ON THE SCHWARZ-CHRISTOFFEL TRANSFORMATION AND p-VALENT FUNCTIONS

BY

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1. **Introduction.** It is well known(1) that the function

$$w = f_1(z) = c_1 \int_0^z \prod_{j=1}^{m} (1 - z_j t)^{-\gamma_j} dt + c_2 \tag{1.1}$$

subject to the conditions

$$|z_j| = 1, \qquad j = 1, 2, \cdots, m, \tag{1.2}$$

$$\sum_{j=1}^{m} \gamma_j = 2, \tag{1.3}$$

$$0 < \gamma_j \leqq 2, \qquad j = 1, 2, \cdots, m, \tag{1.4}$$

maps the open unit circle $|z| < 1$ (hereafter denoted by E) onto P the interior of an m-sided convex polygon. The vertices of the polygon are $w_j = f_1(\bar{z}_j)$ and the exterior angle(2) at the vertex w_j is $\gamma_j \pi$. Conversely if P is given, then z_1, $z_2, \cdots, z_m$, c_1, and c_2 can be determined such that (1.1) maps E onto P, and moreover the origin can be carried into any preassigned point of P and the value of $\arg f'(0)$ can be arbitrarily preassigned. The equation (1.1) subject to the conditions (1.2) and (1.3) is one form of the Schwarz-Christoffel transformation(3).

Schwarz(4) stated that the formula (1.1) is easily generalized to the case where P is a multi-sheeted domain bounded by straight lines and containing branch points, and Christoffel(5) considered this generalization in some detail.

Study(6), Loewner(7), Gronwall(8), Bieberbach(9), Paatero(10), and

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(1) Churchill, *Introduction to complex variables and applications*, New York, McGraw-Hill, 1948.

(2) If $1 \leqq \gamma_j \leqq 2$, then $w_j = \infty$. There is no difficulty in extending the concept of an exterior angle to this case. The region P is unbounded but still convex.

(3) The Schwarz-Christoffel transformation is usually given as a function which maps the upper half-plane onto the interior of a polygon. It is easy to obtain (1.1) from the standard form as indicated in §2.

(4) *Ueber einige Abbildungsaufgaben*, J. Reine Angew. Math. vol. 70 (1869) pp. 105–120, or Mathematische Abhandlungen, vol. 2, pp. 65–83, in particular p. 77.

(5) *Ueber die Abbildung einer n-blattrigen einfach Zusammenhängender ebenen Fläche auf einen Kreise*, Göttingen Nachrichten, 1870, pp. 359–369.

(6) *Vorlesungen über ausgewählten Gegenstände der Geometrie*, vol. 2, Leipzig, Teubner, 1913.

(7) *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises*

Robertson[11] have used the Schwarz-Christoffel transformation as a starting point for the derivation of properties of univalent functions. As far as I have been able to discover, it was Robertson who first pointed out that equation (1.1) leads to a very simple proof that $|b_n| \leqq n|b_1|$ for the coefficients of a univalent function in the special case that the image of E is starlike with respect to the origin.

By using Robertson's methods together with a generalization of (1.1) we are able to prove a number of theorems about certain subclasses of the class of p-valent functions.

As a by-product, we obtain two more proofs that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}, \tag{1.5}$$

and we prove the arithmetic identities

$$D' = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} = 2\sum_{m=2}^{\infty} \frac{1}{(2m-1)^2} H_{2m-2}, \tag{1.6}$$

$$E' = \sum_{m=1}^{\infty} \frac{1}{(2m)^3} = \frac{2}{9}\sum_{m=1}^{\infty} \frac{1}{(2m)^2} H_{2m-1}, \tag{1.7}$$

$$F' = \sum_{m=1}^{\infty} \frac{1}{m^3} = \sum_{m=2}^{\infty} \frac{1}{m^2} H_{m-1}, \tag{1.8}$$

where

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}. \tag{1.9}$$

2. **The generalized Schwarz-Christoffel transformation.** The material of this paragraph is either contained or implied in the works of Schwarz and Christoffel. It is included here for completeness. Let

$$g(u) = \prod_{j=1}^{m} (u_j - u)^{-\gamma_j}, \qquad u_1 < u_2 < \cdots < u_m, \tag{2.1}$$

$|z| < 1$, *die durch Funktionen mit nicht verschwindender Ableitung geliefert werden*, Berichte der Gesellschaft die Wissenschaften zu Leipzig vol. 69 (1917) pp. 89–106.

(8) *Sur la déformation dans la représentation conforme*, C. R. Acad. Sci. Paris vol. 162 (1916) pp. 249–252.

(9) *Aufstellung und Beweis des Drehungssatzes für schlichte konforme Abbildungen*, Math. Zeit. vol. 4 (1919) pp. 295–305.

(10) *Über die conforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Annales Academiae Scientiarum Fennicae, ser. A vol. 33 (1931) pp. 1–78.

(11) *On the theory of univalent functions*, Ann. of Math. vol. 37 (1936) pp. 374–408, in particular p. 380.

$$h(u) = \prod_{j=1}^{p-1} (\alpha_j - u)(\bar{\alpha}_j - u), \qquad \Im(\alpha_j) > 0, \tag{2.2}$$

$$w = f_2(z) = c_3 \int_i^z g(u)h(u)du + c_4, \tag{2.3}$$

where the path of integration is subject to the restrictions $\Im(u) \geqq 0$, $u \neq u_j$, $j = 1, 2, \cdots, m$. When $\Im(u) = 0$, arg $g(u)$ is constant and $h(u) > 0$. So $f_2(z)$ maps each segment $u < u_1$, $u_j < u < u_{j+1}$, $u_n < u$, onto some straight line segment. These image segments may be half-rays extending to infinity, or may be full lines. The function $f_2(z)$ is regular in $\Im(z) > 0$, and has critical points at $z = \alpha_j$. Thus $f_2(z)$ maps $\Im(z) > 0$ onto a multi-sheeted region whose boundary consists only of straight line segments, half-rays, and full lines. We shall refer to such regions as multi-sheeted polygons. The function $f_2(z)$ will be regular and univalent in a neighborhood of infinity if we require that

$$\sum_{j=1}^{m} \gamma_j = 2p. \tag{2.4}$$

Conversely[12], if P is any multi-sheeted polygon subject only to the condition that there exists a function $f(z)$ mapping $\Im(z) > 0$ onto P, regular for $\Im(z) > 0$, and regular and univalent in a neighborhood of infinity, then $f(z)$ has the form (2.3) and (2.4) is satisfied[13].

The substitution $u = i(1-t)/(1+t)$ in the integral (2.3) gives

$$w = f(z) = c_5 \int_0^z \prod_{j=1}^{m} (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-1} (t - \beta_j)(1 - \bar{\beta}_j t)dt + c_4, \tag{2.5}$$

where

$$|z_j| = 1, \qquad z_j \neq z_k \text{ if } j \neq k;\, j, k = 1, 2, \cdots, m \tag{2.6}$$

$$|\beta_j| < 1, \qquad j = 1, 2, \cdots, p - 1. \tag{2.7}$$

Since $u = i(1-t)/(1+t)$ maps E onto the half-plane $\Im(u) > 0$, $w = f(z)$ maps E onto a multi-sheeted polygon P. The vertices of P are $w_j = f(\bar{z}_j)$ and the exterior angle at w_j is $\gamma_j \pi$. P has branch points at $w_j^* = f(\beta_j)$, and the number of sheets tied at w_j^* is just one more than the number of times $(t - \beta_j)$ occurs as a factor in the integrand of (2.5). Again if P is any multi-sheeted polygon subject only to the condition that there exists a function $f(z)$, regular in E and mapping E onto P, then $f(z)$ has the form (2.5) and (2.4) is satisfied.

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(12) We omit the proof of this. It is an elementary generalization of the proof in the plane case. See Julia, *Leçons sur la représentation conforme des aires simplement connexes*, Gauthier-Villars, 1931, pp. 67–71.

(13) A still more general form of the Schwarz-Christoffel transformation was obtained independently by D. Gilbarg, *A generalization of the Schwarz-Christoffel transformation*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 609–612.

3. **Some examples.** The function

$$
\begin{aligned}
w = f(z) &= \int_0^z \frac{(t-a)(1-at)}{1-t^4}\,dt \qquad (-1 < a < 1) \\
&= \frac{1}{4}\left\{(1+a^2)\log\frac{1+z^2}{1-z^2} - 2a\log\frac{1+z}{1-z}\right\}
\end{aligned} \tag{3.1}
$$

maps E onto the two-sheeted region comprised of the two infinite strips $|\Im(w)| < \pi(1-a)^2/8$ and $|\Im(w)| < \pi(1+a)^2/8$. It is perhaps interesting to observe the limit case $a=1$. One of the strips disappears and the limit function maps E onto the infinite strip $|\Im(w)| < \pi/2$, slit along the real axis from $-\infty$ to $-2^{-1}\log 2$, the latter being the limit point of the branch point $f(a)$ of (3.1) as $a \to 1$.

Similarly the function

$$
\begin{aligned}
w &= \int_0^z \frac{(t-a)(1-at)}{1+t^4}\,dt \qquad (-1 < a < 1) \\
&= \frac{i}{4}\left\{a2^{1/2}\log\frac{1+i2^{1/2}z-z^2}{1-i2^{1/2}z-z^2} + (1+a^2)\log\frac{1-iz^2}{1+iz^2}\right\}
\end{aligned} \tag{3.2}
$$

maps E onto a region which consists of the two infinite strips $-\pi(1+a^2)/8 < \Re(w) < \pi(1+2^{3/2}a+a^2)/8$ and $-\pi(1+a^2)/8 < \Re(w) < \pi(1-2^{3/2}a+a^2)/8$. The symmetrical position of the two strips in the first example is lacking in the second example.

As a third example, the function

$$
\begin{aligned}
w &= \int_0^z \frac{(t-a)(1-at)(t-b)(1-bt)}{1-t^6}\,dt \qquad (-1 < a, b < 1) \\
&= \frac{1}{6}\left\{(1+a^2+b^2+ab+a^2b^2)\log\frac{1+z^3}{1-z^3} + 3ab\log\frac{1+z}{1-z}\right. \\
&\qquad \left. + (a+b)(1+ab)\log\frac{1-2z^2+z^4}{1+z^2+z^4}\right\}
\end{aligned} \tag{3.3}
$$

maps E onto a region comprised of the three infinite strips

$$
|\Im(w)| < \frac{\pi}{12}(1+a)^2(1+b)^2,
$$

$$
|\Im(w)| < \frac{\pi}{12}\{(1+a^2)(1+b^2) - 2ab\},
$$

$$
|\Im(w)| < \frac{\pi}{12}(1-a)^2(1-b)^2.
$$

A slightly different example is the function

$$(3.4)\qquad w = \int_0^z \frac{t}{(1-t)(1+t)^3}\,dt = \frac{1}{8}\log\frac{1+z}{1-z} - \frac{z}{4(1+z)^2},$$

which maps E onto the region formed by the two half-planes $\Im(w) > -\pi/16$ and $\Im(w) < \pi/16$ joined at the branch point $w=0$. These two half-planes overlap to cover doubly the strip about the real axis of width $\pi/8$, the rest of the

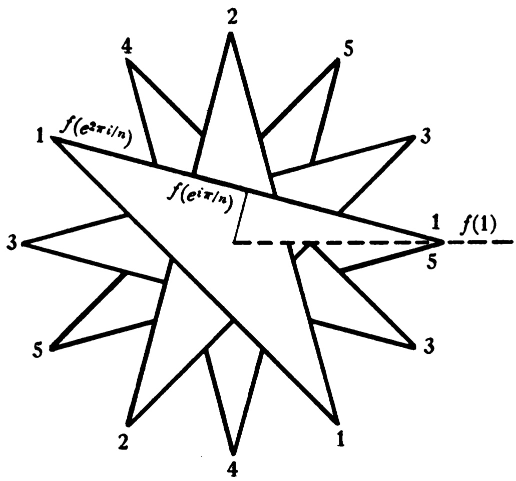

FIG. 1

plane being covered once. One should note that (3.4) is obtained by adding two functions, one of which maps E onto a strip and the other maps E onto a slit plane.

Finally we observe that the function

$$(3.5)\qquad \begin{aligned} w = f(z) &= \int_0^z \frac{t^{p-1}}{(1-t^n)^{2p/n}}\,dt \\ &= \frac{z^p}{p} + \sum_{m=1}^{\infty} \frac{z^{mn+p}}{m!(mn+p)} \prod_{k=0}^{m-1}\left(\frac{2p}{n}+k\right) \end{aligned}$$

maps E onto a regular p-sheeted n-gon. The case $n=12$, $p=5$ is shown in Fig. 1. The number placed by the vertex denotes the sheet in which that vertex lies, when the positive real axis is taken as the tie-line. Of course these numbers are not uniquely determined.

4. **The arithmetic identities.** For the example function (3.5), it is easy to see, either directly from the integral, or by a consideration of the symmetry of the image region, that $f(e^{i\pi/n})$ is a point bisecting the line segment joining $f(1)$ and $f(e^{i2\pi/n})$. So

$$(4.1)\qquad |f(e^{i\pi/n})| = f(1)\cos(p\pi/n),$$

for all positive integers p and n such that $0 < 2p/n < 1$.

If we use the infinite series form of the function, multiply by p, and introduce the new variable $\zeta = 2p/n$,

$$
\begin{aligned}
(4.2)\qquad & 1+\sum_{m=1}^{\infty}\frac{(-1)^m\zeta}{m!(2m+\zeta)}\prod_{k=0}^{m-1}(\zeta+k) \\
&\qquad = \cos(\zeta\pi/2)\left\{1+\sum_{m=1}^{\infty}\frac{\zeta}{m!(2m+\zeta)}\prod_{k=0}^{m-1}(\zeta+k)\right\}.
\end{aligned}
$$

Each side is an analytic function of ζ, for a sufficiently restricted ζ, and since the two functions coincide on the everywhere dense set of rationls $0<\zeta=2p/n<1$, (4.2) is an identity in ζ and we may equate coefficients of like powers of ζ in the power series expansion. For ζ^2 this gives

$$
(4.3)\qquad \frac{\pi^2}{8}=\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2},
$$

from which, by a well known trick, one can obtain (1.5). Equating coefficients of ζ^3 gives (1.6). Other identities can be obtained by using the coefficients of higher powers of ζ, but these appear to be quite complicated and of little interest.

By dissecting the regular p-sheeted n-gon into $2n$ triangles, it is easy to see that the area of that figure is

$$
(4.4)\qquad A = nf(1)\,|\,f(e^{i\pi/n})\,|\,\sin(p\pi/n).
$$

On the other hand, by a well known formula[14] for the area of the image of E, applied to (3.5), we have

$$
(4.5)\quad A=\pi\sum_{m=1}^{\infty}m\,|\,a_m\,|^2=\pi\left\{\frac{1}{p}+\sum_{m=1}^{\infty}\frac{1}{(mn+p)(m!)^2}\prod_{k=0}^{m-1}\left(\frac{2p}{n}+k\right)^2\right\}.
$$

If we equate (4.4) and (4.5), multiply by $2p^2/n$, and introduce the new variable ζ, we find

$$
\begin{aligned}
(4.6)\qquad & \pi\left\{\zeta+\sum_{m=1}^{\infty}\frac{\zeta^2}{(2m+\zeta)(m!)^2}\prod_{k=0}^{m-1}(\zeta+k)^2\right\} \\
&\qquad = 2\sin(\zeta\pi/2)\left\{1+\sum_{m=1}^{\infty}\frac{\zeta}{(2m+\zeta)m!}\prod_{k=0}^{m-1}(\zeta+k)\right\} \\
&\qquad\quad \cdot\left\{1+\sum_{m=1}^{\infty}\frac{(-1)^m\zeta}{(2m+\zeta)m!}\prod_{k=0}^{m-1}(\zeta+k)\right\}.
\end{aligned}
$$

Again we have coincidence of two analytic functions of ζ for an everywhere dense set of rational ζ, in the interval $0<\zeta<1$, and hence (4.6) is an identity. If we equate coefficients of ζ^3 in the power series expansions of (4.6) we have

(14) Pólya-Szegö, *Aufgaben und Lehrsätze*, vol. 1, New York, Dover, 1945, p. 109.

$$\frac{\pi^3}{12} = 2\pi \sum_{m=1}^{\infty} \frac{1}{(2m)^2}, \tag{4.7}$$

and hence (1.5). If we equate coefficients of ζ^4, we obtain (1.7). Finally we can combine (4.1) and (4.4) to obtain

$$A = n\{f(1)\}^2 \cos (p\pi/n) \sin (p\pi/n) = 2^{-1}n\{f(1)\}^2 \sin (2p\pi/n). \tag{4.8}$$

This together with (4.5) yields

$$\begin{aligned} \pi\left\{\zeta + \sum_{m=1}^{\infty} \frac{\zeta^2}{(2m+\zeta)(m!)^2} \prod_{k=0}^{m-1} (\zeta+k)^2\right\} \\ = \sin \zeta\pi \left\{1 + \sum_{m=1}^{\infty} \frac{\zeta}{(2m+\zeta)m!} \prod_{k=0}^{m-1} (\zeta+k)\right\}^2, \end{aligned} \tag{4.9}$$

once more an identity in ζ. Equating coefficients of ζ^4, we obtain (1.8).

The three equations (1.6), (1.7), and (1.8) are certainly not independent. For if we replace the multipliers 2, 2/9, and 1 by unknowns, d, e, and f, it is easy to see that

$$\frac{D'}{d} + \frac{E'}{e} = \frac{F'}{f}, \tag{4.10}$$

and since $E' = F'/8$ and $D' = 7F'/8$,

$$\frac{7}{d} + \frac{1}{e} = \frac{8}{f} \cdot \tag{4.11}$$

Thus any two of the three equations (1.6), (1.7), and (1.8) would imply the third.

5. **The two subclasses of p-valent functions.** We generalize the idea of a convex region to include certain p-sheeted regions in the following way. Let w_b be a boundary point of a p-sheeted region R and let $w_b(r)$ be the subregion of R consisting of those points w of R in the same sheet with w_b and satisfying the inequality $|w - w_b| < r$. If for every boundary point w_b of R there is an $r > 0$ such that $w_b(r)$ is convex, then R is said to be a locally convex region.

Now if $f(z)$ is regular in E and if $f'(re^{i\theta}) \neq 0$, it will map $|z| < r < 1$ onto a region $R(r)$ whose boundary $f(re^{i\theta})$ is an analytic curve with a continuously turning tangent. Let ψ be the angle of intersection of this tangent with the real axis. The angle ψ is not uniquely determined as a function of θ, but will be so if we fix on one of the possible values of ψ when $\theta = 0$, and determine $\psi(\theta)$ for $0 \leqq \theta < 2\pi$ by continuity. Then $\psi'(\theta) \geqq 0$ if and only if $R(r)$ is a locally convex region. It is well known[15] that $\psi'(\theta) = 1 + \Re(zf''(z)/f'(z))$.

[15] Montel, *Leçons sur les fonctions univalentes ou multivalentes*, Gauthier-Villars, 1933, pp. 11–14.

DEFINITION. The function $f(z)$ is said to be an element of the class $C(p)$, p a positive integer, if it is regular in E, if $f(0)=0$, and if there is a $\rho<1$ such that for all r in the interval $\rho<r<1$

$$G(r,\theta) = 1 + \Re\left(re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) > 0, \qquad 0 \leqq \theta \leqq 2\pi, \tag{5.1}$$

and

$$\int_0^{2\pi} G(r,\theta)d\theta = 2\pi p. \tag{5.2}$$

If $f(z)\in C(p)$, it maps $|z|<r<1$ onto a locally convex region. Furthermore $f'(z)$ has exactly $p-1$ roots in the circle $|z|<r$, multiple roots being counted in accordance with their multiplicities. For if ν is the number of these roots,

$$\begin{aligned} 2\pi p &= \int_0^{2\pi} G(r,\theta)d\theta = \Re\oint_{z=re^{i\theta}}\left(1+\frac{zf''(z)}{f'(z)}\right)d\theta \\ &= 2\pi + \Re\,\frac{1}{i}\oint\frac{f''(z)}{f'(z)}\,dz = 2\pi + 2\pi\nu. \end{aligned}$$

Finally $f(z)$ is at most p-valent in E. For the contour integral which gives the number of roots of $f(z)-c$ is just the variation of arg $(f(z)-c)$ along the contour, divided by 2π. But the bounding curve of $R(r)$ for a function of class $C(p)$ is a curve which turns continuously in a counterclockwise manner as θ runs from 0 to 2π, and the total number of complete turns is exactly p. Thus the variation in arg $(f(z)-c)$ cannot exceed $2\pi p$. It is obvious that $f(z)\in C(p)$ may be divalent.

To generalize the concept of a plane region starlike with respect to a point, to include certain p-sheeted regions, we consider the line joining a boundary point w_b with the given point. If as w_b describes the boundary of R, the line turns continuously in a counterclockwise direction (or continuously in a clockwise direction) then R is said to be starlike with respect to the given point[16].

Now let $F(z)$ be regular in E and $F(re^{i\theta})\neq 0$ for $\rho<r<1$. If $\phi=\arg F(re^{i\theta})$, ϕ is not uniquely determined as a function of θ, but will be so if we fix on one of the possible values of ϕ when $\theta=0$, and determine $\phi(\theta)$ for $0\leqq\theta<2\pi$ by continuity. Then $\phi'(\theta)>0$ if and only if $R(r)$ is starlike with respect to the origin. It is well known[15] that $\phi'(\theta)=\Re(zF'(z)/F(z))$.

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[16] This concept of a generalized starlike function has been used previously by the following authors: Obrechkoff, Bull. Sci. Math. (2) vol. 60 (1935) pp. 36–42; Ozaki, Science Reports of the Tokyo Bunrika Daigaku Section A, 2 No. 32 and 36 (1936) and 4 No. 77 (1941); Robertson, Ann. of Math. vol. 38 (1937) pp. 770–783 and vol. 42 (1941) pp. 829–838, Duke Math. J. vol. 12 (1945) pp. 669–684; Biernacki, Mathematica Timisoara vol. 23 (1949) pp. 54–59.

DEFINITION. The function $F(z)$ is said to be an element of the class $S(p)$, p a positive integer, if it is regular in E, if $F(0)=0$, and if there is a ρ such that for all r in the interval $\rho<r<1$

$$H(r, \theta) = \Re\left(\frac{re^{i\theta}F'(re^{i\theta})}{F(re^{i\theta})}\right) > 0, \qquad 0 \leqq \theta \leqq 2\pi, \tag{5.3}$$

and

$$\int_0^{2\pi} H(r, \theta)d\theta = 2\pi p. \tag{5.4}$$

If $F(z)\in S(p)$, it maps $|z|<r<1$ onto a region starlike with respect to the origin. Just as (5.2) implied that $f'(z)$ had $p-1$ roots in E, the condition (5.4) implies that $F(z)$ has p roots in E. To see that $F(z)$ is not more than p-valent in E consider $\delta(\lambda)=\Delta$ arg $(F(re^{i\theta})-\lambda c)$, where Δ denotes the change as θ runs from 0 to 2π. Let c be fixed and let λ vary from 0 to 1. We already have $\delta(0)=2\pi p$. But $\delta(\lambda)$ is always an integer multiple of 2π, and is a continuous function of λ except when $\lambda c=w_b$, a boundary point of the image of $|z|<r$. At such points $\delta(\lambda)$ jumps $\pm 2\pi$. By (5.3) and (5.4) there are exactly p such boundary points for every c. Finally we remark that for $|c|$ sufficiently large, $\delta(1)=0$, so that the jumps in $\delta(\lambda)$ must be -2π and $0\leqq\delta(\lambda)\leqq 2\pi p$ for all λ and c.

For $p=1$, $C(p)$ and $S(p)$ are the classical univalent functions, convex and starlike respectively, and $S(1)\supset C(1)$. It is worth noting that for $p\geqq 2$, $S(p)\not\supset C(p)$.

LEMMA 1[17]. *Let $c\neq 0$ be an arbitrary constant. If $f(z)\in C(p)$, then $F(z)=czf'(z)\in S(p)$, and conversely if $F(z)\in S(p)$, then $f(z)\in C(p)$, where*

$$f(z) = c\int_0^z \frac{F(t)}{t}\,dt. \tag{5.5}$$

Proof. For $f(z)$ and $F(z)$ related as indicated,

$$z\frac{F'(z)}{F(z)} = z\frac{cf'(z) + czf''(z)}{czf'(z)} = 1 + z\frac{f''(z)}{f'(z)}, \tag{5.6}$$

so that (5.3) implies (5.1) and (5.4) implies (5.2), and conversely.

LEMMA 2. *Let $f(z)=z^q+\cdots$, with critical points $\beta_1, \beta_2, \cdots, \beta_{p-q}\neq 0$ in E, be an element of $C(p)$, and suppose further that $f(z)$ is regular for $|z|=1$. There is a sequence of functions of the form*

([17]) First proved in the univalent case by J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. vol. 17 (1915) pp. 12–22. See also Montel, loc. cit.

$$(5.7)\quad f_m(z) = A_m \int_0^z t^{q-1} \prod_{j=1}^{m} (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-q} \left(1 - \frac{t}{\beta_j^{(m)}}\right)(1 - \bar{\beta}_j^{(m)} t)dt$$

with $|z_j| = 1$, $|\beta_j^{(m)}| < 1$, $0 < \gamma_j < 1$ *for* $j = 1, 2, \cdots, m$, *and*

$$(5.8)\quad \sum_{j=1}^{m} \gamma_j = 2p,$$

such that, as $m \to \infty$, $\beta_j^{(m)} \to \beta_j$, $A_m \to q$, *and* $f_m(z) \to f(z)$ *uniformly for* $|z| \leq r < 1$.

The proof of this lemma is analogous to the one given by Robertson[18] and will be omitted. We can apply this lemma to functions of class $C(p)$ since if $f(z) \in C(p)$, then $f(rz)/r^q$ will satisfy the conditions for the lemma for every r, $\rho < r < 1$. Finally in view of the convergence properties we may consider not (5.7) but the simplified version

$$(5.9)\quad f_m(z) = q \int_0^z t^{q-1} \prod_{j=1}^{m} (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-q} \left(1 - \frac{t}{\beta_j}\right)(1 - \bar{\beta}_j t)dt.$$

LEMMA 3. *Let*

$$\prod_{j=1}^{m} (1 - z_j t)^{-\gamma_j} = 1 + \sum_{n=1}^{\infty} c_n t^n,$$

where z_j *and* γ_j *are subject to the conditions of Lemma* 2. *Then* $|c_n| \leq C_{n+2p-1,2p-1}$, *with equality if and only if* $z_1 = z_2 = \cdots = z_m$.

Proof. Clearly $c_n = c_n(z_1, z_2, \cdots, z_m)$ is a homogeneous polynomial of nth degree with positive coefficients. Hence a maximum occurs when all z_j are equal. The value of the maximum is easily obtained by setting all $z_j = 1$. Thus $C_{n+2p-1,2p-1}$ is just the sum of the coefficients of the polynomial. To see that this is the only case in which equality occurs, suppose without loss of generality that $z_1 = 1$ and $z_k = e^{i\theta} \neq 1$. Since $c_n(z_1, z_2, \cdots, z_m)$ contains the terms z_1^n and $z_1^{n-1} z_k$ both with positive coefficients, it follows that in this case $|c_n| < C_{n+2p-1,2p-1}$.

THEOREM 1. *Let* $f(z) \in C(p)$, *of the form*

$$(5.10)\quad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \qquad 1 \leq q \leq p,$$

having $p - q$ *critical points* $\beta_1, \beta_2, \cdots, \beta_{p-q} \neq 0$ *in* E. *Then*

$$(5.11)\quad |f(re^{i\theta})| \leq f_M(r), \qquad 0 \leq r < 1,$$

and

[18] Loc. cit. footnote 11, pp. 376–377.

$$|a_n| \leqq A_n, \qquad n = q+1, q+2, \cdots, \tag{5.12}$$

where

$$\begin{aligned} f_M(z) &= q\int_0^z \frac{t^{q-1}}{(1-t)^{2p}} \prod_{j=1}^{p-q}\left(1+\frac{t}{|\beta_j|}\right)(1+t|\beta_j|)dt\,. \\ &= z^q + \sum_{n=q+1}^{\infty} A_n z^n. \end{aligned} \tag{5.13}$$

The bounds (5.11) *and* (5.12) *are sharp, since* $f_M(z)\in C(p)$. *The extremal function* $f_M(z)$ *maps* E *onto a region* R_M *consisting of* $p-1$ *full planes and a half-plane,* $\Re(w)<f_M(-1)$ *if* p *is odd,* $\Re(w)>f_M(-1)$ *if* p *is even.*

Proof. The function $f(z)$ may be approximated by a sequence of functions of the form (5.9). Without loss of generality set $z_1=1$. Then by Lemma 3, the maximum coefficients in the power series for the first product in (5.9) occur when $z_2=z_3=\cdots=z_m=1$. These coefficients are then positive, and hence in combining the second product with the first, maximal coefficients are obtained by replacing β_j by $-|\beta_j|$ for $j=1, 2, \cdots, p-q$. Then (5.9) becomes (5.13) and the inequality (5.12) is established. The inequality (5.11) can be obtained by a similar argument, but it is simpler to observe that (5.11) is a consequence of (5.12), since all the coefficients of $f_M(z)$ are positive.

Since $G(1,\theta)=0$ for $f_M(z)$ and since further $f_M'(z)$ does not vanish on $|z|=1$, the boundary of R consists of a single straight line. The reality of the coefficients implies that the line is symmetric about the real axis, that is, orthogonal to the real axis. Finally by noting that $f_M'(-1)$ has the sign of $(-1)^{p-1}$, the position of the half-plane is easily determined.

THEOREM 2. *Let* $F(z)\in S(p)$ *of the form* (5.10) *have* $p-q$ *roots* $\beta_1, \beta_2, \cdots, \beta_{p-q}\neq 0$ *in* E. *Then*

$$|F(re^{i\theta})| \leqq F_M(r), \qquad 0\leqq r<1, \tag{5.14}$$

and

$$|a_n| \leqq B_n, \qquad n = q+1, q+2, \cdots, \tag{5.15}$$

where

$$F_M(z) = \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|) = z^q + \sum_{n=q+1}^{\infty} B_n z^n. \tag{5.16}$$

The bounds (5.14) *and* (5.15) *are sharp, since* $F_M(z)\in S(p)$. *The extremal function* $F_M(z)$ *maps* E *onto a region* R_M *consisting of* $p-1$ *full planes and one plane with a single radial slit.*

Proof. The inequality (5.15) follows from (5.12) by applying Lemma 1 and Theorem 1 to the function $F(z)=zf'(z)/q$, where $f(z)$ satisfies the conditions of Theorem 1. The inequality (5.14) is a consequence of (5.15). Since $H(1,\theta)=0$ for $F_M(z)$ the boundary of R_M consists of radial lines. But the only root of $F'_m(z)$ on $|z|=1$ is the simple root at $z=-1$, and the only singularity is the pole at $z=+1$. Hence the boundary of R_M consists of a single radial line.

There are some special cases of these two theorems which are worth mentioning. Let us suppose that instead of fixing $|\beta_j|$ for the critical points of $f(z)$ or the roots of $F(z)$, we merely require that there is a $\rho>0$ such that $|\beta_j| \geqq \rho$ for $j=1, 2, \cdots, p-q$. Then since $|\bar{\beta}_j+\beta_j^{-1}| \leqq |\beta_j|+|\beta_j|^{-1} \leqq \rho+\rho^{-1}$, it is easy to see that the extremalizing functions of Theorems 1 and 2 must be replaced by

$$f_M(z) = q\int_0^z \frac{t^{q-1}}{(1-t)^{2p}}\left(1+\frac{t}{\rho}\right)^{p-q}(1+\rho t)^{p-q}dt, \tag{5.17}$$

and

$$F_M(z) = \frac{z^q}{(1-z)^{2p}}\left(1+\frac{z}{\rho}\right)^{p-q}(1+\rho z)^{p-q}, \tag{5.18}$$

with the same conclusions holding.

In the special case that $q=p$, we have

$$f_M(z) = p\int_0^z \frac{t^{p-1}}{(1-t)^{2p}}dt = \sum_{n=p}^{\infty} \frac{p}{n} C_{n+p-1,2p-1}z^n \tag{5.19}$$

and

$$F_M(z) = \frac{z^p}{(1-z)^{2p}} = \sum_{n=p}^{\infty} C_{n+p-1,2p-1}z^n, \tag{5.20}$$

again with the same conclusions holding.

This last special case does not require the Schwarz-Christoffel transformation for the proof. For any function $F(z)=z^p+\cdots \in S(p)$ can be expressed as $\{G(z)\}^p$, where $G(z)=z+\cdots \in S(1)$, and since $z/(1-z)^2$ is the extremalizing function for $S(1)$, (5.20) follows. This bound $|a_n| \leqq C_{n+p-1,2p-1}$ was obtained previously by Robertson[19] for a larger class of functions.

Each of the integrals (5.13), (5.17), and (5.19) may be expressed in terms of a finite number of the elementary functions. However, the last one (5.19) can be obtained in a simple way directly from the mapping properties of the function.

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(19) *A representation of all analytic functions in terms of functions with positive real parts*, Ann, of Math. vol. 38 (1937) pp. 770–783, in particular p. 778, inequality (5.8). See also *Star center points of multivalent functions*, Duke Math. J. vol. 12 (1945) pp. 669–684, in particular p. 681, inequality (6.9).

For simplicity let p be odd and let $s(z)$ be defined by the following properties. $s(z)$ maps E onto a region S consisting of $p-1$ full planes and a half-plane $\Re(w)>-1$, all tied at $w=0$. Thus $s(z)$ has a pth order root at $z=0$. Let $s^{(p)}(0)>0$. The function $s(z)$ is now determined uniquely. The symmetry of S about the real axis shows that the interval $-1<z<1$ goes into a real segment, and since p is odd and $s'(z)\neq 0$ for $z\neq 0$, this segment is the half-line $-1<w<\infty$. So $z=1$ is the only singularity of $s(z)$ for $|z|\leqq 1$. By the Schwarz reflection principle we can continue $s(z)$ across the circle $|z|=1$. The reflection of S across the line $\Re(w)=-1$ shows that $s(z)$ maps $|z|>1$ on a region S^* consisting of $p-1$ full planes and a half-plane $\Re(w)<-1$ all tied at $w=-2$, the image of $z=\infty$. Thus $s(z)$ is regular in the entire complex plane with the exception of the point $z=1$, and maps the plane on a region consisting of $2p-1$ sheets. Therefore $s(z)$ is a rational function of degree $2p-1$, and since $s(z)$ takes every value $2p-1$ times, $z=1$ is a pole of order $2p-1$ and

$$s(z)=\frac{a_pz^p+a_{p+1}z^{p+1}+\cdots+a_{2p-1}z^{2p-1}}{(1-z)^{2p-1}}. \tag{5.21}$$

On the other hand, consideration of S^* shows that $s_1(z)=-2-s(1/z)$ also maps $|z|<1$ onto S and takes $-1<z<1$ onto the half-line $-1<w<\infty$. Hence $s(z)=s_1(z)$ or $s(z)+s(1/z)=-2$. Using this with (5.21) gives

$$s(z)=\frac{-2\sum_{n=p}^{2p-1}(-1)^nC_{2p-1,n}z^n}{(1-z)^{2p-1}}. \tag{5.22}$$

But except for a magnification the image of E for $s(z)$ is the same as that given by (5.19). Therefore for p odd

$$f_M(z)=p\int_0^z\frac{t^p}{(1-t)^{2p}}dt=\frac{(-1)^p\sum_{n=p}^{2p-1}(-1)^nC_{2p-1,n}z^n}{C_{2p-1,p}(1-z)^{2p-1}}. \tag{5.23}$$

A similar argument can be given to show that (5.23) also holds for p even.

The form of the function (5.18) can also be obtained directly from the mapping properties. For suppose $s(z)$ maps E onto a region S consisting of $p-1$ full planes and a plane with a radial slit along the real axis. The z-plane is rotated so that $s(1)=\infty$. The function is to have a qth order root at $z=0$ ($q\geqq 1$) and a $(p-q)$th order root at $z=-\rho$, $0<\rho<1$. Finally we may require that $s^{(q)}(0)>0$, which because of the symmetry of S implies that the interval $0\leqq z<1$ goes into $0\leqq w<\infty$. Reflection across the radial slit gives S^* just a duplication of S, and $s(z)=s(1/z)$. Just as before $s(z)$ is a rational function of degree $2p$, with a pole of order $2p$ at $z=1$ and a root of order $p-q$ at $-1/\rho$. So

$$s(z) = c\frac{z^q(1 + z/\rho)^{p-q}(1 + \rho z)^{p-q}}{(1 - z)^{2p}}u(z), \tag{5.24}$$

where $u(z)$ is a polynomial of degree not greater than q. Using $s(z)=s(1/z)$ with (5.24) yields $u(z)=1$ and $s(z)$ is identical with (5.18) when $c=1$. A similar argument can be given to obtain (5.16).

THEOREM 3. *Let $F(z)\in S(p)$ of the form* (5.10) *have $p-q$ roots $\beta_1, \beta_2, \cdots, \beta_{p-q}$ such that $0<|\beta_j|\leqq\rho<1$ for $j=1, 2, \cdots, p-q$. Then for $\rho\leqq r\leqq 1$*

$$\tag{5.25}\begin{aligned}|F(re^{i\theta})| &\geqq \frac{r^q}{(1 + r)^{2p}}\prod_{j=1}^{p-q}(1 - r|\beta_j|)\left(\frac{r}{|\beta_j|} - 1\right)\\ &\geqq \frac{r^q(1 - \rho r)^{p-q}(r/\rho - 1)^{p-q}}{(1 + r)^{2p}}.\end{aligned}$$

The bounds are sharp, the first equality occuring for the function (5.16) *and the second for the function* (5.18) *when $z=-r$.*

Proof. We first note that for $|\beta_j|\leqq\rho\leqq r=|z|\leqq 1$, $|(1-z/\beta_j)(1-\bar{\beta}_jz)| = |(\beta_j-z)(\bar{\beta}_j^{-1}-z)\bar{\beta}_j/\beta_j| \geqq (r-|\beta_j|)(|\beta_j|^{-1}-r)$, with equality if and only if $\arg z=\arg\beta_j$. Using Lemma 1 and Lemma 2 in an obvious fashion the theorem follows at once.

COROLLARY. *If $F(z)=z^p+\cdots\in S(p)$, then*

$$|F(re^{i\theta})| \geqq \frac{r^p}{(1 + r)^{2p}}, \qquad 0 \leqq r \leqq 1, \tag{5.26}$$

and this inequality is sharp.

This can also be obtained directly from the theorem for univalent functions.

THEOREM 4. *Let $f(z)=z+\cdots\in C(p)$ have critical points $\beta_1, \beta_2, \cdots, \beta_{p-1}\neq 0$ in E. Let r_1 be the least positive root of*

$$0 = 1 - r\left[\frac{2p}{1 + r} + \sum_{j=1}^{p-1}\frac{1}{|\beta_j| - r} + \frac{|\beta_j|}{1 - r|\beta_j|}\right] = J_1(r). \tag{5.27}$$

Then $f(rz)/r\in C(1)$ for every r, $0\leqq r\leqq r_1$. The function

$$f_M(z) = \int_0^z\frac{1}{(1 + t)^{2p}}\prod_{j=1}^{p-1}\left(1 - \frac{t}{|\beta_j|}\right)(1 - t|\beta_j|)dt \tag{5.28}$$

shows that the upper bound r_1 cannot be increased.

Proof. Let $\{f_m(z)\}$ be a sequence of functions of the form (5.9) with $q=1$ which converges to $f(z)$. For each $f_m(z)$

$$(5.29)\qquad 1+\frac{zf''_m(z)}{f'_m(z)} = 1+\sum_{j=1}^{m}\gamma_j\frac{zz_j}{1-zz_j} - \sum_{j=1}^{p-1}\frac{z}{\beta_j-z}+\frac{z\bar{\beta}_j}{1-z\bar{\beta}_j}.$$

For $|z|=r<1, \Re(zz_j/(1-zz_j))\geqq -r/(1+r)$. For $|z|\leqq r<\min\{|\beta_1|, |\beta_2|, \cdots, |\beta_{p-1}|\}$,

$$\left|\Re\sum_{j=1}^{p-1}\frac{z}{\beta_j-z}+\frac{z\bar{\beta}_j}{1-z\bar{\beta}_j}\right| \leqq \left|\sum_{j=1}^{p-1}\frac{z}{\beta_j-z}+\frac{z\bar{\beta}_j}{1-z\bar{\beta}_j}\right| \leqq \sum_{j=1}^{p-1}\frac{r}{|\beta_j|-r}+\frac{r|\beta_j|}{1-r|\beta_j|}.$$

Therefore

$$(5.30)\qquad 1+\Re\left(\frac{zf''_m(z)}{f'_m(z)}\right)\geqq J_1(r)\geqq 0$$

for $0\leqq|z|\leqq r_1$.

THEOREM 5. *Let $F(z)=z+\cdots\in S(p)$ have roots $\beta_1, \beta_2, \cdots, \beta_{p-1}\neq 0$ in E. Let r_1 be defined as in Theorem 4. Then $F(rz)/r\in S(1)$ for every r, $0\leqq r\leqq r_1$.*

Proof. Let $f(z)$ be defined by (5.5). Then for each sequence $\{f_m(z)\}$ converging to $f(z)$, the sequence $\{F_m(z)=zf'_m(z)\}$ converges to $F(z)$. Finally

$$\Re\left(z\frac{F'_m(z)}{F_m(z)}\right) = 1+\Re\left(z\frac{f''_m(z)}{f'_m(z)}\right)\geqq J_1(r)\geqq 0$$

for $0\leqq|z|\leqq r_1$.

Theorems 4 and 5 are special cases of Theorems 6 and 7 respectively. The proofs are similar and so are omitted.

THEOREM 6. *Let $f(z)=z^q+\cdots\in C(p)$ have critical points $\beta_1, \beta_2, \cdots, \beta_{p-q}\neq 0$ in E. Let r_q be the least positive root of*

$$(5.31)\qquad 0 = q - r\left[\frac{2p}{1+r}+\sum_{j=1}^{p-q}\frac{1}{|\beta_j|-r}+\frac{|\beta_j|}{1-r|\beta_j|}\right] = J_q(r).$$

Then $f(rz)/r^q\in C(q)$ for every r in the interval $0\leqq r\leqq r_q$. The function (5.13) *shows that the upper bound r_q cannot be increased.*

THEOREM 7. *Let $F(z)=z^q+\cdots\in S(p)$ have roots $\beta_1, \beta_2, \cdots, \beta_{p-q}\neq 0$ in E. Let r_q be defined as in Theorem 6. Then $F(rz)/r^q\in S(q)$ for every r, $0\leqq r\leqq r_q$. The function* (5.16) *shows that the upper bound r_q cannot be increased.*

Since $J_q(r)$ is a decreasing function of $|\beta_j|$ for $r<|\beta_j|<1$, we obtain bounds for r_q by solving the cubic

$$0 = q - r\left[\frac{2p}{1+r} + (p-q)\left(\frac{1}{y-r} + \frac{y}{1-ry}\right)\right] \tag{5.32}$$

for its least positive root. This gives

$$\rho(y) = \frac{p(1+y)^2 - 2qy - (1+y)(p^2(1+y)^2 - 4pqy)^{1/2}}{2qy}, \tag{5.33}$$

so that if $0 < m \leqq |\beta_j| \leqq M < 1$ for $j = 1, 2, \cdots, p-q$, then

$$\frac{qm}{p(1+m)^2} \leqq \rho(m) \leqq r_q \leqq \rho(M). \tag{5.34}$$

6. **The coefficient problem.** It has recently[20] been conjectured that if

$$F(z) = \sum_{n=1}^{\infty} b_n z^n \tag{6.1}$$

is p-valent in E, then for $n = p+1, p+2, \cdots$

$$|b_n| \leqq \sum_{k=1}^{p} \frac{2k(n+p)!}{(n^2-k^2)(p+k)!(p-k)!(n-p-1)!} |b_k|. \tag{6.2}$$

For $p = 2$, $n = 3$, this gives the conjecture that

$$|b_3| \leqq 5|b_1| + 4|b_2|. \tag{6.3}$$

THEOREM 8. *Let $F(z) \in S(2)$ have the form* (6.1), *and let all the coefficients b_n be real. Then* (6.3) *is valid and this inequality is sharp for every pair $|b_1|$, $|b_2|$, not both zero.*

Proof. We may assume $|b_1| \neq 0$, for if $b_1 = 0$ (6.3) is a special case of Theorem 2 with $p = q = 2$. By Lemma 1 and Lemma 2, we have a sequence of functions $F_m(z)$ of the form

$$\begin{aligned} F_m(z) &= c_1 z\left(1 - \frac{z}{\beta}\right)(1 - \bar{\beta}z)\prod_{j=1}^{m}(1 - z_j z)^{-\gamma_j} \\ &= c_1 z\left(1 - \frac{z}{\beta}\right)(1 - \bar{\beta}z)\sum_{n=0}^{\infty} A_n z^n = \sum_{n=1}^{\infty} c_n z^n, \end{aligned} \tag{6.4}$$

where

$$\sum_{j=1}^{m} \gamma_j = 4, \qquad 0 < \gamma_j, \qquad z_j = \cos\theta_j + i \sin\theta_j. \tag{6.5}$$

The sequence $F_m(z)$ converges to $F(z)$ and $c_n \to b_n$ as $m \to \infty$. Since $b_1 \neq 0$,

[20] *On some determinants related to p-valent functions*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 175–192.

$\beta \neq 0$, and since all coefficients are real, the single critical point must lie on the real axis, that is $\beta = \bar{\beta}$. From (6.4)

$$\begin{aligned} A_0 &= 1, \\ A_1 &= \sum_{j=1}^{m} \gamma_j z_j, \\ A_2 &= \frac{1}{2} \sum_{j=1}^{m} \gamma_j z_j^2 + \frac{1}{2}\left(\sum_{j=1}^{m} \gamma_j z_j \right)^2, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} c_2 &= A_1 c_1 - c_1(\beta + \beta^{-1}), \\ c_3 &= A_2 c_1 - A_1 c_1(\beta + \beta^{-1}) + c_1, \end{aligned} \tag{6.7}$$

from which

$$c_3 = c_1(1 + A_2 - A_1^2) + A_1 c_2. \tag{6.8}$$

It is clear from the conditions (6.5) that $|A_1| \leqq 4$. We need only prove that $|1 + A_2 - A_1^2| \leqq 5$. Since all the coefficients are real, the image of E under $f(z)$ will be symmetric about the real axis. In selecting a polygon for approximation, we may require that this polygon is also symmetric about the real axis. Then in the sums (6.6), the z_j occur in conjugate pairs and each element of a conjugate pair is multiplied by the same γ_j. So

$$\begin{aligned} A_2 - A_1^2 &= \frac{1}{2} \sum_{j=1}^{m} \gamma_j z_j^2 - \frac{1}{2}\left(\sum_{j=1}^{m} \gamma_j z \right)^2 \\ &= \frac{1}{2} \sum_{j=1}^{m} \gamma_j (2 \cos^2 \theta_j - 1) - \frac{1}{2}\left(\sum_{j=1}^{m} \gamma_j \cos \theta_j \right)^2 \\ &= -2 + \sum_{j=1}^{m} \gamma_j \cos^2 \theta_j - \frac{1}{2}\left(\sum_{j=1}^{m} \gamma_j \cos \theta_j \right)^2. \end{aligned} \tag{6.9}$$

By Cauchy's inequality,

$$\sum_{j=1}^{m} \gamma_j \cos^2 \theta_j - \frac{1}{4}\left(\sum_{j=1}^{m} \gamma_j \cos \theta_j \right)^2 \geqq 0,$$

so, using (6.5), we have $-6 \leqq A_2 - A_1^2 \leqq 2$ or

$$-5 \leqq 1 + A_2 - A_1^2 \leqq 3. \tag{6.10}$$

We shall see when we have proved Theorem 12 that (6.3) is sharp for every pair $|b_1|$, $|b_2|$, and that the inequalities of Theorems 9, 10, and 11 are also sharp in the same sense.

Using Lemma 1 and Theorem 8 we have immediately the following theorem.

THEOREM 9. *Let $f(z) \in C(2)$ have the form*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{6.11}$$

and let all coefficients be real. Then

$$|a_3| \leqq \frac{5}{3}|a_1| + \frac{8}{3}|a_2|, \tag{6.12}$$

and this inequality is sharp for every pair $|a_1|$, $|a_2|$ not both zero.

Notice that the conjecture (6.2) for p-valent functions suggests the conjecture that

$$|a_n| \leqq \sum_{k=1}^{p} \frac{2k^2(n+p)!}{n(n^2-k^2)(p+k)!(p-k)!(n-p-1)!}|a_k| \tag{6.13}$$

for functions of class $C(p)$ of the form (6.11). This of course gives (6.12) for $p=2$ and $n=3$. It also seems reasonable to conjecture that the bounds obtained in Theorems 2, 3, and 7 are valid for all functions regular and p-valent in E.

The same methods yield the following extensions of Theorems 8 and 9.

THEOREM 10. *Let $F(z) \in S(p)$ be of the form* (6.1) *with $b_1 = b_2 = \cdots = b_{p-2} = 0$, and let all coefficients be real. Then*

$$|b_{p+1}| \leqq (2p+1)(p-1)|b_{p-1}| + 2p|b_p|, \tag{6.14}$$

and this inequality is sharp for every pair $|b_{p-1}|$, $|b_p|$ not both zero.

THEOREM 11. *Let $f(z) \in C(p)$ be of the form* (6.11) *with $a_1 = a_2 = \cdots = a_{p-2} = 0$, and let all the coefficients be real. Then*

$$|a_{p+1}| \leqq \frac{(2p+1)(p-1)^2}{p+1}|a_{p-1}| + \frac{2p^2}{p+1}|a_p|, \tag{6.15}$$

and this inequality is sharp for every pair $|a_{p-1}|$, $|a_p|$ not both zero.

The inequalities (6.14) and (6.15) are special cases of the conjectures (6.2) and (6.13) respectively.

It has been shown[20] that for every set $|b_1|, |b_2|, \cdots, |b_p|$ not all zero, there is a p-valent function with b_n satisfying (6.2) with the equality sign. All of these p-valent functions are of the type described in the following theorem and so are of class $S(p)$. From these we can obtain functions of class $C(p)$ for which the equality sign holds in (6.13) for every set $|a_1|$, $|a_2|, \cdots, |a_p|$ not all zero.

THEOREM 12. *Let*

$$f(z) = P(u) = \sum_{k=1}^{p} (-1)^k a_k u^k, \qquad u = \frac{z}{(1-z)^2}, \tag{6.16}$$

where $a_k \geqq 0$, $k=1, 2, \cdots, p$, and for at least one k, $a_k > 0$. Then $f(z)$ is an element of class $S(p)$ and, moreover, maps E onto a region R consisting of $p-1$ full planes and a plane with a single radial slit.

Proof. It will be simpler and completely equivalent to consider $g(z) = f(-z)$. Then

$$g(z) = a_1 v + a_2 v^2 + \cdots + a_p v^p, \qquad v = \frac{z}{(1+z)^2}, \tag{6.17}$$

and for $g(z)$

$$\begin{aligned} H(r, \theta) &= \Re\left(z \frac{g'(z)}{g(z)}\right) = \Re\left(\frac{a_1 + 2a_2 v + \cdots + p a_p v^{p-1}}{a_1 + a_2 v + \cdots + a_p v^{p-1}} \frac{z}{v} \frac{dv}{dz}\right) \\ &= \Re\left(Q(v) \frac{z}{v} \frac{dv}{dz}\right). \end{aligned} \tag{6.18}$$

Now if $z = e^{i\theta}$, then $v \geqq 1/4$, $Q(v) > 0$, and

$$T(z) = \frac{z}{v} \frac{dv}{dz} = \frac{1-z}{1+z} = i \frac{\cos\theta - 1}{\sin\theta}. \tag{6.19}$$

Therefore $G(1, \theta) = \Re(Q(v) T(e^{i\theta})) = 0$, and the boundary of R consists of radial slits. Since $g'(z)$ has only a single simple root on $|z| = 1$, there is only a single slit.

The function $h(z) = zg'(z)/g(z)$ is regular in some ring domain $\rho < |z| < 1$. To determine $\Re(h(z))$ for $|z| < 1$, we examine more closely $h(z)$ for $|z| = 1$.

A simple computation shows that

$$Q'(v) = \frac{\sum_{\mu > \nu = 1}^{p} a_\mu a_\nu (\mu - \nu)^2 v^{\nu + \mu - 3}}{(a_1 + a_2 v + \cdots + a_p v^{p-1})^2}. \tag{6.20}$$

If only one coefficient a_k is different from zero, then $Q(v)$ is a positive constant. Otherwise $Q'(v) > 0$ for $v \geqq 1/4$.

Let $z = e^{i\theta}$ and let θ vary from 0 to π. Then $T(z)$ runs along the imaginary axis from 0 to $-i\infty$, $v(e^{i\theta})$ runs along the real axis from 1/4 to $+\infty$, and $Q(v)$ is either a positive constant or a monotonically increasing positive function. Thus $h(z)$ maps this arc of the unit circle in a one-to-one manner on the negative imaginary axis, a counterclockwise direction for z, corresponding to a downward direction for $h(z)$. As θ runs from π to 2π, $T(z)$ runs along the imaginary axis from $+\infty$ to $1/4$, and $Q(v)$ is either a positive constant or a monotonically decreasing positive function. Thus $h(z)$ maps this arc in a one-to-one manner on the positive imaginary axis, with the same correspondance of directions as before. Since a regular function is region-preserving there exists a ρ such that for $\rho < |z| < 1$

$$H(r, \theta) = \Re(h(z)) > 0,$$

and hence $g(z) \in S(p)$.

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