

THE (ϕ, k) RECTIFIABLE SUBSETS OF n SPACE

BY

HERBERT FEDERER

1. Introduction. There are many measure functions on the set of all subsets of the Euclidean plane which generalize the idea of arc length in some intuitively acceptable manner. Loosely speaking these functions are termed linear measures. A. S. Besicovitch, A. P. Morse, and J. F. Randolph have made an extensive study of the local geometric properties of those subsets of the plane whose linear measure is finite. (See [MR1], [BE2], [BE3]⁽¹⁾.) These writers have succeeded in giving local characterizations of those sets which lie on a countable number of rectifiable arcs, except for a set of linear measure zero. Such sets have been called countably ϕ rectifiable, where ϕ is the linear measure in question. It has further been shown by Besicovitch that the local geometric properties of plane sets, and their rectifiability properties in the large, are intimately connected with the one-dimensional Lebesgue measures of the perpendicular projections of these sets onto straight lines in all possible directions. (See [BE3].)

As indicated by the title, this paper attempts to generalize the above mentioned results to those measure functions over Euclidean n space which may loosely be termed k -dimensional because they are intuitively reasonable extensions of the classical area of rectifiable k -dimensional surfaces. With this in mind we frequently consider a measure ϕ over n space, and a subset A of n spaces such that $\phi(A) < \infty$ and

$$0 < \limsup_{r \rightarrow 0^+} r^{-k} \phi(A \cap K_x^r) < \infty$$

for ϕ almost all x in A , where K_x^r is the sphere with center x and radius r .

Among other local geometric concepts we use the notion of (ϕ, k) restrictedness. By saying that a set A is (ϕ, k) restricted at a point x , we mean roughly that a suitable configuration of $(n-k)$ -dimensional flat spaces through x is relatively free of points of A , in terms of the measure ϕ and the dimension k . The precise definition is an immediate extension of the corresponding notion for linear measures and plane sets. The same is true of the concept of (ϕ, k) rectifiability. A set A is (ϕ, k) rectifiable if and only if ϕ almost all of A lies on a countable number of rectifiable k -dimensional surfaces. The connection between (ϕ, k) restrictedness, a local concept, and (ϕ, k) rectifiability, a property in the large, is established by methods closely patterned after those

Presented to the Society, April 27, 1946; received by the editor June 14, 1946.

⁽¹⁾ References in brackets are to the bibliography at the end of the paper.

which were used by Morse and Randolph for linear measures in the plane.

A much more difficult problem is presented by the generalization of the projection properties, and the associated local concepts, which Besicovitch treated so successfully for plane measurable sets of finite Carathéodory linear measure. It appears necessary to consider the k -dimensional Lebesgue measures of the perpendicular projections of a subset of n space into almost all k -dimensional subspaces of n space. The phrase "almost all" in the last sentence must be interpreted in terms of a suitable measure over the set of all these subspaces which is invariant under orthogonal transformations of n space. This leads to certain integrals with respect to the Haar measure over the orthogonal group. With this machinery it is possible to extend all the definitions and theorems of Besicovitch, which deal with projection properties, to k -dimensional measures over n space. The main object of this paper is to present the solution of these problems.

This paper does not discuss the generalizations of two concepts which are significant in the theory of linear measures, namely directionality and regularity. It would be an easy matter to give a reasonable definition of the notion of (ϕ, k) directionality of a set A at a point x and to prove its equivalence, ϕ almost everywhere, to (ϕ, k) restrictedness, just as for linear measures. With regularity the state of affairs is quite different. To say that a set A is (ϕ, k) regular at a point x should certainly mean that

$$0 < \lim_{r \rightarrow 0} r^{-k} \phi(A \cap K_x^r) < \infty.$$

It is still easy to prove that every (ϕ, k) rectifiable set is (ϕ, k) regular at ϕ almost all of its points; but the converse, which is known to hold for linear measures, presents, in the general case, a difficult and as yet unsolved problem. The existing treatment for the case of linear measures depends heavily on the proposition that every plane continuum of finite Carathéodory linear measure is a rectifiable curve. The corresponding proposition for k -dimensional measures is false. Nöbeling has shown how to construct a subset of 3 space which is homeomorphic to a 2 sphere, has finite 2-dimensional Hausdorff measure, but is not Hausdorff 2 rectifiable; that is, this set does not lie on a countable number of 2-dimensional rectifiable surfaces except for a subset of 2 dimensional Hausdorff measure zero. (See [N2].) Hence it appears that new geometric methods will be needed to answer the immediate questions connected with the concept of (ϕ, k) regularity.

The definitions are listed in §2, and the main general results may be found in §9. Among them is the following theorem: Let A be a subset of n space whose k -dimensional Hausdorff measure is finite. Then the k -dimensional integral geometric Favard measure of A does not exceed its Hausdorff measure. A necessary and sufficient condition for equality is that A be contained in a countable number of rectifiable (Lipschitzian) k -dimensional surfaces,

except for a subset of Hausdorff measure zero. Sherman obtained this result in the special case where $n=2$ and $k=1$. (See [SH].)

The complete solution of the problem of measure for non-parametric two-dimensional surfaces in three space is given in §10 by a method which combines the general results of this paper with a recently discovered ingenious technique of Besicovitch and some previous work of the author. (See [BE5], [F1], [F4].)

It has been shown elsewhere that the results of this paper can be used to prove the Gauss-Green theorem for any bounded open subset of n space whose boundary has finite $(n-1)$ -dimensional Hausdorff measure. (See [F3], [F4, 4.8].)

An effort has been made to keep this paper as nearly self-contained as possible, except for references to general measure theory. The extensive bibliography is not intended as a list of material preliminary to this paper, but as a fairly complete survey of the literature in the field of k -dimensional measures over n space.

2. Definitions.

2.1 DEFINITION. If A and B are sets, then

$$(A \cap B), (A \cup B), (A - B)$$

are the product, union and difference of A and B .

If $A(x)$ is a set for each $x \in F$, then

$$\bigcup_{x \in F} A(x) = E_t [t \in A(x) \text{ for some } x \in F],$$

$$\bigcap_{x \in F} A(x) = E_t [t \in A(x) \text{ for every } x \in F].$$

If F is a family of sets, then

$$\sigma(F) = \bigcup_{x \in F} x, \quad \pi(F) = \bigcap_{x \in F} x.$$

If $0 \leq g(x) \leq \infty$ for each x in a countable set C , then

$$\sum_{x \in C} g(x)$$

denotes the obvious numerical sum, finite or infinite.

If f and g are functions, then $(g:f)$ is the function h such that $h(x) = g[f(x)]$ for every x .

If f is a function and S is a set, then

$$f^*(S) = E_y [y = f(x) \text{ for some } x \in S].$$

If f is a function, S is a set, and y is a point, then

$$N(f, S, y)$$

is the number (possibly ∞) of points $x \in S$ for which $f(x) = y$.

If f is a function and A is a set, then $(f|A)$ is the function with domain $(A \cap \text{domain } f)$ such that $(f|A)(x) = f(x)$ for $x \in (A \cap \text{domain } f)$.

If x is a point then

$$\text{sng } x = E_y [y = x]$$

is the (singletonic) set whose only element is x .

2.2 DEFINITION. We say ϕ is a *measure over* B if and only if ϕ is a function whose domain is the set of all subsets of B and which satisfies the conditions:

(i) $0 \leq \phi(S) \leq \infty$ for $S \subset B$,

(ii) $\phi(0) = 0$,

(iii) $\phi(S) \leq \phi(T)$ whenever $S \subset T \subset B$,

(iv) $\phi[\sigma(F)] \leq \sum_{S \in F} \phi(S)$ for every countable family F of subsets of B .

Following Carathéodory [C], we say a set S is ϕ measurable if and only if $S \subset B$ and

$$\phi(T) = \phi(T \cap S) + \phi(T - S) \quad \text{whenever } T \subset B.$$

If f is a ϕ measurable function and S is a ϕ measurable set, then the (Lebesgue) integral of f relative to ϕ over the set S is denoted by

$$\int_S f(x) d\phi x.$$

2.3 DEFINITION. Euclidean n space is denoted by E_n and we write $x = (x_1, x_2, \dots, x_n)$ for $x \in E_n$.

$$x \cdot y = \sum_{i=1}^n x_i y_i \quad \text{for } x \in E_n, y \in E_n.$$

$$|x| = (x \cdot x)^{1/2} \quad \text{for } x \in E_n.$$

\mathcal{L}_n is the ordinary n -dimensional Lebesgue measure over E_n .

$$\alpha(n) = \mathcal{L}_n \left\{ E_n \cap E_x [|x| < 1] \right\}.$$

The diameter of a set $S \subset E_n$ is denoted by

$$\text{diam } S.$$

2.4 DEFINITION. Suppose m and n are positive integers.

A function L on E_m to E_n is said to be *linear* if and only if L is continuous and

$$L(x + y) = L(x) + L(y) \quad \text{for } x \in E_m, y \in E_m.$$

We make no distinction between L and its matrix. The j th column ($j=1, 2, \dots, m$) of L is a point of E_n and will be denoted by L^j . Similarly

the i th row ($i=1, 2, \dots, n$) of L is a point of E_m and will be denoted by L_i . Consequently

$$L^j = ((L^j)_1, (L^j)_2, \dots, (L^j)_n) \in E_n,$$

$$L_i = ((L_i)_1, (L_i)_2, \dots, (L_i)_m) \in E_m,$$

with $(L^j)_i = (L_i)_j$ for $j=1, 2, \dots, m$ and $i=1, 2, \dots, n$. We shall use the notation

$$L_i^j = (L^j)_i = (L_i)_j \quad \text{for } j = 1, 2, \dots, m \text{ and } i = 1, 2, \dots, n.$$

The set of all linear functions on E_m to E_n (matrices with m columns and n rows) will be denoted by M_n^m .

We shall frequently use the following fact:

If $A \in M_n^m$ and $B \in M_m^n$, then $(B:A)^j = B(A^j)$.

If, in particular, we take $m=n$ and $A=I$, the unit matrix in M_n^n (the identical transformation of E_n), then the points I^1, I^2, \dots, I^n are the fundamental base vectors of E_n , and $B^j = B(I^j)$.

We further define

$$\|L\| = \sup_{x \in E_m, |x|=1} |L(x)|$$

whenever $L \in M_n^m$.

2.5 DEFINITION. If $L \in M_n^m$, then its conjugate $\bar{L} \in M_m^n$ is defined by the relation

$$\bar{L}_i^j = L_j^i \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n.$$

Consequently

$$\bar{L}^j = L_j \quad \text{for } j = 1, 2, \dots, n,$$

$$\bar{L}_i = L^i \quad \text{for } i = 1, 2, \dots, m.$$

We shall frequently use the following relation:

$$(A:B)_i = \overline{(A:B)^i} = \overline{(B:A)^i} = \bar{B}(\bar{A}^i) = \bar{B}(A_i).$$

2.6 DEFINITION. If $A \in M_n^m$ and $k \leq m$, $r \leq n$ are positive integers, then

$$(A \left| \begin{smallmatrix} k \\ r \end{smallmatrix} \right.)$$

is the matrix in M_r^k such that

$$(A \left| \begin{smallmatrix} k \\ r \end{smallmatrix} \right.)_i^j = A_i^j \quad \text{for } i = 1, 2, \dots, r \text{ and } j = 1, 2, \dots, k.$$

Thus $(A \left| \begin{smallmatrix} k \\ r \end{smallmatrix} \right.)$ is the upper left-hand minor of A with r rows and k columns. We further define

$$P_A^k = (A \left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right.) \quad \text{for } A \in M_n^n, k = 1, 2, \dots, n.$$

(The *projecting functions* of [F2, 2.1] are exactly the functions P_A^k with A orthogonal.)

2.7 DEFINITION. The determinant of a square matrix A will be denoted by

$$\det A.$$

Now suppose $L \in M_n^k$ with $k \leq n$. Then

$$\Delta(L) = \left\{ \sum_{x \in S} (\det A_x)^2 \right\}^{1/2},$$

where $x \in S$ if and only if x is a set of k integers between 1 and n , and A_x is the minor of L which is made up from the k rows whose indices are elements of x .

2.8 DEFINITION. Suppose $k \leq n$ are positive integers and f is a function on E_k to E_n . Then the function

$$Jf$$

is defined as follows: The domain of Jf is the set of all points of approximate differentiability of f and for each such point x we let

$$Jf(x) = \Delta(L)$$

where L is the approximate differential of f at x .

2.9 DEFINITION. We let

$$G_n = M_n^n \cap E_A [|A(x)| = |x| \text{ for } x \in E_n]$$

be the set of all *orthogonal transformations* of n space. Then G_n is a *compact topological group* with respect to the operation, \cdot , of superposition, and with respect to the obvious topology.

Hence G_n carries exactly one *Haar measure*, ϕ_n , for which $\phi_n(G_n) = 1$.

In this paper the following properties of the measure ϕ_n will be used:

Every closed subset of G_n is ϕ_n measurable.

If f is a ϕ_n measurable function on G_n , then

$$\int_{G_n} f(\bar{R}) d\phi_n R = \int_{G_n} f(R) d\phi_n R,$$

and $S \in G_n$ implies

$$\int_{G_n} f(S:R) d\phi_n R = \int_{G_n} f(R:S) d\phi_n R = \int_{G_n} f(R) d\phi_n R.$$

This follows from the fact that G_n is compact, hence unimodular. (See [W8].)

2.10 DEFINITION. If $k \leq n$ are positive integers, then

$$\beta(n, k) = \int_{G_n} |\det(R|_k^k)| d\phi_n R.$$

2.11 REMARK. It is well known that

$$\alpha(n) = \frac{2^n}{n!} \Gamma\left(\frac{1}{2}\right)^{n-1} \Gamma\left(\frac{n+1}{2}\right),$$

and it was proved in [F4, 5.4] that

$$\beta(n, k) = \frac{\alpha(k) \cdot \alpha(n-k)}{\alpha(n) \cdot \binom{n}{k}} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)},$$

where Γ has its classical meaning.

2.12 DEFINITION. If n is a positive integer, then \mathcal{U}_n is the set of all measures over E_n , and \mathcal{U}'_n is the set of those measures $\phi \in \mathcal{U}_n$ which have the property that every closed subset of E_n is ϕ measurable.

2.13 DEFINITION. If n is a positive integer, then

\mathcal{M}_n = the set of all subsets of E_n ,

\mathcal{A}_n = the set of all analytic subsets of E_n ,

\mathcal{B}_n = the set of all Borel subsets of E_n ,

\mathcal{F}_n = the set of all those subsets of E_n which are of type F_σ ,

\mathcal{G}_n = the set of all open subsets of E_n ,

\mathcal{I}_n = the set of all connected open subsets of E_n ,

\mathcal{C}_n = the set of all convex open subsets of E_n ,

\mathcal{S}_n = the set of all open spheres of E_n .

Observe that

$$\mathcal{S}_n \subset \mathcal{C}_n \subset \mathcal{I}_n \subset \mathcal{G}_n \subset \mathcal{F}_n \subset \mathcal{B}_n \subset \mathcal{A}_n \subset \mathcal{M}_n.$$

2.14 DEFINITION. If $r > 0$, $F \subset \mathcal{M}_n$, and g is a function for which

$$0 \leq g(S) \leq \infty \quad \text{whenever } S \in F,$$

then

$$\mathcal{E}_n^r(g, F)$$

is the function on \mathcal{M}_n such that $A \in \mathcal{M}_n$ implies

$$\{\mathcal{E}_n^r(g, F)\}(A) = \inf_{G \in \mathcal{B}} \sum_{S \in G} g(S),$$

where $G \in \mathcal{B}$ if and only if G is a countable subfamily of F for which

$$A \subset \sigma(G) \quad \text{and} \quad \text{diam } S < r \quad \text{whenever } S \in G.$$

In this connection we remind the reader that the infimum of the empty set is ∞ . Hence, if B is vacuous, then $\{\Xi_n^r(g, F)\}(A) = \infty$. Further, every numerical sum over the empty set is zero. Hence we infer from the relation $0 \subset \sigma(0)$ that $\{\Xi_n^r(g, F)\}(0) = 0$.

2.15 DEFINITION. If $F \subset \mathfrak{M}_n$ and g is a function for which

$$0 \leq g(S) \leq \infty \quad \text{whenever } S \in F,$$

then

$$\Xi_n(g, F)$$

is the function on \mathfrak{M}_n such that $A \in \mathfrak{M}_n$ implies

$$\{\Xi_n(g, F)\}(A) = \lim_{r \rightarrow 0^+} \{\Xi_n^r(g, F)\}(A).$$

We observe that this limit is a number (possibly ∞) because

$$\{\Xi_n^r(g, F)\}(A) \leq \{\Xi_n^s(g, F)\}(A)$$

whenever $r > s > 0$.

It may be shown that $\Xi_n(g, F) \in \mathfrak{U}_n'$.

This idea for the construction of a measure, $\Xi_n(g, F)$, from a merely non-negative function, g , whose domain includes a given family, F , of subsets of E_n , is due to *Carathéodory*. (See [C], [H].) Together with his algebraic characterization of measurability (see 2.2), it plays a fundamental role in measure theory.

2.16 DEFINITION. If $k \leq n$ are positive integers, then

$$\gamma_n^k(S) = \sup_{R \in \mathfrak{G}_n} \mathcal{L}_k[P_R^{k*}(S)] \quad \text{for } S \in \mathfrak{M}_n,$$

$$\zeta_n^k(S) = \beta(n, k)^{-1} \int_{G_n} \mathcal{L}_k[P_R^{k*}(S)] d\phi_n R \quad \text{for } S \in \mathfrak{M}_n,$$

$$\chi_n^k(S) = \alpha(k) 2^{-k} (\text{diam } S)^k \quad \text{for } S \in \mathfrak{M}_n.$$

2.17 REMARK. Clearly the domain of the function ζ_n^k is the family of all those sets S for which the integrand

$$\mathcal{L}_k[P_R^{k*}(S)]$$

is ϕ_n measurable with respect to R . We shall show that this is the case whenever $S \in \mathfrak{S}_n$ (or, more generally, whenever $S \in \mathfrak{A}_n$). For this purpose we fix such a set S , let f be the function on the cartesian product space $(E_n \times G_n)$ to the cartesian product space $(E_k \times G_n)$ such that $f(x, R) = [P_R^k(x), R]$, and let

$$T = (E_k \times G_n) \cap \underset{(y,R)}{E} [y = P_R^k(x) \text{ for some } x \text{ in } S].$$

Now $(S \times G_n)$ is a subset of type F_σ (or, more generally, analytic subset) of the space $(E_n \times G_n)$, and f is a continuous function. Hence the set $T = f^*(S \times G_n)$ is a subset of type F_σ (or, more generally, analytic subset) of the space $(E_k \times G_n)$, and the characteristic function t of T is measurable with respect to the product measure of \mathcal{L}_k and ϕ_n . It follows that

$$\begin{aligned} 0 &\leq \int_{G_n} \int_{E_k} t(y, R) d\mathcal{L}_k y d\phi_n R \\ &= \int_{G_n} \mathcal{L}_k \{ P_R^{k*}(S) \} d\phi_n R = \zeta_n^k(S) \beta(n, k). \end{aligned}$$

Thus we have proved that $\mathfrak{S}_n \subset \text{domain } \zeta_n^k$ (or, more generally, that $\mathfrak{A}_n \subset \text{domain } \zeta_n^k$).

2.18 DEFINITION. If $k \leq n$ are positive integers, then

$$\begin{aligned} \mathfrak{S}_n^k &= \Xi_n(\mathfrak{S}_n, \chi_n^k), & \mathfrak{H}_n^k &= \Xi_n(\mathfrak{M}_n, \chi_n^k), \\ \mathfrak{C}_n^k &= \Xi_n(\mathfrak{C}_n, \gamma_n^k), & \mathfrak{F}_n^k &= \Xi_n(\mathfrak{X}_n, \gamma_n^k), \\ \mathfrak{I}_n^k &= \Xi_n(\mathfrak{B}_n, \gamma_n^k), & \mathfrak{G}_n^k &= \Xi_n(\mathfrak{C}_n, \zeta_n^k), \\ \mathfrak{Y}_n^k &= \Xi_n(\mathfrak{A}_n, \zeta_n^k). \end{aligned}$$

The measure \mathfrak{C}_n^k is *Carathéodory k measure over n space*. (See [C].) Both \mathfrak{H}_n^k and \mathfrak{S}_n^k were introduced by Hausdorff (see [H]). We shall, however, apply the term *Hausdorff measure* exclusively to \mathfrak{H}_n^k , and refer to \mathfrak{S}_n^k as *sphere measure*. The function \mathfrak{I}_n^k is *Gross measure* (see [G1], [G2]), and the measure \mathfrak{G}_n^k is named after *Gillespie*, at whose suggestion it was introduced, for the case $k = n - 1$, by Morse and Randolph. (See [MR2].) Finally \mathfrak{Y}_n^k is *Favard measure* (see [FA1]), and the measure \mathfrak{F}_n^k was introduced by the author of this paper. (See [F1], [F2]; also note that \mathfrak{F}_n^k was denoted by \mathcal{L}_n^k in [F4].)

Only \mathfrak{S}_n^k and \mathfrak{H}_n^k are essentially used in this paper. The other measures are listed here for comparison, and will be mentioned in certain supplementary remarks. We note that the usual methods of defining the measures \mathfrak{I}_n^k and \mathfrak{Y}_n^k are different from the above, but lead to the same functions.

2.19 REMARK. Suppose $k < n$ are positive integers.

Let U be the function on E_{k+1} to E_k such that

$$U(y) = (y_1, \dots, y_k) \in E_k \quad \text{for } y \in E_{k+1}.$$

Clearly

$$P_R^k = (U : P_R^{k+1}) \quad \text{for } R \in G_n.$$

Furthermore the Fubini theorem implies that

$$\mathcal{L}_{k+1}(T) \leq \mathcal{L}_k[U^*(T)] \cdot \text{diam } T$$

for every \mathcal{L}_{k+1} measurable subset T of E_{k+1} .

Consequently $\mathcal{L}_{k+1}[P_R^{k+1*}(S)] \leq \mathcal{L}_k[P_R^k(S)] \cdot \text{diam } S$ for every set $S \in \mathfrak{S}_n$ (more generally, every set $S \in \mathfrak{A}_n$).

From this we infer that

$$\begin{aligned} \gamma_n^{k+1}(S) &\leq \gamma_n^k(S) \cdot \text{diam } S, \\ \zeta_n^{k+1}(S) &\leq [\beta(n, k)/\beta(n, k + 1)] \cdot \zeta_n^k(S) \cdot \text{diam } S, \end{aligned}$$

for every set $S \in \mathfrak{S}_n$ (more generally, $S \in \mathfrak{A}_n$). These inequalities are similar to the relation $\chi_n^{k+1}(S) = [\alpha(k+1)/2\alpha(k)] \cdot \chi_n^k(S) \cdot \text{diam } S$, which holds for every set $S \in \mathfrak{M}_n$.

It follows that

$$\begin{aligned} \{\Xi_n^r(\gamma_n^{k+1}, F)\}(A) &\leq \{\Xi_n^r(\gamma_n^k, F)\}(A) \cdot r, \\ \{\Xi_n^r(\zeta_n^{k+1}, F)\}(A) &\leq [\beta(n, k)/\beta(n, k + 1)] \cdot \{\Xi_n^r(\zeta_n^k, F)\}(A) \cdot r, \end{aligned}$$

for $A \in \mathfrak{M}_n$ and $F \subset \mathfrak{S}_n$ (more generally, $F \subset \mathfrak{A}_n$), while

$$\{\Xi_n^r(\chi_n^{k+1}, F)\}(A) \leq [\alpha(k+1)/2\alpha(k)] \cdot \{\Xi_n^r(\chi_n^k, F)\}(A) \cdot r$$

for $A \in \mathfrak{M}_n$ and $F \subset \mathfrak{M}_n$.

Letting $r \rightarrow 0$ we obtain, for each set $A \in \mathfrak{M}_n$, the results:

$$\begin{aligned} \mathfrak{S}_n^k(A) < \infty &\text{ implies } \mathfrak{S}_n^{k+1}(A) = 0, \\ \mathfrak{H}_n^k(A) < \infty &\text{ implies } \mathfrak{H}_n^{k+1}(A) = 0, \\ \mathfrak{C}_n^k(A) < \infty &\text{ implies } \mathfrak{C}_n^{k+1}(A) = 0, \\ \mathfrak{F}_n^k(A) < \infty &\text{ implies } \mathfrak{F}_n^{k+1}(A) = 0, \\ \mathfrak{I}_n^k(A) < \infty &\text{ implies } \mathfrak{I}_n^{k+1}(A) = 0, \\ \mathfrak{G}_n^k(A) < \infty &\text{ implies } \mathfrak{G}_n^{k+1}(A) = 0, \\ \mathfrak{J}_n^k(A) < \infty &\text{ implies } \mathfrak{J}_n^{k+1}(A) = 0. \end{aligned}$$

Hence each of the seven sequences

$$\begin{aligned} \mathfrak{S}_n, \mathfrak{S}_n^2, \dots, \mathfrak{S}_n^n, & \quad \mathfrak{H}_n^1, \mathfrak{H}_n^2, \dots, \mathfrak{H}_n^n, \\ \mathfrak{C}_n^1, \mathfrak{C}_n^2, \dots, \mathfrak{C}_n^n, & \quad \mathfrak{F}_n^1, \mathfrak{F}_n^2, \dots, \mathfrak{F}_n^n, \\ \mathfrak{I}_n^1, \mathfrak{I}_n^2, \dots, \mathfrak{I}_n^n, & \quad \mathfrak{G}_n^1, \mathfrak{G}_n^2, \dots, \mathfrak{G}_n^n, \\ & \quad \mathfrak{J}_n^1, \mathfrak{J}_n^2, \dots, \mathfrak{J}_n^n \end{aligned}$$

provides us with a scale of measure theoretic dimension types in the sense of

Hausdorff (see [H]). To say that a set A has measure theoretic dimension ϕ means simply that $0 < \phi(A) < \infty$. Each of the seven scales satisfies the intuitive requirement that the $(k+1)$ dimensional measure of a set is zero whenever its k -dimensional measure is finite. None of these scales are complete, because there are sets which do not have any of the measures of our scales as dimension type.

Since it is possible to define a measure \mathfrak{H}_n^t for any non-negative number t just as though t were an integer [the definition of $\alpha(t)$ can be extended to all such t in terms of the gamma function], the discrete scale $\mathfrak{H}_n^1, \mathfrak{H}_n^2, \dots, \mathfrak{H}_n^n$ can be extended to a continuous scale, which contains all the measures \mathfrak{H}_n^t for $0 \leqq t \leqq n$. But even this continuous Hausdorff scale is not complete. For instance Besicovitch asserts the existence of a set $A \subset E_3$ which is homeomorphic to a 2 cell and for which $0 < \mathfrak{H}_3^2(A) < \infty$, but $\mathfrak{H}_3^t(A) = \infty$ for $0 \leqq t < 3$ and $\mathfrak{H}_3^3(A) = 0$. (See [BE5, Addendum].)

However the topological dimension of a set $A \subset E_n$ is determined by a knowledge of all the numbers $\mathfrak{H}_{2n+1}^1(B), \mathfrak{H}_{2n+1}^2(B), \dots, \mathfrak{H}_{2n+1}^n(B)$ for every subset B of E_{2n+1} which is homeomorphic to A . (See [HW, VII].)

The last property has not been proved for any but the Hausdorff scale⁽²⁾. This is one of several reasons why Hausdorff measure may be more interesting than the other six functions mentioned above. However it has been shown that each definition leads to a consistent dimension scale of its own.

Many problems which concern the relations between the various scales are still unsolved. Some recent comments on this subject are due to Besicovitch. (See [BE5].)

2.20 DEFINITION. For $x \in E_n$ and $r > 0$ we denote

$$K_x = K(x, r) = E_n \cap E [|y - x| < r],$$

$$C_x^r = C(x, r) = E_n \cap E_y^y [|y - x| \leqq r].$$

2.21 DEFINITION. If $m \leqq n$ are positive integers, $x \in E_n$, and $R \in G_n$, then

$$\square_n^m(R, x) = E_n \cap E_y [|y - x| = |P_R^m(y - x)|].$$

Further, if $\eta > 0$, then

$$\diamond_n^m(R, \eta, x) = E_n \cap E_y [|y - x| < (1 + \eta^2)^{1/2} |P_R^m(y - x)|].$$

2.22 DEFINITION. If $k \leqq n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$, and $x \in E_n$, then

⁽²⁾ *Added in proof.* In a paper entitled *Dimension and measure*, which will appear in this journal, it is shown that the connection with topological dimension is not restricted to Hausdorff measure, but holds for a large class of measures including all those listed here.

$$\begin{aligned} \mathfrak{O}_n^k(\phi, A, x) &= \alpha(k)^{-1} \limsup_{r \rightarrow 0^+} r^{-k} \phi(A \cap K_x^r), \\ \mathfrak{Q}_n^k(\phi, A, x) &= \alpha(k)^{-1} \liminf_{r \rightarrow 0^+} r^{-k} \phi(A \cap K_x^r), \\ \mathfrak{O}_n^k(\phi, A, x) &= \alpha(k)^{-1} \lim_{r \rightarrow 0^+} r^{-k} \phi(A \cap K_x^r). \end{aligned}$$

2.23 REMARK. The (*upper, lower*) (ϕ, k) densities of A at x , which were defined in 2.22, are equal to the similar (upper, lower) limits which are obtained by replacing the open spheres K_x^r by closed spheres C_x^r .

This fact will be useful in §3.

2.24 DEFINITION. If $k < n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$, $R \in G_n$, $0 < \eta < \infty$, $0 < r < \infty$, $x \in E_n$, then

$$\nabla_n^k(\phi, A, R, \eta, r, x) = \alpha(k)^{-1} (\eta r)^{-k} \phi[A \cap \mathfrak{D}_n^{n-k}(R, \eta, x) \cap K(x, r)].$$

2.25 DEFINITION. If $k < n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$, $R \in G_n$, $x \in E_n$, then

$$\mathfrak{O}_n^k(\phi, A, R, x) = \limsup_{\eta \rightarrow 0^+} \{ \limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \}.$$

2.26 DEFINITION. If $k < n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$, $x \in E_n$, then $W_n^k(\phi, A, x)$ is the set of all those orthogonal transformations $R \in G_n$ for which

$$\limsup_{(\eta, r) \rightarrow (0, 0)} \nabla_n^k(\phi, A, R, \eta, r, x) = \infty.$$

2.27 DEFINITION. If $k < n$ are positive integers, $A \subset E_n$, $x \in E_n$, then

$$V_n^k(A, x)$$

is the set of all those orthogonal transformations $R \in G_n$ for which

$$x \text{ is a clusterpoint of } [A \cap \mathfrak{Q}_n^{n-k}(R, x)].$$

2.28 DEFINITION. Suppose $k < n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$, $x \in E_n$. Then we say

A is (ϕ, k) restricted at x

if and only if $\mathfrak{O}_n^k(\phi, A, x) > 0$ and we can find R and η such that $R \in G_n$, $0 < \eta < \infty$ and

$$\mathfrak{O}_n^k\{\phi, [A \cap \mathfrak{D}_n^{n-k}(R, \eta, x)], x\} = 0.$$

2.29 REMARK. The proposition

$$\odot_n^k \{ \phi, [A \cap \diamond_n^{n-k}(R, \eta, x)], x \} = 0$$

is *equivalent* to the statement that

$$\lim_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \epsilon, r, x) = 0 \quad \text{for } 0 < \epsilon \leq \eta,$$

which obviously *implies* the relation

$$\varnothing_n^k(\phi, A, R, x) = 0.$$

2.30 DEFINITION. A function f on a subset of E_k to E_n is said to be *Lipschitzian* if and only if there is a number M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for every two points x and y in the domain of f .

The number M is usually called a *Lipschitz constant* of f .

2.31 DEFINITION. Suppose $k \leq n$ are positive integers, $A \subset E_n$. Then we say

A is k rectifiable

if and only if there is a Lipschitzian function whose domain is a *bounded* subset of E_k and whose range is A .

2.32 DEFINITION. Suppose $k \leq n$ are positive integers, $A \subset E_n$. Then we say

A is countably k rectifiable

if and only if there is a countable family F of k rectifiable sets such that $A = \sigma(F)$.

2.33 REMARK. If F is a *finite* family of k rectifiable sets, then $\sigma(F)$ is a k rectifiable set.

A set A is countably k rectifiable if and only if there is a Lipschitzian function whose domain is a subset of E_k and whose range is A .

The closure of every k rectifiable set is k rectifiable.

2.34 DEFINITION. Suppose $k \leq n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$. Then we say

A is (\phi, k) rectifiable

if and only if corresponding to each $\epsilon > 0$ there is a k rectifiable subset B of A for which $\phi(A - B) < \epsilon$.

2.35 DEFINITION. Suppose $k \leq n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$. Then we say

A is countably (\phi, k) rectifiable

if and only if there is a countably rectifiable subset B of A for which $\phi(A - B) = 0$.

2.36 REMARK. Every (ϕ, k) rectifiable set is countably (ϕ, k) rectifiable.

If $\phi \in \mathcal{U}'_n$, $\phi(A) < \infty$ and A is countably (ϕ, k) rectifiable, then A is (ϕ, k) rectifiable.

2.37 DEFINITION. Suppose $k \leq n$ are positive integers, $\phi \in \mathcal{U}_n$, $A \subset E_n$. Then we say

A is positively (ϕ, k) unrectifiable

if and only if $\phi(A) > 0$ and there is no k rectifiable subset B of A for which $\phi(B) > 0$.

3. (ϕ, k) densities. The theorems of this section are straightforward generalizations of known results. (See [MR1], [BE2], [J], [SI].) Their proofs are given here for completeness.

3.1 THEOREM. If $\phi \in \mathcal{U}'_n$, $A \subset E_n$, $B \subset E_n$, $0 < \lambda < \infty$, and $\vartheta_n^k(\phi, A, x) > \lambda$ for $x \in B$, then

$$\lambda \mathcal{S}_n^k(B) \leq \phi(A).$$

Proof. We assume $\phi(A) < \infty$, and choose $\epsilon > 0$.

Let F be the family of all sets S such that

$$S = C_x^r, \quad x \in B, \quad 0 < 10r < \epsilon,$$

$$\lambda \chi_n^k(S) \leq \phi(A \cap S).$$

Since F covers B in the sense of Vitali, we use [M1, 3.10] to select a disjoint subfamily G of F for which

$$B - \sigma(G) \subset \bigcup_{S \in H} S'$$

whenever $H \subset G$ and $G - H$ is finite.

Here $S' = C(x, 5r)$ whenever $S = C(x, r)$.

For every such H we have

$$\begin{aligned} \lambda \cdot \left\{ \mathcal{E}_n^k(\chi_n, \mathcal{C}_n) \right\}(B) &\leq \lambda \cdot \left\{ \sum_{S \in G} \chi_n^k(S) + \sum_{S \in H} \chi_n^k(S') \right\} \\ &= \lambda \cdot \left\{ \sum_{S \in G} \chi_n^k(S) + 5^k \sum_{S \in H} \chi_n^k(S) \right\} \\ &\leq \sum_{S \in G} \phi(A \cap S) + 5^k \sum_{S \in H} \phi(A \cap S) \\ &= \phi[A \cap \sigma(G)] + 5^k \phi[A \cap \sigma(H)]. \end{aligned}$$

But $\phi[A \cap \sigma(H)]$ can be made as small as we please by proper choice of H .

Consequently

$$\lambda \cdot \{ \Xi_n^\epsilon(\chi_n, \mathfrak{C}_n) \} (B) \leq \phi[A \cap \sigma(G)] \leq \phi(A),$$

and we let $\epsilon \rightarrow 0$ to complete the proof.

3.2 THEOREM. *If $\phi \in \mathcal{U}'_n$, $A \subset E_n$, $B \in \mathfrak{B}_n$, $\phi(A - B) < \infty$, then $\mathfrak{O}_n^k(\phi, A - B, x) = 0$ for \mathfrak{S}_n^k almost all x in B .*

Proof. (By contradiction.) Let ψ be the element of \mathcal{U}'_n such that

$$\psi(S) = \phi(A \cap S) \quad \text{for } S \subset E_n.$$

Select λ and X such that $0 < \lambda < \infty$, $X \subset B$, $\mathfrak{S}_n^k(X) > 0$ and

$$\mathfrak{O}_n^k(\psi, E_n - B, x) = \mathfrak{O}_n^k(\phi, A - B, x) > \lambda \quad \text{for } x \in X.$$

Next we use [MR1, 3.6] or [MA2, 4.10] to secure a closed subset C of $(E_n - B)$ for which

$$\psi(D) < \lambda \mathfrak{S}_n^k(X) \quad \text{with } D = (E_n - B) - C.$$

Since

$$\mathfrak{O}_n^k(\psi, E_n - B, x) = \mathfrak{O}_n^k(\psi, D, x) \quad \text{for } x \in B$$

we may apply 3.1 to obtain the relation

$$\lambda \mathfrak{S}_n^k(X) \leq \psi(D),$$

which is false.

The proof is complete.

3.3 REMARK. Suppose $\phi \in \mathcal{U}'_n$, $A \subset E_n$, $B \in \mathfrak{B}_n$, $\phi(A - B) < \infty$, and

$$D = E_n \cap E_x [\mathfrak{O}_n^k(\phi, A - B, x) = 0].$$

The preceding theorem assures us that

$$\mathfrak{S}_n^k(B - D) = 0,$$

while the following five propositions hold for every $x \in D$:

- (1) $\mathfrak{O}_n^k(\phi, A \cap B, x) = \mathfrak{O}_n^k(\phi, A, x)$,
- (2) $\mathfrak{Q}_n^k(\phi, A \cap B, x) = \mathfrak{Q}_n^k(\phi, A, x)$,
- (3) $\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A \cap B, R, \eta, r, x) = \limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x)$ whenever $R \in G_n$ and $0 < \eta < \infty$,
- (4) $\mathfrak{O}_n^k(\phi, A \cap B, R, x) = \mathfrak{O}_n^k(\phi, A, R, x)$ whenever $R \in G_n$,
- (5) $(A \cap B)$ is (ϕ, k) restricted at x if and only if A is (ϕ, k) restricted at x .

3.4 REMARK. Suppose $\phi \in \mathcal{U}'_n$ and $A \subset E_n$. Let ρ be the set of all rational numbers. Then:

- (1) $[\phi(A \cap K_x^r) / \alpha(k)r^k]$ is lower semicontinuous with respect to (x, r) .
- (2) $\inf_{0 < r < \delta} [\phi(A \cap K_x^r) / \alpha(k)r^k]$ is upper semicontinuous with respect to x , for each $\delta > 0$.
- (3) $\sup_{0 < r < \delta} [\phi(A \cap K_x^r) / \alpha(k)r^k]$ is lower semicontinuous with respect to x , for each $\delta > 0$.
- (4) $\mathcal{O}_n^k(\phi, A, x)$ is Borel measurable with respect to x .
- (5) $\mathcal{O}_n^k(\phi, A, x)$ is Borel measurable with respect to x .
- (6) $\nabla_n^k(\phi, A, R, \eta, r, x)$ is lower semicontinuous with respect to (R, η, r, x) .
- (7) $\sup_{0 < r < \delta} \nabla_n^k(\phi, A, R, \eta, r, x) = \sup_{r \in \rho, 0 < r < \delta} \nabla_n^k(\phi, A, R, \eta, r, x)$.
- (8) $\sup_{0 < r < \delta, 0 < \eta < \delta} \nabla_n^k(\phi, A, R, \eta, r, x) = \sup_{r \in \rho, 0 < r < \delta, \eta \in \rho, 0 < \eta < \delta} \nabla_n^k(\phi, A, R, \eta, r, x)$.
- (9) $\lim \sup_{r \rightarrow 0+} \nabla_n^k(\phi, A, R, \eta, r, x)$ is Borel measurable with respect to (R, η, x) .
- (10) $\lim \sup_{(r, \eta) \rightarrow (0, 0)} \nabla_n^k(\phi, A, R, \eta, r, x)$ is Borel measurable with respect to (R, x) .
- (11) $\sup_{0 < \eta < \delta} \lim \sup_{r \rightarrow 0+} \nabla_n^k(\phi, A, R, \eta, r, x) = \sup_{\eta \in \rho, 0 < \eta < \delta} \lim \sup_{r \rightarrow 0+} \nabla_n^k(\phi, A, R, \eta, r, x)$.
- (12) $\mathcal{O}_n^k(\phi, A, R, x)$ is Borel measurable with respect to (R, x) .

We shall not give the proofs of these propositions, but call attention to the following simple fact: If f and g are numerically-valued functions on the open interval I , f is monotone, g is continuous and nonvanishing, then

$$\sup_{t \in I} \frac{f(t)}{g(t)} = \sup_{t \in (I \cap \rho)} \frac{f(t)}{g(t)}.$$

3.5 THEOREM. If $\mathcal{S}_n^k(A) < \infty$, then $\mathcal{O}_n^k(\mathcal{S}_n^k, A, x) \leq 1$ for \mathcal{S}_n^k almost all x in E_n .

Proof. (By contradiction.) The statement (5) of 3.4 enables us to select λ and B such that $1 < \lambda < \infty$, $B \in \mathfrak{B}_n$, $\mathcal{S}_n^k(B) > 0$ and $\mathcal{O}_n^k(\mathcal{S}_n^k, A, x) > \lambda$ for $x \in B$.

Defining

$$D = B \cap E_z [\mathcal{O}_n^k(\mathcal{S}_n^k, A - B, x) = 0],$$

we infer from 3.2 that $\mathcal{S}_n^k(B - D) = 0$. Consequently

$$\begin{aligned} \mathcal{O}_n^k[\mathcal{S}_n^k, (A - B \cap D), x] &= 0 && \text{for } x \in D, \\ \mathcal{O}_n^k(\mathcal{S}_n^k, A \cap B \cap D, x) &> \lambda && \text{for } x \in (B \cap D), \end{aligned}$$

and 3.1 assures us that

$$\lambda \mathcal{S}_n^k(B \cap D) \leq \mathcal{S}_n^k(A \cap B \cap D),$$

which is false because $\lambda > 1$, $\mathcal{S}_n^k(B \cap D) = \mathcal{S}_n^k(B) > 0$, and $\mathcal{S}_n^k(A \cap B \cap D) \leq \mathcal{S}_n^k(A) < \infty$.

3.6 THEOREM. If $\phi \in \mathcal{U}_n$, $A \in \mathcal{E}_n$, $0 < \lambda < \infty$, and $\phi_n^k(\phi, A, x) < \lambda$ for $x \in A$, then

$$\phi(A) \leq 2^k \lambda \mathcal{I}_n^k(A).$$

Proof. Otherwise

$$\phi(A) > 2^k \lambda \mathcal{I}_n^k(A)$$

and we can find a number $\delta > 0$ and a set $B \subset A$ for which

$$\phi(B) > 2^k \lambda \mathcal{I}_n^k(B)$$

and $x \in B$ implies

$$\phi(A \cap C_x^r) < \lambda \chi_n^k(C_x^r) \quad \text{whenever } 0 < r < \delta.$$

Now suppose $0 < \epsilon < \delta/5$.

Choose a countable family $F \subset \mathcal{M}_n$ for which $B \subset \sigma(F)$, $\text{diam } S < \epsilon$ and $(S \cap B) \neq \emptyset$ for $S \in F$, and such that

$$\sum_{S \in F} \chi_n^k(S) < \mathcal{I}_n^k(B) + \epsilon.$$

With each set $S \in F$ associate a point $x(S) \in (S \cap B)$, and note that

$$B \subset \bigcup_{S \in F} C[x(S), \text{diam } S].$$

Hence we may use [M1, 3.10] to select a subfamily G of F such that

$$C[x(S), \text{diam } S] \cap C[x(T), \text{diam } T] = \emptyset \quad \text{for } S \in G, T \in G, S \neq T,$$

and

$$B \subset \bigcup_{S \in G} C[x(S), \text{diam } S] \cup \bigcup_{S \in H} C[x(S), 5 \text{ diam } S]$$

whenever $H \subset G$ and $(G - H)$ is finite.

For every such H we have

$$\begin{aligned} \phi(B) &\leq \sum_{S \in G} \phi\{B \cap C[x(S), \text{diam } S]\} + \sum_{S \in H} \phi\{B \cap C[x(S), 5 \text{ diam } S]\} \\ &\leq \lambda \cdot \left\{ \sum_{S \in G} \chi_n^k\{C[x(S), \text{diam } S]\} + \sum_{S \in H} \chi_n^k\{C[x(S), 5 \text{ diam } S]\} \right\} \\ &= \lambda 2^k \left\{ \sum_{S \in G} \chi_n^k(S) + 5^k \sum_{S \in H} \chi_n^k(S) \right\} \\ &\leq \lambda 2^k \left\{ \mathcal{I}_n^k(B) + \epsilon + 5^k \sum_{S \in H} \chi_n^k(S) \right\}. \end{aligned}$$

By suitable choice of H it is possible to make $\sum_{S \in H} \chi_n^k(S)$ as small as we please. Consequently

$$\phi(B) \leq \lambda 2^k \{ \mathfrak{I}_n^k(B) + \epsilon \}.$$

Next we let $\epsilon \rightarrow 0$ to conclude

$$\phi(B) \leq \lambda 2^k \mathfrak{I}_n^k(B).$$

The proof is complete.

3.7 THEOREM. *If $A \subset E_n$, then $\mathfrak{O}_n^k(\mathfrak{I}_n^k, A, x) \geq 2^{-k}$ for \mathfrak{I}_n^k almost all x in A .*

Proof. In case the intersection of A with every closed sphere has infinite \mathfrak{I}_n^k measure, we have $\mathfrak{O}_n^k(\mathfrak{I}_n^k, A, x) = \infty$ for all x in E_n , and the conclusion of the theorem holds.

Hence, if the theorem failed to be true, we could find λ and B such that $0 < \lambda < 2^{-k}$, $B \subset A$, $0 < \mathfrak{I}_n^k(B) < \infty$, and $\mathfrak{O}_n^k(\mathfrak{I}_n^k, A, x) < \lambda$ for $x \in B$.

Applying Theorem 3.6 and the relation $\lambda 2^k < 1$, we obtain

$$\mathfrak{I}_n^k(B) \leq \lambda 2^k \mathfrak{I}_n^k(B) < \mathfrak{I}_n^k(B).$$

The proof is complete.

3.8 REMARK. *If $\phi \in \mathfrak{U}_n$, $A \subset E_n$, $\mathfrak{I}_n^k(A) = 0$ and $\mathfrak{O}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then $\phi(A) = 0$.*

This proposition is a consequence of Theorem 3.6.

3.9 REMARK. Suppose the condition

$$\mathfrak{O}_n^k(\phi, A - B, x) < \infty \quad \text{for } \phi \text{ almost all } x \text{ in } B$$

is satisfied in addition to the hypotheses of 3.3.

It follows from 3.8 that $\phi(B - D) = 0$, hence

$$D = E_n \cap E_x \left[\mathfrak{O}_n^k(\phi, A - B \cap D, x) = 0 \right].$$

This shows that B may be replaced by $(B \cap D)$ in the propositions (1) to (5) of 3.3.

4. (ϕ, k) restricted sets. The methods of this section are quite similar to those of [MR1, 7].

4.1 LEMMA. *If $k < n$ are positive integers, $A \subset E_n$, $T \in G_n$, $0 < M < \infty$, and*

$$(A - \text{sng } x) \subset \diamond_n^k(T, M, x) \quad \text{for } x \in A,$$

then the inverse relation of the function $(P_T^k|A)$ is a Lipschitzian function with domain $P_T^{k}(A)$ and range A .*

Proof. If x and y are distinct points of A , then

$$|y - x| < (1 + M^2)^{1/2} |P_T^k(y) - P_T^k(x)|.$$

Hence $(P_T^k|A)$ is univalent, its inverse is a function with domain $P_T^{k*}(A)$ and range A , and with Lipschitz constant $(1 + M^2)^{1/2}$.

4.2 LEMMA. *If $k < n$ are positive integers, $x \in E_n, R \in G_n, T \in G_n, T_i = R_{n-i+1}$ for $i = 1, 2, \dots, k, 0 < \eta < \infty$, then*

$$\diamond_n^{n-k}(R, \eta, x) = E_n \cap E_y [|P_T^k(y - x)| < \eta |P_R^{n-k}(y - x)|].$$

$$\diamond_n^k(T, \eta^{-1}, x) = E_n \cap E_y [|P_T^k(y - x)| > \eta |P_R^{n-k}(y - x)|].$$

Proof. Since R is an orthogonal transformation we have

$$\begin{aligned} |z|^2 &= |R(z)|^2 = \sum_{i=1}^n [R_i \cdot z]^2 = \sum_{i=1}^{n-k} [R_i \cdot z]^2 + \sum_{i=1}^k [T_i \cdot z]^2 \\ &= |P_R^{n-k}(z)|^2 + |P_T^k(z)|^2 \end{aligned} \quad \text{for } z \in E_n.$$

Hence the two relations

$$\begin{aligned} |y - x|^2 &< (1 + \eta^2) |P_R^{n-k}(y - x)|^2, \\ |P_T^k(y - x)|^2 &< \eta^2 |P_R^{n-k}(y - x)|^2 \end{aligned}$$

are equivalent, and the first statement of the lemma is proved. Similarly the three inequalities

$$\begin{aligned} |y - x|^2 &< (1 + \eta^{-2}) |P_T^k(y - x)|^2, \\ |P_R^{n-k}(y - x)|^2 &< \eta^{-2} |P_T^k(y - x)|^2, \\ |P_T^k(y - x)| &> \eta |P_R^{n-k}(y - x)| \end{aligned}$$

are equivalent. This proves the second proposition of the lemma.

4.3 THEOREM. *If*

(i) $k < n$ are positive integers,

(ii) $A \subset E_n, R \in G_n, 0 < \eta < \infty$,

(iii) corresponding to each $x \in A$ there is a number $r > 0$ such that $[A \cap \diamond_n^{n-k}(R, \eta, x) \cap K_x^r] = 0$,

(iv) $T \in G_n, T_i = R_{n-i+1}$ for $i = 1, 2, \dots, k$,

then there is a countable family F for which $A = \sigma(F)$ and $S \in F$ implies that the inverse relation of the function $(P_T^k|S)$ is a Lipschitzian function with domain $P_T^{k*}(S)$ and range S .

Proof. For each positive integer m let A_m be the set of all those points $x \in A$ for which

$$A \cap \diamond_n^{n-k}(R, \eta, x) \cap K(x, m^{-1}) = 0,$$

and select a countable family Q_m of open spheres, each with radius $(2m)^{-1}$, for which $\sigma(Q_m) = E_n$.

Let F be the family of all sets of the form $(U \cap A_m)$ where $U \in Q_m$ and m is a positive integer. Obviously F is countable and $\sigma(F) = A$.

Now suppose $x \in (T \cap A_m) = S \in F$.

Then $y \in (S - \text{sng } x)$ implies $y \in K(x, m^{-1})$, hence

$$y \notin \diamond_n^{n-k}(R, \eta, x)$$

and we use Lemma 4.2 to infer

$$y \in \diamond_n^k[T, (2\eta)^{-1}, x].$$

This means that

$$(S - \text{sng } x) \subset \diamond_n^k[T, (2\eta)^{-1}, x] \quad \text{for } x \in S.$$

Use of 4.1 completes the proof.

4.4 REMARK. Every set A which satisfies the conditions (i), (ii), (iii) of Theorem 4.3 is countably k rectifiable.

4.5 THEOREM. If

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n, A \subset E_n, R \in G_n, 0 < \eta < 1, 0 < \delta < \infty, 0 < \mu < \infty,$
- (iii) $\nabla_n^k(\phi, A, R, \eta, r, x) \leq \mu$ for $x \in A, 0 < r < \delta,$
- (iv) B is the set of those points of A for which

$$[A \cap \diamond_n^{n-k}(R, r, x) \cap K_x^r] \neq 0 \quad \text{whenever } r > 0,$$

then $(A - B)$ is a countably k rectifiable set and

$$\phi(B \cap K_x^r) \leq (2 \cdot 10^{2k}) \mu \alpha(k) r^k$$

for $x \in E_n$ and $0 < r < \delta/(14)$.

Proof. Suppose $a \in E_n$ and $0 < \rho < \delta/(14)$.

Define $T \in G_n$ by the relation

$$T_i = R_{n-i+1} \quad \text{for } i = 1, 2, \dots, n$$

and abbreviate

$$P_T^k(x) = x', \quad P_R^{n-k}(x) = x'' \quad \text{for } x \in E_n.$$

Let $\epsilon = \eta/(12), X = (B \cap K_a^\rho)$, and, for each point $x \in X$, let $h(x)$ be the supremum of numbers of the form $|y'' - x''|$ with

$$y \in [A \cap \diamond_n^{n-k}(R, \epsilon, x) \cap K_a^p].$$

We use (iv) to infer that

$$0 < h(x) \leq 2\rho \qquad \text{for } x \in X,$$

and associate with each point $x \in X$ a point

$$\bar{x} \in [A \cap \diamond_n^{n-k}(R, \epsilon, x) \cap K_a^p]$$

for which $12|\bar{x}'' - x''| > 11h(x)$.

For each $x \in X$ we further define

$$Q(x) = E_n \cap E_y [|y' - x'| < 5\epsilon h(x), |y'' - x''| < 5h(x)],$$

and divide the remainder of the proof into five parts.

Part 1. If $x \in X$, then

$$\{A \cap K_a^p \cap E_y [|y' - x'| < 5\epsilon h(x)]\} \subset Q(x).$$

Proof. Otherwise pick $y \in (A \cap K_a^p)$ with

$$|y' - x'| < 5\epsilon h(x), \quad |y'' - x''| \geq 5h(x),$$

hence $|(y-x)'| < \epsilon |(y-x)''|$ and Lemma 4.2 implies

$$y \in \diamond_n^{n-k}(R, \epsilon, x).$$

Consequently

$$\infty > h(x) \geq |y'' - x''| \geq 5h(x) > 0,$$

which is false.

Part 2. If $x \in X$, then

$$Q(x) \subset [\diamond_n^{n-k}(R, \epsilon, x) \cup \diamond_n^{n-k}(R, \epsilon, \bar{x})].$$

Proof. Otherwise we could find a point y for which

$$\begin{aligned} |y' - x'| < 5\epsilon h(x), \quad |y' - x'| \geq \eta |y'' - x''|, \\ |y' - \bar{x}'| \geq \eta |y'' - \bar{x}''|, \end{aligned}$$

by virtue of Lemma 4.2. From these inequalities we infer

$$\begin{aligned} 0 < 11\epsilon h(x) < 12\epsilon |\bar{x}'' - x''| &= \eta |\bar{x}'' - x''| \\ &\leq \eta |\bar{x}'' - y''| + \eta |y'' - x''| \leq |y' - \bar{x}'| + |y' - x'| \\ &\leq |x' - \bar{x}'| + 2|y' - x'| < \epsilon |x'' - \bar{x}''| + 10\epsilon h(x) \\ &\leq \epsilon h(x) + 10\epsilon h(x) = 11\epsilon h(x), \end{aligned}$$

which is false.

Part 3. If $x \in X$, then $K[x', \epsilon h(x)] \subset K[a', \rho(1+2\epsilon)]$.

Proof. If $y \in E_k$, $|y - x'| < \epsilon h(x)$, then

$$\begin{aligned} |y - a'| &\leq |y - x'| + |x' - a'| < \epsilon h(x) + |x - a| \\ &\leq \epsilon(2\rho) + \rho = \rho(1 + 2\epsilon). \end{aligned}$$

Part 4. If $x \in X$, then

$$\phi[A \cap Q(x)] \leq 2 \cdot (84)^k \cdot \mu \mathcal{L}_k \{K[x', \epsilon h(x)]\}.$$

Proof. Certainly $y \in Q(x)$ implies

$$\begin{aligned} |y - x| &\leq |y' - x'| + |y'' - x''| \leq 5(\epsilon + 1)h(x) < 6h(x), \\ |y - \bar{x}| &\leq |y - x| + |x - \bar{x}| \\ &\leq 5(\epsilon + 1)h(x) + |x' - \bar{x}'| + |x'' - \bar{x}''| \\ &\leq 6(\epsilon + 1)h(x) < 7h(x), \end{aligned}$$

because $6\epsilon = \eta/2 < 1$. Hence

$$Q(x) \subset \{K[x, 7h(x)] \cap K[\bar{x}, 7h(x)]\}$$

with $7h(x) < 14\rho < \delta$. From Part 2 we infer that

$$\begin{aligned} [A \cap Q(x)] &\subset \{A \cap \diamond_n^{n-k}(R, \eta, x) \cap K[x, 7h(x)]\} \\ &\quad \cup \{A \cap \diamond_n^{n-k}(R, \eta, \bar{x}) \cap K[\bar{x}, 7h(x)]\} \end{aligned}$$

and use the condition (iii) to reach the conclusion

$$\begin{aligned} \phi[A \cap Q(x)] &\leq 2\mu\alpha(k)[7h(x)\eta]^k \\ &= 2 \cdot 7^k \cdot (12)^k \cdot \mu\alpha(k)[\epsilon h(x)]^k = 2 \cdot (84)^k \cdot \mu \mathcal{L}_k \{K[x', \epsilon h(x)]\}. \end{aligned}$$

Part 5. $\phi(X) \leq 2 \cdot (10)^{2k} \mu\alpha(k) \rho^k$.

Proof. Since

$$P_T^{k*}(X) = \bigcup_{x \in X} \text{sng } x' \subset \bigcup_{x \in X} K[x', \epsilon h(x)],$$

with $2\epsilon h(x) \leq 4\epsilon\rho$ for $x \in X$, we may use [M1, 3.5] to obtain a subset Y of X for which

$$P_T^{k*}(X) \subset \bigcup_{x \in Y} K[x', 5\epsilon h(x)]$$

and

$$\{K[x', \epsilon h(x)] \cap K[y', \epsilon h(y)]\} = \emptyset$$

whenever x and y are distinct points of Y .

Next we apply Part 1 and obtain

$$\begin{aligned}
 X &= X \cap E_y [y' \in P_T^{k*}(X)] \\
 &= X \cap E_y \left\{ y' \in \bigcup_{x \in Y} K[x', 5\epsilon h(x)] \right\} \\
 &= \bigcup_{x \in Y} X \cap E_y \{ y' \in K[x', 5\epsilon h(x)] \} \\
 &= \bigcup_{x \in Y} X \cap E_y [|y' - x'| < 5\epsilon h(x)] \\
 &\subset \bigcup_{x \in Y} A \cap Q(x).
 \end{aligned}$$

Combining the last relation with Parts 4 and 3 we reach the conclusion

$$\begin{aligned}
 \phi(X) &\leq \sum_{x \in Y} \phi[A \cap Q(x)] \\
 &\leq 2 \cdot (84)^k \mu \sum_{x \in Y} \mathcal{L}_k \{ K[x', \epsilon h(x)] \} \\
 &\leq 2 \cdot (84)^k \mu \mathcal{L}_k \{ K[a', \rho(1 + 2\epsilon)] \} \\
 &= 2 \cdot (84)^k \cdot (1 + 2\epsilon)^k \mu \alpha(k) \rho^k \\
 &\leq 2 \cdot (84)^k (7/6)^k \mu \alpha(k) \rho^k \\
 &\leq 2 \cdot (10)^{2k} \mu \alpha(k) \rho^k.
 \end{aligned}$$

4.6 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n', A \subset E_n, \phi(A) < \infty$,
- (iii) whenever $R \in G_n, 0 < \eta < 1, \mu > 0$ and m is a positive integer, the set

$$T(R, \eta, \mu, m)$$

is defined to consist of those points $x \in E_n$ for which $0 < r < 1/m$ implies

$$(2 \cdot 10^{2k}) \nabla_n^k(\phi, A, R, \eta, r, x) \leq \mu < \mathfrak{O}_n^k(\phi, A, x) < \infty,$$

- (iv) D is a dense subset of G_n, ρ is the set of all positive rational numbers, ρ' is the set of all positive rational numbers less than 1, then each set of the form $[A \cap T(R, \eta, \mu, m)]$ is countably (ϕ, k) rectifiable, and

$$\bigcup_{R \in G_n} \bigcup_{0 < \eta < 1} \bigcup_{\mu > 0} \bigcup_{m=1}^{\infty} T(R, \eta, \mu, m) = \bigcup_{R \in D} \bigcup_{\eta \in \rho'} \bigcup_{\mu \in \rho} \bigcup_{m=1}^{\infty} T(R, \eta, \mu, m).$$

Proof. We suppose $S \in G_n, 0 < \epsilon < 1, u > 0, m$ is a positive integer, and divide the proof of the theorem into two parts.

Part 1. $[A \cap T(S, \epsilon, u, m)]$ is countably (ϕ, k) rectifiable.

Proof. From 3.4 we see that

$$T(S, \epsilon, u, m) \in \mathfrak{B}_n.$$

Let B be the set of those points $x \in T(S, \epsilon, u, m)$ for which $r > 0$ implies

$$[T(S, \epsilon, u, m) \cap \diamond_n^{n-k}(S, r, x) \cap K_x^r] \neq 0.$$

From 4.5 we infer that $[T(S, \epsilon, u, m) - B]$ is a countably k rectifiable set and that

$$(1) \quad \vartheta_n^k(\phi, B, x) \leq u \quad \text{for } x \in B.$$

It is easy to see that, for each positive integer j , the set of all those points $x \in E_n$ for which

$$[T(S, \epsilon, u, m) \cap \diamond_n^{n-k}(S, j^{-1}, x) \cap K(x, j^{-1})] \neq 0$$

is open. Hence B is the intersection of $T(S, \epsilon, u, m)$ with a set of type G_δ . Consequently

$$B \in \mathfrak{B}_n.$$

It follows from 3.9 that

$$\vartheta_n^k(\phi, A \cap B, x) = \vartheta_n^k(\phi, A, x) > u \quad \text{for } \phi \text{ almost all } x \text{ in } (A \cap B).$$

Hence (1) implies $\phi(A \cap B) = 0$.

This completes the proof of Part 1.

Part 2.

$$T(S, \epsilon, u, m) \subset \bigcup_{R \in G_n} \bigcup_{\eta \in \rho'} \bigcup_{\mu \in \rho} \bigcup_{m=1}^{\infty} T(R, \eta, \mu, m).$$

Proof. Suppose $x \in T(S, \epsilon, u, m)$.

Choose μ, η, R in such a way that

$$\begin{aligned} \mu \in \rho, \quad u < \mu < \vartheta_n^k(\phi, A, x), \\ \eta \in \rho', \quad 0 < \eta < \epsilon, \quad u\epsilon^k \leq \mu\eta^k, \\ R \in D, \quad (1 + \eta^2)^{1/2} < (1 + \epsilon^2)^{1/2} \cdot (1 - \|R - S\|). \end{aligned}$$

Next we check that

$$\diamond_n^{n-k}(R, \eta, x) \subset \diamond_n^{n-k}(S, \epsilon, x)$$

by taking a point y in the first set and computing:

$$\begin{aligned} |y - x| &< (1 + \eta^2)^{1/2} |P_R^{n-k}(y - x)| \\ &\leq (1 + \eta^2)^{1/2} \{ |P_S^{n-k}(y - x)| + \|R - S\| \cdot |y - x| \}, \\ |y - x| &< (1 - \|R - S\|)^{-1} (1 + \eta^2)^{1/2} |P_S^{n-k}(y - x)| \\ &\leq (1 + \epsilon^2)^{1/2} |P_S^{n-k}(y - x)|. \end{aligned}$$

Hence y belongs to the second set, the inclusion relation is proved, and we infer from it that

$$\begin{aligned} (2 \cdot 10^{2k}) \nabla_n^k(\phi, A, R, \eta, r, x) &\leq (\epsilon/\eta)^k (2 \cdot 10^k) \nabla_n^k(\phi, A, S, \epsilon, r, x) \\ &\leq (\epsilon/\eta)^k u \leq \mu < \vartheta_n^k(\phi, A, x) \end{aligned}$$

whenever $0 < r < (1/m)$.

Hence $x \in T(R, \eta, \mu, m)$.

The proof is complete.

4.7 THEOREM. *If*

(i) $k < n$ are positive integers,

(ii) $\phi \in \mathcal{U}'_n, A \subset E_n, \phi(A) < \infty,$

(iii) corresponding to ϕ almost all x in A we can find R and η such that $R \in G_n, 0 < \eta < 1,$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) < (2 \cdot 10^{2k})^{-1} \vartheta_n^k(\phi, A, x) < \infty,$$

then A is (ϕ, k) rectifiable.

Proof. Defining $T(R, \eta, \mu, m)$ as in 4.6, we easily check the relation

$$\phi \left\{ A - \bigcup_{R \in G_n} \bigcup_{0 < \eta < 1} \bigcup_{\mu > 0} \bigcup_{m=1}^{\infty} T(R, \eta, \mu, m) \right\} = 0.$$

Since the set D of 4.6 (iv) may be chosen countable, we are assured by 4.6 that A is countably (ϕ, k) rectifiable. Reference to 2.36 completes the proof.

5. (ϕ, k) rectifiable sets.

5.1 LEMMA. *If*

(i) f is a function on E_k to $E_n,$

(ii) C is a compact subset of $E_k,$

(iii) f is univalent on $C,$ and continuous relative to $C,$

(iv) f has the nonsingular differential L at the point $x \in C,$

(v) $R \in G_n$ with $R_i \cdot L^i = 0$ for $i = 1, 2, \dots, n - k,$ and $j = 1, 2, \dots, k,$

(vi) $0 < M < \infty,$

then there is a number $r > 0$ for which

$$\{f^*(C) \cap \diamond_n^{n-k}[R, M, f(x)] \cap K[f(x), r]\} = 0.$$

Proof. Let L' be the inverse of L and choose ϵ such that

$$\epsilon > 0, \quad 0 < \epsilon (\|L'\|^{-1} - \epsilon)^{-1} \leq (1 + M^2)^{-1/2}.$$

Choose $\delta > 0$ so that

$$|y - x| < \delta \quad \text{implies} \quad |f(y) - f(x) - L(y - x)| \leq \epsilon |y - x|,$$

which in turn implies

$$\begin{aligned} |P_R^{n-k}[f(y) - f(x)]| &\leq \epsilon |y - x| \leq \epsilon (\|L'\|^{-1} - \epsilon)^{-1} |f(y) - f(x)|, \\ f(y) &\in \diamond_n^{n-k}[R, M, f(x)], \end{aligned}$$

because $(P_R^{n-k}:L)$ is a null matrix.

The function $(f|C)$ has a continuous inverse function g . Choose $r > 0$ so that

$$z \in \{f^*(C) \cap K[f(x), r]\} \quad \text{implies} \quad |g(z) - x| < \delta,$$

which in turn implies

$$z = f[g(z)] \in \diamond_n^{n-k}[R, M, f(x)].$$

The proof is complete.

5.2 THEOREM. *If k, m, n are positive integers, $k \leq m, k \leq n, A \subset E_m, B \subset E_n, 0 < M < \infty$, and f is such a function with domain A and range B that*

$$|f(x) - f(y)| \leq M |x - y| \quad \text{whenever } x \in A, y \in A,$$

then

$$\mathfrak{H}_n^k(B) \leq M^k \cdot \mathfrak{H}_m^k(A).$$

Proof. Note that $[\text{diam } f^*(S)] \leq M \cdot [\text{diam } S]$ whenever $S \subset A$, and apply the definition of \mathfrak{H}_n^k .

5.3 REMARK. Saks has shown by an example (see [SS2]) that it is in general not possible to replace \mathfrak{H}_n^k by Γ_n^k in the conclusion of Theorem 5.2. It may easily be seen, by a slight modification of Saks' example, that \mathfrak{J}_n^k behaves just as badly as Γ_n^k , but the corresponding problem for the other measure functions which were defined in 2.18 is still unsolved.

5.4 COROLLARY. *If B is a k rectifiable subset of E_n then $\mathfrak{H}_n^k(B) < \infty$.*

5.5 REMARK. From 5.2 it follows that *Hausdorff measure is invariant under isometries*. This fact will later be used as follows:

Consider the k sphere

$$\mathfrak{S} = E_{k+1} \cap E_x [|x| = 1],$$

and the group G_{k+1} whose elements map \mathfrak{S} isometrically onto itself. The transformation group G_{k+1} is compact, and transitive with respect to \mathfrak{S} . Hence it is known (see [L], [W9]) that every measure ϕ over \mathfrak{S} which satisfies the conditions:

- (i) all Borel sets of \mathfrak{S} are ϕ measurable,
- (ii) $\phi(X) = \phi[R^*(X)]$ whenever $X \subset \mathfrak{S}, R \in G_{k+1}$,
- (iii) $\phi(\mathfrak{S}) = \mathfrak{H}_{k+1}^k(\mathfrak{S})$,

is determined for every Borel set $X \subset S$ by the equation

$$\phi(X) = \mathfrak{I}C_{k+1}^k(X).$$

This implies in particular that

$$\mathfrak{I}C_{k+1}^k(S) \cdot \int_{G_{k+1}} f(S_1) d\phi_{k+1}S = \int_S f(x) d\mathfrak{I}C_{k+1}^k x$$

for every Borel measurable numerically valued function f on S .

5.6 REMARK. If n is a positive integer, then $\mathfrak{I}C_n^n = \mathcal{L}_n$.

For a simple proof we refer the reader to [SD].

5.7 LEMMA. If f is a Lipschitzian function on E_k to E_n , $A \subset E_k$, and f has a singular differential at each point of A , then $\mathfrak{I}C_n^k[f^*(A)] = 0$.

Proof⁽³⁾. We assume, without loss of generality, that $\mathcal{L}_k(A) < \infty$, and let M be a Lipschitz constant of f .

Choose $\eta > 0$.

For $x \in A$, let $Df(x)$ be the differential of f at x , and, for $m = 1, 2, 3, \dots$, define

$$U_m(x) = \sup_{0 < |z-x| < 1/m} \frac{|f(z) - f(x) - [Df(x)](z-x)|}{|z-x|}.$$

Since

$$U_m(x) \rightarrow 0 \text{ as } m \rightarrow \infty \text{ for each } x \in A,$$

we may use Egoroff's theorem to select a subset C of A for which $\mathcal{L}_k(A-C) \leq (\eta/2M^k)$ and

$$U_m(x) \rightarrow 0, \text{ uniformly for } x \in C, \text{ as } m \rightarrow \infty.$$

Thus f is uniformly differentiable on C . From the standard theorem on uniform convergence it also follows that the (matrix-valued) function Df is continuous on C .

In view of the arbitrary nature of η , the theorem is a consequence of the second of the two parts into which we divide the remainder of the proof.

Part 1. If $x \in C$, then

$$\lim_{r \rightarrow 0^+} \frac{\mathfrak{I}C_n^k[f^*(C \cap K_x^r)]}{\mathcal{L}_k(K_x^r)} = 0.$$

Proof. Choose $\epsilon > 0$.

Since the matrix $Df(x)$ is singular, we can pick an orthogonal transforma-

⁽³⁾ This proof is similar to a construction given by Kolmogoroff in [K, §7]. His argument suffers from an unfortunate inaccuracy, which can easily be avoided.

tion $R \in G_k$ for which $[Df(x)](R^k) = \theta$, the zero vector of E_n .

Next choose a number $\delta > 0$ such that

$$(1) \quad |f(z) - f(y) - [Df(y)](z - y)| \leq \epsilon |z - y|$$

whenever $y \in C$, $z \in E_k$, $|z - y| < \delta$, and such that

$$(2) \quad \|Df(z) - Df(x)\| < \epsilon$$

whenever $z \in C$ and $|z - x| < \delta$.

Let g be the function on E_k to E_k such that

$$g(z) = x + \sum_{i=1}^{k-1} (z_i - x_i)R^i + [(z_k - x_k)/\epsilon]R^k \quad \text{for } z \in E_k,$$

and let $h = (f \circ g)$.

Observe that $y \in E_k$, $z \in E_k$ implies

$$(3) \quad |g(y) - g(z)|^2 = \sum_{i=1}^{k-1} (y_i - z_i)^2 + [(y_k - z_k)/\epsilon]^2,$$

$$(4) \quad |g(y) - g(z)| \leq (1 + \epsilon^{-1}) |y - z|.$$

Now suppose $0 < r < \delta/2$. Let

$$F = E_k \cap E \left[g(z) \in (C \cap K_x^r) \right],$$

and use (3) to check that $z \in F$ implies $|z_j - x_j| < r$ for $j = 1, 2, \dots, k-1$ and $|z_k - x_k| < \epsilon r$. It follows that

$$(5) \quad \mathcal{L}_k(F) \leq 2^k r^k \epsilon.$$

We abbreviate

$$L_x = Df[g(z)] \quad \text{for } z \in F,$$

and check that $y \in F$, $z \in F$ implies $g(z) \in C$ with $|g(y) - g(z)| < 2r < \delta$; hence (1), (2), (4), and the relation $L_x = Df[g(x)] = Df(x)$ yield the inequalities

$$\begin{aligned} |h(y) - h(z)| &= |f[g(y)] - f[g(z)]| \\ &\leq |L_x[g(y) - g(z)]| + \epsilon |g(y) - g(z)| \\ &\leq |L_x[g(y) - g(z)]| + \{ \|L_x - L_x\| [g(y) - g(z)] \} + \epsilon |g(y) - g(z)| \\ &\leq |L_x[g(y) - g(z)]| + \{ \|L_x - L_x\| + \epsilon \} |g(y) - g(z)| \\ &\leq \left| L_x \left\{ \sum_{i=1}^{k-1} (y_i - z_i)R^i + [(y_k - z_k)/\epsilon]R^k \right\} \right| + 2\epsilon(1 + \epsilon^{-1}) |y - z| \\ &= \left| L_x \left\{ \sum_{i=1}^{k-1} (y_i - z_i)R^i \right\} \right| + 2(1 + \epsilon) |y - z| \\ &\leq [\|L_x\| + 2(1 + \epsilon)] |y - z|. \end{aligned}$$

Hence 5.2 and 5.6 imply that

$$\mathfrak{H}_n^k[h^*(F)] \leq [\|L_x\| + 2(1 + \epsilon)]^k \mathcal{L}_k(F),$$

and we use (5) to conclude

$$\begin{aligned} \frac{\mathfrak{H}_n^k[f^*(C \cap K_x^r)]}{\mathcal{L}_k(K_x^r)} &= \frac{\mathfrak{H}_n^k[h^*(F)]}{\alpha(k)r^k} \leq \frac{[\|L_x\| + 2(1 + \epsilon)]^k 2^k r^k \epsilon}{\alpha(k)r^k} \\ &= \frac{[\|L_x\| + 2(1 + \epsilon)]^k 2^k}{\alpha(k)} \epsilon. \end{aligned}$$

In view of the freedom with which r was chosen we have

$$\limsup_{r \rightarrow 0^+} \frac{\mathfrak{H}_n^k[f^*(C \cap K_x^r)]}{\mathcal{L}_k(K_x^r)} \leq \frac{[\|L_x\| + 2(1 + \epsilon)]^k 2^k}{\alpha(k)} \epsilon.$$

But ϵ was an arbitrary positive number. Hence we may let $\epsilon \rightarrow 0$ to complete the proof of Part 1.

Part 2. $\mathfrak{H}_n^k[f^*(A)] < \eta$.

Proof. Using Part 1 and the Vitali covering theorem, we select a disjointed family F of open spheres for which

$$\mathcal{L}_k[C - \sigma(F)] = 0, \quad \mathcal{L}_k[\sigma(F)] \leq 2\mathcal{L}_k(C)$$

and $S \in F$ implies $\mathfrak{H}_n^k[f^*(C \cap S)] \leq \{ \eta / [2 + 4\mathcal{L}_k(C)] \} \mathcal{L}_k(S)$.

We use 5.2 in checking that

$$\begin{aligned} \mathfrak{H}_n^k[f^*(A)] &\leq \mathfrak{H}_n^k\{f^*[A - C \cap \sigma(F)]\} + \mathfrak{H}_n^k\{f^*[C \cap \sigma(F)]\} \\ &\leq M^k \mathcal{L}_k[A - C \cap \sigma(F)] + \sum_{S \in F} \mathfrak{H}_n^k[f^*(C \cap S)] \\ &\leq M^k \{ \mathcal{L}_k(A - C) + \mathcal{L}_k[C - \sigma(F)] \} + \{ \eta / [2 + 4\mathcal{L}_k(C)] \} \sum_{S \in F} \mathcal{L}_k(S) \\ &\leq (\eta/2) + \{ \eta / [2 + 4\mathcal{L}_k(C)] \} \mathcal{L}_k[\sigma(F)] \leq \eta. \end{aligned}$$

The proof is complete.

5.8 THEOREM. *If $\phi \in \mathcal{U}'$, $\phi(A) < \infty$, A is (ϕ, k) rectifiable, and $\mathfrak{D}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then A is (ϕ, k) restricted at ϕ almost all of its points.*

Proof. Let $\epsilon > 0$.

Choose λ and B such that $0 < \lambda < \infty$, $B \in \mathfrak{B}_n$, $\mathfrak{D}_n^k(\phi, A, x) < \lambda$ for $x \in B$, and $\phi(A - B) < \epsilon$.

Next select F, f and M such that F is a compact subset of E_k , f is a function on E_k to E_n , $0 < M < \infty$, $|f(x) - f(y)| \leq M|x - y|$ whenever $x \in E_k, y \in E_k$, and $\phi[A - f^*(F)] < \epsilon$.

Let D be the set of all those points of F at which the function f is differentiable. Then $\mathcal{L}_k(F - D) = 0$. (See [RAD].) The set D has, by [MF, 5.1 and 3.1], an \mathcal{L}_k measurable subset X such that $f^*(X) = f^*(D)$ and f is univalent on X . Let S be the set of those points of X at which the differential of f is nonsingular. Let C be a closed subset of S for which $\mathcal{L}_k(S - C) < \epsilon / (2^k \lambda M^k)$.

Further let

$$\begin{aligned} P &= E_n \cap E \left[\vartheta_n^k(\phi, A, x) > 0 \right], \\ Q &= B \cap f^*(C) \cap P, \\ Z &= Q \cap E \left[\vartheta_n^k(\phi, A - Q, x) = 0 \right]. \end{aligned}$$

The theorem is a consequence of the last two of the six parts into which we divide the remainder of the argument.

Part 1. $\mathfrak{I}_n^k[f^*(F) - f^*(C)] < \epsilon / (2^k \lambda)$.

Proof. First we check that

$$\begin{aligned} f^*(F) &= f^*(D) \cup f^*(F - D) = f^*(X) \cup f^*(F - D) \\ &= f^*(C) \cup f^*(S - C) \cup f^*(X - S) \cup f^*(F - D). \end{aligned}$$

Next we use the relations

$$\mathcal{L}_k(S - C) < \epsilon / (2^k \lambda M^k), \quad \mathcal{L}_k(F - D) = 0,$$

together with 5.2 and 5.7 to infer that

$$\mathfrak{I}_n^k[f^*(S - C)] < \epsilon / (2^k \lambda), \quad \mathfrak{I}_n^k[f^*(F - D)] = 0, \quad \mathfrak{I}_n^k[f^*(X - S)] = 0.$$

Consequently

$$\mathfrak{I}_n^k[f^*(F) - f^*(C)] \leq \epsilon / (2^k \lambda) + 0.$$

This proves Part 1.

Part 2. $\phi[A \cap B \cap f^*(F) - f^*(C)] < \epsilon$.

Proof. Use Part 1 and Theorem 3.6.

Part 3. $\phi[A \cap f^*(F) - P] = 0$.

Proof. Since $\mathfrak{I}_n^k[f^*(F)] < \infty$, Part 3 is a consequence of Theorem 3.6.

Part 4. $\phi(A \cap Q - Z) = 0$.

Proof. $Q \in \mathfrak{B}_n$, hence Part 4 follows from Remark 3.9.

Part 5. $\phi(A - Z) < 3\epsilon$.

Proof. From the relation

$$\begin{aligned} (A - Z) &\subset (A - Q) \cup (A \cap Q - Z) \subset (A - B) \cup [A - f^*(F)] \\ &\quad \cup [A \cap B \cap f^*(F) - f^*(C)] \cup [A \cap f^*(F) - P] \cup (A \cap Q - Z) \end{aligned}$$

we conclude that

$$\phi(A - Z) < \epsilon + \epsilon + \epsilon + 0 + 0.$$

This proves Part 5.

Part 6. If $x \in Z$, then A is (ϕ, k) restricted at x .

Proof. Certainly $\mathfrak{O}_n^k(\phi, A, x) > 0$.

By 5.1 we can further select $R \in G_n$ and $r > 0$ such that

$$[f^*(C) \cap \diamond_n^{n-k}(R, 1, x) \cap K(x, r)] = 0,$$

hence

$$\begin{aligned} [A \cap \diamond_n^{n-k}(R, 1, x) \cap K(x, r)] &\subset [A - f^*(C)] \subset (A - Q), \\ \mathfrak{O}_n^k[\phi, A \cap \diamond_n^{n-k}(R, 1, x), x] &\leq \mathfrak{O}_n^k(\phi, A - Q, x) = 0. \end{aligned}$$

The proof is complete.

5.9 THEOREM. *If f is a Lipschitzian function on E_k to E_n and X is an \mathcal{L}_k measurable subset of E_k , then*

$$\begin{aligned} \int_{E_n} N(f, X, y) d\mathfrak{I}_n^k y &= \int_X Jf(x) d\mathcal{L}_k x \\ &= \beta(n, k)^{-1} \int_{G_n} \int_{E_n} N[(P_R^k : f), X, z] d\mathcal{L}_k z d\phi_n R. \end{aligned}$$

5.10 REMARK. Theorem 5.9 is analogous to [F2, 4.5] and [F3, 4.5], where the measure Φ_n^k was used in place of the measure \mathfrak{I}_n^k .

Keeping f and X as in 5.9, suppose $\phi \in \mathcal{U}_n$ and consider the statements:

- (1) $\int_{E_n} N(f, X, y) d\phi y = \int_X Jf(x) d\mathcal{L}_k x,$
- (2) $\int_X Jf(x) d\mathcal{L}_k x = \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N[(P_R^k : f), X, z] d\mathcal{L}_k z d\phi_n R.$

The second proposition was proved in [F3, 4.5]. It does not involve the measure ϕ .

From [F2, 4.5] we know that (1) holds if $\phi = \Phi_n^k$. (Note that Φ_n^k was called Φ in [F2], and \mathcal{L}_n^k in [F3].) The proofs given in [F2] apply also—without any change whatever—to the cases $\phi = \mathcal{C}_n^k$ and $\phi = \Gamma_n^k$.

To obtain a proof of (1) for $\phi = \mathfrak{I}_n^k$ from our proof of (1) for $\phi = \Phi_n^k$, we need only replace the lemma [F2, 4.4], which is used in [F2, 4.5], by the Lemma 5.7 of this paper.

Furthermore we may replace \mathfrak{I}_n^k in 5.7 by \mathfrak{S}_n^k , because every set of Hausdorff measure zero has sphere measure zero. The remainder of the proof of [F2, 4.5] applies to \mathfrak{S}_n^k just as well as to $\Phi_n^k, \Gamma_n^k, \mathcal{C}_n^k$, or \mathfrak{I}_n^k .

Hence (1) holds if ϕ is any one of the five measures $\mathfrak{I}_n^k, \mathfrak{S}_n^k, \mathcal{C}_n^k, \Phi_n^k, \Gamma_n^k$.

In 5.11 to 5.13 it will be shown that the same is true for the functions \mathcal{G}_n^k and \mathcal{Y}_n^k .

5.11 REMARK. If $A \in \mathfrak{A}_n$, then

$$\mathcal{F}_n^k(A) = \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N(P_R^k, A, y) d\mathcal{L}_k y d\phi_n R.$$

To prove this we use [F1, 2.4] and [F1, 4.1].
 Choose a sequence Q of partitions of A such that

$$Q_j \subset \mathfrak{A}_n \quad \text{for } j = 1, 2, 3, \dots,$$

$$\lim_{j \rightarrow \infty} \left\{ \sup_{S \in Q_j} (\text{diam } S) \right\} = 0,$$

$$\lim_{j \rightarrow \infty} \left\{ \sum_{S \in Q_j} \zeta_n^k(S) \right\} = \mathcal{F}_n^k(A).$$

Denoting

$$f_j(R) = \sum_{S \in Q_j} \mathcal{L}_k [P_R^k(S)] \quad \text{for } R \in G_n, j = 1, 2, 3, \dots,$$

we infer from [F1, 4.1] that

$$\lim_{j \rightarrow \infty} f_j(R) = \int_{E_k} N(P_R^k, A, y) d\mathcal{L}_k y \quad \text{for } R \in G_n,$$

$$f_j(R) \leq \int_{E_k} N(P_R^k, A, y) d\mathcal{L}_k y \quad \text{for } R \in G_n, j = 1, 2, 3, \dots.$$

Next we see that

$$\sum_{S \in Q_j} \zeta_n^k(S) = \beta(n, k)^{-1} \int_{G_n} f_j(R) d\phi_n R \quad \text{for } j = 1, 2, 3, \dots.$$

From these relations and from Fatou's lemma it follows that

$$\begin{aligned} \mathcal{F}_n^k(A) &= \beta(n, k)^{-1} \lim_{j \rightarrow \infty} \int_{G_n} f_j(R) d\phi_n R \\ &\leq \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N(P_R^k, A, y) d\mathcal{L}_k y d\phi_n R \\ &= \beta(n, k)^{-1} \int_{G_n} \lim_{j \rightarrow \infty} f_j(R) d\phi_n R \\ &\leq \beta(n, k)^{-1} \lim_{j \rightarrow \infty} \int_{G_n} f_j(R) d\phi_n R \\ &= \mathcal{F}_n^k(A). \end{aligned}$$

This completes the proof.

5.12 REMARK. If $B \in \mathfrak{A}_k$ and f is a continuous function on B to E_n , then

$$\int_{E_n} N(f, B, z) d\mathfrak{F}_n^k z = \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N[(P_R^k:f), B, y] d\mathcal{L}_k y d\phi_n R.$$

It follows from [F1, 4.1] that the left integrand is an analytically measurable function. Let

$$\begin{aligned} A_j &= E_n \cap \frac{E}{z} [N(f, B, z) = j] \quad \text{for } j = 1, 2, 3, \dots, \\ A_\infty &= E_n \cap \frac{E}{z} [N(f, B, z) = \infty]. \end{aligned}$$

Whenever $R \in G_n$ and $y \in E_k$ we have

$$N[(P_R^k:f), B, y] = \sum_{j=1} j \cdot N(P_R^k, A_j, y) + \infty \cdot N(P_R^k, A_\infty, y),$$

where $\infty \cdot 0 = 0$, in accordance with the usual conventions of measure theory. From 5.11 and the last equation it follows that

$$\begin{aligned} \int_{E_n} N(f, B, z) d\mathfrak{F}_n^k z &= \sum_{j=1} j \cdot \mathfrak{F}_n^k(A_j) + \infty \cdot \mathfrak{F}_n^k(A_\infty) \\ &= \beta(n, k)^{-1} \int_{G_n} \int_{E_k} \left\{ \sum_{j=1} j \cdot N(P_R^k, A_j, y) + \infty \cdot N(P_R^k, A_\infty, y) \right\} d\mathcal{L}_k y d\phi_n R \\ &= \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N[(P_R^k:f), B, y] d\mathcal{L}_k y d\phi_n R. \end{aligned}$$

The proof is complete.

5.13 REMARK. It will now be shown that the proposition (1) of 5.10 holds for $\phi = \mathfrak{F}_n^k$ and for $\phi = \mathcal{G}_n^k$.

The proof of the inequality

$$(1a) \quad \int_{E_n} N(f, X, y) d\phi y \leq \int_X Jf(x) d\mathcal{L}_k x$$

for $\phi = \mathfrak{F}_n^k$ and $\phi = \mathcal{G}_n^k$ is quite similar to its proof, in [F2, 4.5], for $\phi = \Phi_n^k$.

The Lemma 5.7 of this paper remains true if \mathfrak{A}_n^k is replaced by \mathfrak{F}_n^k or by \mathcal{G}_n^k , because every set of \mathfrak{A}_n^k measure zero has \mathfrak{F}_n^k and \mathcal{G}_n^k measure zero. Hence the lemma [F2, 4.4], which is a step in the derivation of (1a) for $\phi = \Phi_n^k$, is true also for $\phi = \mathfrak{F}_n^k$ and for $\phi = \mathcal{G}_n^k$.

Next consider that portion of the proof of the lemma [F2, 4.3] which concerns itself with the inequality

$$\int_{E_n} N(f, A, y) d\phi y \leq \mu\mathcal{L}_k(A)$$

for $\phi = \Phi_n^k$. To obtain a proof of the same inequality for $\phi = \mathcal{F}_n^k$ and $\phi = \mathcal{G}_n^k$, we modify Part 1 of [F2, 4.3] as follows:

The open sphere $\bar{\alpha}$ is replaced by a convex open set α' which is defined like this: Let

$$h(z) = g_s(x) + L(z - x) \quad \text{for } z \in E_k.$$

Whenever u and v are positive numbers, let $W(u, v)$ be the set of all those points of E_n whose distance from the set $\{(\text{range } h) \cap C[g_s(x), u]\}$ is less than v . Then

$$\lim_{v \rightarrow 0^+} \zeta_n^k[W(1, v)] = \zeta_n^k\{(\text{range } h) \cap C[g_s(x), 1]\} = \alpha(k)$$

and we pick v_1 so that

$$\zeta_n^k[W(1, v_1)] \leq (1 + \epsilon \|U\|)^k \cdot \alpha(k),$$

which implies

$$\zeta_n^k[W(u, uv_1)] \leq u^k (1 + \epsilon \|U\|)^k \cdot \alpha(k) \quad \text{for } u > 0.$$

Now choose α and ρ as in [F2, 4.3], except for taking ρ so small that

$$g_s^*(\alpha) \subset W(\rho, \rho v_1),$$

and define

$$\alpha' = W(\rho, \rho v_1).$$

Then $\zeta_n^k(\alpha') < \mu \mathcal{L}_k(\alpha)$, and the remainder of the proof of (1a) for $\phi = \mathcal{F}_n^k$ and $\phi = \mathcal{G}_n^k$ proceeds just like the proof of (1a) for Φ_n^k .

To prove the opposite inequality

$$(1b) \quad \int_{E_n} N(f, X, y) d\phi y \geq \int_X Jf(x) d\mathcal{L}_k x,$$

we choose a set $B \in \mathcal{S}_k$ for which

$$B \subset X, \quad \mathcal{L}_k(X - B) = 0.$$

From 5.12 and the statement (2) of 5.10 we infer that

$$\begin{aligned} \int_{E_n} N(f, X, y) d\phi y &\geq \int_{E_n} N(f, B, y) d\phi y \\ &\geq \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N[(P_R^k : f), B, z] d\mathcal{L}_k z d\phi_n R \\ &= \int_B Jf(x) d\mathcal{L}_k x = \int_X Jf(x) d\mathcal{L}_k x, \end{aligned}$$

for $\phi = \mathcal{F}_n^k$ and also for $\phi = \mathcal{G}_n^k$, because $\mathcal{G}_n^k(S) \cong \mathcal{F}_n^k(S)$ for every set $S \subset E_n$. This completes the proof.

5.14 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $A \in \mathfrak{B}_n$, A is k rectifiable,

then

$$\mathcal{H}_n^k(A) = \beta(n, k)^{-1} \int_{G_n} \int_{E_k} N(P_R^k, A, z) d\mathcal{L}_k z d\phi_n R.$$

Proof. Select a Lipschitzian function f whose domain is a Borel set of E_k and whose range is A . Next use [MF, 5.1] to ascertain an \mathcal{L}_k measurable subset B of the domain of f , such that $f^*(B) = A$ and f is univalent on B . Then

$$\begin{aligned} N(f, B, y) &= 1 && \text{for } y \in E_n, \\ N[(P_R^k : f), B, z] &= N(P_R^k, A, z) && \text{for } z \in E_k, R \in G_n. \end{aligned}$$

Use 5.9 to complete the proof.

5.15 REMARK. The measure \mathcal{H}_n^k may be replaced in Theorem 5.14 by any one of the six functions $\mathcal{S}_n^k, \mathcal{C}_n^k, \mathcal{F}_n^k, \Gamma_n^k, \mathcal{G}_n^k$ and \mathcal{J}_n^k . This is a consequence of 5.10 and 5.13.

5.16 REMARK. The equation (1) has been proved for several measures ϕ by Kolmogoroff and by Nöbeling. (See [K], [N1].)

6. On differentiation.

6.1 SECTIONAL ASSUMPTIONS. *Throughout §6 we fix a space \mathcal{S} and a distance function ρ , which metrizes the set \mathcal{S} . We use the notation*

$$c(x, r) = \mathcal{S} \cap \bigcap_y E_y [\rho(y, x) \leq r] \quad \text{for } x \in \mathcal{S}, r > 0.$$

We denote by \mathfrak{M} the class of all those measures over \mathcal{S} relative to which all closed subsets of \mathcal{S} are measurable. We further assume that we are given a fixed measure $\mu \in \mathfrak{M}$, and a number λ , which satisfy the regularity condition

$$\mu[c(x, 5r)] < \lambda \mu[c(x, r)] \quad \text{for } x \in \mathcal{S}, r > 0.$$

We also suppose that every subset of \mathcal{S} is contained in a Borel set of equal μ measure.

6.2 REMARK. We shall later apply the results of this section by taking

$$\mathcal{S} = E_{k+1} \cap \bigcap_x E_x [|x| = 1],$$

ρ the usual metric, and $\mu = \mathcal{H}_{k+1}^k$.

The regularity condition, with suitable λ , may be proved by the following elementary argument:

Since congruent sets have equal \mathcal{H}_{k+1}^k measure, we consider only spheres with center $a = (0, 0, \dots, 0, 1) \in \mathcal{S}$. Define $x' = (x_1, x_2, \dots, x_k) \in E_k$ for

$x \in \mathfrak{S}$, and check that

$$(x_{k+1} - y_{k+1})(x_{k+1} + y_{k+1}) = (y' - x') \cdot (x' + y') \quad \text{for } x \in \mathfrak{S}, y \in \mathfrak{S}.$$

Hence $|x_{k+1} + y_{k+1}| > 0$ implies

$$|x' - y'| \leq |x - y| \leq \{1 + (|x' + y'| / |x_{k+1} + y_{k+1}|)\} |x' - y'|.$$

Letting $t(r) = 1 + r/(1-r)$ for $0 < r < 1$, we see that $|x' - y'| \leq |x - y| \leq t(r)|x' - y'|$ whenever x and y are in $c(a, r)$ with $0 < r < 1$. From 5.2 it follows that

$$\mathfrak{C}_k^k \{C[a', rt(r)^{-1}]\} \leq \mathfrak{C}_{k+1}^k [c(a, r)] \leq t(r)^k \mathfrak{C}_k^k [C(a', r)].$$

Hence 5.6 implies

$$t(r)^{-k} \leq \alpha(k)^{-1} r^{-k} \mathfrak{C}_{k+1}^k [c(a, r)] \leq t(r)^k$$

for $0 < r < 1$. Consequently

$$\lim_{r \rightarrow 0+} \alpha(k)^{-1} r^{-k} \mathfrak{C}_{k+1}^k [c(a, r)] = 1.$$

We can therefore pick a number $\epsilon > 0$ such that

$$\mathfrak{C}_{k+1}^k [c(a, 5r)] < (2 \cdot 5^k) \mathfrak{C}_{k+1}^k [c(a, r)] \quad \text{for } 0 < r < \epsilon.$$

Furthermore $r \geq \epsilon$ implies

$$\mathfrak{C}_{k+1}^k [c(a, 5r)] < 2\mathfrak{C}_{k+1}^k (\mathfrak{S}) \leq \{2\mathfrak{C}_{k+1}^k (\mathfrak{S}) / \mathfrak{C}_{k+1}^k [c(a, \epsilon)]\} \mathfrak{C}_{k+1}^k [c(a, r)].$$

Hence the required regularity condition will be satisfied if we take λ to be the greater of the two numbers $(2 \cdot 5^k)$ and $\{2 \cdot \mathfrak{C}_{k+1}^k (\mathfrak{S}) / \mathfrak{C}_{k+1}^k [c(a, \epsilon)]\}$.

6.3 DEFINITION. For $\psi \in \mathfrak{M}$ and $x \in \mathfrak{S}$ we define

$$\delta^* \psi(x) = \limsup_{r \rightarrow 0+} \frac{\psi [c(x, r)]}{\mu [c(x, r)]},$$

$$\delta_i \psi(x) = \liminf_{r \rightarrow 0+} \frac{\psi [c(x, r)]}{\mu [c(x, r)]},$$

$$\delta \psi(x) = \lim_{r \rightarrow 0+} \frac{\psi [c(x, r)]}{\mu [c(x, r)]}.$$

6.4 REMARK. If $\psi \in \mathfrak{M}$, then $\delta^* \psi$ and $\delta_i \psi$ are Borel measurable functions.

To prove this, define

$$f_r(x) = \lim_{s \rightarrow r+} \psi [c(x, r)], \quad g_r(x) = \lim_{s \rightarrow r+} \mu [c(x, r)],$$

and let R be the set of all positive rational numbers. It is easily seen that f_r and g_r are upper semicontinuous functions on \mathfrak{S} , for each $r > 0$, and that

$$\delta^*\psi(x) = \limsup_{R \ni r \rightarrow 0} [f_r(x)/g_r(x)],$$

$$\delta_*\psi(x) = \liminf_{R \ni r \rightarrow 0} [f_r(x)/g_r(x)].$$

6.5 LEMMA. If $\psi \in \mathfrak{M}$, A is a Borel set, $\epsilon > 0$, $0 < t < \infty$, and

$$\delta_*\psi(x) \leq t \quad \text{for } x \in A,$$

then

$$\psi(A) \leq t\mu(A)$$

and there is an open set U for which

$$A \subset U \quad \text{and} \quad \psi(U - A) < \epsilon.$$

Proof. We assume that A is bounded, and suppose

$$t < u < \infty.$$

Denote $S' = c(x, 5r)$ whenever $S = c(x, r)$.

Let F be the family of all the closed spheres S for which $\psi(S) \leq u\mu(S)$, $\text{diam } S < 1$, and suppose

$$Q = \bigcup_{S'} [S'' \in F].$$

The theorem follows from Parts 2 and 3 below by letting $u \rightarrow t+$.

Part 1. If B is a Borel set, $B \subset A$, and $\eta > 0$, then there is an open set U for which

$$B \subset U \quad \text{and} \quad \psi(U) \leq \lambda^2 u \mu(B) + \eta.$$

Proof. Since the Borel set B is contained in an open sphere of finite μ measure, we obtain from [RM1, 3.7] an open set V for which

$$B \subset V \quad \text{and} \quad \mu(V) \leq \mu(B) + \eta/(\lambda^2 u).$$

Since the family

$$Q \cap \bigcup_{S'} [S \subset V]$$

covers B , it has, by [M1, 3.5], a disjoint subfamily G for which

$$B \subset \bigcup_{S \in G} S'.$$

Letting

$$U = \bigcup_{S \in G} (\text{Interior } S''),$$

we see that U is an open set for which $B \subset U$ and

$$\psi(U) \leq \sum_{S \in G} \psi(S'') \leq u \sum_{S \in G} \mu(S'') < u\lambda^2 \sum_{S \in G} \mu(S) \leq u\lambda^2 \mu(V) \leq u\lambda^2 \mu(B) + \eta.$$

Part 2. There is an open set U for which

$$A \subset U \quad \text{and} \quad \psi(U - A) < \epsilon.$$

Proof. From Part 1 we know that the Borel set A is contained in an open set of finite ψ measure. Use of [RM1, 3.7] completes the proof.

Part 3. $\psi(A) \leq u\mu(A) + (u - t)$.

Proof. Use [RM1, 3.7] to obtain an open set V for which

$$A \subset V \quad \text{and} \quad \mu(V) \leq \mu(A) + (u - t)/u.$$

Since the family

$$F \cap \frac{E}{S} [S \subset V]$$

covers A in the sense of Vitali, it has, by [M1, 4.1], a countable disjointed subfamily G for which

$$\mu(B) = 0, \quad \text{where} \quad B = A - \sigma(G).$$

It follows from Part I that $\psi(B) = 0$, hence

$$\begin{aligned} \psi(A) &= \psi[\sigma(G) \cap A] \leq \sum_{S \in G} \psi(S) \leq u \sum_{S \in G} \mu(S) \\ &\leq u\mu(V) \leq u\mu(A) + (u - t). \end{aligned}$$

6.6 LEMMA. If $A \subset S, B \subset S, B$ is a Borel set, $v > 0$, and

$$\limsup_{\eta \rightarrow 0^+} \frac{\mu[B \cap c(x, \eta)]}{\mu[c(x, \eta)]} \geq v \quad \text{for } x \in A$$

then

$$\mu(A - B) = 0.$$

Proof. We assume $0 < v < 1$, and by 6.4 we may just as well assume that A is a Borel set, with $\mu(A) < \infty$.

Let ψ be the element of \mathfrak{M} such that

$$\psi(X) = \mu[X \cap (A - B)] \quad \text{for } X \subset S.$$

Then $x \in A$ implies

$$\begin{aligned} v + \delta_i \psi(x) &\leq \limsup_{\eta \rightarrow 0^+} \frac{\mu[B \cap c(x, \eta)]}{\mu[c(x, \eta)]} + \liminf_{\eta \rightarrow 0^+} \frac{\mu[(A - B) \cap c(x, \eta)]}{\mu[c(x, \eta)]} \leq 1, \\ \delta_i \psi(x) &\leq 1 - v. \end{aligned}$$

Since $(A - B)$ is a Borel set, we may apply 6.5 to infer that

$$\begin{aligned}\mu(A - B) &= \psi(A - B) \leq (1 - v)\mu(A - B) < \infty, \\ 0 &\leq -v\mu(A - B), \quad \mu(A - B) = 0.\end{aligned}$$

The proof is complete.

6.7 REMARK. By methods similar to those of 6.5 one may prove the following theorem, which was discovered independently by Anthony P. Morse (1940) and the author (1943).

If $\psi \in \mathfrak{M}$ and

$$F = \mathfrak{S} \cap E_x [\delta_x \psi(x) < \infty],$$

then

- (i) $\delta\psi(x) = \infty$ for $x \in (\mathfrak{S} - F)$,
- (ii) $\delta\psi(x) < \infty$ for μ and ψ almost all x in F ,
- (iii) $\psi(X) = \psi(X - F) + \int_{(X \cap F)} \delta\psi(x) d\mu x$ for every Borel set $X \subset \mathfrak{S}$.

This is a generalization of the decomposition theorem of de la Vallée Poussin for functions of bounded variation. (See [SS1, pp. 127 and 128].)

We neither prove nor use this result in this paper.

6.8 THEOREM. If

- (i) $\psi_r \in \mathfrak{M}$ for $r > 0$,
- (ii) $p(x, \eta, r) = \psi_r[c(x, \eta)] / \mu[c(x, \eta)]$ for $x \in \mathfrak{S}$, $\eta > 0$, $r > 0$,
- (iii) $T \subset \mathfrak{S}$, $0 < t \leq 1$, and

$$\limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} p(x, \eta, r) \right\} > t \quad \text{for } x \in T,$$

$$(iv) \quad H = \mathfrak{S} \cap E_x \left[\limsup_{(\eta, r) \rightarrow (0, 0)} p(x, \eta, r) = \infty \right],$$

(v) for each $r > 0$, Y_r is such a subset of \mathfrak{S} that

$$\psi_r(\mathfrak{S} - Y_r) = 0,$$

(vi) for each positive integer n , the set $Z_n = \bigcup_{0 < r < 1/n} Y_r$ is the union of a Borel set and a set of μ measure zero,

(vii) $Z = \bigcap_{n=1}^{\infty} Z_n$,

then $\mu[T - (H \cup Z)] = 0$.

Proof. For each positive integer n , and each $r > 0$, define

$$Q(n, r) = \mathfrak{S} \cap E_x [\delta^r \psi_r(x) \leq n].$$

Further, for $n = 1, 2, 3, \dots$, let H_n be the set of those points x in \mathfrak{S} for which there exist y, η, r such that

$$\begin{aligned}y &\in \mathfrak{S}, \quad 0 < \eta < 1/n, \quad 0 < r < 1/n, \\ x &\in c(y, \eta), \quad p(y, \eta, r) \geq n.\end{aligned}$$

The theorem is a consequence of the last of the nine parts into which we

divide the remainder of the proof.

Part 1. If $c(a, u) \subset c(b, v) \subset c(a, 5u)$, then

$$p(a, u, r) \leq \lambda p(b, v, r) \quad \text{and} \quad p(b, v, r) \leq \lambda p(a, 5u, r).$$

Proof. $\psi_r[c(a, u)] \leq \psi_r[c(b, v)] \leq \psi_r[c(a, 5u)]$,

$$\lambda^2 \mu[c(a, u)] > \lambda \mu[c(a, 5u)] \geq \lambda \mu[c(b, v)] \geq \lambda \mu[c(a, u)] > \mu[c(a, 5u)],$$

$$\lambda^{-2} \frac{\psi_r[c(a, u)]}{\mu[c(a, u)]} \leq \lambda^{-1} \frac{\psi_r[c(b, v)]}{\mu[c(b, v)]} \leq \frac{\psi_r[c(a, 5u)]}{\mu[c(a, 5u)]}.$$

This proves Part 1.

Part 2. $H = \bigcap_{n=1}^{\infty} H_n$.

Proof. The inclusion

$$H \subset \bigcap_{n=1}^{\infty} H_n$$

is evident. (Take $y = x$ in the definition of H_n .)

On the other hand, the relation

$$(1) \quad x \in H_n$$

implies the existence of y, η, r for which

$$(2) \quad \begin{aligned} 0 < \eta < 1/n, \quad 0 < r < 1/n, \\ y \in \mathcal{S}, \quad x \in c(y, \eta), \quad p(y, \eta, r) \geq n. \end{aligned}$$

Consequently

$$c(y, \eta) \subset c(x, 2\eta) \subset c(y, 5\eta),$$

and Part 1 implies

$$(3) \quad p(x, 2\eta, r) \geq n/\lambda.$$

Hence (1) implies that (2) and (3) hold for some numbers η and r .

Since this is true for every positive integer n , we infer

$$\bigcap_{n=1}^{\infty} H_n \subset H.$$

The proof of Part 2 is complete.

Part 3. If n is a positive integer, then H_n is the union of a set of type F_σ and a set of μ measure zero.

Proof. From the definition of H_n we obtain a family F of closed spheres for which $H_n = \sigma(F)$.

Now the Vitali covering theorem [M1, 4.1] yields such a countable disjoint subfamily G of F that

$$[\sigma(F) - \sigma(G)] = 0.$$

But $\sigma(G)$ is a Borel set of type F_σ , and

$$\sigma(F) = \sigma(G) \cup [\sigma(F) - \sigma(G)].$$

Hence Part 3 is proved.

Part 4. If $0 < \eta \leq \epsilon < \infty$, $x \in c(a, \epsilon)$, and

$$p(a, 5\epsilon, r) \leq n \leq p(x, \eta, r),$$

then we can find a number u for which $\eta \leq u \leq 2\epsilon$, $c(x, \eta) \subset c(x, u) \subset c(a, 5\epsilon)$, $n \leq p(x, u, r) \leq n\lambda$.

Proof. In case $p(x, \eta, r) \leq n\lambda$, take $u = \eta$.

Otherwise $p(x, \eta, r) > n\lambda$, and

$$V = E_v [0 < v \leq 4\epsilon, p(x, v, r) > n\lambda] \neq 0.$$

In this case we take $u = \sup V$, note that

$$\lim_{v \rightarrow \eta^+} p(x, v, r) = p(x, \eta, r) > n\lambda,$$

hence $0 < \eta < u \leq 4\epsilon$, and

$$c(x, \eta) \subset c(x, u) \subset c(a, 5\epsilon).$$

Next we define

$$S = E_y [\rho(y, x) < u]$$

and observe that

$$\lim_{v \rightarrow u^-} \psi_r [c(x, v)] = \psi_r(S) \leq \psi_r [c(x, u)],$$

$$\lambda \lim_{v \rightarrow u^-} \mu [c(x, v)] = \lambda \mu(S) \geq \lambda \mu [c(x, u/5)] > \mu [c(x, u)],$$

which we combine with the definition of u to infer that

$$n = (n\lambda)/\lambda \leq \lim_{v \rightarrow u^-} p(x, v, r)/\lambda < p(x, u, r).$$

On the other hand we have the relation

$$c(a, \epsilon) \subset c(x, v) \subset c(a, 5\epsilon) \quad \text{for } 2\epsilon \leq v \leq 4\epsilon,$$

and infer from Part 1 that

$$p(x, v, r) \leq \lambda p(a, 5\epsilon, r) \leq \lambda n \quad \text{for } 2\epsilon \leq v \leq 4\epsilon.$$

Consequently $u \leq 2\epsilon$ and

$$p(x, u, r) = \lim_{v \rightarrow u^+} p(x, v, r) \leq \lambda n.$$

This completes the proof of Part 4.

Part 5. If n is a positive integer, $a \in S$, $0 < r < (1/n)$, $0 < \epsilon < (1/5n)$, and $p(a, 5\epsilon, r) \leq n$, then

$$\psi_r[c(a, \epsilon) - Q(n, r)] \leq n\lambda^2\mu[H_n \cap c(a, 5\epsilon)].$$

Proof. Let X be the set of all those points $x \in c(a, \epsilon)$ for which there is a number u such that $u \leq 2\epsilon$,

$$c(x, u) \subset c(a, 5\epsilon), \quad n \leq p(x, u, r) \leq n\lambda.$$

Also, for each $x \in X$, pick such a number u , and denote it by $f(x)$. Hence $x \in X$ implies

$$f(x) \leq 2\epsilon, \quad c[x, f(x)] \subset c(a, 5\epsilon), \quad n \leq p[x, f(x), r] \leq n\lambda.$$

Next we use Part 4 and the definition of $Q(n, r)$ to infer that

$$[c(a, \epsilon) - Q(n, r)] \subset X.$$

On the other hand the definition of H_n implies

$$\bigcup_{x \in X} c[x, f(x)] \subset [H_n \cap c(a, 5\epsilon)].$$

Since $f(x)/5 < \epsilon$ for $x \in X$, and

$$X \subset \bigcup_{x \in X} c[x, f(x)/5],$$

we may use the covering theorem [M1, 3.5] to obtain a subset X' of X for which

$$X \subset \bigcup_{x \in X'} c[x, f(x)],$$

and $c[x, f(x)/5] \cap c[y, f(y)/5] = 0$ whenever x and y are distinct points of X' .

Inasmuch as

$$\bigcup_{x \in X'} c[x, f(x)] \subset c(a, 5\epsilon)$$

and every closed sphere has positive, finite μ measure, the set X' must be countable. Hence we compute

$$\begin{aligned} \psi_r[c(a, \epsilon) - Q(n, r)] &\leq \psi_r(X) \leq \sum_{x \in X'} \psi_r\{c[x, f(x)]\} \\ &\leq n\lambda \sum_{x \in X'} \mu\{c[x, f(x)]\} \leq n\lambda^2 \sum_{x \in X'} \mu\{c[x, f(x)/5]\} \\ &= n\lambda^2\mu \left\{ \bigcup_{x \in X'} c[x, f(x)/5] \right\} \leq n\lambda^2\mu[H_n \cap c(a, 5\epsilon)]. \end{aligned}$$

This proves Part 5.

Part 6. If n is a positive integer, $a \in S$, $0 < r < (1/n)$, and $\epsilon > 0$, then

$$\psi_r[c(a, \epsilon) \cap Q(n, r)] \leq n\mu[c(a, \epsilon) \cap Z_n].$$

Proof. We use 6.4 and 6.5 to compute:

$$\begin{aligned} \psi_r[c(a, \epsilon) \cap Q(n, r)] &= \psi_r[c(a, \epsilon) \cap Z_n \cap Q(n, r)] \\ &\leq n\mu[c(a, \epsilon) \cap Z_n \cap Q(n, r)] \leq n\mu[c(a, \epsilon) \cap Z_n]. \end{aligned}$$

Part 7. If n is a positive integer, and $a \in T$, then

$$\limsup_{r \rightarrow 0^+} \frac{\mu[c(a, \eta) \cap (H_n \cup Z_n)]}{\mu[c(a, \eta)]} \geq \frac{t}{2n\lambda^3}.$$

Proof. Pick $s > 0$. Use (iii) to select ϵ such that

$$0 < 5\epsilon < s, \quad \epsilon < (1/5n), \quad \limsup_{r \rightarrow 0^+} p(a, \epsilon, r) > t.$$

Next pick a number r such that $0 < r < (1/n)$, $p(a, \epsilon, r) > t$. Now we distinguish two cases:

If $p(a, 5\epsilon, r) \leq n$, we use Parts 5 and 6 to conclude that

$$\begin{aligned} t\mu[c(a, 5\epsilon)] &< t\lambda\mu[c(a, \epsilon)] < \lambda\psi_r[c(a, \epsilon)] \\ &\leq \lambda\{\psi_r[c(a, \epsilon) - Q(n, r)] + \psi_r[c(a, \epsilon) \cap Q(n, r)]\} \\ &\leq \lambda\{n\lambda^2\mu[c(a, 5\epsilon) \cap H_n] + n\mu[c(a, \epsilon) \cap Z_n]\} \\ &\leq 2n\lambda^3\mu[c(a, 5\epsilon) \cap (H_n \cup Z_n)], \\ (1) \quad &\frac{[c(a, 5\epsilon) \cap (H_n \cup Z_n)]}{[c(a, 5\epsilon)]} \geq \frac{t}{2n\lambda^3}. \end{aligned}$$

In case, however, $p(a, 5\epsilon, r) > n$, we have $c(a, 5\epsilon) \subset H_n$, which implies

$$\mu[c(a, 5\epsilon) \cap (H_n \cup Z_n)] = \mu[c(a, 5\epsilon)] \geq [t/(2n\lambda^3)]\mu[c(a, 5\epsilon)],$$

so that (1) holds in all cases.

Since $0 < 5\epsilon < s$, and s was an arbitrary positive number, the proof of Part 7 is complete.

Part 8. If n is a positive integer, then $\mu[T - (H_n \cup Z_n)] = 0$.

Proof. Use Part 7, Part 3, assumption (vi), and Lemma 6.6.

Part 9. $\mu[T - (H \cup Z)] = 0$.

Proof. The relations

$$H_{n+1} \subset H_n, \quad Z_{n+1} \subset Z_n \quad \text{for } n = 1, 2, 3, \dots,$$

imply that

$$(H \cup Z) = \bigcap_{n=1}^{\infty} (H_n \cup Z_n).$$

Consequently

$$\begin{aligned} [T - (H \cup Z)] &= \left[T - \bigcap_{n=1}^{\infty} (H_n \cup Z_n) \right] \\ &= \bigcup_{n=1}^{\infty} [T - (H_n \cup Z_n)] \end{aligned}$$

and we apply Part 8 to complete the proof.

7. On (ϕ, k) unrestricted sets. In this section we generalize Besicovitch's results on the "condensation lines" of plane sets of finite linear measure. (See [BE3].) Many of our geometrical constructions were suggested by Besicovitch's methods.

7.1 THEOREM. *If*

(i) A and B are sets and f is a function such that $(\text{domain } f) \subset A$, $(\text{range } f) \subset B$,

(ii) ϕ is a measure over A ,

(iii) ψ is the function on the set of all subsets of B such that

$$\psi(S) = \phi \left\{ E_{\frac{x}{2}} [f(x) \in S] \right\} \quad \text{for } S \subset B,$$

then ψ is a measure over B and every set $S \subset B$, for which the set

$$E_{\frac{x}{2}} [f(x) \in S]$$

is ϕ measurable, is ψ measurable.

This is a direct consequence of the definitions given in 2.2.

7.2 LEMMA. *If*

(i) $k < n$ are positive integers,

(ii) $A \in \mathfrak{S}_n$ (more generally $A \in \mathfrak{A}_n$),

(iii) B is the set of those ordered pairs (R, x) for which $R \in V_n^+(A, x)$,

then B is a Borel (or, more generally, analytic) subset of the cartesian product space $(G_n \times E_n)$.

Proof. For pairs (i, j) of positive integers we define the subsets $S(i, j)$ of the product space $(G_n \times E_n \times E_n)$ by the relation

$$S(i, j) = E_{(R,x,y)} [y \in \{A \cap \square_n^{n-k}(R, x) \cap C(x, i^{-1}) - K(x, j^{-1})\}].$$

Then $S(i, j)$ is of type F_σ (or, more generally, an analytic set), and so is its projection

$$B(i, j) = (G_n \times E_n) \cap E_{(R,x)} [(R, x, y) \in S(i, j) \text{ for some } y \in E_n].$$

But

$$B = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B(i, j).$$

The proof is complete.

7.3 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}'_n, A \in \mathcal{G}_n$ (more generally, $A \in \mathcal{A}_n$),
- (iii) $a \in E_n, 0 < t \leq 1, B \subset G_n$, and $\mathcal{O}_n^k(\phi, A, R, a) > t$ for $R \in B$,

then

$$\phi_n \{ B - [V_n^k(A, a) \cup W_n^k(\phi, A, a)] \} = 0.$$

Proof. We denote

$$Q_j^+(R, \eta) = E_n \cap E_x \left[\left\{ \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 \right\}^{1/2} < (n-k)^{1/2} \eta (x-a) \cdot R_j \right],$$

$$Q_j^-(R, \eta) = E_n \cap E_x \left[\left\{ \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 \right\}^{1/2} < -(n-k)^{1/2} \eta (x-a) \cdot R_j \right],$$

$$\rho_j^+(R, \eta, r) = \alpha(k)^{-1} (\eta r)^{-k} \phi [A \cap Q_j^+(R, \eta) \cap K_a^r],$$

$$\rho_j^-(R, \eta, r) = \alpha(k)^{-1} (\eta r)^{-k} \phi [A \cap Q_j^-(R, \eta) \cap K_a^r],$$

whenever $R \in G_n, \eta > 0, r > 0$, and j is an integer between 1 and $(n-k)$. For all such j we also define

$$F_j^+ = G_n \cap E_R \left[\limsup_{\eta \rightarrow 0^+} \left\{ \limsup_{r \rightarrow 0^+} \rho_j^+(R, \eta, r) \right\} \geq t/2(n-k) \right],$$

$$F_j^- = G_n \cap E_R \left[\limsup_{\eta \rightarrow 0^+} \left\{ \limsup_{r \rightarrow 0^+} \rho_j^-(R, \eta, r) \right\} \geq t/2(n-k) \right],$$

let h_j^+ be the characteristic function of the set

$$\{ F_j^+ - [V_n^k(A, a) \cap W_n^k(\phi, A, a)] \},$$

and let h_j^- be the characteristic function of the set

$$\{ F_j^- - [V_n^k(A, x) \cap W_n^k(\phi, A, x)] \}.$$

It may be shown that F_j^+, F_j^- are Borel sets, and that h_j^+, h_j^- are Borel measurable functions (or, more generally, analytically measurable functions, in case A is an analytic set). The outline of a proof is given in 3.4 and 7.2.

We further let I be the identity matrix in G_n , and define the following mapping on G_{k+1} to G_n :

With each matrix $S \in G_{k+1}$ we associate the matrix $*S \in G_n$ by the relations

$$\begin{aligned}
 *S_i^j &= I_i^j && \text{for } i < n - k \text{ or } j < n - k, \\
 *S_i^j &= S_{i-n+k+1}^{j-n+k+1} && \text{for } i \geq n - k \text{ and } j \geq n - k,
 \end{aligned}$$

where i and j are integers between 1 and n .

The remainder of the argument is divided into eight parts.

Part 1. If $R \in G_n$ and $\eta > 0$, then

$$\diamond_n^{n-k}(R, \eta, a) \subset \bigcup_{j=1}^{n-k} [Q_j^+(R, \eta) \cup Q_j^-(R, \eta)].$$

Proof. The denial of this proposition implies (see 4.2) the existence of a point x for which

$$(1) \quad \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 < \eta^2 \sum_{i=1}^{n-k} [(x-a) \cdot R_i]^2$$

but

$$\left\{ \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 \right\}^{1/2} \geq (n-k)^{1/2} \eta | (x-a) \cdot R_j |$$

for $j=1, 2, \dots, n-k$. Squaring and adding the last $(n-k)$ inequalities we obtain

$$(2) \quad (n-k) \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 \geq (n-k) \eta^2 \sum_{j=1}^{n-k} [(x-a) \cdot R_j]^2.$$

Since (1) and (2) are contradictory, the proof of Part 1 is complete.

Part 2. $B \subset \bigcup_{j=1}^{n-k} (F_j^+ \cup F_j^-)$.

Proof. The denial of this proposition implies the existence of a point $R \in B$ for which

$$\limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} [\rho_j^+(R, \eta, r) + \rho_j^-(R, \eta, r)] \right\} < t/(n-k)$$

for $j=1, 2, \dots, n-k$.

Using the hypothesis of the theorem, the definitions 2.24, 2.25, and Part 1, we conclude

$$\begin{aligned}
 t &\leq \mathcal{O}_n^k(\phi, A, R, a) \\
 &\leq \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} \sum_{j=1}^{n-k} [\rho_j^+(R, \eta, r) + \rho_j^-(R, \eta, r)] \right\} \\
 &\leq \sum_{j=1}^{n-k} \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} [\rho_j^+(R, \eta, r) + \rho_j^-(R, \eta, r)] \right\} < t,
 \end{aligned}$$

which is the contradiction required to prove Part 2.

Part 3. If u and v are integers between 1 and $(n - k)$, then

$$\int_{G_n} h_u^+(R) d\phi_n R = \int_{G_n} h_v^-(R) d\phi_n R.$$

Proof. (Note that u and v may be equal.) Let U be the matrix such that

$$\begin{aligned} U_i &= I_i && \text{for } i \neq u \text{ and } i \neq v, \\ U_u &= -I_v, && U_v = -I_u. \end{aligned}$$

Evidently $U \in G_n$. From 2.5 we see that $R \in G_n$ implies

$$\begin{aligned} (U:R)_i &= R_i && \text{for } i \neq u \text{ and } i \neq v, \\ (U:R)_u &= -R_v, && (U:R)_v = -R_u, \end{aligned}$$

which in turn implies $Q_u^+(R, \eta) = Q_v^-[(U:R), \eta]$, $\rho_u^+(R, \eta, r) = \rho_v^-[(U:R), \eta, r]$, for any two positive numbers η and r . Hence

$$(1) \quad R \in F_u^+ \text{ if and only if } (U:R) \in F_v^-.$$

Furthermore

$$\begin{aligned} \square_n^{n-k}[(U:R), a] &= \square_n^{n-k}[R, a], \\ \diamond_n^{n-k}[(U:R), \eta, a] &= \diamond_n^{n-k}[R, \eta, a], \end{aligned}$$

for $R \in G_n$ and $\eta > 0$. Consequently

$$(2) \quad R \in V_n^k(A, a) \text{ if and only if } (U:R) \in V_n^k(A, a),$$

$$(3) \quad R \in W_n^k(\phi, A, a) \text{ if and only if } (U:R) \in W_n^k(\phi, A, a).$$

From (1), (2), and (3) we conclude that

$$h_u^+(R) = h_v^-(U:R) \quad \text{for } R \in G_n,$$

which we combine with the fact that ϕ_n is the Haar measure of G_n to obtain the desired relation

$$\int_{G_n} h_u^+(R) d\phi_n R = \int_{G_n} h_v^-(U:R) d\phi_n R = \int_{G_n} h_v^-(R) d\phi_n R.$$

Part 4. If $R \in G_n$, $S \in G_{k+1}$, $\eta > 0$, then

$$\begin{aligned} (*S:R)_i &= R_i && \text{for } i = 1, 2, \dots, n - k - 1, \\ (*S:R)_{n-k} &= \sum_{j=-}^{k+1} S_1^j R_{n-k-1+j}, \end{aligned}$$

$$\sum_{i=n-k}^n [(x - a) \cdot (*S:R)_i]^2 = \sum_{i=n-k}^n [(x - a) \cdot R_i]^2 \quad \text{for } x \in E_n,$$

$Q_{n-k}^+[(*S:R), \eta]$ is the set of all those points $x \in E_n$ for which

$$\left\{ \sum_{i=n-k}^n [(x - a) \cdot R_i]^2 \right\}^{1/2} < [(n - k)\eta^2 + 1]^{1/2} \cdot [(x - a) \cdot (*S:R)_{n-k}].$$

Proof. For $i = 1, 2, \dots, n - k - 1$ the relations

$$(*S:R)_i = \bar{R}(*S_i) = \bar{R}(I_i) = R_i$$

follow from 2.5. This proves the first statement. Similarly

$$\begin{aligned} (*S:R)_{n-k} &= \bar{R}(*S_{n-k}) = \bar{R}\left(\sum_{j=1}^{k+1} S_1^j I_{n-k-1+j}\right) \\ &= \sum_{j=1}^{k+1} S_1^j \bar{R}(I_{n-k-1+j}) = \sum_{j=1}^{k+1} S_1^j R_{n-k-1+j} \end{aligned}$$

and the second equation is proved.

From the first equation we see that $x \in E_n$ implies

$$\sum_{i=1}^{n-k-1} [(x - a) \cdot (*S:R)_i]^2 = \sum_{i=1}^{n-k-1} [(x - a) \cdot R_i]^2,$$

whereas

$$\begin{aligned} \sum_{i=1}^n [(x - a) \cdot (*S:R)_i]^2 &= |x - a|^2, \\ \sum_{i=1}^n [(x - a) \cdot R_i]^2 &= |x - a|^2, \end{aligned}$$

because $(*S:R)$ and R are orthogonal transformations. From the last three equations, the third statement of Part 4 follows at once.

Now, if x is a point of either set mentioned in the fourth statement, then clearly $(x - a) \cdot (*S:R)_{n-k} > 0$. Hence we may assume this relation in the proof of the fourth statement, which consists in checking that each inequality in the following sequence is equivalent to the succeeding:

$$\begin{aligned} \left\{ \sum_{i=n-k+1}^n [(x - a) \cdot (*S:R)_i]^2 \right\}^{1/2} &< (n - k)^{1/2} \eta [(x - a) \cdot (*S:R)_{n-k}], \\ \sum_{i=n-k+1}^n [(x - a) \cdot (*S:R)_i]^2 &< (n - k)\eta^2 [(x - a) \cdot (*S:R)_{n-k}]^2, \\ \sum_{i=n-k}^n [(x - a) \cdot (*S:R)_i]^2 &< [1 + (n - k)\eta^2] [(x - a) \cdot (*S:R)_{n-k}]^2, \end{aligned}$$

$$\left\{ \sum_{i=n-k}^n [(x-a) \cdot (*S:R)_i]^2 \right\}^{1/2} < [1 + (n-k)\eta^2]^{1/2} [(x-a) \cdot (*S:R)_{n-k}].$$

This completes the proof of the fourth statement, and of Part 4.

Part 5. If $R \in G_n$ and $\eta > 0$, then

$$Q_{n-k}^+(R, \eta) \subset \diamond_n^{n-k} [R, (n-k)^{1/2}\eta, a].$$

Proof. Suppose $x \in Q_{n-k}^+(R, \eta)$. Then

$$\begin{aligned} \sum_{i=n-k+1}^n [(x-a) \cdot R_i]^2 &< (n-k)\eta^2 [(x-a) \cdot R_{n-k}]^2 \\ &\leq [(n-k)^{1/2}\eta]^2 \sum_{i=n-k}^n [(x-a) \cdot R_i]^2. \end{aligned}$$

This proves Part 5.

Part 6. If $R \in G_n$, then

$$\int_{G_{k+1}} h_{n-k}^+(*S:R) d\phi_{k+1}S = 0.$$

Proof. We fix the notations

$$\begin{aligned} \mathfrak{S} &= E_{k+1} \cap E_y [|y| = 1], \\ c(y, \eta) &= \mathfrak{S} \cap E_z [|z-y| \leq \eta] \quad \text{for } x \in \mathfrak{S}, \eta > 0. \end{aligned}$$

Let f be the function with domain

$$D = E_n \cap E_x \left[\sum_{i=n-k}^n [(x-a) \cdot R_i]^2 > 0 \right],$$

and to \mathfrak{S} , such that

$$f(x) = \frac{[(x-a) \cdot R_{n-k}, (x-a) \cdot R_{n-k+1}, \dots, (x-a) \cdot R_n]}{\left\{ \sum_{i=n-k}^n [(x-a) \cdot R_i]^2 \right\}^{1/2}}.$$

For each $r > 0$, let

$$Y_r = f^* [A \cap D \cap K_a^r],$$

and define the measure ψ_r over \mathfrak{S} by the relation

$$\psi_r(Y) = r^{-k} \phi \{ A \cap K_a^r \cap D \cap E_x [f(x) \in Y] \} \quad \text{for } Y \subset \mathfrak{S}.$$

We further let

$$p(y, \eta, r) = \frac{\psi_r[c(y, \eta)]}{\mathfrak{H}_{k+1}^k[c(y, \eta)]} \quad \text{for } y \in S, \eta > 0, r > 0,$$

$$T = S \cap E_y \left[\limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} p(y, \eta, r) \right\} > t/2(n - k + 1)^{k+1} \right]$$

$$H = S \cap E_y \left[\limsup_{(\eta, r) \rightarrow (0, 0)} p(y, \eta, r) = \infty \right],$$

$$Z_j = \bigcup_{0 < r < 1/j} Y_r \quad \text{for } j = 1, 2, 3, \dots,$$

$$Z = \bigcap_{j=1} Z_j,$$

and denote

$$s(\eta) = \{2 - 2[(n - k)\eta^2 + 1]^{-1/2}\}^{1/2} \quad \text{for } \eta > 0.$$

The remainder of the proof of Part 6 consists in the consideration of nine auxiliary propositions.

FIRST PROPOSITION. *If $r > 0$, then ψ_r is such a measure over S that every closed subset of S is ψ_r measurable.*

We simply combine the facts that f is continuous, every closed subset of E_n is ϕ measurable, and Theorem 7.2, to obtain a proof of the first proposition.

SECOND PROPOSITION. *If j is a positive integer, then*

$$Z_j = Y_{1/j} \in \mathfrak{S}_{k+1} \quad (\text{more generally, } \in \mathfrak{A}_{k+1}).$$

First we note that

$$K(a, 1/j) = \bigcup_{0 < r < 1/j} K(a, r),$$

hence, applying f^* , we obtain

$$Y_{1/j} = \bigcup_{0 < r < 1/j} Y_r = Z_j.$$

Now f is continuous and the set

$$D \cap A \cap K(a, 1/j)$$

is a bounded F_σ (an analytic set). Consequently the set

$$Z_j = f^*[D \cap A \cap K(a, 1/j)]$$

is an F_σ (an analytic set).

THIRD PROPOSITION. $\mathfrak{H}_{k+1}^k[T - (H \cup Z)] = 0$.

Evidently

$$\psi_r(S - Y_r) = 0 \quad \text{for } r > 0.$$

We combine this fact with the first and second propositions and 6.2 to check that the hypotheses of 6.1 and 6.8 are satisfied. The third proposition follows at once from Theorem 6.8.

FOURTH PROPOSITION. *If $S \in G_{k+1}$ and $\eta > 0$, then*

$$Q_{n-k}^+[(S; R), \eta] = D \cap \bigcap_x \{f(x) \in K[S_1, s(\eta)]\}.$$

Consider the inequalities

- (1) $\left\{ \sum_{i=n-k}^n [(x - a) \cdot R_i]^2 \right\}^{1/2} < [(n - k)\eta^2 + 1]^{1/2} [(x - a) \cdot (S; R)_{n-k}],$
- (2) $\sum_{i=n-k}^n [(x - a) \cdot R_i]^2 > 0.$
- (3) $|f(x) - S_1| < s(\eta).$

From Part 4 we know that (1) is equivalent to the statement " $x \in Q_{n-k}^+[(S; R), \eta]$," and (2) is equivalent to the statement " $x \in D$."

Now evidently (1) implies (2).

Hence we can prove the fourth proposition by showing that (1) and (3) are equivalent, provided that (2) holds.

Consequently we assume (2).

We infer, from Part 4, that

$$\begin{aligned} (x - a) \cdot (S; R)_{n-k} &= \left\{ \sum_{i=1}^n [(x - a) \cdot R_i] R_i \right\} \cdot \left\{ \sum_{j=1}^{k+1} S_1^j R_{n-k-1+j} \right\} \\ &= \sum_{j=1}^{k+1} [(x - a) \cdot R_{n-k-1+j}] S_1^j \\ &= [S_1 \cdot f(x)] \left\{ \sum_{i=n-k}^n [(x - a) \cdot R_i]^2 \right\}^{1/2}, \end{aligned}$$

and, from the relation $|S_1| = |f(x)| = 1$, we see that

$$|f(x) - S_1|^2 = 2[1 - S_1 \cdot f(x)].$$

Consequently (1) is equivalent to the inequalities

$$\begin{aligned} 1 &< [(n - k)\eta^2 + 1]^{1/2} [S_1 \cdot f(x)], \\ 1 - [S_1 \cdot f(x)] &< 1 - [(n - k)\eta^2 + 1]^{-1/2}, \\ |f(x) - S_1|^2 &< [s(\eta)]^2, \end{aligned}$$

which is equivalent to (3).

This proves the fourth proposition.

FIFTH PROPOSITION. *If $S \in G_{k+1}$, then*

$$E_x [f(x) = S_1] \subset D \cap \square_n^{n-k} [(*S:R), a].$$

Use the fourth proposition and Part 4 to verify that

$$\begin{aligned} E_x [f(x) = S_1] &\subset \bigcap_{\eta > 0} (D \cap E_x \{f(x) \in K[S_1, s(\eta)]\}) \\ &= \bigcap_{\eta > 0} \{D \cap Q_{n-k}^+ [(*S:R), \eta]\} \\ &\subset D \cap E_x \left\{ \sum_{i=n-k+1}^n [(x - a) \cdot (*S:R)_i]^2 = 0 \right\} \\ &= D \cap \square_n^{n-k} [(*S:R), a]. \end{aligned}$$

SIXTH PROPOSITION. *There is a function g on \mathfrak{S} such that*

$$h_{n-k}^+ (*S:R) = g(S_1) \quad \text{for } S \in G_{k+1}.$$

To prove this, we observe that $h_{n-k}^+ (*S:R) = 1$ or 0 according to whether or not the following three conditions are satisfied:

- (1) $\limsup_{\eta \rightarrow 0^+} \left\{ \limsup_{r \rightarrow 0^+} \rho_{n-k}^+ [(*S:R), \eta, r] \right\} \geq t/2(n - k),$
- (2) a is not a clusterpoint of $\{A \cap \square_n^{n-k} [(*S:R), a]\},$
- (3) $\limsup_{(\eta, r) \rightarrow (0, 0)} \alpha(k)^{-1} (\eta r)^{-k} \phi \{A \cap \diamond_n^{n-k} [(*S:R), \eta, a] \cap K_a^r\} < \infty.$

Now Part 4 implies that the sets

$$Q_{n-k}^+ [(*S:R), \eta], \quad \square_n^{n-k} [(*S:R), a], \quad \diamond_n^{n-k} [(*S:R), \eta, a]$$

are completely determined by R, η, r and S_1 . But R is fixed throughout Part 6, and the variables “ η ” and “ r ” are bound in (1) and (3). From this the sixth proposition follows at once.

SEVENTH PROPOSITION. *If $y \in \mathfrak{S}$, then*

$$\begin{aligned} \lim_{\eta \rightarrow 0^+} \frac{\alpha(k)\eta^k}{\mathfrak{C}_{k+1}^k \{c[y, s(\eta)]\}} &= (n - k)^{-k/2}, \\ \lim_{\eta \rightarrow 0^+} \frac{\alpha(k)\eta^k}{\mathfrak{C}_{k+1}^k \{c[y, s(\eta)/2]\}} &= 2^k(n - k)^{-k/2}. \end{aligned}$$

It is easy to verify that

$$\lim_{\eta \rightarrow 0} \eta/s(\eta) = (n - k)^{-1/2}.$$

Furthermore it was shown in 6.2 that

$$\lim_{r \rightarrow 0} \mathfrak{C}_{k+1}^k \{c[y, r]\} / [\alpha(k)r^k] = 1.$$

From these two relations the seventh proposition follows at once.

EIGHTH PROPOSITION. *If $g(y) > 0$, then $y \in [T - (H \cup Z)]$.*

We select $S \in G_{k+1}$ so that $y = S_1$. Then

$$h_{n-k}^+(*S:R) = g(y) = 1,$$

hence $(*S:R) \in F_{n-k}^+$, and we use the seventh and fourth proposition to infer

$$\begin{aligned} \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} p(y, \eta, r) \right\} &\geq \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} \frac{\psi_r[S \cap K(y, \eta)]}{\mathfrak{C}_{k+1}^k[c(y, \eta)]} \right\} \\ &= \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} \frac{\psi_r\{S \cap K[S_1, s(\eta)]\}}{\mathfrak{C}_{k+1}^k\{c[y, s(\eta)]\}} \right\} \\ &= \limsup_{\eta \rightarrow 0+} \left\{ \frac{\alpha(k)\eta^k}{\mathfrak{C}_{k+1}^k\{c[y, s(\eta)]\}} \limsup_{r \rightarrow 0+} \frac{\psi_r\{S \cap K[S_1, s(\eta)]\}}{\alpha(k)\eta^k} \right\} \\ &= (n - k)^{-k/2} \limsup_{\eta \rightarrow 0+} \left\{ \limsup_{r \rightarrow 0+} \rho_{n-k}^+[*S:R, \eta, r] \right\} \\ &\geq t/2(n - k)^{k/2+1} > t/2(n - k + 1)^{k+1}. \end{aligned}$$

Consequently $y \in T$.

Furthermore $(*S:R) \in W_n^k(\phi, A, a)$, and we again use the fourth and seventh propositions, as well as Part 5, to conclude

$$\begin{aligned} \limsup_{(\eta,r) \rightarrow (0,0)} p(y, \eta, r) &= \limsup_{(\eta,r) \rightarrow (0,0)} \frac{\psi_r[c(y, \eta)]}{\mathfrak{C}_{k+1}^k[c(y, \eta)]} \leq \limsup_{(\eta,r) \rightarrow (0,0)} \frac{\psi_r[S \cap K(S_1, 2\eta)]}{\mathfrak{C}_{k+1}^k[c(y, \eta)]} \\ &= \limsup_{(\eta,r) \rightarrow (0,0)} \frac{\psi_r\{S \cap K[S_1, s(\eta)]\}}{\mathfrak{C}_{k+1}^k\{c[y, s(\eta)/2]\}} \\ &= \limsup_{(\eta,r) \rightarrow (0,0)} \frac{\alpha(k)\eta^k}{\mathfrak{C}_{k+1}^k\{c[y, s(\eta)/2]\}} \rho_{n-k}^+[*S:R, \eta, r] \\ &\leq 2^k(n - k)^{-k/2} \limsup_{(\eta,r) \rightarrow (0,0)} \frac{\phi\{A \cap \diamond_n^{n-k}[*S:R, (n - k)^{1/2}\eta, a] \cap K_a^r\}}{\alpha(k)\eta^k r^k} \\ &= 2^k \limsup_{(\eta,r) \rightarrow (0,0)} \nabla_n^k[\phi, A, (*S:R), \eta, r, a] < \infty. \end{aligned}$$

Consequently $y \notin H$.

Last we use the fact that $(*S:R) \in V(A, a)$ to prove that $y \in Z$. Suppose, in fact, that $y \in Z$. Then

$$y \in Z_j = Y_{1/j} \quad \text{for } j = 1, 2, 3, \dots,$$

in virtue of the second proposition. Hence the definition of $Y_{1/j}$ implies the existence of points x^1, x^2, x^3, \dots such that

$$x_j \in A \cap K(a, 1/j), \quad f(x^j) = y = S_1,$$

for $j = 1, 2, 3, \dots$. Hence

$$\lim_{j \rightarrow \infty} x^j = a,$$

while the fifth proposition implies

$$x^j \in A \cap \square_n^{n-k}[*S:R), a] \quad \text{for } j = 1, 2, 3, \dots$$

This contradicts the fact that $(*S:R) \in V(A, a)$.

Consequently $y \notin Z$.

The proof of the eighth proposition is complete.

NINTH PROPOSITION. $\int_{G_{k+1}} h_{n-k}^+(*S:R) d\phi_{k+1}S = 0$.

Use the sixth proposition, 5.5, the eighth and the third proposition to check that

$$\begin{aligned} \int_{G_{k+1}} h_{n-k}^+(*S:R) d\phi_{k+1}S &= \int_{G_{k+1}} g(S_1) d\phi_{k+1}S \\ &= \int_S g(y) d\mathcal{H}_{k+1}^k y \leq \mathcal{H}_{k+1}^k [T - (H \cup Z)] = 0. \end{aligned}$$

This completes the proof of the ninth proposition, and of Part 6.

Part 7. $\int_{G_n} h_{n-k}^+(R) d\phi_n R = 0$.

Proof. Since $S \in G_{k+1}$ implies

$$\int_{G_n} h_{n-k}^+(R) d\phi_n R = \int_{G_n} h_{n-k}^+(*S:R) d\phi_n R,$$

we may use Part 6 to infer

$$\begin{aligned} \int_{G_n} h_{n-k}^+(R) d\phi_n R &= \int_{G_{k+1}} \int_{G_n} h_{n-k}^+(*S:R) d\phi_n R d\phi_{k+1}S \\ &= \int_{G_n} \int_{G_{k+1}} h_{n-k}^+(*S:R) d\phi_{k+1}S d\phi_n R = 0. \end{aligned}$$

The interchange of the order of integration is justified because $h_{n-k}^+(*S:R)$

is Borel measurable with respect to (S, R) on the product space $(G_{k+1} \times G_n)$.

Part 8. $\phi_n \{B - [V_n^k(A, a) \cup W_n^k(\phi, A, a)]\} = 0$.

Proof. It follows from Part 2 that

$$B - [V_n^k(A, a) \cup W_n^k(\phi, A, a)] \subset \bigcup_{j=1}^{n-k} \{F_j^+ - [V_n^k(A, a) \cup W_n^k(\phi, A, a)]\} \\ \cup \bigcup_{j=1}^{n-k} \{F_j^- - [V_n^k(A, a) \cup W_n^k(\phi, A, a)]\},$$

and we may use the Parts 3 and 7 to conclude

$$\phi_n \{B - [V_n^k(A, a) \cup W_n^k(\phi, A, a)]\} \leq \sum_{j=1}^{n-k} \int_{G_n} h_j^+(R) d\phi_n R + \sum_{j=1}^{n-k} \int_{G_n} h_j^-(R) d\phi_n R \\ = 2(n - k) \int_{G_n} h_{n-k}^+(R) d\phi_n R = 0.$$

An immediate consequence of 7.3 is the following theorem.

7.4 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}'_n, A \in \mathcal{G}_n$ (more generally, $A \in \mathcal{A}_n$),
- (iii) $x \in E_n$ and $\mathcal{O}_n^k(\phi, A, R, x) > 0$ for ϕ_n almost all R in G_n ,

then

$$\phi_n \{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} = 0.$$

7.5 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) A is a G_δ subset of $E_n, \mathcal{I}_n^k(A) = 0$,
- (iii) $a \in E_n$ and

$$Z = G_n \cap \frac{E}{R} [\{A \cap \square_n^{n-k}(R, a) - \text{sng } a\} \neq \emptyset],$$

then

$$\phi_n(Z) = 0.$$

Proof. We denote

$$L_{j,m}^+(R) = A \cap \square_n^{n-k}(R, a) \cap \frac{E}{z} [(x - a) \cdot R_j > m^{-1}],$$

$$L_{j,m}^-(R) = A \cap \square_n^{n-k}(R, a) \cap \frac{E}{z} [(x - a) \cdot R_j < -m^{-1}],$$

whenever $R \in G_n, m$ is a positive integer, and j is an integer between 1 and $n - k$. For all such m and j we further define

$$\Lambda_{j,m}^+ = G_n \cap \frac{E}{R} [L_{j,m}^+(R) \neq \emptyset],$$

$$\Lambda_{j,m}^- = G_n \cap E_R [L_{j,m}^-(R) \neq 0],$$

and let $\lambda_{j,m}^+$ and $\lambda_{j,m}^-$ be the characteristic functions of the sets $\Lambda_{j,m}^+$ and $\Lambda_{j,m}^-$.

We define I and $*S$, for $S \in G_{k+1}$, as in the proof of Theorem 7.3, and divide the remainder of the argument into six parts.

Part 1. $Z \subset \bigcup_{m=1}^\infty \bigcup_{j=1}^{n-k} (\Lambda_{j,m}^+ \cup \Lambda_{j,m}^-)$.

Proof. The relation $x \in \{A \cap \square_n^{n-k}(R, a) - \text{sng } a\}$ implies

$$0 < |x - a|^2 = |P_R^{n-k}(x - a)|^2 = \sum_{i=1}^{n-k} [(x - a) \cdot R_i]^2,$$

hence

$$x \in \bigcup_{m=1}^\infty \bigcup_{j=1}^{n-k} [L_{j,m}^+(R) \cup L_{j,m}^-(R)].$$

This proves Part 1.

Part 2. If m is a positive integer and j is a positive integer between 1 and $n - k$, then $\Lambda_{j,m}^+$ is an analytic subset of G_n .

Proof. The subset

$$(G_n \times E_n) \cap E_{(R,x)} [x \in L_{j,m}^+(R)]$$

of the product space $(G_n \times E_n)$ is readily seen to be of class G_δ . Hence its projection

$$\Lambda_{j,m}^+ = G_n \cap E_R [x \in L_{j,m}^+(R) \text{ for some } x \text{ in } E_n]$$

is an analytic subset of G_n .

Part 3. If u and v are integers between 1 and $(n - k)$, and m is a positive integer, then

$$\int_{G_n} \lambda_{u,m}^+(R) d\phi_n R = \int_{G_n} \lambda_{v,m}^-(R) d\phi_n R.$$

Proof. Define U as in 7.3, Part 3, and check that

$$L_{u,m}^+(R) = L_{v,m}^-(U:R) \quad \text{for } R \in G_n.$$

It follows that

$$\lambda_{u,m}^+(R) = \lambda_{v,m}^-(U:R) \quad \text{for } R \in G_n.$$

Consequently

$$\int_{G_n} \lambda_{u,m}^+(R) d\phi_n R = \int_{G_n} \lambda_{v,m}^-(U:R) d\phi_n R = \int_{G_n} \lambda_{v,m}^-(R) d\phi_n R.$$

Part 4. If $R \in G_n$ and m is a positive integer, then

$$\int_{G_{k+1}} \lambda_{n-k,m}^+(*S:R) d\phi_{k+1} S = 0.$$

Proof. Define \mathfrak{S} and f as in 7.3, Part 6, and let

$$B = A \cap E_x \left[\sum_{i=n-k}^n [(x - a) \cdot R_i]^2 > m^{-2} \right].$$

The remainder of the proof of Part 4 consists in the consideration of four auxiliary propositions.

FIRST PROPOSITION. $\mathfrak{I}_{k+1}^k [f^*(B)] = 0.$

If x and y are points of B , then

$$f(x) = \frac{\xi}{|\xi|}, \quad f(y) = \frac{\eta}{|\eta|},$$

where

$$\begin{aligned} \xi &= [(x - a) \cdot R_{n-k}, \dots, (x - a) \cdot R_n] \in E_{k+1}, \\ \eta &= [(y - a) \cdot R_{n-k}, \dots, (y - a) \cdot R_n] \in E_{k+1}. \end{aligned}$$

Consequently

$$\begin{aligned} |f(x) - f(y)| &= \frac{||\eta|\xi - |\xi|\eta||}{|\xi| \cdot |\eta|} = \frac{||\eta|(\xi - \eta) + \eta(|\eta| - |\xi|)|}{|\xi| \cdot |\eta|} \\ &\leq \frac{2|\xi - \eta|}{|\xi|} \leq 2m|\xi - \eta| \leq 2m|x - y|. \end{aligned}$$

Therefore it follows from 5.2 that

$$\mathfrak{I}_{k+1}^k [f^*(B)] \leq (2m)^k \mathfrak{I}_n^k(B) \leq (2m)^k \mathfrak{I}_n^k(A) = 0.$$

This proves the first proposition.

SECOND PROPOSITION. *There is a function F on \mathfrak{S} such that*

$$\lambda_{n-k,m}^+(*S:R) = F(S_1) \quad \text{for } S \in G_{k+1}.$$

From 7.3, Part 4, we infer that the set $L_{n-k,m}^+(*S:R)$ is completely determined by R and S_1 . But R is fixed throughout Part 4 of the present proof. Hence the second proposition is established.

THIRD PROPOSITION. *If $F(y) > 0$, then $y \in f^*(B)$.*

Select $S \in G_{k+1}$ so that $y = S_1$. Then

$$\lambda_{n-k,m}^+(*S:R) = F(S_1) = 1,$$

and we can find a point

$$x \in L_{n-k,m}^+(*S:R).$$

From 7.3, Part 4, we see that $u = 1, 2, \dots, k+1$ implies

$$\begin{aligned} (x - a) \cdot R_{n-k-1+u} &= \sum_{i=1}^{n-k} [(x - a) \cdot (*S:R)_i] [(*S:R)_i \cdot R_{n-k-1+u}] \\ &= [(x - a) \cdot (*S:R)_{n-k}] \sum_{j=1}^{k+1} S_1^j (R_{n-k-1+j} \cdot R_{n-k-1+u}) \\ &= | (x - a) \cdot (*S:R)_{n-k} | S_1^u, \end{aligned}$$

because $(x - a) \cdot (*S:R)_{n-k} > m^{-1} > 0$.

It follows that

$$\sum_{u=1}^{k+1} [(x - a) \cdot R_{n-k-1+u}]^2 = [(x - a) \cdot (*S:R)_{n-k}]^2 > m^{-2},$$

hence $x \in B$, and that $f(x) = S_1$.

Consequently $y = S_1 \in f^*(B)$.

FOURTH PROPOSITION. $\int_{G_{k+1}} \lambda_{n-k,m}^+(*S:R) d\phi_{k+1} S = 0$.

With the help of the first three propositions, and of 5.5, we compute

$$\begin{aligned} \int_{G_{k+1}} \lambda_{n-k,m}^+(*S:R) d\phi_{k+1} S &= \int_{G_{k+1}} F(S_1) s d\phi_{k+1} S \\ &= \int_S F(y) d\mathfrak{C}_{k+1}^k y \leq \mathfrak{C}_{k+1}^k [f^*(B)] = 0. \end{aligned}$$

The proof of Part 4 is complete.

Part 5. If m is a positive integer, then

$$\int_{G_n} \lambda_{n-k,m}^+(R) d\phi_n R = 0.$$

Proof. We use Part 2 and Part 4 to check that

$$\begin{aligned} \int_{G_n} \lambda_{n-k,m}^+(R) d\phi_n R &= \int_{G_{k+1}} \int_{G_n} \lambda_{n-k,m}^+(*S:R) d\phi_n R d\phi_{k+1} S \\ &= \int_{G_n} \int_{G_{k+1}} \lambda_{n-k,m}^+(*S:R) d\phi_{k+1} S d\phi_n R = 0. \end{aligned}$$

Part 6. $\phi_n(Z) = 0$.

Proof. From Part 1, Part 3 and Part 5 we infer that

$$\begin{aligned} \phi_n(Z) &\leq \sum_{m=1}^{\infty} \sum_{j=1}^{n-k} \int_{G_n} [\lambda_{j,m}^+(R) + \lambda_{j,m}^-(R)] d\phi_n R \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{n-k} 2 \int_{G_n} \lambda_{n-k,m}^+(R) d\phi_n R = 0. \end{aligned}$$

The proof is complete.

7.6 REMARK. In the proof of 7.5 we found it necessary to make use of the theory of analytic sets. Since it is the only essential part of the paper in which we are at present unable to avoid this theory, it would be particularly interesting if another, simpler proof were discovered for Theorem 7.5.

7.7 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}'_n, A \subset E_n, \phi(A) < \infty,$
- (iii) $B \subset A, B \in \mathcal{B}_n, \phi(A - B) = 0,$
- (iv) $\mathfrak{C}_n^k(\phi, A, x) > 0$ for $x \in A,$

then $x \in E_n$ implies

$$\phi_n[V_n^k(A, x) - V_n^k(B, x)] = 0, \quad W_n^k(\phi, B, x) = W_n^k(\phi, A, x).$$

Proof. From (iii) and 3.3 we infer that

$$\begin{aligned} \mathfrak{C}_n^k[\phi, (A - B), x] &= 0 && \text{for } x \in E_n, \\ \mathfrak{C}_n^k[\phi, (A - B), x] &= \mathfrak{C}_n^k[\phi, A, x] && \text{for } \mathfrak{C}_n^k \text{ almost all } x \text{ in } (A - B). \end{aligned}$$

Hence (iv) assures us that

$$\mathfrak{C}_n^k(A - B) = 0.$$

Now clearly

$$[V_n^k(A, x) - V_n^k(B, x)] \subset G_n \cap \frac{E}{\mathbb{R}} [\{(A - B) \cap \square_n^{n-k}(R, x) - \text{sng } x\} \neq \emptyset],$$

and the first statement of our present theorem follows from 7.5, because every set of \mathfrak{C}_n^k measure zero is contained in a G_δ of \mathfrak{C}_n^k measure zero.

Since the second statement of our theorem is obvious, the proof is complete.

8. **Projection properties.** We follow Besicovitch's lead in linking the projection properties of sets with their local geometric characteristics. Again our methods were suggested by [BE3].

8.1 LEMMA. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}'_n, A \subset E_n, \phi(A) < \infty,$
- (iii) $R \in G_n, S \in G_n, S_i = R_{n-i+1}$ for $i = 1, 2, \dots, k,$

(iv) $B = A \cap E_x [R \in W_n^k(\phi, A, x)],$

then

$$\mathcal{L}_k [P_S^{k*}(B)] = 0.$$

Proof. Let ψ be the function on the set of all subsets of E_k such that

$$\psi(Y) = \phi \{ A \cap E_x [P_S^k(x) \in Y] \} \quad \text{for } Y \subset E_k.$$

From 7.1 we infer that $\psi \in \mathcal{U}_k'$. Hence the standard theorem on differentiation with respect to Lebesgue measure (see [SS1, 5.4] or [M1, 8.12]) assures us that

$$(1) \quad \limsup_{t \rightarrow 0^+} \frac{\psi [K(y, t)]}{\mathcal{L}_k [K(y, t)]} < \infty \quad \text{for } \mathcal{L}_k \text{ almost all } y \text{ in } E_k.$$

Now suppose $x \in B, y = P_S^k(x)$. Lemma 4.2 implies

$$[\diamond_n^{n-k}(R, \eta, x) \cap K(x, r)] \subset E_x [P_S^k(z) \in K(y, \eta r)]$$

whenever $\eta > 0, r > 0$. Consequently

$$\begin{aligned} \infty &= \limsup_{(\eta, r) \rightarrow (0, 0)} \nabla_n^k(\phi, A, R, \eta, r, x) \\ &\leq \limsup_{(\eta, r) \rightarrow (0, 0)} \frac{\psi [K(y, \eta r)]}{\mathcal{L}_k [K(y, \eta r)]} = \limsup_{t \rightarrow 0^+} \frac{\psi [K(y, t)]}{\mathcal{L}_k [K(y, t)]}. \end{aligned}$$

Since y is an arbitrary point of the set $P_S^{k*}(B)$, we infer from (1) that $\mathcal{L}_k [P_S^{k*}(B)] = 0$.

8.2 LEMMA. *If*

(i) $k < n$ are positive integers

(ii) A is an \mathcal{H}_n^k measurable subset of $E_n, \mathcal{H}_n^k(A) < \infty, R \in G_n,$

then

$$\int_{E_k} N(P_R^k, A; y) d\mathcal{L}_k y \leq \mathcal{H}_n^k(A).$$

Proof. We use the notation of [F1, 2.4].

Let F be the family of all those subsets of A which are the union of a compact set and a set of \mathcal{H}_n^k measure zero. Let I be the identity map of E_n . It follows from 5.2 and 5.6 that

$$\mathcal{L}_k [P_R^{k*}(S)] \leq \mathcal{H}_n^k [I^*(S)] \quad \text{for } S \in F.$$

Consequently (see [F1, 2.4])

$$V_F(P_R^k, A, \mathcal{L}_k) \leq V_F(I, A, \mathfrak{C}_n^k).$$

Hence [F1, 4.1] implies

$$\int_{E_k} N(P_R^k, A, y) d\mathcal{L}_k y \leq \int_{E_n} N(I, A, x) d\mathfrak{C}_n^k x = \mathfrak{C}_n^k(A).$$

The proof is complete.

8.3 LEMMA. *If*

- (i) $k < n$ are positive integers,
- (ii) A is an \mathfrak{C}_n^k measurable subset of E_n , $\mathfrak{C}_n^k(A) < \infty$,
- (iii) $R \in G_n, S \in G_n, S_i = R_{n-i+1}$ for $i = 1, 2, \dots, k$,
- (iv) $B = A \cap E_x [R \in V_n^k(A, x)]$,

then

$$\mathcal{L}_k [P_S^{k*}(B)] = 0.$$

Proof. From 8.2 we know that

$$(1) \quad N(P_S^k, A, y) < \infty \quad \text{for } \mathcal{L}_k \text{ almost all } y \text{ in } E_k.$$

Now suppose $x \in B, y = P_S^k(x)$. Then

$$\square_n^{n-k}(R, x) = E_x [P_S^k(z) = y],$$

which implies $N(P_S^k, A, y) = \infty$.

Since y was an arbitrary point of the set $P_S^{k*}(B)$, we infer from (1) that $\mathcal{L}_k [P_S^{k*}(B)] = 0$.

8.4 REMARK. The Theorems 8.2 and 8.3 remain true if \mathfrak{C}_n^k is replaced by any $\phi \in \mathcal{U}_n'$ for which

$$\phi(S) \geq \mathcal{L}_k [P_R^{k*}(S)] \quad \text{whenever } S \subset E_n, R \in G_n.$$

For instance $\mathfrak{S}_n^k, \mathfrak{C}_n^k, \Phi_n^k, \Gamma_n^k$ are such measures, while \mathfrak{Y}_n^k is not.

8.5 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n', A \in \mathfrak{G}_n$ (more generally, $A \in \mathfrak{A}_n$), $\phi(A) < \infty, B \in \mathfrak{B}_n, B \subset A$,
- (iii) $\lambda > 0$ and $\mathfrak{C}_n^k(\phi, A, x) \geq \lambda$ for $x \in B$,
- (iv) $\phi_n \{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} = 0$ for ϕ almost all x in B ,

then

$$\mathcal{L}_k [P_R^{k*}(B)] = 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

Proof. Let

$$V = (G_n \times B) \cap E_{(R,x)} [R \in V_n^k(A, x)],$$

$$W = (G_n \times B) \cap \bigcap_{(R,x)} E [R \in W_n^k(\phi, A, x)].$$

The sets V and W are, as indicated in 7.2 and 3.4, Borel sets (or, more generally, analytic sets) of the cartesian product space $(G_n \times B)$. Hence they are measurable with respect to the product measure of ϕ_n and ϕ , and the Fubini theorem applies to their characteristic functions. Since (iv) assures us that

$$\phi_n \left\{ G_n - \int_R E [(R, x) \in (V \cup W)] \right\} = 0 \quad \text{for } \phi \text{ almost all } x \text{ in } B,$$

we may conclude that

$$\phi \left\{ B - \int_x E [(R, x) \in (V \cup W)] \right\} = 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

Now define the matrix $U \in G_n$ by the relation

$$U_i = I_{n-i+1} \quad \text{for } i = 1, 2, \dots, n,$$

where I is the identity matrix of G_n , and let Z be the set of all those orthogonal transformations $R \in G_n$ for which

$$\phi \left\{ B - \int_x E [R \in V_n^k(A, x) \cup W_n^k(\phi, A, x)] \right\} = 0.$$

We know that $\phi_n(G_n - Z) = 0$, and ϕ_n is the Haar measure of G_n . Consequently

$$\phi_n \left\{ G_n - \int_R E [(U:R) \in Z] \right\} = 0,$$

and we complete the proof of the theorem by verifying the following statement.

STATEMENT. *If $(U:R) \in Z$, then $\mathcal{L}_k [P_R^{k*}(B)] = 0$.*

Proof. Let

$$B_1 = B \cap \int_x E [(U:R) \in V_n^k(A, x)],$$

$$B_2 = B \cap \int_x E [(U:R) \in W_n^k(\phi, A, x)],$$

and check that

$$(U:R)_i = \bar{R}(U_i) = \bar{R}(I_{n-i+1}) = R_{n-i+1} \quad \text{for } i = 1, 2, \dots, n.$$

Since $\phi(B_2) < \infty$, Lemma 8.1 implies

$$(1) \quad \mathcal{L}_k [P_R^{k*}(B_2)] = 0.$$

Next $\phi(A) < \infty$, hence (iii) and 3.1 imply $\mathcal{H}_n^k(A) < \infty$, and we use 8.3 to conclude that

$$(2) \quad \mathcal{L}_k [P_R^{k*}(B_1)] = 0.$$

Finally $\phi[B - (B_1 \cup B_2)] = 0$, hence (iii), 3.3, 3.1 imply $\mathfrak{H}_n^k[B - (B_1 \cup B_2)] = 0$, and 8.2 assures us that

$$(3) \quad \mathcal{L}_k\{P_R^{k*}[B - (B_1 \cup B_2)]\} = 0.$$

Combining (1), (2), and (3), we see that the proof is complete.

8.6 REMARK. If the measure ϕ satisfies the condition

$$\phi(S) \geq \mathcal{L}_k[P_R^{k*}(S)] \quad \text{whenever } S \subset E_n, R \in G_n,$$

then the hypothesis (iii) of Theorem 8.5 can be replaced by the weaker assumption

$$\mathfrak{O}_n^k(\phi, A, x) > 0 \quad \text{for } x \in B.$$

Only slight changes are required in the proof. In fact our weakened assumption is still sufficient, in conjunction with 3.3 and 3.1, to assure us that $\mathfrak{H}_n^k[B - (B_1 \cup B_2)] = 0$, while the relation $\mathcal{L}_k[P_R^{k*}(B_1)] = 0$ follows from 8.4.

8.7 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n', A \subset E_n, \phi(A) < \infty, B \in \mathfrak{B}_n,$
- (iii) $\mathfrak{O}_n^k(\phi, A, x) > 0$ for $x \in B,$
- (iv) $\mathfrak{O}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in $B,$
- (v) for ϕ almost all x in B we have

$$\mathfrak{O}_n^k(\phi, A, R, x) > 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n,$$

then

$$\mathcal{L}_k[P_R^{k*}(A \cap B)] = 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

Proof. For each positive integer m let

$$B_m = B \cap E \left[m^{-1} \leq \mathfrak{O}_n^k(\phi, A, x) \leq m \right],$$

and use 3.3 with [MR1, 3.6] to secure a set $C_m \in \mathfrak{G}_n$ for which $C_m \subset B_m,$ $\mathfrak{H}_n^k(B_m - C_m) = 0$ and

$$\mathfrak{O}_n^k(\phi, A - B_m, x) = 0 \quad \text{for } x \in C_m.$$

From (iv) and 3.6 we see that $\phi(A \cap B_m - C_m) = 0$, hence

$$\mathfrak{O}_n^k(\phi, A - C_m, x) = 0 \quad \text{for } x \in C_m,$$

which implies

$$\mathfrak{O}_n^k(\phi, A \cap C_m, x) \geq m^{-1} \quad \text{for } x \in C_m,$$

and for ϕ almost all x in C_m we have

$$\mathcal{O}_n^k(\phi, C_m, R, x) > 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

It follows from 7.4 that

$$\phi_n \{G_n - [V_n^k(C_m, x) \cup W_n^k(\phi, C_m, x)]\} = 0$$

for ϕ almost all x in C_m .

Hence 8.5 implies

$$\mathcal{L}_k[P_R^{k*}(C_m)] = 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

Letting

$$D = B \cap E_x [\mathcal{O}_n^k(\phi, A, x) = \infty],$$

we see that $\phi(D) = 0$, hence $\mathcal{H}_n^k(D) = 0$ in virtue of 3.1.

Consequently

$$\begin{aligned} \mathcal{L}_k[P_R^{k*}(A \cap B)] &\leq \sum_{m=1}^{\infty} \{ \mathcal{L}_k[P_R^{k*}(C_m)] + \mathcal{H}_n^k(B_m - C_m) \} + \mathcal{H}_n^k(D) \\ &= \sum_{m=1}^{\infty} \mathcal{L}_k[P_R^{k*}(C_m)] = 0 \end{aligned}$$

for ϕ_n almost all R in G_n .

The proof is complete.

8.8 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n, A \in \mathcal{B}_n, \phi(A) > 0$,
- (iii) $\mathcal{O}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A ,
- (iv) $\mathcal{L}_k[P_R^{k*}(A)] = 0$ for ϕ_n almost all R in G_n ,

then A is positively (ϕ, k) unrectifiable.

Proof. Otherwise A would contain a rectifiable set $B \in \mathcal{B}_n$ for which $\phi(B) > 0$.

From (iv) and 5.14 we infer that $\mathcal{H}_n^k(B) = 0$.

Hence (iii) and 3.6 imply $\phi(B) = 0$.

The proof is complete.

8.9 THEOREM. *If*

- (i) $k < n$ are positive integers,
- (ii) $\phi \in \mathcal{U}_n', A \subset E_n, \phi(A) < \infty$,
- (iii) $\mathcal{O}_n^k(\phi, A, x) < \infty$ for ϕ almost all x ,
- (iv) A is positively (ϕ, k) unrectifiable,

then A has a subset B for which $\phi(A - B) = 0$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2 \cdot 10^{2k})^{-1} \bar{\sigma}_n^k(\phi, A, x)$$

whenever $x \in B$, $R \in G_n$ and $0 < \eta < 1$.

Proof. Using the notation of 4.6, we denote

$$C = \bigcup_{R \in G_n} \bigcup_{0 < \eta < 1} \bigcup_{\mu > 0} \bigcup_{m=1}^{\infty} T(R, \eta, \mu, m).$$

From (iv) and Theorem 4.6 we infer that

$$\phi[A \cap T(R, \eta, \mu, m)] = 0 \quad \text{for } R \in D, \eta \in \rho', \mu \in \rho, m = 1, 2, 3, \dots,$$

hence $\phi(A \cap C) = 0$, and that $x \in (A - C)$ implies either

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2 \cdot 10^{2k})^{-1} \bar{\sigma}_n^k(\phi, A, x)$$

whenever $R \in G_n$ and $0 < \eta < 1$ or

$$\bar{\sigma}_n^k(\phi, A, x) = \infty.$$

Consequently the set

$$B = \{(A - C) - E_x [\bar{\sigma}_n^k(\phi, A, x) = \infty]\}$$

fills the requirements of the theorem.

9. Summary of the main general results. The results which are collected in this section reflect the basic differences in the geometric properties of (ϕ, k) rectifiable and unrectifiable sets.

9.1 THEOREM. *If $k < n$ are positive integers, $\phi \in \mathcal{U}_n'$, $A \subset E_n$, $\phi(A) < \infty$ and $\bar{\sigma}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then the following four propositions are equivalent:*

- (1) A is (ϕ, k) rectifiable,
- (2) A is (ϕ, k) restricted at ϕ almost all of its points,
- (3) corresponding to ϕ almost all x in A there is an $R \in G_n$ such that

$$\bar{\sigma}_n^k(\phi, A, R, x) = 0 < \bar{\sigma}_n^k(\phi, A, x),$$

- (4) corresponding to ϕ almost all x in A we can find R and η such that $R \in G_n$, $0 < \eta < 1$, and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) < (2 \cdot 10^{2k})^{-1} \bar{\sigma}_n^k(\phi, A, x).$$

Proof. (1) implies (2) by 5.8. (2) implies (3) by 2.29. (3) implies (4) by 2.25. (4) implies (1) by 4.7.

9.2 THEOREM. If $k < n$ are positive integers, $\phi \in \mathbb{U}'_n, A \in \mathfrak{B}_n, 0 < \phi(A) < \infty, \bar{\sigma}_n^k(\phi, A, x) > 0$ for $x \in A$, and $\bar{\sigma}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then the following four propositions are equivalent:

- (1) A is positively (ϕ, k) unrectifiable,
- (2) A has a subset B for which $\phi(A - B) = 0$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2 \cdot 10^{2k})^{-1} \bar{\sigma}_n^k(\phi, A, x)$$

whenever $x \in B, R \in G_n$ and $0 < \eta < 1$,

- (3) A has a subset B for which $\phi(A - B) = 0$ and $x \in B$ implies

$$\bar{\sigma}_n^k(\phi, A, R, x) > 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n,$$

- (4) $\mathcal{L}_k[P_R^{k*}(A)] = 0$ for ϕ_n almost all R in G_n .

Proof. (1) implies (2) by 8.9. (2) implies (3) by 2.25. (3) implies (4) by 8.7. (4) implies (1) by 8.8.

9.3 THEOREM. If $k < n$ are positive integers, $\phi \in \mathbb{U}'_n, A \in \mathfrak{B}_n, 0 < \phi(A) < \infty, \lambda > 0, \bar{\sigma}_n^k(\phi, A, x) > \lambda$ for $x \in A$, and $\bar{\sigma}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then the following five propositions are equivalent:

- (1) A is positively (ϕ, k) unrectifiable,
- (2) A has a subset B for which $\phi(A - B) = 0$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2 \cdot 10^{2k})^{-1} \bar{\sigma}_n^k(\phi, A, x),$$

whenever $x \in B, R \in G_n$ and $0 < \eta < 1$,

- (3) A has a subset B for which $\phi(A - B) = 0$ and $x \in B$ implies

$$\bar{\sigma}_n^k(\phi, A, R, x) > 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n,$$

- (4) $\phi_n \{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} = 0$ for ϕ almost all x in A ,

- (5) $\mathcal{L}_k[P_R^{k*}(A)] = 0$ for ϕ_n almost all R in G_n .

Proof. (3) implies (4) by 7.4. (4) implies (5) by 8.5.

Reference to 9.2 completes the proof.

9.4 THEOREM. If $k < n$ are positive integers, $\phi \in \mathbb{U}'_n$, every subset of E_n is contained in a Borel set of equal ϕ measure, A is a ϕ measurable subset of $E_n, 0 < \phi(A) < \infty, \lambda > 0, \bar{\sigma}_n^k(\phi, A, x) > \lambda$ for $x \in A$, and $\bar{\sigma}_n^k(\phi, A, x) < \infty$ for ϕ almost all x in A , then the propositions (1), (2), (3), (4), (5) of 9.3 are equivalent.

Proof. Using [MR1, 3.6] we find a set $B \in \mathfrak{S}_n$ for which $B \subset A$ and $\phi(A - B) = \mathfrak{I}_n^k(A - B) = 0$.

We use 7.7 to check that

$$\bar{\sigma}_n^k(\phi, A, x) = \bar{\sigma}_n^k(\phi, B, x),$$

$$\begin{aligned} \nabla_n^k(\phi, A, R, \eta, r, x) &= \nabla_n^k(\phi, B, R, \eta, r, x), \\ \varnothing_n^k(\phi, A, R, x) &= \varnothing_n^k(\phi, B, R, x), \\ \phi_n\{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} &= \phi_n\{G_n - [V_n^k(B, x) \cup W_n^k(\phi, B, x)]\}, \\ \mathcal{L}_k[P_R^{k*}(A)] &= \mathcal{L}_k[P_R^{k*}(B)], \end{aligned}$$

whenever $x \in E_n, R \in G_n, \eta > 0, r > 0$.

Hence B satisfies the hypothesis of Theorem 9.3, and each of the propositions (1), (2), (3), (4), (5) of 9.3 applied to B is equivalent to the same proposition applied to A . Since these propositions are equivalent for the set B , they must also be equivalent for the set A .

9.5 REMARK. If the measure ϕ satisfies the condition

$$\phi(S) \geq \mathcal{L}_k[P_R^{k*}(S)] \quad \text{whenever } S \subset E_n, R \in G_n,$$

then the hypothesis

$$\varnothing_n^k(\phi, A, x) > \lambda > 0 \quad \text{for } x \in A,$$

which occurs in 9.3 and 9.4, may be replaced by the weaker assumption

$$\varnothing_n^k(\phi, A, x) > 0 \quad \text{for } x \in A.$$

This follows from 8.6.

9.6 THEOREM. *If*

- (i) $k < n$ are positive integers,
 - (ii) $\phi \in \mathcal{U}_n'$, every subset of E_n is contained in a Borel set of equal ϕ measure, and $\phi(S) \geq \mathcal{L}_k[P_R^{k*}(S)]$ whenever $S \subset E_n, R \in G_n$,
 - (iii) A is a ϕ measurable subset of $E_n, \phi(A) < \infty$,
- then A can be decomposed into three ϕ measurable sets A_1, A_2, A_3 for which

$$A = A_1 \cup A_2 \cup A_3, \quad A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = 0,$$

and for which the following ten propositions hold:

- (1) $0 < \varnothing_n^k(\phi, A, x) < \infty$ for $x \in A_1$,
- (2) A_1 is (ϕ, k) rectifiable,
- (3) A is (ϕ, k) restricted at (ϕ, k) almost all points of A_1 ,
- (4) $\phi_n\{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} > 0$ for ϕ almost all x in A_1 ,
- (5) $0 < \varnothing_n^k(\phi, A, x) < \infty$ for $x \in A_2$,
- (6) either $\phi(A_2) = 0$ or A_2 is positively (ϕ, k) unrectifiable,
- (7) $x \in A_2$ implies

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2.10^{2k})^{-1} \varnothing_n^k(\phi, A, x),$$

whenever $R \in G_n$ and $0 < \eta < 1$,

- (8) $\phi_n \{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} = 0$ for ϕ almost all x in A_2 ,
- (9) $\mathcal{L}_k[P_R^{k*}(A_2)] = 0$ for ϕ_n almost all R in G_n ,
- (10) $x \in A_3$ implies that either $\odot_n^k(\phi, A, x) = 0$ or $\oslash_n^k(\phi, A, x) = \infty$.

Proof. Let A_1 be the set of all those points $x \in A$ for which $0 < \oslash_n^k(\phi, A, x) < \infty$ and we can find R and η such that $R \in G_n, 0 < \eta < 1$, and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) < (2.10^{2k})^{-1} \oslash_n^k(\phi, A, x).$$

Let A_2 be the set of all those points $x \in A$ for which $0 < \oslash_n^k(\phi, A, x) < \infty$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A, R, \eta, r, x) \geq (2.10^{2k})^{-1} \oslash_n^k(\phi, A, x)$$

whenever $R \in G_n$ and $0 < \eta < 1$.

Let A_3 be the set of all those points $x \in A$ for which either $\odot_n^k(\phi, A, x) = 0$ or $\oslash_n^k(\phi, A, x) = \infty$.

Clearly the statements (1), (5), (7) and (10) are true. The ϕ measurability of the sets A_1, A_2 and A_3 is a consequence of 3.4 and 4.6.

The remainder of the proof is divided into two parts.

I. Proof of (2), (3) and (4). According to Theorem 4.6 there is a set $B \in \mathfrak{B}_n$ for which $A_1 = (A \cap B)$. It follows from 3.3 and 3.9 that $\odot_n^k(\phi, A - A_1, x) = 0$ for ϕ almost all x in A_1 . Corresponding to each such x we can find R and η such that $R \in G_n, 0 < \eta < 1$, and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A_1, R, \eta, r, x) < (2.10^{2k})^{-1} \oslash_n^k(\phi, A_1, x).$$

Consequently 9.1 implies (2), and that A_1 is (ϕ, k) restricted at ϕ almost all points of A_1 . Hence (3) follows at once.

Next we choose a set $C \in \mathfrak{B}_n$ for which $A \subset C$ with $\phi(C) < \infty$, and let D be the set of all those points x for which

$$\phi_n \{G_n - [V_n^k(C, x) \cup W_n^k(\phi, C, x)]\} = 0.$$

Inferring from 3.4 and 7.2 that D is a ϕ measurable set, we select a set $F \in \mathfrak{F}_n$ for which $F \subset (A_1 \cap D)$ and $\phi[(A_1 \cap D) - F] = 0$. Then $x \in F$ implies

$$\oslash_n^k(\phi, C, x) \geq \oslash_n^k(\phi, A, x) > 0, \quad \phi_n \{G_n - [V_n^k(C, x) \cup W_n^k(\phi, C, x)]\} = 0.$$

Hence 8.6 assures us that

$$\mathcal{L}_k[P_R^{k*}(F)] = 0 \quad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

Using 9.5 we conclude that F has no rectifiable subset of positive ϕ measure. Inasmuch as F is a subset of the (ϕ, k) rectifiable set A_1 , this implies $\phi(F) = 0$. Hence $\phi(A_1 \cap D) = 0$ and we conclude that

$\phi_n \{G_n - [V_n^k(A, x) \cup W_n^k(\phi, A, x)]\} \geq \phi_n \{G_n - [V_n^k(C, x) \cup W_n^k(\phi, C, x)]\} > 0$
 for ϕ almost all x in A_1 . This proves (4).

II. **Proof of (6), (8), and (9).** Clearly $\mathcal{O}_n^k(\phi, A - A_2, x) = 0$ for \mathcal{H}_n^k and ϕ almost all x in A_2 . Hence (7) implies that A_2 has a subset B for which $\phi(A_2 - B) = 0$ and

$$\limsup_{r \rightarrow 0^+} \nabla_n^k(\phi, A_2, R, \eta, r, x) \geq (2 \cdot 10^{2k})^{-1} \mathcal{O}_n^k(\phi, A_2, x)$$

whenever $x \in B, R \in G_n, 0 < \eta < 1$.

Now 9.5 implies (6), (9), and the proposition

$$\phi_n \{G_n - [V_n^k(A_2, x) \cup W_n^k(\phi, A_2, x)]\} = 0$$

for ϕ almost all x in A_2 . Clearly (8) follows from the last relation.

The proof is complete.

9.7 **THEOREM.** *If $k < n$ are positive integers, A is an \mathcal{H}_n^k measurable subset of E_n , and $\mathcal{H}_n^k(A) < \infty$, then*

$$\mathcal{H}_n^k(A) \geq \beta(n, k)^{-1} \int_{G_n} \int_{E_n} N(P_R^k, A, y) d\mathcal{L}_k y d\phi_n R = \mathcal{J}_n^k(A),$$

and a necessary and sufficient condition for equality is that A be (\mathcal{H}_n^k, k) rectifiable.

Proof. Choose A_1, A_2 and A_3 in accordance with 9.6. Then we have

$$\begin{aligned} \mathcal{H}_n^k(A_1) &= \mathcal{J}_n^k(A_1), && \text{by 5.9 and 5.11,} \\ \mathcal{J}_n^k(A_2) &= 0, && \text{by (9) of 9.6,} \\ \mathcal{H}_n^k(A_3) &= 0, && \text{by 3.5 and 3.7.} \end{aligned}$$

Consequently

$$\begin{aligned} \mathcal{H}_n^k(A) &= \mathcal{H}_n^k(A_1) + \mathcal{H}_n^k(A_2), \\ \mathcal{J}_n^k(A) &= \mathcal{J}_n^k(A_1), \\ \mathcal{H}_n^k(A) &= \mathcal{J}_n^k(A) + \mathcal{H}_n^k(A_2). \end{aligned}$$

The theorem follows immediately from the last equation and the statements (2) and (6) of 9.6.

9.8 **REMARK.** Theorem 9.7 was proved for the special case $k = 1, n = 2$ by Sherman, whose treatment is based on Besicovitch's results on the projection properties of irregular plane sets of finite linear measure. (See [SH], [BE3].)

10. **The area of nonparametric 2-dimensional surfaces.** This section contains the complete solution of the problem of measure for nonparametric two-

dimensional surfaces in three space. The method is based on the general results of the preceding section, on a recently discovered ingenious technique of A. S. Besicovitch, and on previous work of the author. (See [BE5], [F1], [F4].)

10.1 LEMMA. *If $k < n$ are positive integers, $A \subset E_n$, $\psi = \Xi_n^\infty(\chi_n^k, \mathfrak{M}_n)$, then $\mathfrak{C}_n^k(\psi, A, x) \geq 2^{-k}$ for \mathfrak{C}_n^k almost all x in A .*

Proof. Suppose B is a bounded subset of A , $0 < \lambda < 2^{-k}$, $0 < \delta < \infty$ and

$$\psi(A \cap C_x^r) < \lambda \chi_n^k(C_x^r)$$

whenever $x \in B$, $0 < r < \delta$.

Now suppose $0 < \epsilon < \delta/5$ and $\phi = \Xi_n^\epsilon(\chi_n^k, \mathfrak{M}_n)$.

From the boundedness of B it follows that $\phi(B) < \infty$. In an attempt to prove that $\phi(B) = 0$ we assume the contrary, $\phi(B) > 0$, recall that $\lambda 2^k < 1$, and select a countable family $F \subset \mathfrak{M}_n$ for which $B \subset \sigma(F)$, $S \in F$ implies $\text{diam } S < \epsilon$ and $(S \cap B) \neq \emptyset$, and for which

$$\sum_{S \in F} \chi_n^k(S) < (\lambda 2^k)^{-1/2} \phi(B).$$

Next we choose G and H as in the proof of 3.6. Since

$$\begin{aligned} \psi(X) &= \phi(X) && \text{whenever } X \in \mathfrak{M}_n, \text{ diam } X < \epsilon, \\ 5 \text{ diam } S &< \epsilon && \text{for } S \in G, \end{aligned}$$

we have

$$\begin{aligned} \phi(B) &\leq \sum_{S \in G} \phi\{B \cap C[x(S), \text{diam } S]\} + \sum_{S \in H} \phi\{B \cap C[x(S), 5 \text{ diam } S]\} \\ &= \sum_{S \in G} \psi\{B \cap C[x(S), \text{diam } S]\} + \sum_{S \in H} \psi\{B \cap C[x(S), 5 \text{ diam } S]\} \\ &= \lambda 2^k \left\{ (\lambda 2^k)^{-1/2} \phi(B) + 5^k \sum_{S \in H} \chi_n^k(S) \right\}. \end{aligned}$$

By suitable choice of H we can make the second term as small as we please. Consequently $0 < \phi(B) \leq (\lambda 2^k)^{1/2} \phi(B) < \infty$, which is false.

This shows that $\phi(B) = 0$.

Recalling the arbitrary nature of ϵ we infer that

$$\{\Xi_n^\epsilon(\chi_n^k, \mathfrak{M}_n)\}(B) = 0 \quad \text{whenever } 0 < \epsilon < \delta/5.$$

Hence $\mathfrak{C}_n^k(B) = 0$.

The proof is complete.

10.2. *Sectional assumptions.* Throughout the remainder of §10 we fix a continuous numerically valued function f on E_2 , let \bar{f} be the function on E_2

to E_3 such that $\bar{f}(x) = [x_1, x_2, f(x)] \in E_3$ for $x \in E_2$, and let $A = \text{range } \bar{f}$.

We denote

$$x' = (x_1, x_2) \text{ for } x \in E_3, \quad S' = \bigcup_{x \in S} \text{sng } x' \text{ for } S \subset E_3,$$

and define $\psi = \Xi_3^\infty(\mathcal{X}_3^2, \mathcal{M}_3)$, $\psi' = \Xi_2^\infty(\mathcal{X}_2^1, \mathcal{M}_2)$.

We further let ξ^1 and ξ^2 be the functions on E_2 to E_2 such that

$$\xi^1(x) = [x_1, f(x)], \quad \xi^2(x) = [x_2, f(x)] \quad \text{for } x \in E_2,$$

and let η_1 and η_2 be the functions such that

$$\eta_1(T) = \int_{E_2} N(\xi^1, T, y) d\mathcal{L}_2 y, \quad \eta_2(T) = \int_{E_2} N(\xi^2, T, y) d\mathcal{L}_2 y,$$

whenever T is a subset of E_2 for which the integrand is \mathcal{L}_2 measurable with respect to y .

10.3 LEMMA. *If $S \subset E_3$, then*

$$\psi(S) \leq \pi \cdot (\text{diam } S) \cdot \psi'(S').$$

Proof. Consider any decomposition of the form

$$S' = \bigcup_{i=1}^\infty A_i.$$

For each positive integer i the set

$$S \cap E_{\frac{x}{i}} [x' \in A_i]$$

is contained in $s(i)$ cylindrical sets $B_i^1, B_i^2, \dots, B_i^{s(i)}$, such that

$$[s(i) - 1](\text{diam } A_i) < (\text{diam } S),$$

$$(\text{diam } B_i^j) = 2^{1/2}(\text{diam } A_i) \quad \text{for } j = 1, 2, \dots, s(i).$$

Consequently

$$\begin{aligned} \psi(S) &\leq \sum_{i=1}^\infty \sum_{j=1}^{s(i)} (\pi/4)(\text{diam } B_i^j)^2 \\ &= \sum_{i=1}^\infty s(i)(\pi/2)(\text{diam } A_i)^2 \\ &\leq (\pi/2) \sum_{i=1}^\infty [(\text{diam } S) + (\text{diam } A_i)](\text{diam } A_i) \\ &\leq \pi \cdot (\text{diam } S) \sum_{i=1}^\infty (\text{diam } A_i). \end{aligned}$$

Since the partition of S' was arbitrary, the proof is complete.

10.4 REMARK. *If S is a rectangle, then*

$$\eta_1(S) = H_1(f, S), \quad \eta_2(S) = H_2(f, S),$$

where H_1 and H_2 are defined as in [SS1, p. 174].

This follows from [F1, 5.4] and [F1, 6.2, Part 2].

10.5 REMARK. *If $S \in \mathfrak{B}_2$, then*

$$\eta_1(S) \leq \mathfrak{F}_3^2[\bar{f}^*(S)], \quad \eta_2(S) \leq \mathfrak{F}_3^2[\bar{f}^*(S)], \quad \mathcal{L}_2(S) \leq \mathfrak{F}_3^2[\bar{f}^*(S)].$$

If S is a rectangle, these inequalities follow from 10.4, [F1, 5.5] and [F4, 6.14]. Applying the usual limiting processes, we see that they hold for every Borel set S .

10.6 LEMMA. *If V and W are Borel measurable functions on E_1 to E_1 , $a < b$, $V(u) < W(u)$ for $a < u < b$, and*

$$S = E_2 \cap \underset{(u,v)}{E} [a < u < b, V(u) < v < W(u)],$$

then

$$\int_a^b [T_{v=V(u)}^{W(u)} \bar{f}(u, v)] d\mathcal{L}_1 u \leq 2\mathfrak{F}_3^2[\bar{f}^*(S)].$$

Proof. By the methods of [F1, 6.2, Parts 1 and 2] it may be readily seen that

$$\eta_1(S) = \int_a^b [T_{v=A(u)}^{B(u)} f(u, v)] d\mathcal{L}_1 u.$$

From this and 10.5 we conclude:

$$\begin{aligned} \int_a^b [T_{v=V(u)}^{W(u)} \bar{f}(u, v)] d\mathcal{L}_1 u &\leq \int_a^b [T_{v=V(u)}^{W(u)} v] d\mathcal{L}_1 u + \int_a^b [T_{v=V(u)}^{W(u)} f(u, v)] d\mathcal{L}_1 u \\ &= \mathcal{L}_2(S) + \eta_1(S) \leq 2\mathfrak{F}_3^2[\bar{f}^*(S)]. \end{aligned}$$

The following two lemmas are generalizations of recent results of A. S. Besicovitch. (See [BE5, Lemma 3].)

10.7 LEMMA. *If $x \in E_3$, $0 < r < \infty$, then*

$$\psi' \{ [A \cap K(x, r)]' \} \leq 4 \liminf_{h \rightarrow 0^+} h^{-1} \mathfrak{F}_3^2 [A \cap K(x, r+h) - C(x, r-h)].$$

Proof. The set

$$[A \cap K(x, r)]'$$

is a bounded open subset of E_2 and has a countable number of components. Hence its closure has a countable set of components C_1, C_2, C_3, \dots , that is

$$\text{Closure } \{ [A \cap K(x, r)]' \} = \bigcup_{i=1}^{\infty} C_i,$$

C_i is compact and connected for each positive integer i , $(C_i \cap C_j) = 0$ whenever i and j are distinct positive integers.

We define

$$a_1^i = \inf_{z \in C_i} x_1, \quad b_1^i = \sup_{z \in C_i} x_1, \quad a_2^i = \inf_{z \in C_i} x_2, \quad b_2^i = \sup_{z \in C_i} x_2,$$

for each positive integer i .

For each positive integer n , let $\delta(n)$ be a positive number which is less than any of the distances between two of the sets C_1, C_2, \dots, C_n . Since

$$\psi' \{ [A \cap K(x, r)]' \} \leq \sum_{i=1}^{\infty} (\text{diam } C_i),$$

the lemma follows from the last of the four parts into which we divide the remainder of the proof.

Part 1. If i is a positive integer and $y \in (\text{Boundary } C_i)$, then $|\bar{f}(y) - x| = r$.

Proof. It is easy to see that

$$\begin{aligned} |\bar{f}(z) - x| &< r && \text{for } z \in [A \cap K(x, r)]', \\ |\bar{f}(z) - x| &\geq r && \text{for } z \in [A \cap K(x, r)]'. \end{aligned}$$

Now $y \in C_i \subset \text{Closure } \{ [A \cap K(x, r)]' \}$, and the first inequality assures us that

$$|\bar{f}(y) - x| \leq r.$$

On the other hand y is a boundary point of $\text{Closure } \{ [A \cap K(x, r)]' \}$, because no boundary point of a component of a set is an interior point of the set. Hence y is a limit point of $\{ E_2 - [A \cap K(x, r)]' \}$, and the second inequality implies

$$|\bar{f}(y) - x| \geq r.$$

This proves Part 1.

Part 2. If n is a positive integer and $0 < h < \delta(n)$, then

$$h \sum_{i=1}^n (b_1^i - a_1^i) \leq 2\mathcal{J}_3^2 [A \cap K(x, r+h) - C(x, r-h)].$$

Proof. Whenever i is an integer between 1 and n , and $a_1^i < u < b_1^i$, we define $V_i(u)$ and $W_i(u)$ by the relations:

$$V_i(u) = \sup_v E_v [(u, v) \in C_i],$$

$$W_i(u) = \sup_v E_v [T_{v=V_i(u)}^v \bar{f}(u, v) \leq h].$$

It may be shown that V_i and W_i are upper semi-continuous functions.

From Part 1 we see that $|\bar{f}[u, V_i(u)] - x| = r$.

We may assume that $\mathcal{F}_3^2[A \cap K(x, r+h) - C(x, r-h)] < \infty$, and infer, with the help of 10.6, that $T_{v=V_i(u)}^{W_i(u)} \bar{f}(u, v) = h$ for \mathcal{L}_1 almost all u between a_1^i and b_1^i .

Furthermore $V_i(u) < v < W_i(u)$ implies

$$|v - V_i(u)| \leq |\bar{f}(u, v) - \bar{f}[u, V_i(u)]| \leq T_{v=V_i(u)}^v \bar{f}(u, v) < h < \delta(n).$$

Hence $r-h < |\bar{f}(u, v) - x| < r+h$, and (u, v) does not belong to any one of the sets C_1, C_2, \dots, C_n . Consequently the relation $v = V_j(u)$ cannot hold for any integer j between 1 and n .

It follows that no two intervals of the form $[V_i(u), W_i(u)], [V_j(u), W_j(u)]$ overlap, unless $i=j$.

Defining

$$S_i = E_2 \cap \underset{(u,v)}{E} [a_1^i < u < b_1^i, V_i(u) < v < W_i(u)] \quad \text{for } i = 1, 2, \dots, n,$$

we conclude that the sets S_1, S_2, \dots, S_n are disjoint and that

$$\bigcup_{i=1}^n \bar{f}^*(S_i) \subset [K(x, r+h) - C(x, r-h)].$$

From this and Lemma 10.6 we infer that

$$\begin{aligned} h \sum_{i=1}^n (b_i - a_i) &= \sum_{i=1}^n \int_{a_1^i}^{b_1^i} h d\mathcal{L}_1 u = \sum_{i=1}^n \int_{a_1^i}^{b_1^i} [T_{v=V_i(u)}^{W_i(u)} \bar{f}(u, v)] d\mathcal{L}_1 u \leq 2 \sum_{i=1}^n \mathcal{F}_3^2[\bar{f}^*(S_i)] \\ &= 2\mathcal{F}_3^2 \left\{ \bigcup_{i=1}^n \bar{f}^*(S_i) \right\} \leq 2\mathcal{F}_3^2[A \cap K(x, r+h) - C(x, r-h)]. \end{aligned}$$

The proof of Part 2 is complete.

Part 3. If n is a positive integer and $0 < h < \delta(n)$, then

$$h \sum_{i=1}^n (b_2^i - a_2^i) \leq 2\mathcal{F}_3^2[A \cap K(x, r+h) - C(x, r+h)].$$

Part 3 can be proved exactly like Part 2.

Part 4. If n is a positive integer, then

$$\sum_{i=1}^n (\text{diam } C_i) \leq 4 \liminf_{r \rightarrow 0+} h^{-1} \mathcal{F}_3^2[A \cap K(x, r+h) - C(x, r-h)].$$

Proof. From the Parts 3 and 4 we infer that $0 < h < \delta(n)$ implies

$$\begin{aligned} \sum_{i=1}^n (\text{diam } C_i) &\leq \sum_{i=1}^n (b_1^i - a_1^i) + \sum_{i=1}^n (b_2^i - a_2^i) \\ &\leq 4h^{-1} \mathcal{F}_3^2[A \cap K(x, r+h) - C(x, r-h)]. \end{aligned}$$

The proof is complete.

10.8 LEMMA. *If $x \in E_n$ and $\rho > 0$, then*

$$\psi[A \cap K(x, \rho)] \leq 16\pi \mathcal{F}_3^2[A \cap K(x, 2\rho)].$$

Proof. Let

$$\begin{aligned} U(r) &= \psi' \{ [A \cap K(x, r)]' \} && \text{for } r > 0, \\ V(r) &= \mathcal{F}_3^2[A \cap K(x, r)] && \text{for } r > 0. \end{aligned}$$

Clearly U and V are monotone functions, and Lemma 10.7 assures us that $U(r) \leq 8V'(r)$ whenever $V'(r)$ is a number. Hence Lemma 10.3 implies

$$\begin{aligned} \psi[A \cap K(x, \rho)] &\leq 2\pi\rho U(\rho) \leq 2\pi \int_{\rho}^{2\rho} U(r) dr \leq 16\pi \int_{\rho}^{2\rho} V'(r) dr \\ &\leq 16\pi [V(2\rho) - V(\rho)] \leq 16\pi V(2\rho). \end{aligned}$$

The proof is complete.

10.9 LEMMA. $\mathcal{O}_n^k(\mathcal{F}_3^2, A, x) \geq (256\pi)^{-1}$ for \mathcal{H}_3^2 almost all x in A .

Proof. We use 10.8 and 10.1 to check that

$$\begin{aligned} \mathcal{O}_n^k(\mathcal{F}_3^2, A, x) &= \limsup_{r \rightarrow 0^+} \alpha(k)^{-1} (2r)^{-2} \mathcal{F}_3^2[A \cap K(x, 2r)] \\ &\geq \limsup_{r \rightarrow 0^+} \alpha(k)^{-1} (2r)^{-2} (16\pi)^{-1} \psi[A \cap K(x, r)] \\ &= (64\pi)^{-1} \mathcal{O}_n^k(\psi, A, x) \geq (256\pi)^{-1} \end{aligned}$$

for \mathcal{H}_3^2 almost all x in A .

The proof is complete.

10.10 THEOREM. *If X is a rectangle, $X \subset E_2$, then*

$$\mathcal{H}_3^2[\bar{f}^*(X)] = \mathcal{F}_3^2[\bar{f}^*(X)].$$

Proof. From the definitions of \mathcal{H}_3^2 and \mathcal{F}_3^2 we know that

$$\mathcal{F}_3^2(S) \leq \beta(3, 2)^{-1} \mathcal{H}_3^2(S) \quad \text{for } S \subset E_3,$$

and may consequently assume that $\mathcal{F}_3^2[\bar{f}^*(X)] < \infty$.

Hence 10.9 and 3.1 assure us that

$$\mathcal{H}_3^2[\bar{f}^*(X)] < \infty.$$

Following the proof of 9.7, we next select two \mathcal{H}_3^2 measurable sets A_1 and A_2 for which

$$\begin{aligned} \bar{f}^*(X) &= A_1 \cup A_2, & A_1 \cap A_2 &= 0, \\ \mathcal{H}_3^2[\bar{f}^*(X)] &= \mathcal{H}_3^2[\bar{f}^*(X)] + \mathcal{H}_3^2(A_2), & \mathcal{H}_3^2(A_2) &= 0. \end{aligned}$$

Since A_2 is an \mathcal{H}_3^2 measurable set of finite \mathcal{H}_3^2 measure, we can find a set $B \in \mathcal{B}_3$ for which $A_2 \subset B \subset \bar{f}^*(X)$ and $\mathcal{H}_3^2(B - A_2) = 0$. Consequently $\mathcal{H}_3^2(B - A_2) = 0$ and $\mathcal{H}_3^2(B) = 0$.

From 10.9 we know that

$$\bar{\sigma}_n^k[\mathcal{H}_3^2, \bar{f}^*(X), x] \geq (256\pi)^{-1} \text{ for } \mathcal{H}_3^2 \text{ almost all } x \text{ in } B.$$

Hence 3.2 implies

$$\bar{\sigma}_n^k(\mathcal{H}_3^2, B, x) \geq (256\pi)^{-1} \text{ for } \mathcal{H}_3^2 \text{ almost all } x \text{ in } B.$$

Now we use 3.1 and the relation $\mathcal{H}_3^2(B) = 0$ to conclude that $\mathcal{H}_3^2(B) = 0$ which implies $\mathcal{H}_3^2(A_2) = 0$.

This completes the proof.

10.11 REMARK. If X is a rectangle, $X \subset E_2$, and ϕ is any one of the six measures $\mathcal{H}_3^2, \mathcal{S}_3^2, \mathcal{C}_3^2, \Phi_3^2, \Gamma_3^2, \mathcal{G}_3^2$, then

$$\phi[\bar{f}^*(X)] = \mathcal{H}_3^2[\bar{f}^*(X)].$$

In case $\mathcal{H}_3^2[\bar{f}^*(X)] = \infty$, this statement follows immediately from the definition of the measure functions.

In case $\mathcal{H}_3^2[\bar{f}^*(X)] < \infty$, we infer from 10.10 and 9.7 that the set $\bar{f}^*(X)$ is $(\mathcal{H}_3^2, 2)$ rectifiable, and the statement to be proved is a consequence of 5.15.

10.12 REMARK. If X is a rectangle, $X \subset E_2$, $g = (\bar{f}|X)$, and ϕ is any one of the seven measures $\mathcal{H}_3^2, \mathcal{S}_3^2, \mathcal{C}_3^2, \Phi_3^2, \Gamma_3^2, \mathcal{G}_3^2, \mathcal{J}_3^2$, then

$$\phi(\text{range } g) = \mathcal{L}(g) = \mathcal{A}(g),$$

where \mathcal{L} is Lebesgue area and \mathcal{A} is the lower semi-continuous area we defined in [F4, 6.9] in terms of stable multiplicities.

This statement is an immediate consequence of 10.11 and [F4, 6.14].

BIBLIOGRAPHY

A. APPERT

[A] *Mesures normales dans les espaces distanciés*, Bull. Sci. Math. vol. 60 (1936) pp. 329-352 and 368-380.

A. S. BESICOVITCH

[BE1] *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. vol. 98 (1927) pp. 422-464.

- [BE2] *On the fundamental geometrical properties of linearly measurable plane sets of points* (II), Math. Ann. vol. 115 (1938) pp. 296–329.
- [BE3] *On the fundamental geometrical properties of linearly measurable plane sets of points* (III), Math. Ann. vol. 116 (1939) pp. 349–357.
- [BE4] *On the Kolmogoroff maximum and minimum measures*, Math. Ann. vol. 113 (1936) pp. 416–423.
- [BE5] *On the definition and value of the area of a surface*, Quart. J. Math. Oxford Ser. vol. 16 (1945) pp. 86–102.
- [BE6] *On tangents to general sets of points*, Fund. Math. vol. 22 (1934) pp. 49–53.
- A. S. BESICOVITCH and P. A. P. MORAN
- [BM] *The measure of product and cylinder sets*, Proc. London Math. Soc. vol. 20 (1945) pp. 110–120.
- A. S. BESICOVITCH AND G. WALKER
- [BW] *On the density of irregular linearly measurable sets of points*, Proc. London Math. Soc. vol. 32 (1931) pp. 142–153.
- C. CARATHÉODORY
- [C] *Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs*, Nachr. Ges. Wiss. Göttingen (1914) pp. 404–426.
- D. R. DICKINSON
- [D] *Study of extreme cases with respect to the densities of irregular linearly measurable plane sets of points*, Math. Ann. vol. 116 (1939) pp. 359–373.
- S. EILENBERG
- [EI] *Continua of finite linear measure*. II, Amer. J. Math. vol. 66 (1944) pp. 425–427.
- S. EILENBERG AND O. G. HARROLD, JR.
- [EH] *Continua of finite linear measure*. I, Amer. J. Math. vol. 65 (1943) pp. 137–146.
- TH. ESTERMANN
- [E] *Über Carathéodory's und Minkowski's Verallgemeinerungen des Längenbegriffs*, Abh. Math. Sem. Hamburgischen Univ. vol. 4 (1925) pp. 73–116.
- J. FAVARD
- [FA1] *Une définition de la longueur et de l'aire*, C. R. Acad. Sci. Paris vol. 194 (1932) pp. 344–346.
- [FA2] *La longueur et l'aire d'après Minkowski*, Bull. Soc. Math. France vol. 61 (1933) pp. 63–84.
- H. FEDERER
- [F1] *Surface area*. I, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 420–437.
- [F2] *Surface area*. II, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 438–456.
- [F3] *The Gauss-Green theorem*, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 44–76.
- [F4] *Coincidence functions and their integrals*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 441–466.
- J. GILLIS
- [GI1] *Note on the projection of irregular linearly measurable plane sets of points*, Fund. Math. vol. 26 (1936) pp. 228–233.
- [GI2] *On linearly measurable plane sets of points of upper density $1/2$* , Fund. Math. vol. 22 (1934) pp. 57–69.
- [GI3] *A theorem on irregular linearly measurable sets of points*, J. London Math. Soc. vol. 10 (1935) pp. 234–240.
- [GI4] *On linearly measurable sets of points*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie vol. 28 (1936) pp. 49–70.
- [GI5] *On the projection of irregular linearly measurable plane sets of points*, Proc. Cambridge Philos. Soc. vol. 30 (1934) pp. 47–54.

W. GROSS

- [G1] *Über das Flächenmass von Punktmengen*, Monatshefte für Mathematik und Physik vol. 29 (1918) pp. 145–176.
 [G2] *Über das lineare Mass von Punktmengen*, Monatshefte für Mathematik und Physik vol. 29 (1918) pp. 177–193.

F. HAUSDORFF

- [H] *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1918) pp. 157–179.

W. HUREWICZ AND H. WALLMAN

- [HW] *Dimension theory*, Princeton Mathematical Series, Princeton University Press, 1941.

O. JANZEN

- [JA] *Über einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt*, Inaugural dissertation, Königsberg, 1907.

R. L. JEFFERY

- [J] *Sets of k extent in n -dimensional space*, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 629–647.

A. KOLMOGOROFF

- [K] *Beiträge zur Masstheorie*, Math. Ann. vol. 107 (1932) pp. 351–366.

L. H. LOOMIS

- [L] *Abstract congruence and the uniqueness of Haar measure*, Ann. of Math. vol. 46 (1945) pp. 348–355.

S. MAZURKIEWICZ AND S. SAKS

- [MS] *Sur les projections d'un ensemble fermé*, Fund. Math. vol. 8 (1926) pp. 109–113.

K. MENGER

- [MK] *Entwurf einer neuen Theorie des Masses*, Ergebnisse eines Mathematischen Kolloquiums vol. 2 (1932) pp. 6–10.

D. S. MILLER

- [MD] *Carathéodory and Gillespie linear measure*, Duke Math. J. vol. 11 (1944) pp. 139–145.

H. MINKOWSKI

- [MI] *Über die Begriffe Länge, Oberfläche und Volumen*, Jber. Deutschen Math. Verein. vol. 9 (1900) pp. 115–121.

G. W. MORGAN

- [MG] *The density directions of irregular linearly measurable plane sets*, Proc. London Math. Soc. vol. 38 (1935) pp. 481–494.

A. P. MORSE

- [MA1] *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235.
 [MA2] *The role of internal families in measure theory*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 723–728.

A. P. MORSE AND H. FEDERER

- [MF] *Some properties of measurable functions*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 270–277.

A. P. MORSE AND J. F. RANDOLPH

- [MR1] *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305.
 [MR2] *Gillespie measure*, Duke Math. J. vol. 6 (1940) pp. 408–419.

G. NÖBELING

- [N1] *Über den Flächeninhalt dehmingbeschränkter Flächen*, Math. Zeit. vol. 48 (1943) pp. 747–771.
 [N2] *Über die Flächenmasse im Euklidischen Raum*, Math. Ann. vol. 118 (1943) pp. 687–701.

- [N3] *Über die Struktur der rektifizierbaren Kontinuen*, J. Reine Angew. Math. vol. 184 (1942) pp. 91–115.
- [N4] *Über die Länge der Euklidischen Kontinuen*. I, Jber. Deutschen Math. Verein. vol. 52 (1942) pp. 132–160.
- [N5] *Über die Länge der Euklidischen Kontinuen*. II, Jber. Deutschen Math. Verein. vol. 52 (1942) pp. 189–197.
- [N6] *Über die Länge der Euklidischen Kontinuen*. III, Monatshefte für Mathematik und Physik vol. 50 (1942) pp. 282–287.
- L. PONTRJAGIN AND L. SCHNIRELMAN
 [PS] *Sur une propriété métrique de la dimension*, Ann of Math. vol. 33 (1932) pp. 156–162.
- H. RADEMACHER
 [RAD] *Über partielle und totale Differenzierbarkeit*. I, Math. Ann. vol. 79 (1919) pp. 340–359.
- J. F. RANDOLPH
 [RA1] *Carathéodory measure and a generalization of the Gauss-Green lemma*, Trans. Amer. Math. Soc. vol. 38 (1935) pp. 531–548.
- [RA2] *On generalizations of length and area*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 268–274.
- [RA3] *The Vitali covering theorem for Carathéodory linear measure*, Ann. of Math. vol. 40 (1939) pp. 299–308.
- [RA4] *Some properties of sets of the Cantor type*, J. London Math. Soc. vol. 16 (1941) pp. 38–42.
- S. SAKS
 [SS1] *Theory of the integral*, Warsaw, 1937.
- [SS2] *Remarque sur la mesure linéaire des ensembles plans*, Fund. Math. vol. 9 (1927) pp. 16–24.
- A. SARD
 [SD] *The equivalence of n -measure and Lebesgue measure in E_n* , Bull. Amer. Math. Soc. vol. 49 (1943) pp. 758–759.
- J. P. SCHAUDER
 [SCH] *The theory of surface measure*, Fund. Math. vol. 8 (1926) pp. 1–48.
- S. SHERMAN
 [SH] *A comparison of linear measures in the plane*, Duke Math. J. vol. 9 (1942) pp. 1–9.
- W. SIERPINSKI
 [SI] *Sur la densité linéaire des ensembles plans*, Fund. Math. vol. 9 (1927) pp. 172–185.
- H. STEINHAUS
 [ST1] *Sur la portée pratique et théorique de quelques théorèmes sur la mesure des ensembles de droites*, C. R. Congrès des Mathématiciens des pays Slaves, Warsaw, 1929, pp. 348–354.
- [ST2] *Zur Praxis der Rektifikation und zum Längenbegriff*, Sächsische Akademie zu Leipzig, Berichte vol. 82 (1930) pp. 120–130.
- E. SZPILRAIN
 [SZ] *La dimension et la mesure*, Fund. Math. vol. 28 (1937) pp. 81–89.
- A. WEIL
 [W] *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, vol. 869, Hermann, Paris, 1938.
- W. H. YOUNG AND G. C. YOUNG
 [YY] *The theory of sets of points*, Cambridge University Press, 1906, chap. 13.

BROWN UNIVERSITY,
 PROVIDENCE, R. I.