THE MEAN CONVERGENCE OF ORTHOGONAL SERIES. II

BY
HARRY POLLARD

Let $w(x)$ be a weight function on $(-1, 1)$ and $p_n(x)$ the corresponding orthonormal polynomials [1, p. 25](1). Then

$$\int_{-1}^{1} p_m(x)p_n(x)w(x)dx = \delta_{mn} \qquad (m, n = 0, 1, \cdots).$$

The development of a function $f(x)$ defined on this interval takes the form

$$f(x) \sim \sum_{0}^{\infty} a_n p_n(x), \tag{1}$$

where

$$a_n = \int_{-1}^{1} p_n(x)f(x)w(x)dx. \tag{2}$$

The problem I wish to consider is this. Let L_w^p denote the class of functions $f(x)$ which are measurable, and for which

$$\int_{-1}^{1} |f(x)|^p w(x)dx < \infty. \tag{3}$$

For what values of p is it true that for all $f(x)$ in L_w^p we have convergence in the mean of expansion (1), that is,

$$\lim_{N\to\infty} \int_{-1}^{1} \left| f(x) - \sum_{0}^{N} a_n p_n(x) \right|^p w(x)dx = 0? \tag{4}$$

The range of values of p depends on the nature of the function $w(x)$. If $w(x) \equiv (1-x^2)^{-1/2}$, in which case the $p_n(x)$ are the normalized Tchebichef polynomials, the range is $1 < p < \infty$; this follows from M. Riesz' classical theorem on the mean convergence of Fourier series. Recently I have considered the problem for $w(x) \equiv 1$, that is, for Legendre series [3]. In this case the range turns out to be $4/3 < p < 4$, with failure of mean convergence for some functions in L^p if $1 \leqq p < 4/3$ or $p > 4$.

The purpose of this paper is to bridge the gap between the two results by considering the problem for the general weight function $w(x) \equiv (1-x^2)^{\lambda-1/2}$,

Presented to the Society, November 30, 1946; received by the editors February 25, 1947.

(1) Numbers in brackets refer to the bibliography at the end of the paper.

$\lambda \geqq 0$; the corresponding $p_n(x)$ are the normalized ultraspherical polynomials [1]. It will be established (§8) that (4) *holds for all* $f(x)$ *in* L_w^p *if*

$$2 - \frac{1}{\lambda + 1} < p < 2 + \frac{1}{\lambda}, \tag{5}$$

but not if $1 \leqq p < 2 - 1/(\lambda+1)$ *or* $p > 2 + 1/\lambda$(2). For $\lambda = 0$ this is consistent with Riesz' theorem, for $\lambda = 1/2$ with my own.

If in $(1-x^2)^{\lambda-1/2}$ we replace x by $x\lambda^{-1/2}$, and take the limit as $\lambda \to \infty$, the weight function becomes e^{-x^2}, and the corresponding polynomials those of Hermite on $(-\infty, \infty)$. The inequality (5) suggests that in this case p-mean convergence holds only for $p = 2$. This, and a similar result for Laguerre series, will be confirmed by suitable counterexamples.

A large part of the work carries over to more general weight functions than $(1-x^2)^{\lambda-1/2}$. We shall therefore first obtain a set of sufficient conditions on $w(x)$ in order that the property (4) hold for all $f(x)$ in L_w^p. Later it will be established that if (5) is true, then these conditions are actually fulfilled by the special weight function.

It is possible to generalize the whole problem by considering distributions $d\alpha(x)$ rather than $w(x)dx$. Except for $p = 2$, the more general problem appears to be very difficult, and the present methods seem to offer little towards its solution.

1. **The hypotheses on $w(x)$.** Let $p > 1$ be fixed. We shall assume that $w(x)$ satisfies hypotheses (H1)–(H8) given below. As they stand they are rather complicated, but are in precisely the form we need. In later sections a simpler set of conditions, which imply these, will be given.

(H1) *$w(x)$ is integrable and positive almost everywhere*;

$$\int_{-1}^{1} \frac{|\log w(x)|}{(1-x^2)^{1/2}}\,dx < \infty; \tag{H2}$$

(H3) *the corresponding orthonormal polynomials $p_n(x)$ satisfy the condition*

$$(1 - x^2)^{1/4} w(x)^{1/2} |p_n(x)| \leqq A,$$

where A is independent of n and x;

(H4) *if $(1-x^2)^{-1}q_n(x)$ denotes the polynomials orthonormal with respect to $(1-x^2)w(x)$, then*

$$(1 - x^2)^{-1/4} w(x)^{1/2} |q_n(x)| \leqq A;$$

$$\| w(x)^{1/p-1/2}(1 - x^2)^{-1/4} \|_p < \infty; \tag{H5}$$

$$\| w(x)^{1/2-1/p}(1 - x^2)^{-1/4} \|_{p'} < \infty. \tag{H6}$$

Here $p' = p/(p-1)$ and $\|g(x)\|_p = (\int_{-1}^{1} |g(x)|^p dx)^{1/p}$.

---

(2) Except when $\lambda = 0$, the end values are in doubt. If $\lambda = 0$, it is known that the theorem fails for $p = 1$ and $p = \infty$. See, for example, [3]. The theorem was announced by the author, without proof, in the note [3a].

(H7) *The kernels*

$$K_{\pm}(x, y) = \left| \frac{((1 - y^2)/(1 - x^2))^{\pm 1/4}(w(y)/w(x))^{1/2-1/p} - 1}{x - y} \right|$$

have the property that

$$\int_{-1}^{1} K_{\pm}(x, y)g(y)dy$$

belong to $L^p(-1, 1)$ *whenever* $g(y)$ *does.*

It is easy to prove, although we shall not use the fact, that if the preceding hypotheses are satisfied for some $p>1$, they are also satisfied when p is replaced by p'. This is to be expected in view of the fact that mean convergence for p implies mean convergence for p' [4, p. 108].

In the following sections we shall prove that if (H1)–(H7) hold, then (4) is true for all $f(x)$ in L^p_w.

2. **The partial sums.** The kernel $k_N(x, y)$ is given by [1, p. 42]

$$k_N(x, y) = \sum_0^N p_n(x)p_n(y) = u_N \frac{p_{N+1}(x)p_N(y) - p_N(x)p_{N+1}(y)}{x - y},$$

where $u_n = k_n/k_{n+1}$, and $k_n > 0$ is the coefficient of x^n in the polynomial $p_n(x)$.

The polynomials $q_n(x)$ defined in the preceding section are given by(3)

$$\begin{aligned} q_n(x) &= (k_n/k_{n+2})^{1/2}\{(p_{n+1}\bar{p}_n - p_n\bar{p}_{n+1})(p_{n+2}\bar{p}_{n+1} - p_{n+1}\bar{p}_{n+2})\}^{-1/2} \\ &\quad \begin{vmatrix} p_{n+2}(x) & p_{n+1}(x) & p_n(x) \\ p_{n+2} & p_{n+1} & p_n \\ \bar{p}_{n+2} & \bar{p}_{n+1} & \bar{p}_n \end{vmatrix} \\ &= A_n p_n(x) + B_n p_{n+1}(x) + C_n p_{n+2}(x). \end{aligned} \tag{2.1}$$

We write p_n for $p_n(1)$ and $\bar{p}_n$ for $p_n(-1)$.

A straightforward computation shows that

$$k_n(x, y) = \frac{u_n}{A_n} \frac{p_{n+1}(x)q_n(y) - q_n(x)p_{n+1}(y)}{x - y} + \frac{u_n}{A_n} \frac{C_n}{u_{n+1}} k_{n+1}(x, y),$$

so that

$$k_n(x, y) - \frac{u_n}{A_n} \frac{C_n}{u_{n+1}} \{k_n(x, y) + p_{n+1}(x)p_{n+1}(y)\} = \frac{u_n}{A_n} \frac{p_{n+1}(x)q_n(y) - q_n(x)p_{n+1}(y)}{x - y}.$$

(3) This formula is contained in [5, pp. 138–139] with some inessential differences, due to another normalization of the polynomials.

Hence we have the formula

$$k_n(x, y) = \left(1 - \frac{u_n}{u_{n+1}} \frac{C_n}{A_n}\right)^{-1} \frac{u_n}{A_n} p_{n+1}(x) \frac{q_n(y)}{x - y}$$
$$- \left(1 - \frac{u_n}{u_{n+1}} \frac{C_n}{A_n}\right)^{-1} \frac{u_n}{A_n} q_n(x) \frac{p_{n+1}(y)}{x - y}$$
$$+ \left(1 - \frac{u_n}{u_{n+1}} \frac{C_n}{A_n}\right)^{-1} \frac{u_n}{u_{n+1}} \frac{C_n}{A_n} p_{n+1}(x)p_{n+1}(y).$$

Now suppose that $f(x) \in L_w^p$. Then the partial sums $s_n(f; x)$ of (1) are given by $\int_{-1}^1 k_n(x, y)f(y)w(y)dy$, so that almost everywhere

$$\begin{aligned} s_n(f; x) = {} & \frac{u_nD_n}{A_n} p_{n+1}(x) \int_{-1}^{1} \frac{w(y)q_n(y)f(y)}{x - y} dy \\ & - \frac{u_nD_n}{A_n} q_n(x) \int_{-1}^{1} \frac{w(y)p_{n+1}(y)f(y)}{x - y} dy \\ & + D_n \frac{u_n}{u_{n+1}} \frac{C_n}{A_n} a_{n+1}p_{n+1}(x), \end{aligned} \tag{2.2}$$

where $D_n = (1 - (u_n/u_{n+1})(C_n/A_n))^{-1}$. The convergence almost everywhere of the improper integrals, as Cauchy principal values, follows from the fact that $w(x)f(x)$ is integrable [6, p. 132]. The appearance of a_{n+1} in the last term is explained by the definition (2).

LEMMA 2.1. *Under the hypotheses of* §1 *the following are true*:

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1, \qquad \lim_{n\to\infty} \frac{u_n}{A_n} = 1, \qquad \lim_{n\to\infty} \frac{C_n}{A_n} = -1.$$

LEMMA 2.2. *Under the hypotheses of* §1, *the coefficients* a_n *defined by* (2) *are bounded.*

Before proving the lemmas we apply them to (2.2). By virtue of (H3), (H4) and the lemmas

$$\begin{aligned} | s_n(f; x) | \leqq {} & A \left| (1 - x^2)^{-1/4}w(x)^{-1/2} \int_{-1}^{1} \frac{w(y)q_n(y)f(y)}{x - y} dy \right| \\ & + A \left| (1 - x^2)^{1/4}w(x)^{-1/2} \int_{-1}^{1} \frac{w(y)p_{n+1}(y)f(y)}{x - y} dy \right| \\ & + A(1 - x^2)^{-1/4}w(x)^{-1/2}. \end{aligned} \tag{2.3}$$

A is a constant independent of n and x.

3. **Proofs of the lemmas.** To prove Lemma 2.1 we begin with the observa-

tion that by virtue of (H1) and (H2)

$$k_n \sim \pi^{-1/2} 2^n \exp \left\{ -\frac{1}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{(1-x^2)^{1/2}} \, dx \right\}.$$

This formula can be found in [1, p. 302]. It follows that $\lim_{n\to\infty} u_n = 1/2$.

Now, by (2.1)

$$\frac{C_n}{A_n} = \left(\frac{\bar{p}_n}{\bar{p}_{n+1}} - \frac{p_n}{p_{n+1}} \right) \left(\frac{p_{n+2}}{p_{n+1}} - \frac{\bar{p}_{n+2}}{\bar{p}_{n+1}} \right)^{-1}$$

and

$$A_n = (k_n/k_{n+2})^{1/2} \left(\frac{p_{n+2}}{p_{n+1}} - \frac{\bar{p}_{n+2}}{\bar{p}_{n+1}} \right)^{1/2} \left(\frac{p_n}{p_{n+1}} - \frac{\bar{p}_n}{\bar{p}_{n+1}} \right)^{-1/2}.$$

The conclusion of the lemma will now follow if we can prove that

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1, \qquad \lim_{n\to\infty} \frac{\bar{p}_{n+1}}{\bar{p}_n} = -1. \tag{3.1}$$

By [1, p. 46] $p_{n+1}/p_n > 0$ and $\bar{p}_{n+1}/\bar{p}_n < 0$. We shall prove the first part of (3.1); the second part follows by a similar argument. The $p_n(x)$ satisfy a recursion formula [1, p. 41 (3.2.1)]

$$p_n(x) = (b_n x + c_n) p_{n-1}(x) - d_n p_{n-2}(x),$$

where [1, p. 303] $b_n \to 2$, $c_n \to 0$, $d_n \to 1$. Take $x = 1$ and let $\lambda_n = p_n/p_{n-1}$. Then

$$\lambda_n = b_n + c_n - d_n/\lambda_{n-1}.$$

Since $0 < \lambda_n < b_n + c_n$, at least for large n, the λ_n are bounded. Also

$$d_n = (b_n + c_n - 2)\lambda_{n-1} + (2 - \lambda_n)\lambda_{n-1}.$$

Hence

$$\lim_{n\to\infty} (2 - \lambda_n)\lambda_{n-1} = 1. \tag{3.2}$$

Then $2 - \lambda_n$ is positive for large n.

We now use the fact that for positive sequences a_n and b_n

$$\liminf a_n b_n \leqq \liminf a_n \limsup b_n \leqq \limsup a_n b_n.$$

Then, by (3.2)

$$1 \leqq \liminf\, (2 - \lambda_n) \limsup \lambda_{n-1} \leqq 1,$$
$$1 \leqq \limsup\, (2 - \lambda_n) \limsup \lambda_{n-1} \leqq 1.$$

It follows that

$$(2 - \limsup \lambda_n) \limsup \lambda_n = 1,$$
$$(2 - \liminf \lambda_n) \liminf \lambda_n = 1,$$

so $\limsup \lambda_n = \liminf \lambda_n = \lim \lambda_n = 1$, as was to be proved.

To prove Lemma 2.2 note that by (2)

$$|a_n| = \left| \int_{-1}^{1} w^{1/p} f \cdot w^{1/p'} p_n(y) dy \right| \leqq \| w^{1/p} f \|_p \| w^{1/p'} p_n(y) \|_{p'}.$$

Hence, by (3) and (H3)

$$|a_n| \leqq A \| w^{1/p' - 1/2} (1 - y^2)^{-1/4} \|_{p'}.$$

Now apply (H6).

4. **Norms of the partial sums.** We wish to prove next that

$$\limsup_{n \to \infty} \| w(x)^{1/p} s_n(f; x) \|_p < \infty \tag{4.1}$$

for all $f(x)$ satisfying (3). According to (2.3), Minkowski's inequality and (H5)

$$\begin{aligned} \| w(x)^{1/p} s_n(f; x) \|_p \leqq A & \left\| (1 - x^2)^{-1/4} w(x)^{1/p - 1/2} \int_{-1}^{1} \frac{w(y) q_n(y) f(y)}{x - y} dy \right\|_p \\ & + A \left\| (1 - x^2)^{1/4} w(x)^{1/p - 1/2} \int_{-1}^{1} \frac{w(y) p_{n+1}(y) f(y)}{x - y} dy \right\|_p \\ & + A, \end{aligned}$$

where A is independent of n.

To establish (4.1) it is enough to prove that each of the first two terms on the right-hand side of (4.2) is bounded in n; call them I_n and J_n respectively.

Now let

$$A_n(y) = (1 - y^2)^{-1/4} w(y)^{1/2} q_n(y).$$

According to (H4) these functions are uniformly bounded in n and y. Since

$$\begin{aligned} (1 - x^2)^{1/4} w(x)^{1/p - 1/2} w(y) q_n(y) f(y) = {} & w^{1/p}(y) f(y) A_n(y) \\ & + w^{1/p}(y) f(y) A_n(y) \left\{ \left(\frac{1 - y^2}{1 - x^2} \right)^{1/4} \left(\frac{w(y)}{w(x)} \right)^{1/2 - 1/p} - 1 \right\}, \end{aligned}$$

we have

$$\begin{aligned} I_n \leqq A & \left\| \int_{-1}^{1} \frac{w^{1/p}(y) f(y) A_n(y)}{x - y} dy \right\|_p \\ & + A \left\| \int_{-1}^{1} \frac{w^{1/p}(y) f(y) A_n(y)}{x - y} \left\{ \left(\frac{1 - y^2}{1 - x^2} \right)^{1/4} \left(\frac{w(y)}{w(x)} \right)^{1/2 - 1/p} - 1 \right\} dy \right\|_p \end{aligned} \tag{4.2}$$

Since $w^{1/p}(y)f(y)A_n(y) \in L^p(-1, 1)$, the first term on the right is [6, p. 132] less than or equal to

$$A\|w^{1/p}fA_n\|_p < A,$$

in view of the boundedness of $A_n(y)$. The second term of (4.2) is less than or equal to

$$A\left\|\int_{-1}^{1} w^{1/p}(y)\,|f(y)|\,K_+(x, y)dy\right\|_p,$$

which is finite, by (H7). Hence I_n is bounded.

A similar argument applies to J_n. Let

$$B_n(y) = (1 - y^2)^{1/4}w(y)^{1/2}p_{n+1}(y).$$

By (H3), they are uniformly bounded. Now

$$(1 - x^2)^{1/4}w(x)^{1/p-1/2}w(y)p_{n+1}(y)f(y) = w^{1/p}(y)f(y)B_n(y)$$
$$+ w^{1/p}(y)f(y)B_n(y)\left\{\left(\frac{1 - y^2}{1 - x^2}\right)^{-1/4}\left(\frac{w(y)}{w(x)}\right)^{1/2-1/p} - 1\right\}.$$

The rest of the proof proceeds as for I_n, with the substitution of $K_-(x, y)$ for $K_+(x, y)$.

5. **The main theorem.** We are now in a position to prove the following theorem.

THEOREM 5.1. *If $w(x)$ satisfies the hypotheses* (H1)–(H7) *for some value of $p>1$, then* (4) *is true for all $f(x)$ in L_w^p.*

$s_n(f; x)$ can be regarded as a sequence of linear operators on the space L_w^p, normed by $\|w^{1/p}f\|_p$. According to (4.1), which we may write

$$\limsup_{n\to\infty}\left(\int_{-1}^{1} |s_n(f; x)|^p w(x)dx\right)^{1/p} < \infty, \tag{5.1}$$

the sequence of operators satisfies the hypotheses of a theorem of Banach [4, p. 80] from which we can infer the existence of a constant A_p such that

$$\int_{-1}^{1} |s_n(f; x)|^p w(x)dx \leqq A_p^p\int_{-1}^{1} |f(x)|^p w(x)dx, \tag{5.2}$$

for all f in L_w^p.

We wish to prove that $s_n(f; x)\to f$, for all f in L_w^p, convergence being according to the norm of L_w^p. Now $s_n(f; x)$ certainly converges to f for all *polynomials* f. Hence by (5.2) and another theorem of Banach [4, p. 79] $s_n(f; x)$ converges (in L_w^p) for *all* f. To prove that it actually converges to f, we proceed as follows. Call the limit g. Then

$$\lim_{n\to\infty}\int_{-1}^{1}|s_n(f;x)-g(x)|^p w(x)dx=0,$$

so for each m

$$\lim_{n\to\infty}\int_{-1}^{1}p_m(x)\{s_n(f;x)-g(x)\}w(x)dx=0.$$

From the orthogonality of the $p_m(x)$ with respect to $w(x)$ it follows that

$$\int_{-1}^{1}\{f(x)-g(x)\}p_m(x)w(x)dx=0\quad(m=0,1,2,\cdots).$$

Hence $f(x)\equiv g(x)$.

6. **Reduction of the hypotheses.** We shall now specialize $w(x)$ to the form

$$w(x)=t(x)(1-x)^\alpha(1+x)^\beta \qquad \alpha\geqq-1/2,\ \beta\geqq-1/2, \tag{6.1}$$

where $t(x)$ is *positive, has a continuous derivative, and*

$$t'(x+h)-t'(x)=O(\ln^{-2}|h|)\qquad(h\to 0),$$

uniformly in $-1\leqq x\leqq 1$. The class of such $w(x)$ will be denoted by $\boldsymbol{B}$.

For such a weight function Bernstein [5, pp. 267–268] has shown that the corresponding $p_n(x)$ satisfy $(1-x^2)^{1/4}w(x)^{1/2}|p_n(x)|\leqq A$. Obviously $(1-x^2)w(x)$ also belongs to $\boldsymbol{B}$ so that

$$(1-x^2)^{1/4}\{(1-x^2)w(x)\}^{1/2}\frac{|q_n(x)|}{1-x^2}\leqq A,$$

where $q_n(x)/(1-x^2)$ are the polynomials orthonormal with respect to $(1-x^2)w(x)$. It follows that $w(x)$ satisfies hypotheses (H1) through (H4) of §1.

It is also easy to verify that (H5) and (H6) are satisfied by a function $w(x)$ in $\boldsymbol{B}$, provided

$$4\max\left\{\frac{\alpha+1}{2\alpha+3},\frac{\beta+1}{2\beta+3}\right\}<p<4\min\left\{\frac{\alpha+1}{2\alpha+1},\frac{\beta+1}{2\beta+1}\right\}. \tag{6.2}$$

We illustrate with (H5); (H6) can be treated the same way. Since $t(x)$ is bounded

$$|w(x)^{1/p-1/2}(1-x^2)^{-1/4}|\leqq A|(1-x)^{\alpha(1/p-1/2)-1/4}(1+x)^{\beta(1/p-1/2)-1/4}|$$

which belongs to L^p if

$$p\left[\alpha\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{4}\right]>-1,\quad p\left[\beta\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{4}\right]>-1$$

or

$$p < \frac{4(\alpha + 1)}{2\alpha + 1}, \qquad p < \frac{4(\beta + 1)}{2\beta + 1}.$$

We may therefore deduce from Theorem 5.1 the following simpler criterion for mean convergence.

THEOREM 6.1. *If $w(x) \in \boldsymbol{B}$, and satisfies* (H7) *for a value of p in the range* (6.2), *then* (4) *holds for all $f(x)$ in L^p_w.*

7. **A lemma.** In order to show that the conditions of Theorem 6.1 are fulfilled by the special function $(1-x^2)^{\lambda-1/2}$, $\lambda \geqq 0$, we shall need the following lemma(4).

LEMMA 7.1. *If $-1<c<1$, $c<1/p<c+1$, $p>1$ and $f(x)$ is a non-negative function of $L^p(-1, 1)$, then the function*

$$\int_{-1}^{1} \left| \frac{((1-y^2)/(1-x^2))^c - 1}{x-y} \right| f(y)dy$$

also belongs to $L^p(-1, 1)$.

It is enough to prove that if $g(x) \geqq 0$, $g(x) \in L^{p'}(-1, 1)$, then

$$\int_{-1}^{1} g(x)dx \int_{-1}^{1} | \cdots | f(y)dy \tag{7.1}$$

converges. The expression denoted by the dots is

$$\left\{ \left(\frac{1-y^2}{1-x^2} \right)^c - 1 \right\} (x-y)^{-1}.$$

(7.1) can be written

$$\int_{-1}^{1}\int_{-1}^{1} g(x) | \cdots |^{1/p'} f(y) | \cdots |^{1/p} \left(\frac{1-y^2}{1-x^2} \right)^{-1/pp'} \left(\frac{1-y^2}{1-x^2} \right)^{1/pp'} dxdy,$$

which by Hölder's inequality for double integrals is less than or equal to

$$\left\{ \int_{-1}^{1} g^{p'}dx \int_{-1}^{1} | \cdots | \left(\frac{1-y^2}{1-x^2} \right)^{-1/p} dy \right\}^{1/p'}$$

$$\cdot \left\{ \int_{-1}^{1} f^p dy \int_{-1}^{1} | \cdots | \left(\frac{1-y^2}{1-x^2} \right)^{1/p'} dx \right\}^{1/p}.$$

The proof of convergence is complete provided the inner integral of each

(4) The lemma is stated in the note [3a] without proof.

iterated integral is a bounded function. But it is easy to see from the hypotheses of the lemma that each inner integral is of the form

$$\int_{-1}^{1}\left|\frac{((1-t^2)/(1-s^2))^a-((1-t^2)/(1-s^2))^b}{s-t}\right|ds,$$

where $0<a<1$, $0<b<1$. That such a function is bounded in $-1<t<1$ was established in [3, §13].

8. **The ultraspherical case.** Now let $w(x)=(1-x^2)^{\lambda-1/2}$, $\lambda\geqq 0$. This is obviously in the class $\boldsymbol{B}$. The inequalities (6.2) reduce to

$$(8.1)\qquad 2-\frac{1}{\lambda+1}<p<2+\frac{1}{\lambda},$$

since $\alpha=\beta=\lambda-1/2$. In order to establish the mean convergence property (4) for this range of p, it is enough to verify (H7), for then Theorem 6.1 is applicable.

Now the kernels of (H7) are for this function $w(x)$

$$\left|\frac{((1-y^2)/(1-x^2))^c-1}{x-y}\right|,$$

where $c=\pm 1/4+(\lambda-1/2)(1/2-1/p)$. By Lemma 7.1, (H7) is verified provided $c<1/p<c+1$. But this is a consequence of (8.1). Hence we have established the following theorem.

THEOREM 8.1. *If $f(x)$ is measurable and satisfies the condition*

$$\int_{-1}^{1}|f(x)|^p(1-x^2)^{\lambda-1/2}dx<\infty,$$

where $\lambda\geqq 0$, and $2-1/(\lambda+1)<p<2+1/\lambda$, then the expansion of $f(x)$ in the ultraspherical series(5)

$$f(x)\sim\sum b_nP_n^{(\lambda)}(x)$$

converges to $f(x)$ in the weighted pth mean:

$$\lim_{N\to\infty}\int_{-1}^{1}\left|f(x)-\sum_{0}^{N}b_nP_n^{(\lambda)}(x)\right|^p(1-x^2)^{\lambda-1/2}dx=0.$$

9. **Counterexamples.** We shall now show that mean convergence fails for $w(x)=(1-x)^{\lambda-1/2}$, $\lambda>0$, if

$$1\leqq p<2-\frac{1}{\lambda+1}\quad or\quad p>2+\frac{1}{\lambda}\cdot$$

(5) For these polynomials, which, apart from constants, can be identified with the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $\alpha=\lambda-1/2$, see [1, p. 80].

Since $2-1/(\lambda+1)$ and $2+1/\lambda$ are conjugate, we may confine ourselves to the first range of p [4, p. 108]. Certainly for such p the function

$$f(x) = (1-x)^{-1/2(\lambda+1)} = (1-x)^{-\alpha/2-3/4}, \qquad \alpha = \lambda - 1/2,$$

belongs to L_w^p.

The expansion of $f(x)$ takes the form

$$f(x) \sim \sum a_n P_n^{(\alpha,\alpha)}(x),$$

where [1, p. 249] for some $A>0$,

$$a_n > An^{-2(-\alpha/2-3/4)-\alpha-1} = An^{1/2} \qquad (n \to \infty).$$

To show the failure of mean convergence it is enough to establish that

$$\text{(9.1)} \qquad \liminf_{n\to\infty} a_n \left\{ \int_{-1}^{1} (1-x^2)^{\lambda-1/2} \left| P_n^{(\alpha,\alpha)}(x) \right|^p dx \right\}^{1/p} > 0.$$

Now, since $p \geqq 1$,

$$\begin{aligned} a_n \left\| (1-x^2)^{(\lambda-1/2)1/p} P_n^{(\alpha,\alpha)}(x) \right\|_p &\geqq An^{1/2} \int_{-1}^{1} (1-x^2)^{\lambda-1/2} \left| P_n^{(\alpha,\alpha)}(x) \right| dx \\ &\geqq An^{1/2} \int_0^1 (1-x)^\alpha \left| P_n^{(\alpha,\alpha)}(x) \right| dx \\ &> A, \end{aligned}$$

by [1, p. 168 (7.34.1)]. From this (9.1) follows, and the proof is complete.

As we stated earlier, except when $\lambda=0$, the status of the values $p=2-1/(\lambda+1)$, $p=2+1/\lambda$ is in doubt. Analogy with the case $\lambda=0$ suggests failure of the property at these extreme values, but I am unable to confirm this.

10. **Hermite and Laguerre series.** The Hermitian series of a function $f(x)$ defined on $(-\infty, \infty)$ takes the form

$$\text{(10.1)} \qquad f(x) \sim \sum_0^\infty a_n H_n(x)$$

where [1, p. 238]

$$a_n = \pi^{-1/2} 2^{-n} (n!)^{-1} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x) dx.$$

We may ask whether the condition

$$\text{(10.2)} \qquad \int_{-\infty}^{\infty} |f(x)|^p e^{-x^2} dx < \infty,$$

$f(x)$ measurable, guarantees that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|f(x)-\sum_{0}^{N}a_nH_n(x)\right|^p e^{-x^2}dx=0. \tag{10.3}$$

For $p=2$ it is known that this is the case, for example because the moment problem

$$\mu_n=\int_{-\infty}^{\infty}x^ne^{-x^2}dx$$

is determined [7, p. 61]. We shall show that if $1\leqq p<2$ (and hence if $p>2$) that (10.3) may fail.

Choose c so that $1/2<c<1/p$, and let $f(x)=e^{cx^2}$. Then (10.2) is satisfied. To compute the a_n observe that [1, p. 102 (5.5.7)]

$$\begin{aligned}\sum_{0}^{\infty}a_nw^n &= \pi^{-1/2}\int_{-\infty}^{\infty}e^{-(1-c)x^2}\sum_{0}^{\infty}\frac{w^nH_n(x)}{2^nn!}dx\\ &= \pi^{-1/2}\int_{-\infty}^{\infty}e^{-(1-c)x^2}e^{xw-w^2/2}dx\\ &= K\exp\left\{\frac{c}{1-c}\frac{w^2}{4}\right\}.\end{aligned}$$

Then

$$\begin{aligned}a_{2n+1}&=0,\\ a_{2n}&=K\left(\frac{c}{1-c}\right)^n\frac{1}{4^n}\frac{1}{n!}.\end{aligned} \tag{10.4}$$

We shall prove that

$$\limsup_{n\to\infty}a_{2n}\left\{\int_{-\infty}^{\infty}e^{-x^2}\left|H_{2n}\right|^pdx\right\}^{1/p}=\infty,$$

from which the failure of (10.3) follows. We have

$$\begin{aligned}\left(\int_{-\infty}^{\infty}e^{-x^2}\left|H_{2n}(x)\right|^pdx\right)^{1/p} &\geqq A\int_{-\infty}^{\infty}e^{-x^2}\left|H_{2n}(x)\right|dx\\ &\geqq A\int_{\pi}^{2\pi}e^{-x^2}\left|H_{2n}(x)\right|dx\\ &\geqq A\frac{(2n)!}{n!}\int_{\pi}^{2\pi}e^{-x^2/2}\left|\cos\{(2n+1)^{1/2}x\}\right|dx,\end{aligned}$$

where A is positive [1, p. 194 (8.22.8)]. Combining this with (10.4) yields

$$b_n = a_{2n}\left\{\int_{-\infty}^{\infty} e^{-x^2} \mid H_{2n}(x) \mid^p dx\right\}^{1/p} \geqq \frac{A}{n}\left(\frac{c}{1-c}\right)^n \int_{(2n+1)^{1/2}\pi}^{2(2n+1)^{1/2}\pi} \mid \cos y \mid dy.$$

Now let $n \to \infty$ through those values of n for which $(2n+1)^{1/2}$ is an integer N_n. Then

$$b_{N_n} \geqq \frac{A}{N_n}\left(\frac{c}{1-c}\right)^{N_n} \int_{N_n\pi}^{2N_n\pi} \mid \cos y \mid dy \geqq A\left(\frac{c}{1-c}\right)^{N_n}.$$

Since $c > 1/2$, $\lim_{n\to\infty} b_{N_n} = \infty$, and the proof is complete.

For Laguerre series [1, p. 238] a similar result is true for $f(x) = e^{cx}$, $1/2 < c < 1/p$. The details follow the same lines as in the preceding cases, with the substitution of [1, p. 192 (8.22.1)] for [1, p. 194 (8.22.8)].

References

1. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939.

2. M. Riesz, *Sur les fonctions conjugées*, Math. Zeit. vol. 27 (1927) pp. 218–244.

3. H. Pollard, *The mean convergence of orthogonal series*. I, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 387–403.

3a. ———, *The mean convergence of orthogonal series of polynomials*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 8–10.

4. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1933.

5. S. Bernstein, *Sur les polynômes orthogonaux relatifs à un segment fini*, Journal de Mathématiques (9) vol. 9 (1930) pp. 127–177; vol. 10 (1931) pp. 219–286.

6. E. C. Titchmarsh, *The Fourier integral*, Oxford, 1937.

7. J. A. Shohat and J. D. Tamarkin, *The moment problem*, Mathematical Surveys, vol. 1, New York, 1943.

Cornell University,
Ithaca, N.Y.