THE WEIERSTRASS E-FUNCTION IN THE CALCULUS OF VARIATIONS

BY
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1. **Introduction.** The present paper is the first of a set of three papers concerned primarily with the isoperimetric problem of Bolza. This problem is one of the most general problems in the calculus of variations and can be described as follows: Consider a class of arcs

$$C: \qquad a^h, \ y^i(t) \qquad (t^1 \leqq t \leqq t^2;\ h = 1, \cdots, r;\ i = 0, 1, \cdots, n)$$

in ay-space satisfying a set of conditions of the form

$$\phi^\beta(a, y, \dot{y}) = 0 \qquad (\beta = 1, \cdots, m < n), \tag{1.1}$$

$$y^i(t^1) = T^{i1}(a), \qquad y^i(t^2) = T^{i2}(a), \tag{1.2}$$

$$I^\sigma(C) = g^\sigma(a) + \int_C f^\sigma(a, y, \dot{y})dt = 0 \qquad (\sigma = 1, \cdots, s). \tag{1.3}$$

The components a^h of C are constants. We seek to find in this class of arcs one that minimizes a function

$$I(C) = g(a) + \int_C f(a, y, \dot{y})dt. \tag{1.4}$$

The functions ϕ^β, f^σ, f are positively homogeneous in the variables $\dot{y}^i$. The problem here formulated contains as special cases most of the interesting problems in the calculus of variations involving simple integrals(1).

In the present paper we shall develop certain interesting properties of the Weierstrass E-function. These properties are of interest apart from their applications to be found in the papers that follow. We shall be concerned particularly with the concept of E-dominance. This concept can be described briefly as follows: Let $\mathfrak{D}$ be the set of all admissible elements (a, y, p) satisfying the conditions $\phi^\beta(a, y, p) = 0$ and let C_0 be an arc whose elements $(a, y, p) = (a, y, \dot{y})$ are in $\mathfrak{D}$. A function $F(a, y, p)$ will be said to E-dominate a second function $H(a, y, p)$ near C_0 on $\mathfrak{D}$ if there is a neighborhood $\mathfrak{D}_0$ relative to $\mathfrak{D}$ of the elements (a, y, p) on C_0 and a constant $b > 0$ such that the inequality

$$E_F(a, y, p, q) \geqq b \left| E_H(a, y, p, q) \right|$$

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(1) Cf. M. R. Hestenes, *Generalized problem of Bolza*, Duke Math. J. vol. 5 (1939) pp. 309–324. A discussion of several different formulations of the problem of Bolza can be found in this paper.

holds whenever (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$. Here E_F and E_H are the E-functions of F and H respectively. We are especially interested in finding necessary and sufficient conditions that a function $F(a, y, p)$ shall E-dominate the integrand $L(p)=(p^ip^i)^{1/2}$ of the length integral. It is shown below that $L(p)$ is E-dominated by a function of the form

$$F = l^0f + l^\sigma f^\sigma + m^\beta(a, y)\phi^\beta$$

if and only if the arc C_0 satisfies, with the multipliers l^0, l^σ, $m^\beta(a, y)$, the strengthened condition of Weierstrass and the condition of nonsingularity. That the strengthened condition of Weierstrass plus nonsingularity implies that L is E-dominated by a function F of this type has also been established by W. T. Reid in connection with an expansion proof of a sufficiency theorem for parametric problems. His results have not been published as yet. In the present paper we not only show the equivalence of these relations but show further that a function F E-dominates L near C_0 on $\mathfrak{D}$ if and only if there is a function of the form

$$F^* = F + \theta(a, y, p)\phi^\beta\phi^\beta$$

that E-dominates L near C_0 on the class of all admissible elements (a, y, p). The results here given indicate that for a function F possessing only first derivatives the condition that F shall E-dominate L appears to be a natural extension of the strengthened condition of Weierstrass and the condition of nonsingularity.

The concept of E-dominance will be used freely in the two papers that follow. In the first of these it will be used in connection with a theorem of Lindeberg analogous to one given by Reid(2) for the nonparametric case. In the second paper it will be shown that the sufficiency theorems for the problem of Bolza can be obtained from those of the problem of Mayer, a result that does not appear to have been completely justified. Following the method used by Reid we shall show that sufficiency theorems for the isoperimetric problem of Bolza can be obtained from those of the problem of Bolza without isoperimetric conditions. Moreover it will be seen that sufficiency theorems for parametric problems can be obtained from those for nonparametric problems. Interesting results will also be obtained in regard to the regions in which the sufficiency theorems are valid.

The third and final paper of the present series will be devoted to the proof of a sufficiency theorem for a proper strong relative minimum for the isoperimetric problem of Bolza. This sufficiency theorem is essentially one conjectured by McShane(3) with the usual inequality $I(C)>I(C_0)$ $(C\neq C_0)$ replaced by an inequality of the form

(2) W. T. Reid, *Isoperimetric problems of Bolza in nonparametric form*, Duke Math. J. vol. 5 (1939) pp. 675–691.

(3) E. J. McShane, *Sufficient conditions for a weak relative minimum in the problem of Bolza*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379.

$$I(C) - I(C_0) \geqq \min (\epsilon, \epsilon |C, C_0|^2),$$

where $|C, C_0|$ is a suitably defined metric for the class of arcs under consideration. This new inequality enables one to obtain an analogue of Osgood's theorem as a corollary to our sufficiency theorem. One of the interesting features of the method here used is that it is applicable without modification to isoperimetric problems, that is, the method is the same for a problem with isoperimetric side conditions as for one without isoperimetric side conditions. The method used is essentially the one used by McShane in order to establish a sufficiency theorem for a weak relative minimum and later extended by Myers(4) in order to establish a sufficiency theorem for a semistrong relative minimum for the nonparametric problem of Lagrange.

The results given in these three papers are applicable to the nonparametric case as well as to the parametric case.

2. **Preliminary definitions and lemmas.** The present section will be devoted to a description of some of the hypotheses, definitions and notations that will be used in the three papers. We shall use the following notations

$$a = (a^1, \cdots, a^r), \qquad y = (y^0, y^1, \cdots, y^n), \qquad \dot{y} = (\dot{y}^0, \dot{y}^1, \cdots, \dot{y}^n),$$
$$p = (p^0, p^1, \cdots, p^n), \qquad q = (q^0, q^1, \cdots, q^n).$$

If k is a real number, then $kp = (kp^0, kp^1, \cdots, kp^n)$. Repeated indices in a term denote summation with respect to that index. The length $(p^ip^i)^{1/2}$, where $i = 0, 1, \cdots, n$, of the vector p will be denoted by $|p|$. We distinguish between the symbols $|p|$, $|p^i|$. The latter denotes the absolute value of the ith component p^i of p. Similar remarks hold for the symbols $|a|$, $|a^h|$, $|y|$, $|y^i|$, and so on.

We suppose that we have given an open set $\mathfrak{R}$ of $(r+2n+2)$-dimensional points $(a, y, p) \neq (a, y, 0)$ with the property that if (a, y, p) is in $\mathfrak{R}$ so also is (a, y, kp) for every positive number k. An element (a, y, p) will be called *admissible* if it is in $\mathfrak{R}$. By an *admissible subregion* $\mathfrak{R}_0$ of $\mathfrak{R}$ will be meant one such that if (a, y, p) is in $\mathfrak{R}_0$ so also is the element (a, y, kp) $(k > 0)$.

By an *admissible function* $H(a, y, p)$ will be meant a real single-valued function on $\mathfrak{R}$ that satisfies the homogeneity condition

$$H(a, y, kp) = kH(a, y, p) \qquad (k > 0) \tag{2.1}$$

on $\mathfrak{R}$, is continuous, and has continuous first and second derivatives with respect to the variables $p^0, \cdots, p^n$. As a consequence of the relation (2.1) one has on $\mathfrak{R}$ the well known identities

$$H = p^i H_{p^i}, \qquad p^i H_{p^ip^j} = 0 \qquad (i, j = 0, 1, \cdots, n), \tag{2.2}$$

(4) F. G. Myers, *Sufficient conditions for the problem of Lagrange*, Duke Math. J. vol. 10 (1943) pp. 73–97.

and the homogeneity relations

$$
\begin{aligned}
H_{p^i}(a, y, kp) &= H_{p^i}(a, y, p),\\
H_{p^ip^j}(a, y, kp) &= k^{-1}H_{p^ip^j}(a, y, p),
\end{aligned}
\tag{2.3}
$$

where $k>0$. These relations will be used freely.

It will be understood throughout that the functions $f(a, y, p)$, $f^\sigma(a, y, p)$, $\phi^\beta(a, y, p)$ appearing in the formulation of our problem are admissible functions of class C'' on $\mathfrak{R}$. The functions $g(a)$, $g^\sigma(a)$, $T^{i1}(a)$, $T^{i2}(a)$ are assumed to be of class C'' on $\mathfrak{R}$.

An element (a, y, p) in $\mathfrak{R}$ will be said to be *differentially admissible* if $\phi^\beta(a, y, p)=0$ $(\beta=1, \cdots, m)$. It is clear from (2.1) that the class of all differentially admissible elements form an admissible subregion of $\mathfrak{R}$. This subregion will be denoted by $\mathfrak{D}$.

Consider now a rectifiable curve C in ay-space having an absolutely continuous representation

$$
a, \ y(t) \qquad (t^1 \leqq t \leqq t^2), \tag{2.4}
$$

the components $a=(a^1, \cdots, a^r)$ being constants. By virtue of our conventions $y(t)$ represents the set $y^0(t)$, $y^1(t), \cdots, y^n(t)$. Their derivatives $\dot{y}^0(t)$, $\dot{y}^1(t), \cdots, \dot{y}^n(t)$ exist almost everywhere on t^1t^2 and define a vector $\dot{y}(t)$. At the points where $\dot{y}(t)$ is not defined we set $\dot{y}(t)=(0, \cdots, 0)$. We shall consider throughout only the rectifiable arcs (2.4) for which the element $(a, y(t), \dot{y}(t))$ is in $\mathfrak{R}$ for almost all values of t on t^1t^2. By an *admissible arc C* will be meant an arc (2.4) of this type satisfying the differential equations (1.1), the isoperimetric conditions (1.3), and the end conditions (1.2).

We shall center our attention on a particular admissible arc

$$
C_0: \qquad a_0, \ y_0(t) \qquad (t^1 \leqq t \leqq t^2)
$$

of class C''. It will be assumed throughout that C_0 does not intersect itself and that the matrix

$$
\left\| \phi^\beta_{p^i}(a_0, y_0(t), \dot{y}_0(t)) \right\| \qquad (\beta = 1, \cdots, m;\ i = 0, 1, \cdots, n)
$$

has rank m on t^1t^2. An element (a, y, p) will be said to be *on* C_0 if there is a constant $k>0$ and a value t on t^1t^2 such that

$$
a = a_0, \qquad y = y_0(t), \qquad p = k\dot{y}_0(t).
$$

By a *neighborhood of the elements* (a, y, p) *on* C_0 will be meant an admissible subregion $\mathfrak{R}_0$ of $\mathfrak{R}$ containing the elements (a, y, p) on C_0 in its interior. It will be convenient to designate such a neighborhood by the phrase "a *neighborhood* $\mathfrak{R}_0$ of C_0." Similarly by a *neighborhood* $\mathfrak{D}_0$ *of* C_0 *relative to* $\mathfrak{D}$ will be meant the set of all differentially admissible elements in a neighborhood $\mathfrak{R}_0$ of C_0.

The assumption that the arc C_0 is of class C'' is made only for convenience. In the first two papers it would be sufficient to assume that C_0 is of class C'.

In the third paper the conditions imposed on C_0 imply that it is of class C''. It is for this reason that we make our initial assumption that C_0 is of class C''. Similar remarks hold regarding our assumptions concerning the functions $f, f^\sigma, \phi^\beta, g, g^\sigma, T^{i1}, T^{i2}$.

The following lemma will be useful.

LEMMA 2.1. *There exist $n-m$ admissible functions $\phi^\gamma(a, y, p)$ $(\gamma=m+1, \cdots, n)$ of class C'' such that the determinant*

$$\left|\begin{matrix} p^i \\ \phi^\alpha_{p^i} \end{matrix}\right| \qquad (i = 0, 1, \cdots, n; \alpha = 1, \cdots, n) \tag{2.5}$$

is different from zero on C_0. Moreover the equations

$$|r| = c|p|, \qquad \phi^\beta(a, y, r) = 0, \qquad \phi^\gamma(a, y, r) = \phi^\gamma(a, y, p) + v^\gamma \tag{2.6}$$

have solutions $r^i(a, y, p, v, c)$ of class C'' on a neighborhood of the values $(a, y, p, v, c) = (a, y, p, 0, 1)$ on C_0. The solutions satisfy the homogeneity conditions

$$r^i(a, y, kp, kv, c) = kr^i(a, y, p, v, c) \qquad (k > 0).$$

The first statement in the lemma has been established by Bliss(5). The second statement follows from implicit function theorems.

3. **Weak E-dominance.** The present paper is concerned primarily with the properties of the Weierstrass E-function. In this section will be found a description of certain properties of this function that will be useful later. Most of these properties are well known.

By the *Weierstrass E-function* E_F associated with an admissible function F will be meant the function

$$E_F(a, y, p, q) = F(a, y, q) - q^i F_{p^i}(a, y, p). \tag{3.1}$$

In view of the relations (2.1) and (2.3) we have

$$E_F(a, y, kp, k'q) = k'E_F(a, y, p, q) \qquad (k > 0, k' > 0). \tag{3.2}$$

As a consequence of these relations it is clear that we can at will restrict ourselves to *normed sets* (a, y, p) and (a, y, q), that is, to sets for which $|p| = 1$ and $|q| = 1$.

The E-function

$$E_L(p, q) = L(q) - q^i p^i / L(p) \tag{3.3}$$

associated with the integrand

$$L(p) = |p| = (p^i p^i)^{1/2} \tag{3.4}$$

(5) Bliss, *The problem of Mayer with variable end points*, Trans. Amer. Math. Soc. vol. 19 (1918) pp. 312–313.

of the length integral will play a dominant role in this paper. It is easily seen that

$$(3.5) \qquad 0 < E_L(p, q) \leqq 2L(q) \qquad (q \neq kp,\ k > 0).$$

Moreover we have

$$(3.6) \qquad L_{p^i p^j} z^i z^j > 0 \qquad (z \neq \rho p),$$

as can be seen from the identity

$$L_{p^i p^j} z^i z^j = \frac{1}{2L^3} (z^i p^j - z^j p^i)(z^i p^j - z^j p^i).$$

Let F and H be admissible functions and let E_F and E_H be the corresponding E-functions. The function F will be said to *weakly E-dominate H near C_0 on $\mathfrak{D}$* if there is a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ and a constant $b>0$ such that the inequality

$$(3.7) \qquad | E_H(a, y, p, q) | \leqq b E_F(a, y, p, q)$$

holds for every pair of elements (a, y, p) and (a, y, q) in $\mathfrak{D}_0$. Similarly if an inequality of the form (3.7) holds for every pair of elements (a, y, p) and (a, y, q) on a neighborhood $\mathfrak{R}_0$ of C_0, then F will be said to *weakly E-dominate H near C_0 on $\mathfrak{R}$*. It is clear that F weakly E-dominates $H \equiv 0$ near C_0 on $\mathfrak{D}$ if and only if the inequality

$$E_F(a, y, p, q) \geqq 0$$

holds for every pair of elements (a, y, p), (a, y, q) on a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$.

In order to prove certain consequences of weak E-dominance we shall make use of the following well known result.

LEMMA 3.1. *If the inequality*

$$(3.8) \qquad E_F(a, y, p, q) \geqq 0$$

holds whenever (a, y, p) is on C_0 and (a, y, q) is a differentially admissible element on a neighborhood $\mathfrak{R}_0$ of those on C_0, then the inequality

$$(3.9) \qquad F_{p^i p^j} z^i z^j \geqq 0$$

holds on C_0, subject to the conditions

$$(3.10) \qquad z^i \phi^\beta_{p^i} = 0, \qquad z^i \neq \rho p^i.$$

The inequality (3.9) subject to the conditions (3.10) in general does not imply the inequality (3.8). These inequalities are however equivalent in case F is *nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$*, that is, in case the determinant

$$\begin{vmatrix} F_{p^ip^j} & \phi^{\beta}_{p^i} \\ \phi^{\gamma}_{p^j} & 0 \end{vmatrix} \qquad \begin{matrix} (i, j = 0, 1, \cdots, n) \\ (\beta, \gamma = 1, \cdots, m) \end{matrix}$$

has rank $n+m$ on C_0. This result is a consequence of the following:

THEOREM 3.1. *Given an admissible function F the following conditions are equivalent.*

I. *The inequality*

$$F_{p^ip^j}z^iz^j > 0 \tag{3.11}$$

holds on C_0 subject to the conditions (3.10).

II. *The function F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$ and the inequality* (3.9) *holds on C_0 subject to the conditions* (3.10).

III. *The function F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$ and weakly E-dominates $H \equiv 0$ near C_0 on $\mathfrak{D}$.*

IV. *The function F weakly E-dominates $L = |p|$ near C_0 on $\mathfrak{D}$.*

V. *Every admissible function H is weakly E-dominated by F near C_0 on $\mathfrak{D}$.*

VI. *There is a constant c such that the inequality*

$$\overset{*}{F}_{p^ip^j}z^iz^j > 0 \qquad (z \neq \rho p) \tag{3.12}$$

holds on C_0 for every set $z \neq \rho p$, where F^* is the admissible function(6)

$$F^* = F + (c/L)\phi^\beta\phi^\beta. \tag{3.13}$$

VII. *There is an admissible function F^* of the form* (3.13) *that weakly E-dominates L near C_0 on R.*

The equivalence of the first three conditions is well known. The equivalence of the last two follows from the equivalence of the first and fourth for the case when there are no side conditions $\phi^\beta = 0$. We shall accordingly restrict our attention to conditions IV, V, VI. Because of the homogeneity properties (2.3) we can suppose throughout that the vectors p and z occurring in (3.9), (3.11) and (3.12) satisfy the relations

$$p^iz^i = 0, \qquad |z| = 1, \qquad |p| = 1. \tag{3.14}$$

In this case the condition $z \neq \rho p$ is automatically satisfied and hence need not be considered in the arguments given below.

Suppose now that IV holds. Setting $G = F - bL$ $(b > 0)$ it follows from IV that b can be chosen so that the inequality

$$E_G(a, y, p, q) = E_F(a, y, p, q) - bE_L(p, q) \geqq 0$$

holds whenever (a, y, p) and (a, y, q) are on a neighborhood $\mathfrak{D}_0$ of C_0 relative to D. Let Q_F be the Legendre form

(6) Cf. W. T. Reid, loc. cit. p. 679.

$$Q_F = Q_F(a, y, p, z) = F_{p^ip^j}z^iz^j$$

for F and denote the corresponding forms for G and L by Q_G and Q_L respectively. Using Lemma 3.1 we see that the inequality

$$0 \leqq Q_G(a, y, p, z) = Q_F(a, y, p, z) - bQ_L(p, z)$$

is satisfied whenever (a, y, p) is on C_0 and the relations (3.10) and (3.14) hold. Since $Q_L>0$ on this set it follows that $Q_F>0$ also. Consequently condition IV (and hence also V) implies condition I.

We shall show next that condition I implies condition V and hence also condition IV. Let S be the set of points (a, y, p, z) having (a, y, p) on C_0 and satisfying the conditions (3.10) and (3.14). By I we have $Q_F>0$ on S. Consequently if H is an admissible function there is a constant $b_0\geqq 0$ such that $Q_F-bQ_H>0$ on S provided $|b|\leqq b_0$. Consequently if we set $G=F-bH$ we have $Q_G>0$ on S whenever $|b|\leqq b_0$. Since I implies III, the inequality

$$E_G(a, y, p, q) = E_F(a, y, p, q) - bE_H(a, y, p, q) \geqq 0$$

holds whenever $|b|\leqq b_0$ and (a, y, p), (a, y, q) lie in a suitably chosen neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$. It follows that condition I implies condition V and hence IV.

It remains to show that condition I is equivalent to condition VI. To this end let T be the set of points (a, y, p, z) with (a, y, p) on C_0 and satisfying (3.14). Set $H=\phi^\beta\phi^\beta/L$. Since $\phi^\beta=0$ on T we have

$$Q_H = \phi^\beta_{p^i}\phi^\beta_{p^j}z^iz^j$$

on T. Consequently $Q_H=0$ if and only if $\phi^\beta{}_{p^i}z^i=0$, that is, if and only if (a, y, p, z) is on the set S described in the last paragraph. It follows that on S the Legendre form $Q_{F^*}=Q_F+cQ_H$ for the function $F^*=F+cH$ is equal to Q_F. Hence condition VI implies condition I. Conversely if condition I holds, then $Q_F>0$ on S, that is, on the subset of T on which $Q_H=0$. Since Q_F and Q_H are continuous functions on T with $Q_H\geqq 0$ there is a constant $h>0$ such that $Q_F>0$ at all points of T having $Q_H<h$. Choose c so that $ch>m$, where $-m$ is the minimum value of Q_F on T. For this value of c we have $Q_{F^*}=Q_F+cQ_H>0$, as desired. Condition VI is therefore implied by condition I. This completes the proof of Theorem 3.1.

4. **Dominance and E-dominance.** An admissible function F will be said to *dominate a second admissible function H near C_0 on $\mathfrak{D}$* if there is a neighborhood $\mathfrak{F}$ of C_0 in ay-space and a constant $b>0$ such that the inequality

$$|H(a, y, p)| \leqq bF(a, y, p) \tag{4.1}$$

holds for every set (a, y, p) in $\mathfrak{D}$ with (a, y) in $\mathfrak{F}$. If b and $\mathfrak{F}$ can be chosen so that the inequality (4.1) holds for every set (a, y, p) in $\mathfrak{R}$ having (a, y) in $\mathfrak{F}$, then F will be said to *dominate H near C_0 on $\mathfrak{R}$.*

An admissible function F will be said to *E-dominate an admissible function H near C_0 on $\mathfrak{D}$* if there is a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ and a constant $b>0$ such that the inequality

$$(4.2) \qquad |E_H(a, y, p, q)| \leqq bE_F(a, y, p, q)$$

holds whenever (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$. If this inequality holds whenever (a, y, q) is in $\mathfrak{R}$ and (a, y, p) is in a neighborhood $\mathfrak{R}_0$ of C_0, then F will be said to *E-dominate H near C_0 on $\mathfrak{R}$.*

Relations between dominance and E-dominance are given in Theorem 4.1.

THEOREM 4.1. *Let F and H be admissible functions. If F dominates $L=|p|$ and E-dominates H near C_0 on $\mathfrak{D}$, then F dominates H near C_0 on $\mathfrak{D}$. Conversely, if F dominates H and E-dominates L near C_0 on $\mathfrak{D}$, then H is E-dominated by F near C_0 on $\mathfrak{D}$.*

The proof of this result will be given in the next section. Taking $F=L$ one obtains the following corollaries.

COROLLARY 1. *An admissible function H is E-dominated by L near C_0 on $\mathfrak{D}$ if and only if it is dominated by L near C_0 on $\mathfrak{D}$. If $H\equiv 0$ on $\mathfrak{D}$, then H is E-dominated by L near C_0 on $\mathfrak{D}$. Every admissible function H of the form*

$$H(a, y, p) = \lambda^\beta(a, y, p)\phi^\beta(a, y, p)$$

is E-dominated by L near C_0 on $\mathfrak{D}$.

COROLLARY 2. *Suppose that $\mathfrak{R}$ is the set of all elements $(a, y, p)\neq(a, y, 0)$ whose components (a, y) are in a region in ay-space. Then every admissible function H is E-dominated by L near C_0 on $\mathfrak{R}$.*

COROLLARY 3. *Suppose F dominates and E-dominates L near C_0 on $\mathfrak{D}$. Then an admissible function H is E-dominated by F near C_0 on $\mathfrak{D}$ if and only if F dominates H near C_0 on $\mathfrak{D}$.*

A further result of this type is given in the following theorem.

THEOREM 4.2. *If F is an admissible function that E-dominates L near C_0 on $\mathfrak{D}$, there is an admissible function of the form*

$$G(a, y, p) = D^i(a, y)p^i$$

such that $F-G$ dominates and E-dominates L near C_0 on $\mathfrak{D}$. In fact if along an extension of C_0 in $\mathfrak{D}$ the functions $F_{p^i}(a, y, p)$ have continuous derivatives with respect to arc length then G can be chosen to be the integrand of an invariant integral.

The hypothesis in the last statement of the theorem is satisfied, for example, when there is an extension of C_0 that is an extremal. It also holds when F is of class C''.

This theorem will be established in the next section. The principal theorem in this paper is the following:

THEOREM 4.3. *An admissible function F is nonsingular relative to $\phi^1, \cdots, \phi^m$ and E-dominates $\phi^1, \cdots, \phi^m$ near C_0 on $\mathfrak{D}$ if and only if $L = |p|$ is E-dominated by F near C_0 on $\mathfrak{D}$.*

The proof of this result will be given in §6 below. For the case in which there are no side conditions $\phi_\beta = 0$ this result can be stated in the form given in the following corollary.

COROLLARY 1. *The function L is E-dominated by F near C_0 on $\mathfrak{R}$ if and only if the determinant $|F_{p^ip^j}|$ has rank n on C_0 and there is a neighborhood $\mathfrak{R}_0$ of C_0 such that the inequality*

$$E_F(a, y, p, q) \geqq 0 \tag{4.4}$$

holds whenever (a, y, p) is in $\mathfrak{R}_0$ and (a, y, q) is in $\mathfrak{R}$.

THEOREM 4.4. *Suppose that F is of the form*

$$F(a, y, p) = [g^{ij}(a, y)p^ip^j]^{1/2},$$

where $g^{ij}(a, y)\pi^i\pi^j$ is a positive definite quadratic form. Then F dominates and E-dominates L near C_0 on $\mathfrak{R}$. Moreover an admissible function H is E-dominated by F near C_0 on $\mathfrak{D}$ if and only if it is dominated by F near C_0 on $\mathfrak{D}$.

For in this case the inequality (4.4) holds for all (a, y, p) and (a, y, q) on $\mathfrak{R}$. Moreover the determinant $|F_{p^ip^j}|$ has rank n. It follows from the last corollary that L is E-dominated by F near C_0 on $\mathfrak{R}$. The last statement follows readily from the fact that each of the two functions F and L dominates and E-dominates the other.

5. **Proofs of Theorems 4.1 and 4.2.** As a first step observe that since the derivatives F_{p^i}, H_{p^i} are positively homogeneous of order zero in p, there is a neighborhood $\mathfrak{R}_0$ of C_0 and a constant $c>0$ such that the inequalities

$$|F_{p^i}(a, y, p)q^i| \leqq cL(q), \qquad |H_{p^i}(a, y, p)q^i| \leqq cL(q)$$

hold whenever (a, y, p) is in $\mathfrak{R}_0$. Using the relations

$$|H(a, y, q)| \leqq |E_H(a, y, p, q)| + |q^iH_{p^i}(a, y, p)|,$$
$$|E_H(a, y, p, q)| \leqq |H(a, y, q)| + |q^iH_{p^i}(a, y, p)|$$

one obtains the first two of the inequalities

$$|H(a, y, q)| \leqq |E_H(a, y, p, q)| + cL(q), \tag{5.1}$$
$$|E_H(a, y, p, q)| \leqq |H(a, y, q)| + cL(q), \tag{5.2}$$
$$F(a, y, q) \leqq E_F(a, y, p, q) + cL(q), \tag{5.3}$$
$$E_F(a, y, p, q) \leqq F(a, y, q) + cL(q), \tag{5.4}$$

which hold whenever (a, y, p) is in $\mathfrak{R}_0$ and (a, y, q) is in $\mathfrak{R}$. The last two can be established in a similar manner.

Suppose now that F dominates L and E-dominates H near C_0 on $\mathfrak{D}$. Then there is a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ and a constant b such that

$$L(q) \leqq bF(a, y, q), \tag{5.5}$$

$$|E_H(a, y, p, q)| \leqq bE_F(a, y, p, q) \tag{5.6}$$

hold whenever (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$. We can suppose that $\mathfrak{D}_0$ is interior to $\mathfrak{R}_0$. By the use of Lemma 2.1 it is seen that we can select a neighborhood $\mathfrak{F}$ of C_0 in ay-space such that if (a, y) is in $\mathfrak{F}$ there is a value p such that (a, y, p) is in $\mathfrak{D}_0$. Consider therefore an element (a, y, p) in $\mathfrak{D}_0$ with (a, y) in $\mathfrak{F}$ and select any value q such that (a, y, q) is in $\mathfrak{D}$. Using (5.1), (5.6) and (5.4) we see that

$$|H(a, y, q)| \leqq bE_F(a, y, p, q) + cL(q) \leqq bF(a, y, q) + (b + 1)cL(q).$$

It follows from (5.5) that

$$|H(a, y, q)| = b'F(a, y, q) \tag{5.7}$$

where $b' = b + b^2c + bc$. Consequently F dominates H near C_0 on $\mathfrak{D}$, as was to be proved.

Suppose conversely that F dominates H and E-dominates L near C_0 on $\mathfrak{D}$. Let $\mathfrak{F}$ be a neighborhood of C_0 in (a, y)-space and b' be a constant chosen so that the inequality (5.7) holds for every set (a, y, q) in $\mathfrak{D}$ with (a, y) in $\mathfrak{F}$. By Theorem 3.1, parts IV and V, the function F weakly E-dominates H near C_0 on $\mathfrak{D}$. We can accordingly select a constant $b>0$ so that (5.6) holds whenever (a, y, p) and (a, y, q) are in a neighborhood $\mathfrak{D}_1$ of C_0 relative to $\mathfrak{D}$. Choose a neighborhood $\mathfrak{D}_0$ of C_0 whose closure is in $\mathfrak{D}_1$ and whose components (a, y) are in $\mathfrak{F}$. According to our hypothesis that L is E-dominated by F we can diminish $\mathfrak{D}_0$ if necessary so that there is a constant b'' for which the inequality $E_L(p, q) \leqq b''E_F(a, y, p, q)$ holds whenever (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$. Since the closure of $\mathfrak{D}_0$ is interior to $\mathfrak{D}_1$ we can select another constant $c'>0$ effective as in the relation

$$L(q) \leqq c'E_L(p, q) \leqq c''E_F(a, y, p, q) \qquad (c'' = c'b'') \tag{5.8}$$

when (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$ but not in $\mathfrak{D}_1$. Consider now a set (a, y, p) in $\mathfrak{D}_0$ and select q so that (a, y, q) is in $\mathfrak{D}$. If (a, y, q) is in $\mathfrak{D}_1$ then (5.6) holds. Suppose therefore that (a, y, q) is not in $\mathfrak{D}_1$. Combining the relations (5.2), (5.7), (5.3) and (5.8) we obtain

$$\begin{aligned}|E_H(a, y, p, q)| &\leqq b'F(a, y, q) + cL(q)\\ &\leqq b'E_F(a, y, p, q) + c_1L(q) \qquad (c_1 = b'c + c)\\ &\leqq c_2E_F(a, y, p, q) \qquad (c_2 = b' + c_1c'').\end{aligned}$$

Consequently (5.6) holds in this case also provided $b \geqq c_2$. It follows that H is E-dominated by F near C_0 on $\mathfrak{D}$ and Theorem 4.1 is proved.

In order to prove Theorem 4.2 we shall make use of the following lemma.

LEMMA 5.1. *Let $B_i(a, y)$ be a set of continuous functions having continuous derivatives with respect to arc length along an extension of C_0 in $\mathfrak{D}$. There exist functions $D_i(a, y)$ of class C' which coincide with $B_i(a, y)$ along C_0 and which satisfy the relations*

$$\partial D_i/\partial y^j = \partial D_j/\partial y^i \qquad (i, j = 0, 1, \cdots, n). \tag{5.9}$$

In the proof we can suppose that C_0 is part of the y^0-axis since this can be brought about by a nonsingular transformation of class C''. Under this transformation the vector B_i is to be transformed covariantly. Moreover this transformation can be carried out so as to preserve arc length along C_0. Setting $x = y^0$ we then have C_0 given by the set

$$a_h = 0, \qquad 0 \leqq x \leqq l, \qquad y^j = 0 \qquad (j = 1, \cdots, n).$$

The given functions $B_i(a, x, y)$ are such that $B_i(0, x, 0)$ have continuous derivatives. Set

$$D_j(a, x, y) = B_j(0, x, 0) \qquad (j = 1, \cdots, n)$$
$$D_0(a, x, y) = B_0(0, x, 0) + y_j \partial D_j/\partial x.$$

It is clear that $D_i = B_i$ along C_0 and that

$$\partial D_0/\partial y^j = \partial D_j/\partial x, \qquad \partial D_j/\partial y^k = \partial D_k/\partial y^j = 0 \qquad (j, k = 1, \cdots, n).$$

This proves the lemma.

We are now in position to prove Theorem 4.2. To this end let $p^i(a, y)$ be functions of class C' such that $(a, y, p^i(a, y))$ is in $\mathfrak{D}$ when (a, y) is in a neighborhood $\mathfrak{F}$ of C_0 and is on C_0 whenever (a, y) is on C_0. If $\mathfrak{F}$ is taken sufficiently small, the fact that L is E-dominated by F implies the existence of a positive constant c such that

$$E_F(a, y, p(a, y), q) \geqq cE_L(p(a, y), q)$$

holds when (a, y) is in $\mathfrak{F}$ and (a, y, q) is in $\mathfrak{D}$. Setting

$$B_i(a, y) = F_{p^i}(a, y, p(a, y)) - cL_{p^i}(p(a, y))$$

it is seen that this inequality takes the form

$$F(a, y, q) - B_i(a, y)q^i \geqq cL(q). \tag{5.10}$$

Since $E_F = E_{F-G}$ this proves the first statement in Theorem 4.2.

In order to prove the last statement in the theorem observe that the functions $B_i(a, y)$ have continuous derivatives with respect to arc length along an extension of C_0. Select functions D_i related to B_i as described in Lemma 5.1.

Diminish $\mathfrak{F}$ so that

$$|D - B| < c/2$$

on $\mathfrak{F}$. Hence if (a, y) is in $\mathfrak{F}$ and (a, y, q) in $\mathfrak{D}$ we have, by (5.10),

$$F(a, y, q) - D_i q^i \geqq cL(q) - (D_i - B_i)q^i \geqq (c/2)L(q),$$

as was to be proved.

6. **Proof of Theorem** 4.3. If L is E-dominated by F near C_0 in $\mathfrak{D}$, then $\phi^1, \cdots, \phi^m$ are E-dominated by F near C_0 on $\mathfrak{D}$, by virtue of Corollary 1 to Theorem 4.1. From Theorem 3.1 it follows that F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$. Theorem 4.3 will be established if we show conversely that L is E-dominated by F near C_0 on $\mathfrak{D}$ whenever $\phi^1, \cdots, \phi^n$ are E-dominated by F near C_0 on $\mathfrak{D}$ and F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$.

In the remainder of this section we shall assume therefore that F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$ and that there is a constant $b>0$ and a neighborhood $\mathfrak{D}_1$ of C_0 relative to $\mathfrak{D}$ such that the inequality

$$E_F(a, y, p, q) \geqq b\,|\,E_{\phi\beta}(a, y, p, q)\,| = b\,|\,q^i\phi^\beta_{p^i}(a, y, p)\,| \tag{6.1}$$

holds whenever (a, y, p) is in $\mathfrak{D}_1$ and (a, y, q) is in $\mathfrak{D}$. By virtue of Theorem 3.1 the function F weakly E-dominates L near C_0 on $\mathfrak{D}$. Hence we can suppose that $\mathfrak{D}_1$ and b have been chosen so that the relation

$$E_F(a, y, p, q) \geqq bE_L(p, q) \tag{6.2}$$

holds whenever (a, y, p) and (a, y, q) are in $\mathfrak{D}_1$.

Theorem 4.3 will be established if we show that after suitably diminishing b (keeping $b>0$) we can select a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ and interior to $\mathfrak{D}_1$ such that (6.2) holds whenever (a, y, p) is in $\mathfrak{D}_0$ and (a, y, q) is in $\mathfrak{D}$. Suppose this choice cannot be made. Then given a constant $b'>0$ and a neighborhood $\mathfrak{D}_0$ of C_0 in $\mathfrak{D}_1$ the inequality

$$E_F(a, y, p, q) < b'E_L(p, q)$$

holds for a suitably chosen set (a, y, p, q) with (a, y, p) in $\mathfrak{D}_0$ and (a, y, q) in $\mathfrak{D}-\mathfrak{D}_1$. Because of the homogeneity properties of E_F and E_L this set can be chosen so that $|p| = |q| = 1$ and hence such that $E_L(p, q) \leqq 2$. There exists therefore a sequence

$$(a_k, y_k, p_k, q_k) \qquad (k = 1, 2, \cdots)$$

converging to a set (a_0, y_0, p_0, q_0) having the following properties:

$$(a_k, y_k, p_k) \text{ in } \mathfrak{D}_1,\ (a_k, y_k, q_k) \text{ in } \mathfrak{D} - \mathfrak{D}_1,\ (a_0, y_0, p_0) \text{ on } C_0, \tag{6.3}$$

$$|\,p_k\,| = |\,q_k\,| = |\,p_0\,| = |\,q_0\,| = 1, \tag{6.4}$$

$$\lim_{k=\infty} E_F(a_k, y_k, p_k, q_k) = 0. \tag{6.5}$$

The set (a_0, y_0, q_0) need not be in $\mathfrak{D}$. By the use of (6.1) and (6.5) it is seen that

$$q_0^i \phi_{p^i}^\beta(a_0, y_0, p_0) = 0. \tag{6.6}$$

Consider now the functions $r^i(a, y, p, v, c)$ described in Lemma 2.1. If ϵ is sufficiently small the functions

$$r_k^i(v, c) = r^i(a_k, y_k, p_k, v, c) \qquad |v| < \epsilon, \ |c - 1| < \epsilon$$

will be well defined for large values of k and will converge uniformly to

$$r_0^i(v, c) = r^i(a_0, y_0, p_0, v, c).$$

Moreover the elements

$$(a_k, y_k, r_k(v, c)), \qquad (a_0, y_0, r_0(v, c))$$

will be in $\mathfrak{D}_1$. By (6.1) and (6.5) we have

$$\liminf_{k=\infty} \left\{ E_F[a_k, y_k, r_k(v, c), q_k] - E_F(a_k, y_k, p_k, q_k) \right\} \geqq 0$$

and hence, by the definition of E_F,

$$\lim_{k=\infty} q_k^i [F_{p^i}(a_k, y_k, p_k) - F_{p^i}(a_k, y_k, r_k(v, c))] \geqq 0.$$

It follows that

$$q_0^i [F_{p^i}(a_0, y_0, p_0) - F_{p^i}(a_0, y_0, r_0(v, c))] \geqq 0. \tag{6.7}$$

If $q_0 = -p_0$ this relation can be written in the form

$$E_F(a_0, y_0, r_0(v, c), p_0) \leqq 0. \tag{6.8}$$

For $c=1$, $v \neq 0$ we have $|r_0| = 1$, $r_0 \neq p_0$ and (6.2) holds with $(a, y, p, q) = (a_0, y_0, r_0(v, 0), p_0)$, contrary to (6.8). Hence $q_0 \neq -p_0$. We now select

$$c = |p_0 + eq_0|, \ v^\gamma = \phi^\gamma(a_0, y_0, p_0 + eq_0) - \phi^\gamma(a_0, y_0, p_0)$$

where ϕ^γ are the functions defined in Lemma 2.1. Let $r(e)$ be the corresponding values of $r^i(v, c)$. Then $r(0) = p_0$ and by Lemma 2.1

$$|r(e)| = |p_0 + eq_0|,$$

$$\phi^\beta(a_0, y_0, r(e)) = 0, \qquad \phi^\gamma(a_0, y_0, r(e)) = \phi^\gamma(a_0, y_0, p_0 + eq_0).$$

Differentiating these relations with respect to e and setting $e=0$ we find that the derivative r_e^i of r^i satisfies the relation

$$p_0^i r_e^i = p_0^i q_0^i,$$

$$r_e^i \phi_{p^i}^\beta(a_0, y_0, p_0) = 0, \qquad (r_e^i - q_0^i)\phi_{p^i}^\gamma(a_0, y_0, p_0) = 0.$$

Hence by (6.6) we have

$$r_e^i(0) = q_0^i.$$

Consider now the function

$$Q(e) = q_0^i[F_{p^i}(a_0, y_0, p_0) - F_{p^i}(a_0, y_0, r(e))].$$

We have $Q(0)=0$ since $r(0)=p_0$ and $Q(e)\geqq 0$ by (6.7). It follows that $Q'(0)=0$. Since $r_e{}^i(0)=q_0{}^i$ this gives

$$0 = Q'(0) = - q_0^i q_0^j F_{p^ip^j}(a_0, y_0, p_0).$$

Since F weakly E-dominates L near C_0 on $\mathfrak{D}$ and $q_0\neq \pm p_0$ we have by Theorem 3.1 and (6.6)

$$q_0^i q_0^j F_{p^ip^j}(a_0, y_0, p_0) > 0.$$

This contradiction completes the proof of Theorem 4.3.

7. **Further theorems on E-dominance.** The following theorem is of interest.

THEOREM 7.1. *Let $r^i(a, y)$ be a set of $n+1$ continuous functions satisfying the equations*

$$\phi^\beta[a, y, r(a, y)] = 0 \tag{7.1}$$

and having the set $[a, y, r(a, y)]$ on C_0 whenever (a, y) is on C_0. The function $L=|p|$ is E-dominated by F near C_0 on $\mathfrak{D}$ if and only if there is a neighborhood $\mathfrak{F}$ of the points (a, y) on C_0 and a constant $b>0$ such that the inequality

$$E_L(r(a, y), q) \leqq bE_F(a, y, r(a, y), q) \tag{7.2}$$

holds for every set (a, y, q) in $\mathfrak{D}$ with (a, y) in $\mathfrak{F}$.

In order to establish this result suppose first that $\mathfrak{F}$ and b can be chosen so that the inequality (7.2) holds whenever (a, y) is in $\mathfrak{F}$ and (a, y, q) is in $\mathfrak{D}$. By an argument like that used in the proof of Theorem 3.1 it can be seen that the inequality (7.2) implies condition I of Theorem 3.1 and hence that F weakly E-dominates L near C_0 on $\mathfrak{D}$. Hence there is a neighborhood $\mathfrak{D}_1$ of C_0 relative to $\mathfrak{D}$ and a constant $b_1>0$ such that the inequality

$$E_L(p, q) \leqq b_1E_F(a, y, p, q) \tag{7.3}$$

holds whenever (a, y, p) and (a, y, q) are in $\mathfrak{D}_1$.

As a next step in the proof of Theorem 7.1 we write $E_F(a, y, p, q)$ in the form

$$E_F(a, y, p, q) = E_F(a, y, r(a, y), q) + Q(a, y, p, q) \tag{7.4}$$

where

$$Q = q^i[F_{p^i}(a, y, r(a, y)) - F_{p^i}(a, y, p)]. \tag{7.5}$$

Suppose now that L is not E-dominated by F near C_0 on $\mathfrak{D}$. Then there exist a sequence of pairs of elements (a_k, y_k, p_k), (a_k, y_k, q_k) in $\mathfrak{D}$ converging to a pair (a_0, y_0, p_0), (a_0, y_0, q_0) such that (a_0, y_0, p_0) is on C_0 and

$$|p_k| = |q_k| = |r(a_k, y_k)|, \tag{7.6}$$

$$E_F(a_k, y_k, p_k, q_k) < (1/k)E_L(p_k, q_k). \tag{7.7}$$

By (7.6) it follows that $p_0 = r(a_0, y_0)$ and hence by (7.5) that

$$\lim_{k=\infty} Q(a_k, y_k, p_k, q_k) = 0.$$

Combining this result with (7.4) and (7.7) we obtain the inequality

$$\limsup_{k=\infty} E_F(a_k, y_k, r(a_k, y_k), q_k) \leqq 0.$$

As a consequence of this relation we have, by (7.2), $E_L(r(a_0, y_0), q_0) = 0$ and hence also $q_0 = r(a_0, y_0) = p_0$. For large values of k the elements (a_k, y_k, p_k) and (a_k, y_k, q_k) accordingly will be in $\mathfrak{D}_1$ and will satisfy the relations (7.3) and (7.7). This is impossible. The criterion stated in the theorem implies that L is E-dominated by F near C_0 on $\mathfrak{D}$. The converse is immediate and the theorem is established.

THEOREM 7.2. *Suppose that F weakly E-dominates L near C_0 on $\mathfrak{R}$. Then L is E-dominated by F near C_0 on $\mathfrak{D}$ if and only if there is a neighborhood $\mathfrak{R}_1$ of C_0 and a constant $b>0$ such that the inequality*

$$E_L(p, q) \leqq bE_F(a, y, p, q) \tag{7.8}$$

holds whenever (a, y, p) is in $\mathfrak{R}_1$ and (a, y, q) is in $\mathfrak{D}$.

In order to prove this result let $r^i(a, y, p)$ be a set of continuous functions defined on a neighborhood $\mathfrak{R}_0$ of C_0 such that the set $(a, y, r(a, y, p))$ is in $\mathfrak{D}$ and coincides with (a, y, p) when (a, y, p) is on C_0. The existence of such functions is established by the use of Lemma 2.1. As before we write

$$E_F(a, y, p, q) = E_F(a, y, r(a, y, p), q) + Q(a, y, p, q) \tag{7.9}$$

where $Q(a, y, p, q) = q^i[F_{p^i}(a, y, r(a, y, p)) - F_{p^i}(a, y, p)]$.

Suppose now that L is E-dominated by F near C_0 on $\mathfrak{D}$ and that the inequality (7.8) failed to hold as stated. Then there would exist a sequence of pairs of elements (a_k, y_k, p_k), (a_k, y_k, q_k) converging to elements (a_0, y_0, p_0), (a_0, y_0, q_0) such that (a_k, y_k, q_k) is in $\mathfrak{D}$,

$$\begin{gathered} |p_k| = |q_k| = 1, \qquad \lim_{k=\infty} r(a_k, y_k, p_k) = p_0, \\ E_F(a_k, y_k, p_k, q_k) < (1/k)E_L(p_k, q_k) \leqq 2/k. \end{gathered} \tag{7.10}$$

Using (7.9) it follows that

$$\limsup_{k=\infty} E_F(a_k, y_k, r(a_k, y_k, p_k), q_k) \leqq 0.$$

Since L is E-dominated by F near C_0 on $\mathfrak{D}$ and the element $(a_k, y_k, r(a_k, y_k, p_k))$ is in $\mathfrak{D}$ this relation can hold only in case

$$\lim_{k=\infty} E_L[r(a_k, y_k, p_k), q_k] = E_L(p_0, q_0) = 0,$$

that is only in case $q_0 = p_0$. For large values of k the elements (a_k, y_k, p_k) and (a_k, y_k, q_k) are in an arbitrarily small neighborhood $\mathfrak{R}_1$ of C_0 and satisfy the relations (7.10). But this is impossible when F weakly E-dominates L near C_0 on $\mathfrak{R}$. The inequality (7.8) therefore holds as stated. The converse is immediate and the theorem is established.

8. **A consequence of E-dominance.** It was seen in Theorem 3.1 that an admissible function F weakly E-dominates L near C_0 on $\mathfrak{D}$ if and only if it can be modified on the set $\mathfrak{R}-\mathfrak{D}$ so that it weakly E-dominates L near C_0 on $\mathfrak{R}$. This result is also valid for E-dominance, as can be seen by the use of the following theorem.

THEOREM 8.1. *Let F be an admissible function that E-dominates L near C_0 on $\mathfrak{D}$. There exists an admissible function F^* of the form*

$$F^* = F + \theta(a, y, p)\phi^\beta\phi^\beta \tag{8.1}$$

which E-dominates L near C_0 on $\mathfrak{R}$. The function $\theta(a, y, p)$ satisfies the homogeneity condition

$$\theta(a, y, kp) = k^{-1}\theta(a, y, p) \qquad (k > 0) \tag{8.2}$$

on $\mathfrak{R}$ and can be chosen to be of class C^∞.

In order to establish this result we can suppose, by Theorem 3.1, that F has been modified so that F weakly E-dominates L near C_0 on $\mathfrak{R}$. By virtue of Theorem 7.2 we can select a neighborhood $\mathfrak{R}_1$ of C_0 and a constant $c>0$ such that the inequality

$$E_F(a, y, p, q) \geqq cE_L(p, q) \tag{8.3}$$

holds whenever (a, y, p) is in $\mathfrak{R}_1$ and (a, y, q) is in $\mathfrak{D}$ or in $\mathfrak{R}_1$. Select admissible regions $\mathfrak{R}_2, \mathfrak{R}_3, \cdots$ such that the closure of $\mathfrak{R}_j$ is in $\mathfrak{R}_{j+1}$ and such that

$$\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2 + \cdots.$$

Let $\mathfrak{R}_0$ be a neighborhood of C_0 whose closure is in $\mathfrak{R}_1$. By virtue of (8.3) there is a constant $c'>0$ such that the inequality

$$E_F(a, y, p, q) > 2c' \tag{8.4}$$

holds for normed sets (a, y, p) and (a, y, q) having (a, y, p) in $\mathfrak{R}_0$ and (a, y, q) in $\mathfrak{D}$ but not in $\mathfrak{R}_1$. By continuity there is for each integer $j \geqq 1$ a constant

$\delta_j > 0$ such that (8.4) holds subject to the conditions

$$(8.5)\qquad (a, y, p) \text{ in } \mathfrak{R}_0, \quad (a, y, q) \text{ in } \mathfrak{R}_{j+1} - \mathfrak{R}_j, \quad |p| = |q| = 1,$$

$$(8.6)\qquad \phi^\beta(a, y, q)\phi^\beta(a, y, q) \leqq \delta_j.$$

Choose a constant b_j so that the inequality

$$(8.7)\qquad E_F(a, y, p, q) \geqq b_j$$

holds whenever (8.5) is satisfied and let c_j be a positive constant such that

$$(8.8)\qquad b_j + c_j\delta_j > 2c' \qquad (j \text{ not summed}).$$

Let $\theta_j(a, y, p)$ be a function of normed sets (a, y, p) of class c^∞ such that $\theta_j = 0$ on $\mathfrak{R}_{j-1}$, $\theta_j \geqq 0$ on $\mathfrak{R}_j$, $\theta_j = 1$ exterior to $\mathfrak{R}_j$. For an arbitrary set (a, y, p) on $\mathfrak{R}$ we define θ_j by the formula

$$\theta_j(a, y, p) = L^{-1}\theta_j(a, y, p/L) \qquad (L = |p|).$$

The function θ defined by the sum

$$(8.9)\qquad \theta(a, y, p) = c_j\theta_j(a, y, p)$$

can be shown to have the properties described in Theorem 8.1. It is well defined on $\mathfrak{R}$ since at most $j+1$ of the terms in (8.9) are different from zero on $\mathfrak{R}_j$. The relation (8.2) holds. Moreover

$$(8.10)\qquad \theta \geqq c_j \text{ on } \mathfrak{R}_{j+1} - \mathfrak{R}_j \qquad (j \geqq 1).$$

Since $\theta \equiv 0$ on $\mathfrak{R}_0$ the E-function for the function F^* given by (8.1) is expressible in the form

$$(8.11)\quad E_{F^*}(a, y, p, q) = E_F(a, y, p, q) + \theta(a, y, q)\phi^\beta(a, y, q)\phi^\beta(a, y, q)$$

whenever (a, y, p) is in $\mathfrak{R}_0$. Consider now normed sets (a, y, p) and (a, y, q) in $\mathfrak{R}$ with (a, y, p) in $\mathfrak{R}_0$. If (a, y, q) is in $\mathfrak{R}_1$, then (8.3) holds with F replaced by F^*. Suppose therefore that (a, y, q) is in $\mathfrak{R}_{j+1} - \mathfrak{R}_j$ $(j \geqq 1)$. If the inequality (8.6) holds, then by (8.4) and (8.11) we have

$$(8.12)\qquad E_{F^*}(a, y, p, q) > 2c' \geqq c'E_L(p, q).$$

If (8.6) fails to hold, then by (8.7), (8.8), (8.10), (8.11) it is seen that (8.12) still holds. Hence F E-dominates L near C_0 on $\mathfrak{R}$, as was to be proved.

Combining this result with Theorem 4.3 we have the following corollary.

COROLLARY. *If F is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$ and E-dominates $\phi^1, \cdots, \phi^m$ near C_0 on $\mathfrak{D}$, then there exists a function F^* of the form* (8.1) *that E-dominates L near C_0 on $\mathfrak{R}$.*

If in the proof just given we set $c' = 0$, one obtains the following:

THEOREM 8.2. *Let F be an admissible function such that at each element*

(a, y, p) in a neighborhood $\mathfrak{N}_1$ of those on C_0 the inequality

$$E_F(a, y, p, q) > 0$$

holds whenever (a, y, q) is in $\mathfrak{D}$ or in $\mathfrak{N}_1$ and $(q) \neq (kp)$ $(k>0)$. There exists a function F^* of the form (8.1) *such that at each element (a, y, p) in a neighborhood $\mathfrak{N}_0$ of those on C_0 one has*

$$E_{F^*}(a, y, p, q) > 0$$

whenever (a, y, q) is in $\mathfrak{N}$ and $(q) \neq (kp)$ $(k>0)$.

9. **The strengthened condition of Weierstrass.** We now return to the study of the problem described in the introduction. The functions $f(a, y, p)$, $f^\sigma(a, y, p)$, $\phi^\beta(a, y, p)$ are admissible functions of class C''. We form the function

$$F(a, y, p, \lambda, \mu) = \lambda^0 f + \lambda^\sigma f^\sigma + \mu^\beta \phi^\beta. \tag{9.1}$$

The arc C_0

$$\overset{h}{a}_0, \ \overset{i}{y}_0(t) \qquad (t_1 \leqq t \leqq t_2)$$

will be said to satisfy the strengthened condition II_N *of Weierstrass with a set of multipliers*

$$\lambda^0, \lambda^\sigma, \mu^\beta(t) \qquad (\sigma = 1, \cdots, s; \beta = 1, \cdots, m) \tag{9.2}$$

if the following conditions hold: The multipliers λ^0, λ^σ are constants and $\lambda^0 \geqq 0$; the multipliers $\mu^\beta(t)$ are continuous functions of t on t_1t_2; at each element (a, y, p, λ, μ) in a neighborhood N of those on C_0 (λ^0 being held fast) having (a, y, p) in $\mathfrak{D}$ the inequality

$$E(a, y, p, \lambda, \mu, q) \geqq 0 \tag{9.3}$$

holds whenever (a, y, q) is in $\mathfrak{D}$. Here

$$E = F(a, y, q, \lambda, \mu) - q^i F_{p^i}(a, y, p, \lambda, \mu).$$

The arc C_0 together with the set of multipliers (9.2) will be said to be *nonsingular* if the determinant

$$\begin{vmatrix} F_{p^ip^j} & \overset{\beta}{\phi}_{p^i} \\ \overset{\gamma}{\phi}_{p^i} & 0 \end{vmatrix} \qquad \begin{matrix} (i, j = 0, 1, \cdots, n) \\ (\beta, \gamma = 1, \cdots, m) \end{matrix}$$

has rank $m+n$ on C_0.

THEOREM 9.1. *Let l^0, l, $m^\beta(a, y)$ be a set of continuous functions such that on C_0 we have*

$$l^0 = \lambda^0, \qquad l^\sigma = \lambda^\sigma, \qquad m^\beta[a_0, y_0(t)] = \mu^\beta(t) \tag{9.4}$$

where the multipliers on the right are given by the set (9.2). *The arc* C_0 *together with the multipliers* (9.2) *is nonsingular and satisfies the condition* II_N *of Weierstrass if and only if the admissible function*

$$F[a, y, p, l, m(a, y)] \tag{9.5}$$

E-dominates the functions $L = |p|$ *and* f^σ $(\sigma = 1, \cdots, s)$ *near* C_0 *on* $\mathfrak{D}$.

In order to prove this result let $F^*(a, y, p)$ be the function (9.5). We have the identity

$$\begin{aligned} E(a, y, p, \lambda, \mu, q) &= E_{F^*}(a, y, p, q) \\ &\quad + (\lambda^\sigma - l^\sigma)E_{f^\sigma}(a, y, p, q) + (\mu^\beta - m^\beta)E_{\phi^\beta}(a, y, p, q). \end{aligned} \tag{9.6}$$

Suppose now that C_0 together with the multipliers (9.2) is nonsingular and satisfies the condition II_N of Weierstrass. Then F^* is obviously nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$. Moreover by virtue of condition II_N there is a neighborhood $\mathfrak{D}_0$ of C_0 relative to $\mathfrak{D}$ and a constant $b > 0$ such that the left member of (9.6) is positive whenever (a, y, p) is in $\mathfrak{D}_0$, (a, y, q) is in $\mathfrak{D}$ and

$$|\lambda^\sigma - l^\sigma(a, y)| \leqq b, \qquad |\mu^\beta - m^\beta(a, y)| \leqq b.$$

It follows from (9.6) that $E_{F^*} \geqq b|E_{f^\sigma}|$, $E_{F^*} \geqq b|E_{\phi^\beta}|$ in this event. Consequently f^σ and ϕ^β are E-dominated by F^* near C_0 on $\mathfrak{D}$. By Theorem 4.3 the function $L = |p|$ is also E-dominated by F^* near C_0 on $\mathfrak{D}$.

Suppose conversely that L and f^σ are E-dominated by F^* near C_0 on $\mathfrak{D}$. Then, by Theorem 4.3, F^* is nonsingular on C_0 relative to $\phi^1, \cdots, \phi^m$ and consequently C_0 and the set (9.2) are nonsingular. Moreover by the same theorem the functions ϕ^β are E-dominated by F^* near C_0 on $\mathfrak{D}$. By the use of the identity (9.6) it is seen that the condition II_N holds for C_0 and the multipliers (9.2). This completes the proof of the theorem.

THEOREM 9.2. *Let* $p^i(a, y)$ *be a set of continuous functions such that the set* $[a, y, p(a, y)]$ *is on* C_0 *when* (a, y) *is on* C_0 *and such that*

$$\phi^\beta[a, y, p(a, y)] = 0 \qquad (\beta = 1, \cdots, m)$$

on a neighborhood of C_0. *Let* l^0, l, $m^\beta(a, y)$ *be a set of continuous multipliers satisfying the conditions* (9.4) *on* C_0. *Then* C_0 *and the set of multipliers* (9.2) *are nonsingular and satisfy the condition* II_N *if and only if there is a neighborhood* $\mathfrak{F}$ *of* C_0 *in ay-space and a constant* $b > 0$ *such that the inequality*

$$E_L[p(a, y), q] \leqq bE[a, y, p(a, y), l, m(a, y), q]$$

holds for every element (a, y, q) *in* $\mathfrak{D}$ *with* (a, y) *in* $\mathfrak{F}$.

This result is obtained by combining Theorems 9.1 and 7.1.

THEOREM 9.3. *Let* $F(a, y, p)$ *be a function of the form*

$$F(a, y, p) = l^0 f + l^\sigma f^\sigma + m^\beta(a, y)\phi^\beta \tag{9.7}$$

where $l^0 \geqq 0$, l^σ *are constants, and* $m^\beta(a, y)$ *are continuous near* C_0. *Let* $H(a, y, p)$ *be a second admissible function of the form*

$$(a, y, p) = l^0 f + \lambda^\sigma(a, y) f^\sigma + \mu^\beta(a, y)\phi^\beta$$

where $\lambda^\sigma(a, y)$ *and* $\mu^\beta(a, y)$ *are continuous near* C_0. *If* $L = |p|$ *and* f^σ *are E-dominated by* F *near* C_0 *on* $\mathfrak{D}$, *then the function* H *is E-dominated by* F *near* C_0 *on* $\mathfrak{D}$.

This result follows readily if we write H in the form

$$H = F + F^*$$

where

$$F^* = (\lambda^\sigma - l^\sigma) f^\sigma + (\mu^\beta - m^\beta)\phi^\beta.$$

Suppose now that L, f^σ are E-dominated by F near C_0 on $\mathfrak{D}$. Then by Theorem 4.3 the functions ϕ^β are E-dominated by F near C_0 on $\mathfrak{D}$. Consequently F^* is also E-dominated by F near C_0 on $\mathfrak{D}$. The same is true for F and hence also for H, as was to be proved.

COROLLARY 1. *If a function* F *of the form* (9.7) *with* $l^0 > 0$ *E-dominates* L *and* f^σ *near* C_0 *on* $\mathfrak{D}$, *then* f *is E-dominated by* F *near* C_0 *on* $\mathfrak{D}$.

COROLLARY 2. *Let* $l^0 \geqq 0$, l^σ, $m^\beta(a, y)$ *be a set of continuous multipliers such that the function* $F(a, y, p)$ *defined by* (9.7) *E-dominates* $L = |p|$ *and* f^σ *near* C_0 *on* $\mathfrak{D}$. *If* $\bar{l}^0$, $\bar{l}^\sigma$, $\bar{m}^\beta(a, y)$ *is a second set of continuous multipliers such that*

$$\bar{l}^0 = l^0, \qquad \bar{l}^\sigma = l^\sigma, \qquad \bar{m}^\beta(a, y) = m^\beta(a, y)$$

on C_0, *then*

$$\bar{F}(a, y, p) = \bar{l}^0 f + \bar{l}^\sigma f^\sigma + \bar{m}^\beta(a, y)\phi^\beta$$

also E-dominates L *near* C_0 *on* $\mathfrak{D}$. *Moreover an admissible function* $H(a, y, p)$ *is E-dominated by* $\bar{F}(a, y, p)$ *near* C_0 *on* $\mathfrak{D}$ *if and only if it is E-dominated by* $F(a, y, p)$ *near* C_0 *on* $\mathfrak{D}$.

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