the local reference tetrahedron and the point x_3+x_0 are covariantly determined points. We shall show that the form ϕ_2 is a projective analog of the second fundamental form of S_0.

Let π denote the tangent plane to S_0 at x_0 and let η denote the plane determined by the points x_1, x_2, x_3+x_0. The local equations for π and η are $x^3=0$ and $x^3-x^0=0$, respectively.

Let dp denote the differential defined by the relation

$$dp = \phi_2^{1/2},$$

and let C_λ denote a curve of S_0 whose direction at each of its points is defined by

$$du^2/du^1 = \lambda(u^1, u^2).$$

Along this curve we have, therefore,

$$dp/du^1 = (2R\lambda)^{1/2}.$$

Let us regard p as independent variable(8) for points of C_λ, where p is defined by the integral

$$p = \int_{u_0^1}^{u^1} (2R\lambda)^{1/2}du^1.$$

(8) For the geometric interpretation of the intrinsic parameter p see P. O. Bell [1, pp. 532–534].

The general coordinates $x_0(dp)$ of a point X_0 of C_λ "near" to x_0 are given by the expansion

$$x_0(dp) = x_0 + x'dp + x''dp^2/2 + x'''dp^3/3! + \cdots,$$

in which accents indicate differentiation with respect to p. Let P and Q denote the intersections of the line joining the points X_0, x_3 with the planes π and η, respectively. The general coordinates of P, Q may be found to be given by the forms

$$P = x_0(dp) + \mu_1 x_3, \qquad Q = x_0(dp) + \mu_2 x_3,$$

in which μ_1, μ_2 are defined by

$$\mu_1 = -(3\phi_2 + \phi_3)/3! + \cdots, \quad \mu_2 = 1 + (_2),$$

in which we denote by ϕ_3 the cubic differential form $R(\gamma(du^2)^3 + \beta(du^1)^3)$ and by $(_2)$ terms of order $(dp)^2$ at least. The cross-ratio equations

$$(P, x_3, X_0, Q) = (\mu_1, \infty, 0, \mu_2) = (3\phi_2 + \phi_3)/3! + (_4),$$

in which $(_4)$ denotes terms of order $(dp)^4$ at least, may now be readily obtained. Let us call the cross-ratio (P, x_3, X_0, Q) the *projective distance* $D_{\pi X_0}$ *from* π *to* X_0 *with respect to* x_3. We observe that the difference between this projective distance and its principal part is equal to $\phi_3/3!$ plus terms of order $(dp)^4$ at least. Since the curves of Darboux are integral curves of the differential equation $\phi_3 = 0$, we may state our results in the following theorem.

THEOREM 8.1. *The invariant form ϕ_2 is equal to twice the principal part of the projective distance from π to X_0 with respect to x_3. The difference between this projective distance and its principal part is equal to $\phi_3/3!$ plus terms of order at least $(dp)^4$. This difference consists of terms of order at least $(dp)^4$ if the points x_0, X_0 lie on a curve defined by $R = 0$ or on a curve of Darboux. The ratio ϕ_3/ϕ_2 is independent of R and is equal to the projective linear element of S_0.*

Let x^0, x^1, x^2, 1 again represent homogeneous cartesian coordinates of the point x_0 of S_0 and let the coordinates of the point x_3 at infinity on the normal to S_0 at x_0 be represented by the three direction cosines z^0, z^1, z^2 of the normal and $z^3 = 0$. Let a local reference tetrahedron (x_0, x_1, x_2, x_3) with equal units on the three axes through x_0 be established with x_1 and x_2 the intersections of the ideal line (line at infinity) in the plane π with the asymptotic u^1, u^2 tangents to S_0 at x_0. *The form ϕ_2 and the corresponding projective distance $D_{\pi X_0}$ associated with the plane η determined by the ideal line x_1x_2 and the unit point $x_0 + x_3$ are the second fundamental form of S_0 and associated metric distance from* π to X_0, respectively.

9. A theorem on projective geodesics and an application to metric geometry. The projective first fundamental form of S_0 relative to the R-associate of S_0 and the associated projective total curvature of S_0 (characterized in §7) are defined analytically by the equations

$$K ds^2 = \pi(du^1)^2 + x(du^2)^2, \qquad K = -\pi x/R^2.$$

We may, therefore, write

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 \tag{9.1}$$

in which $g_{11} = -R^2/x$, $g_{22} = -R^2/\pi$. We shall call the extremals of the integral invariant

$$\int_{u_0^1}^{u^1} (g_{11} + g_{22}\lambda^2)^{1/2} du^1,$$

in which $\lambda = du^2/du^1$, the projective geodesics of S_0 relative to the R-associate of S_0. Since the above integral invariant is of the form

$$\int_{u_0}^{u} \phi(u, v, v')du, \qquad v' = dv/du,$$

and Euler's equation for the extremals of this integral is known to be

$$\phi_{v'v'}v'' = \phi_v - \phi_{uv'} - \phi_{vv'}v',$$

the equation for these projective geodesics may be found to be

$$\begin{aligned} 2d^2u^2/(du^1)^2 &= (g_{11})_2/g_{22} + (\log g_{11}/(g_{22})^2)_1 du^2/du^1 \\ &\quad + (\log (g_{11})^2/g_{22})_2(du^2/du^1)^2 - [(g_{22})_1/g_{11}](du^2/du^1)^3. \end{aligned} \tag{9.2}$$

The cusp-axis at x_0 of a system of hypergeodesics defined by an equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3, \qquad \text{where} \quad u^1 = u, \ u^2 = v,$$

is known(9) to be a line l' which passes through the points x_0, $x_{12} - ax_1 - bx_2$ where $a = (\theta_2 + C)/2$, $b = (\theta_1 - B)/2$. The cusp-axis at x_0 of the projective geodesics (9.2) is therefore the line l' for which

$$a = (\log R^4\pi/\chi^2)_2/4, \qquad b = (\log R^4\chi/\pi^2)_1/4.$$

Since the condition $b_2 = a_1$ may be expressed in the present case as

$$(\log \pi/\chi)_{12} = 0$$

we have the following theorem.

THEOREM 9.1. *If the Γ' curves of the cusp-axis congruence of the projective geodesics of S_0 relative to the R-associate of S_0 form a conjugate net, or coincide with an asymptotic family, or are indeterminate, the associated projective lines of curvature form an isothermally conjugate net, and conversely.*

(9) Cf. E. P. Lane [1, p. 193].

Since a distinct form (9.1) is associated uniquely with each R-conjugate congruence, the above theorem describes a representative result which is characteristic of every member of the class of conjugate congruences.

An analogous Euclidean geometric theorem may be derived as follows. Let us consider a surface S_0 having the property that its asymptotic net is orthogonal. Let x^0, x^1, x^2, 1 and z^0, z^1, z^2, 0 denote homogeneous rectangular coordinates of a generic point x_0 of S_0 and the point x_3 at infinity on the normal to S_0 at x_0, respectively, in which z^0, z^1, z^2 denote direction cosines of the normal at x_0. Let points x_α, $\alpha=1, 2$, be defined by the relations $x_\alpha = x_{0\alpha}$. It follows that the homogeneous coordinates of the points x_j, $j=0, 1, 2, 3$, satisfy a system of equations of the form $x_{j\alpha} = \Gamma^h{}_{j\alpha} x_h$, in which $\Gamma^h{}_{jk}$, j, α, $h=1, 2$, are the Christoffel symbols of the second kind for the first fundamental form of S_0. According to the hypotheses we have

$$g_{12} = 0, \qquad \Gamma^3{}_{11} = \Gamma^3{}_{22} = 0, \qquad \Gamma^3{}_{12} \neq 0.$$

The geodesics are the extremals of the arc length integral

$$\int_{u_0^1}^{u^1} (g_{11} + g_{22}\lambda^2)^{1/2} du^1$$

whose cusp-axis at x_0 is the normal to S_0. The Γ' curves of the congruence of normals (known as the lines of curvature) form a conjugate net if and only if

$$g_{11} \neq 0, \qquad g_{22} \neq 0, \qquad (\log g_{11}/g_{22})_{12} = 0, \tag{9.3}$$

and conversely. The equation for the lines of curvature then assumes the form

$$g_{11}(du^1)^2 - g_{22}(du^2)^2 = 0.$$

Fulfillment of conditions (9.3) is assured by the two Codazzi equations. Equations (9.3) are necessary and sufficient conditions that the lines of curvature form an isothermally conjugate net(10); moreover (9.3) with the added condition $g_{12}=0$ are the conditions that the parametric net be isothermally orthogonal(11). Hence we have the following theorem.

THEOREM 9.2. *If the asymptotic curves of S_0 form an orthogonal net, it is isothermally orthogonal and the lines of curvature of S_0 form an isothermally conjugate net.*

10. Intrinsic geometry of a tetrad of surfaces. The surfaces S_i of a tetrad are undisturbed by the general transformation (5.1) of independent variables. The generic points of S_i are maintained as the vertices x_i of the local moving reference tetrahedron by a transformation of dependent variables of the form

(10) For the geometric significance of isothermal conjugacy see E. J. Wilczynski [1, pp. 211–221].

(11) See, for example, E. P. Lane [2, p. 236].

$$\bar{x}_h = \bar{\lambda}_h{}^j x_j, \tag{10.1}$$

in which

$$\bar{\lambda}_h{}^j = 0, \qquad h \neq j,$$
$$\bar{\lambda}_h{}^j \neq 0, \qquad h = j.$$

Corresponding to this transformation of dependent variables we have that the normalized cofactors $\bar{\mu}^h{}_j$ of $\bar{\lambda}_h{}^j$ in $|\bar{\lambda}_i{}^h|$ are defined by the relations

$$\bar{\mu}^h{}_j = 0, \qquad h \neq j,$$
$$\bar{\mu}^h{}_j = 1/\bar{\lambda}_h{}^h, \qquad h = j,$$

where $\bar{\lambda}_h$ denotes the function $\bar{\lambda}_h{}^j$ for $h=j$.

The law of transformation of the connection $\Gamma^j{}_{h\alpha}$ under the general transformation combining (5.1), (5.2) and (5.3) is given by (5.16) with $\bar{\lambda}_h{}^j$, $\bar{\mu}^h{}_j$ satisfying the above relations. Incorporating these relations in the law (5.16) we find

$$\bar{\Gamma}^j{}_{h\alpha} = \delta^j{}_h \partial(\log \bar{\lambda}_h)/\partial \bar{u}^\alpha + \bar{\lambda}_h \Gamma^j{}_{h\beta}(\partial u^\beta/\partial \bar{u}^\alpha)/\bar{\lambda}_j$$

in which h and j are not summed. It follows that the linear forms $\phi^j{}_h$ defined by

$$\phi^j{}_h = \Gamma^j{}_{h\alpha} du^\alpha \tag{10.2}$$

are relative invariants of the general transformation combining (5.1) and (10.1). Hence *the directions* defined by

$$\phi^j{}_h = 0, \qquad h \neq j,$$

possess intrinsic geometric significance with respect to the tetrad of surfaces S_i. On comparing the right member of (10.2) with the forms for the relative invariants θ_{01}, θ_{10} of the vertex x_h and face $x^j=0$, we find that the direction defined by $\phi^j{}_h=0$ is that for which these forms vanish. The geometric significance of the vanishing of these forms has already been given in theorem (6.1).

In the remainder of this section we denote a chosen permutation of (0, 1, 2, 3) by (i, j, k, l). An index to be summed through i, j, l will be represented by s and an index to be summed through j, l will be denoted by r. The form $F^k{}_i$ defined by the formula

$$F^k{}_i = -\phi^r{}_i \phi^k{}_r/\phi^k{}_i$$

is a linear differential invariant. To obtain a geometric characterization of this form consider the point X_i whose general homogeneous coordinates $X_i{}^p$ are given by the vector form

$$X_i = x_i(u^1 + du^1, u^2 + du^2),$$

and the point P_i determined by the projection of X_i upon the plane $x^k=0$ from the point x_k. The general homogeneous coordinates of P_i may be represented by the vector form

$$P_i = x_i + \Gamma^s{}_{i\alpha} x_s du^\alpha.$$

The line joining the points x_i and P_i intersects the line joining the points x_j and x_l in the point Q whose general coordinates Q^p are given by the form

$$Q = \Gamma^r{}_{i\alpha} x_r du^\alpha.$$

Since the characteristic of the plane $x^k=0$ corresponding to the direction du^2/du^1 is defined by the equations

$$x^k = 0, \qquad x^r\Gamma^k{}_{r\alpha} du^\alpha = 0,$$

the intersection of the line joining the points x_i and P_i with this characteristic is the point R whose general coordinates are defined by

$$R = x_i - \phi^k{}_i \phi^r{}_i x_r / \phi^r{}_i \phi^k{}_r.$$

Let P_{ki} denote the intersection of the characteristic of the plane $x^k=0$, for the direction $du^2/du^1=\lambda$, with the plane $x^i=0$. The local point coordinates of P_{ki} satisfy the equations

$$(10.3) \qquad x^k = 0, \qquad x^i = 0, \qquad x^r\phi^k{}_r = 0.$$

The proof of the following theorem may now be easily supplied by the reader.

THEOREM 10.1. *The form $F^k{}_i$ is the principal part of the invariant cross-ratio (x_i, Q, P_i, R). The point P_{ki} lies in the plane determined by the line joining the points x_i and x_k and the tangent to S_i at x_i in the direction λ if, and only if, λ is a root of the differential equation $F^k{}_i=0$.*

The linear differential invariants which we shall call the projective linear elements $E^{lj}{}_i$ we define by the relations

$$E^{lj}{}_i = F^l{}_i + F^j{}_i.$$

These elements may be geometrically characterized in the following simple manner. Let O denote the intersection of the line joining x_i and P_i with the line joining P_{jk} and P_{lk}. The validity of the following characterization may be verified by simple calculations.

THEOREM 10.2. *The principal part of the cross-ratio (x_i, Q, P_i, O) is the projective linear element $E^{lj}{}_i$.*

The points P_{jk} and P_{kj} for a given direction λ are clearly collinear with the points x_l and x_i. The curves which correspond to the developables of the congruence of lines joining corresponding points of S_i and S_l are the curves whose directions are characterized by the property that the points x_i, x_l,

$\phi^r{}_i x_r$, $\phi^r{}_l x_r$ are coplanar. This condition may clearly be expressed by the differential equation $\phi^j{}_i\phi^k{}_l=\phi^j{}_l\phi^k{}_i$. This condition is also seen to be the condition that a direction be such that the points P_{jk} and P_{kj} coincide. Hence we have the following theorem.

THEOREM 10.3. *The directions for which the relative invariants*

$$\phi^j{}_i\phi^k{}_l$$

are symmetric in the indices j, k are the directions which correspond to the developables of the congruence of lines joining corresponding points of S_l and S_i. Moreover, for only these directions do the points P_{li} and P_{il} coincide.

For a given invariant direction the cross-ratio $(x_l, x_i, P_{jk}, P_{kj})$ is an absolute invariant which we denote by $R^{jk}{}_{il}$. By a simple calculation we find that

$$R^{jk}{}_{li} = \phi^j{}_l\phi^k{}_i/\phi^j{}_i\phi^k{}_l.$$

Corresponding to a given invariant $R^{jk}{}_{il}$ there are two invariant directions defined by this equation. In particular if $R^{jk}{}_{li}=-1$, the points P_{jk}, P_{kj} are harmonic conjugates with respect to the points x_l, x_i, and in view of the form of (10.3) we have the following theorem.

THEOREM 10.4. *The directions for which P_{jk}, P_{kj} are harmonic conjugates with respect to the points x_l, x_i are the two directions for which the form*

$$\phi^j{}_i\phi^k{}_l$$

is skew-symmetric in j, k.

11. **The projective linear element and Fubini's element of projective arc length.** Let (x_0, x_1, x_2, x_3) denote a moving reference tetrahedron intrinsically connected with the surface S_0 at x_0 in such a manner that the vertices x_1, x_2 are located on the tangents at x_0 to the u^1, u^2 curves, respectively, of S_0. The general homogeneous coordinates of x_α are therefore defined by the forms $x_{0\alpha}=x_\alpha+\Gamma^0{}_{0\alpha}x_0$. The special cases $F^1{}_0$, $F^2{}_0$ of the corresponding invariant forms $F^k{}_i$ are, consequently, given by

$$F^1{}_0 = -(\Gamma^1{}_{21}du^1du^2 + \Gamma^1{}_{22}(du^2)^2)/du^1,\quad F^2{}_0 = -(\Gamma^2{}_{12}du^1du^2 + \Gamma^2{}_{11}(du^1)^2)/du^2,$$

and the special projective linear element $E^{12}{}_0$ is given by

$$E^{12}{}_0 = -(\Gamma^1{}_{22}(du^2)^3 + \Gamma^1{}_{21}(du^2)^2du^1 + \Gamma^2{}_{12}du^2(du^1)^2 + \Gamma^2{}_{11}(du^1)^3)/du^1du^2.$$

The classical projective linear element of a surface S_0 may now be identified as the special case of $E^{12}{}_0$ for which the points x_1, x_2, x_3 are the points x_u, x_v, x_{uv} whose general coordinates are expressed in Fubini's normal coordinates. However, *if these points be replaced by the similarly defined points whose general coordinates are normal with respect to an arbitrarily selected canonical form of Grove, the associated invariant form $E^{21}{}_0$ is again* found to be *the classical pro-*

jective linear element of the surface S_0, and *the invariants* $F^1{}_0$, $F^2{}_0$ *are the elementary linear differential forms of Bompiani*(12). Under the present restrictions the forms $F^1{}_0$, $F^2{}_0$, $E^{12}{}_0$ are given by the simple expressions

$$F^1{}_0 = -\Gamma^1{}_{22}(du^2)^2/du^1, \qquad F^2{}_0 = -\Gamma^2{}_{11}(du^1)^2/du^2, \qquad E^{12}{}_0 = F^1{}_0 + F^2{}_0.$$

The product

$$F^1{}_0F^2{}_0 = \Gamma^1{}_{22}\Gamma^2{}_{11}du^1du^2$$

is Fubini's normal form for the square of his element of projective arc length(13).

12. **Dual characteristics. Systems of hypergeodesics and the first canonical pencil.** Let us establish a one-to-one correspondence between the points of a surface S' and the planes of a surface S by defining the local coordinates x^i and ξ_i of a generic point X of S' and the corresponding tangent plane π of S, respectively, to be single-valued functions of u^1, u^2. The following geometric elements of S' and S are placed in one-to-one correspondence: (i) the points of a curve C_λ of S' and the tangent planes of a developable surface D_λ of S, (ii) the tangent to C_λ at a generic point X and the characteristic line of D_λ in the plane π which corresponds to X, and (iii) the osculating plane of C_λ at X and the ray-point of D_λ in π. These elements of S' and S thus attached to C_λ and D_λ will be called the *characteristics* of C_λ and D_λ respectively. We shall refer to these characteristics as *dual characteristics* with respect to the correspondence between the points of S' and the planes of S.

To define a developable surface D_λ of S and a corresponding curve C_λ of S' we let u^α be functions of an independent parameter t so that local coordinates ξ_i, x^i of a generic plane π of S and the corresponding point X of S' become functions of t. The characteristics of D_λ described in (i), (ii), and (iii) above are defined analytically by

$$\begin{gathered} \xi_i x^i = 0, \\ \xi_i x^i = 0, \qquad x^i\xi_{i,\alpha}du^\alpha/dt = 0, \\ \xi_i x^i = 0, \quad x^i\xi_{i,\alpha}du^\alpha/dt = 0, \quad x^i(\xi_{i,\alpha}d^2u^\alpha/dt^2 + \xi_{i,\alpha,\beta}du^\alpha du^\beta/dt^2) = 0, \end{gathered} \tag{12.1}$$

respectively, in which ξ_i are functions of t. The equations in local plane coordinates ξ_i for the corresponding characteristics of C_λ at X may be written by simply interchanging the roles of ξ_i and x^i in the above equations. They are, therefore,

$$\begin{gathered} x^i\xi_i = 0, \\ x^i\xi_i = 0, \qquad \xi_i x^i{}_{,\alpha}du^\alpha/dt = 0, \\ x^i\xi_i = 0, \quad \xi_i x^i{}_{,\alpha}du^\alpha/dt = 0, \quad \xi_i(x^i{}_{,\alpha}d^2u^\alpha/dt^2 + x^i{}_{,\alpha,\beta}du^\alpha du^\beta/dt^2) = 0 \end{gathered} \tag{12.2}$$

in which x^i are functions of t.

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(12) E. Bompiani [1, pp. 167–173].

(13) G. Fubini and E. Čech [1, pp. 64–69].

Let p and Y denote the tangent plane to S' at X and the point of contact of the plane π with S, respectively, and let the local plane coordinates of p and the local point coordinates of Y be denoted by η_i and y^i respectively. As X varies over S the line joining XY generates a congruence which we denote by Γ_{XY}. A curve C_λ of S' having the property that its osculating plane at a general one of its points X contains the line XY is called a *union-curve*(14) of the congruence Γ_{XY}. The differential equation of the union-curves of a general congruence Γ_{XY} may be obtained by imposing the condition that the plane determined by the third line of equations (12.2) contain the point Y. This condition is clearly given by the determinantal relation

$$(12.3) \qquad (y^i,\ x^i,\ x^i{}_{,\alpha}du^\alpha/dt,\ x^i{}_{,\alpha}d^2u^\alpha/dt^2 + x^i{}_{,\alpha,\beta}du^\alpha du^\beta/dt^2) = 0.$$

The dual of the line XY is the line πp determined by the planes π and p. The dual of a union-curve of a congruence Γ_{XY} is a developable of S having the property that the ray-point of the generic plane π lies in the plane p. Such a developable will be called a *union-developable* of the congruence $\Gamma_{\pi p}$. The curve of S generated by the contact point of π with S as π varies over a union-developable of the congruence $\Gamma_{\pi p}$ will be called an *adjoint union-curve*(15) of the congruence $\Gamma_{\pi p}$. The differential equation of the adjoint union-curves of the congruence $\Gamma_{\pi p}$ is found, by replacing y^i by η_i and x^i by ξ_i, to be

$$(12.4) \qquad (\eta_i,\ \xi_i,\ \xi_{i,\alpha}du^\alpha/dt,\ \xi_{i,\alpha}d^2u^\alpha/dt^2 + \xi_{i,\alpha,\beta}du^\alpha du^\beta/dt^2) = 0.$$

As X varies along a union-curve of the congruence Γ_{XY} the point Y describes a curve of S which we call an *S'-tangeodesic* of S, and as π varies over a union-developable of the congruence $\Gamma_{\pi p}$ the point X describes a curve of S' which we call an *adjoint S-tangeodesic* of S'.

The author has defined in a different manner(16) the systems of ρ- and *σ-tangeodesics* of S. We shall show that the above definition of the S'-tangeodesics of S is equivalent to a generalization of the definition of either of these systems of tangeodesics. Let R_λ denote the ruled surface generated by the line XY as X, Y describe corresponding curves C_λ of S' and S, respectively. If the curve C_λ of S' is a union-curve of the congruence Γ_{XY}, the osculating plane of C_λ at X coincides with the tangent plane to the ruled surface R_λ; the curve C_λ is, therefore, an asymptotic curve of R_λ. Conversely, if the curve C_λ of S' is an asymptotic curve of R_λ, it is a union-curve of the congruence Γ_{XY}. It follows that the following generalization of the definition of the ρ-tangeodesics of S serves to characterize the S'-tangeodesics of S: *An S'-tangeodesic of S is a curve C_λ of S whose associated ruled surface R_λ* (of the congruence Γ_{XY}) *intersects the surface S' in an asymptotic curve of R_λ.*

(14) Union-curves were introduced by P. Sperry [1, p. 214].

(15) This is a generalization of the definition of adjoint-union curves (or dual-union curves) due to G. M. Green [1, p. 140] and P. Sperry [1, p. 222].

(16) P. O. Bell [2, p. 575].

If we let S' and S be surfaces S_h and S_k of the fundamental tetrad, respectively, the local coordinates of X and Y become the Kronecker deltas $\delta^i{}_h$ and $\delta^i{}_k$ respectively. Substituting these deltas for x^i and y^i, respectively, in (12.3) and expanding the determinant yields the following curvilinear differential equation which defines on S_k the S_h-tangeodesics and on S_h the union-curves of the congruence generated by the line $x_h x_k$:

$$(12.5)\qquad \begin{aligned}&(\Gamma^m{}_{h\alpha}\Gamma^n{}_{h\beta} - \Gamma^n{}_{h\alpha}\Gamma^m{}_{h\beta})\frac{du^\alpha d^2u^\beta}{dt^3}\\ &\qquad = (\Gamma^n{}_{h\alpha}\Gamma^m{}_{h\beta,\gamma} - \Gamma^m{}_{h\alpha}\Gamma^n{}_{h\beta,\gamma})\frac{du^\alpha du^\beta du^\gamma}{dt^3},\end{aligned}$$

in which intrinsic differentiation is with respect to upper indices and $(hmnk)$ represents a permutation of (0123).

The equation of the adjoint union-curves of the congruence $\Gamma_{\pi p}$ generated by the line of intersection of the planes π and p defined by $x^h=0$ and $x^k=0$, respectively, may be written by interchanging the upper and lower left index in each of the symbols involving $\Gamma^j{}_{h\alpha}$, $\Gamma^j{}_{h\beta}$, $j=m, n$, in (12.5). The equation is

$$(12.6)\qquad \begin{aligned}&(\Gamma^h{}_{m\alpha}\Gamma^h{}_{n\beta} - \Gamma^h{}_{n\alpha}\Gamma^h{}_{m\beta})\frac{du^\alpha d^2u^\beta}{dt^3}\\ &\qquad = (\Gamma^h{}_{n\alpha}\Gamma^h{}_{m\beta,\gamma} - \Gamma^h{}_{m\alpha}\Gamma^h{}_{n\beta,\gamma})\frac{du^\alpha du^\beta du^\gamma}{dt^3},\end{aligned}$$

in which each intrinsic differentiation is with respect to the lower left index.

Let us determine the equation of the S_3-tangeodesics of S_0 for the special case in which the surface S_3 is the R-associate of S_0. This surface S_3, which has been geometrically characterized in §7, is generated by the point x_3 whose general homogeneous coordinates are given by the form

$$Rx_3 = x_{12} - (\beta\gamma + \theta_{12})x$$

in which x denotes Grove's normal coordinates for x_0 and u^1, u^2 are asymptotic parameters. We find now that equation (12.5) for $h=3$, $k=0$ can be written in the form

$$d\lambda/du^1 = A + B\lambda + C\lambda^2 + D\lambda^3, \quad \text{where} \quad \lambda = du^2/du^1,$$
$$A = -\Gamma^1{}_{31,1}/\Gamma^1{}_{32}, \qquad B = \Gamma^2{}_{31,1}/\Gamma^2{}_{31} - (\Gamma^1{}_{32,1} + \Gamma^1{}_{31,2})/\Gamma^1{}_{32},$$
$$C = (\Gamma^2{}_{31,2} + \Gamma^2{}_{32,1})/\Gamma^2{}_{31} - \Gamma^1{}_{32,2}/\Gamma^1{}_{32}, \qquad D = \Gamma^2{}_{32,2}/\Gamma^2{}_{31},$$

in which intrinsic differentiation is with respect to the upper index. In virtue of the relations (7.2) and the condition $\Gamma^3{}_{12}=R$ we find, on evaluating the coefficients B, C in terms of the coefficients of Grove's canonical form,

$$B = (\log \pi/R\chi)_1 - \gamma\pi/\chi, \qquad C = (\log \pi R/\chi)_2 + \beta\chi/\pi.$$

The cusp-axis of these tangeodesics at x_0 is the line l' which passes through the point x_0 and the point z whose normal coordinates are given by $z=x_{12}-ax_1-bx_2$ in which a and b are defined by

$$a = ((\log \pi R^2/\chi)_2 + \beta\chi/\pi)/2, \qquad b = ((\log \chi R^2/\pi)_1 + \gamma\pi/\chi)/2.$$

The proof of the following theorem may now be readily supplied by the reader.

THEOREM 12.1. *If the Γ' curves of the congruence generated by this cusp-axis form a conjugate net, or coincide with an asymptotic family of S_0, or are indeterminate, then*

$$2(\log \pi/\chi)_{12} + (\gamma\pi/\chi)_2 - (\beta\chi/\pi)_1 = 0,$$

and conversely. If the axis curves of the net of projective lines of curvature of S_0 relative to S_3 form a conjugate net, or coincide with an asymptotic family of S_0, or are indeterminate, then

$$(\log \pi/\chi)_{12} + (\beta\chi/\pi)_1 - (\gamma\pi/\chi)_2 = 0,$$

and conversely. The projective lines of curvature of S_0 relative to S_3 form an isothermally conjugate net if, and only if,

$$(\log \pi/\chi)_{12} = 0.$$

The net of R_λ-derived curves of S_0, where λ is a projective principal direction of S_0 relative to S_3, belongs to class $\mathfrak{C}$ if, and only if,

$$(\beta\chi/\pi)_1 - (\gamma\pi/\chi)_2 = 0\text{[17]}.$$

If any two of these four conditions are fulfilled the other two are also fulfilled.

In conclusion we consider briefly the systems of ρ- and σ-tangeodesics of S_0 for which ρ and σ are points on the asymptotic u^1- and u^2-tangents to S_0 at x_0 whose general coordinates are given by

$$\rho = x_1 - bx_0, \qquad \sigma = x_2 - ax_0, \qquad \text{where } x_{0\alpha} = x_\alpha.$$

The joint-edge at x_0 of the systems of ρ- and σ-tangeodesics of S_0 was found[18] to be the line which passes through the points x_0 and z whose general coordinates in Fubini's normal coordinates satisfy the relation $z=x_{12}-\bar{a}x_1-\bar{b}x_2$ in which $\bar{a}=-a+\psi/2$, $\bar{b}=-b+\phi/2$. Let $\bar{l}$ denote the reciprocal with respect to S_0 at x_0 of the joint-edge of the ρ- and σ-tangeodesics of S_0 at x_0, let l denote the line joining $\rho\sigma$, and let t denote the *first canonical tangent* to S_0 at x_0. Since $\bar{a}$, $\bar{b}$ are given by the formulas

$$\bar{a} = k_1\psi, \qquad \bar{b}_1 = k\phi$$

where $k_1=(1-2k)/2$, if we put

(17) P. O. Bell [3, p. 398].
(18) P. O. Bell [2, p. 576].

$$a = k\psi, \qquad b = k\phi,$$

we have, as a consequence, the following theorem.

THEOREM 12.2. *If the line l joining $\rho\sigma$ is a canonical line l_k, the reciprocal $\bar{l}$ of the joint-edge of the ρ- and σ-tangeodesics of S_0 at x_0 is the canonical line l_{k_1} for which $k_1 = (1-2k)/2$.*

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