ON THE MEASURE OF CARTESIAN PRODUCT SETS

BY

GERALD FREILICH

1. **Introduction.** There are various measure functions defined on the set of all subsets of Euclidean n-space, E^n, which generalize the elementary concepts of length or area, or in general, k-dimensional volume. These measures are called respectively 1-dimensional, 2-dimensional, and k-dimensional measures over E^n. The following question then naturally arises:

For A and B orthogonal subspaces of E^n with dimensions α and $\beta = n - \alpha$, and for $0 \leqq m \leqq \alpha$ and $0 \leqq k \leqq \beta$, *is the $(m+k)$-dimensional measure of the cartesian product of the finitely measurable sets $S \subset A$ and $T \subset B$ equal to the m-dimensional measure of S times the k-dimensional measure of T?*

In the case $n=3$, $\alpha=2$, $\beta=m=k=1$, and T an interval, J. F. Randolph obtained a partial answer for Carathéodory and Gross measures (see [R])(1), A. P. Morse and Randolph obtained the affirmative answer for Gillespie measure (see [MR]), while A. S. Besicovitch and P. A. P. Moran presented a rough outline of a counter example for Hausdorff measure (see [BM]). By using new methods the current paper obtains results for integral-geometric (Favard) measure, Hausdorff measure, and sphere measure.

The cylinder case ($\beta=1$) for Favard measure is answered in the affirmative (Theorem 4.5) and further generalized (Theorem 5.5) to $k=\beta$, thus completely answering the question if $n \leqq 3$. Partial results for the remaining cases are contained in Theorems 5.7 and 5.8.

In 6.10 an example is constructed, using and extending the ideas of Besicovitch and Moran, which answers the question in the negative for both Hausdorff measure (Theorem 6.18) and for sphere measure (Theorem 6.19). Moreover we obtain in this way a set for which the Hausdorff measure differs from the Carathéodory measure.

Finally we obtain for $X \subset E^n$, $k \leqq m \leqq n$, a new formula which expresses $\mathcal{J}_n^k(X)$, in terms of the multiplicity integrals, with respect to $\mathcal{J}_m^k$, of the perpendicular projections of X into m planes, as an integral over the set of all these m planes ($\mathcal{J}_n^k$ is Favard k measure over E^n).

2. **Preliminaries.**

2.1 Definition. We agree that $\infty \cdot 0 = 0 \cdot \infty = 0$.

For S a set, $\gamma(S)$ is the number (possibly ∞) of elements of S. For F a family of sets, $\sigma(F) = \bigcup_{x \in F} x$.

For x a point,

Presented to the Society, April 30, 1949; received by the editors June 29, 1949.

(1) Symbols in brackets refer to the bibliography at the end of the paper.

$$\{x\} = \underset{y}{E}\,[y = x].$$

For f a function, f^{-1} is its inverse, and for X a set, $(f|X)$ is the function with domain $(X \cap \text{domain } f)$ defined by

$$(f | X)(x) = f(x) \quad \text{for} \quad x \in (X \cap \text{domain } f);$$

furthermore

$$N[f, X, x] = \gamma(X \cap \underset{y}{E}\,[f(y) = x]),$$
$$f^*(X) = \underset{y}{E}\,[y = f(x) \text{ for some } x \in X],$$

and for g also a function, $(f:g)$ is the superposition function defined by $(f:g)(x) = f(g(x))$ for all x.

2.2 DEFINITION. E^n is Euclidean n-dimensional vector space with the usual inner product and metric. We denote the origin of E^n by θ^n. If $A \subset E^n$ and $B \subset E^n$, then

$$A + B = E^n \cap \underset{z}{E}\,[z = x + y \text{ for some } x \in A,\ y \in B].$$

Lebesgue n-dimensional measure over E^n is denoted by $\mathcal{L}_n$.

2.3 DEFINITION. If m and n are positive integers, and L is a function on E^m to E^n, then L is linear if and only if

$$L(x + y) = L(x) + L(y) \qquad \text{for } x \in E^m,\ y \in E^m,$$
$$L(\lambda \cdot x) = \lambda \cdot L(x) \qquad \text{for } x \in E^m,\ \lambda \text{ a real number.}$$

If L is a linear function on E^m to E^n, $1 \leqq i \leqq m$, and I^i is the ith unit vector of E^m, then we shall write

$$L(I^i) = L^i = (L^i_1, L^i_2, \cdots, L^i_n).$$

By the matrix of L we mean the matrix with n rows and m columns whose entry in the ith row and jth column $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ is L^j_i. We make no distinction between L and its matrix. If $m \leqq n$, we define

$\Delta(L)$ is the square root of the sum of the squares of the determinants of all m by m minors of L.

If f is a function on E^m to E^n, $1 \leqq m \leqq n$, and $x \in E^m$, then we shall say that L is the differential of f at x if and only if L is a linear function on E^m to E^n such that

$$\lim_{z \to x} \frac{|f(z) - f(x) - L(z - x)|}{|z - x|} = 0.$$

If L is the (unique) differential of f at x, then we let

$$Jf(x) = \Delta(L).$$

2.4 Definition. If $1 \leqq m \leqq n$, then p_n^m is the function on E^n to E^m such that

$$p_n^m(x) = (x_1, x_2, \cdots, x_m) \qquad \text{for } x \in E^n,$$

and η_n^m is the function on E^m to E^n such that

$$\eta_n^m(y) = (y_1, y_2, \cdots, y_m, 0, 0, \cdots, 0) \qquad \text{for } y \in E^m.$$

2.5 Definition. Let G_n be the set of all linear functions R on E^n to E^n such that $|R(x)| = |x|$ for $x \in E^n$. The function ρ defined by the formula

$$\rho(R, S) = \sup_{|x|=1} |R(x) - S(x)| \qquad \text{for } R \in G_n, S \in G_n,$$

metrizes G_n in such a way that G_n is a compact topological group with respect to the operation, :, of superposition. G_n is called the orthogonal group of E^n. We shall denote the identity of G_n by ${}^nI$.

We let ϕ_n denote the unique Haar measure over G_n for which $\phi_n(G_n) = 1$, nonempty open sets are ϕ_n measurable and have positive ϕ_n measure, and the ϕ_n measure of a subset of G_n is the infimum of the ϕ_n measures of open sets containing it.

We shall use the following properties of ϕ_n (see [W, 8]):

If f is a ϕ_n measurable function on G_n, then

$$\int_{G_n} f(R^{-1}) d\phi_n R = \int_{G_n} f(R) d\phi_n R,$$

and for $S \in G_n$,

$$\int_{G_n} f(S:R) d\phi_n R = \int_{G_n} f(R:S) d\phi_n R = \int_{G_n} f(R) d\phi_n R.$$

2.6 Definition. If $1 \leqq k \leqq n$, then

$$\beta(n, k) = \int_{G_n} \Delta(R|_k^k) d\phi_n R,$$

where $(R|_k^k)$ is defined for $R \in G_n$, as the linear function on E^k to E^k such that $(R|_k^k)_j^i = R_j^i$ for $i, j = 1, 2, \cdots, k$.

2.7 Definition. Let

$$Z_n^s = E^n \cap \underset{x}{E}\,[x_i = 0 \text{ for } i = 1, 2, \cdots, n-s],$$

and define the function λ_n^s on the cartesian product $(G_n \times E^{n-s})$ to the set Λ_n^s of all s-dimensional flat subspaces of E^n by the formula

$$\lambda_n^s(R, w) = R^*(Z_n^s + \{\eta_n^{n-s}(w)\}) \quad \text{for } R \in G_n, w \in E^{n-s}.$$

The motivation for this definition rests on the following fact:

Let $\Psi_n^s(F)$ be defined for each suitably restricted real-valued function F on Λ_n^s by the formula

$$\Psi_n^s(F) = \int_{G_n} \int_{E^{n-s}} F(\lambda_n^s(R, w))d\mathcal{L}_{n-s}wd\phi_n R.$$

Then Ψ_n^s is an invariant integral. (See [F3, 4].)

2.8 Definition. We say that g is a gauge over S if and only if S is a metric space,

$$\operatorname{dmn} g \subset \underset{x}{E}\,[x \subset S], \quad \text{and} \quad \operatorname{rng} g \subset \underset{t}{E}\,[0 \leqq t \leqq \infty].$$

If g is a gauge over S, $r > 0$, define the function g_r for $x \subset S$ by the formula

$$g_r(x) = \inf_{G \in B} \sum_{y \in G} g(y),$$

where $G \in B$ if and only if G is such a countable family that $x \subset \sigma G$ and diam $y \leqq r$ whenever $y \in G$.

We shall say that ϕ is generated by g if and only if g is a gauge (over S) and ϕ is the function on

$$\underset{x}{E}\,[x \subset S]$$

defined by

$$\phi(x) = \lim_{r \to 0+} g_r(x) \qquad \text{for } x \subset S.$$

It is easy to check that ϕ is such a measure that closed sets are ϕ measurable.

2.9 Definition. Suppose $1 \leqq k \leqq n$. Let g_1, g_2, g_3, g_4 be the gauges over E^n such that

dmn g_1 is the set of all open spheres of E^n,

dmn g_2 is the set of all subsets of E^n,

$$g_i(x) = (k!)^{-1} \cdot \Gamma(1/2)^{k-1} \cdot \Gamma((k+1)/2) \cdot (\operatorname{diam} x)^k \qquad \text{for } x \in \operatorname{dmn} g_i,\ i = 1, 2,$$

dmn g_3 is the set of all analytic subsets of E^n,

$$g_3(x) = \beta(n, k)^{-1} \int_{G_n} \mathcal{L}_k[(\overset{k}{p}_n : R)^*(x)]d\phi_n R \qquad \text{for } x \in \operatorname{dmn} g_3,$$

dmn g_4 is the set of all convex open subsets of E^n,

$$g_4(x) = \sup_{R \in G_n} \mathcal{L}_k[(\overset{k}{p}_n : R)^*(x)] \qquad \text{for } x \in \operatorname{dmn} g_4.$$

Then we define

$\overset{k}{\mathcal{S}}_n$ is the measure generated by g_1,

$$\mathcal{H}_n^k \text{ is the measure generated by } g_2,$$
$$\mathcal{I}_n^k \text{ is the measure generated by } g_3,$$
$$C_n^k \text{ is the measure generated by } g_4.$$

$\mathcal{S}_n^k$ is sphere k measure over n space, $\mathcal{H}_n^k$ is Hausdorff measure, $\mathcal{I}_n^k$ is integral-geometric (Favard) measure, and C_n^k is Carathéodory measure. We also define $\mathcal{I}_n^0$ by the formula

$$\mathcal{I}_n^0(A) = \gamma(A) \qquad \text{for } A \subset E^n.$$

2.10 Remark. It is quite easy to see that $\mathcal{S}_n^k$, $\mathcal{H}_n^k$, $\mathcal{I}_n^k$, and C_n^k are invariant under rigid motions of E^n, and that they satisfy the condition that any subset of E^n is contained in an analytic set of the same measure.

We shall use the fact that if A is an analytic subset of E^n, $1 \leqq k < n$, then

$$\begin{aligned}\mathcal{I}_n^k(A) &= \beta(n, k)^{-1} \int_{G_n} \int_{E^k} N[(p_n^k : R), A, y] d\mathcal{L}_k y d\phi_n R \\ &= \beta(n, k)^{-1} \int_{G_n} \int_{E^k} \gamma[A \cap \lambda_n^{n-k}(R, y)] d\mathcal{L}_k y d\phi_n R.\end{aligned}$$

(See F2, 5.11] and [F3, 5].) Also it is obvious from 2.9 that

$$\mathcal{I}_n^n = \mathcal{L}_n.$$

2.11 Definition. If f is a function on a subset of E^k to E^n, then f is said to be Lipschitzian if and only if there is a number M such that

$$|f(x) - f(y)| \leqq M \cdot |x - y| \qquad \text{for all } x, y \in \operatorname{dmn} f.$$

We shall make use of the fact that if f is Lipschitzian, $\operatorname{dmn} f \subset E^k$, $\operatorname{rng} f \subset E^n$, then there exists a Lipschitzian function g on E^k to E^n such that $(g | \operatorname{dmn} f) = f$.

Suppose $1 \leqq k \leqq n$, $A \subset E^n$. Then we shall say that A is k rectifiable if and only if there is a Lipschitzian function whose domain is a bounded subset of E^k and whose range is A.

3. **A new formula involving Favard measure.** Though Theorem 3.1 is not used in the remainder of this paper, I feel that it will be useful in future investigations concerning Favard measure.

3.1 Theorem. *If A is an analytic subset of E^n, m and k are such non-negative integers that $k \leqq m \leqq n$, then*

$$\mathcal{I}_n^k(A) = (\beta(m, k)/\beta(n, k)) \cdot \int_{G_n} \int_{E^m} N[(p_n^m : R), A, y] d\mathcal{I}_m^k y d\phi_n R.$$

Proof. With each $S \in G_m$, we associate the orthogonal transformation $\bar{S} \in G_n$ by the formulas

$$\overline{S}^i_j = S^i_j \qquad \text{for } i = 1, \cdots, m; \qquad j = 1, \cdots, m,$$
$$\overline{S}^i_j = 0 \qquad \text{for } i = 1, \cdots, m; \qquad j = m+1, \cdots, n,$$
$$\overline{S}^i_j = \delta^i_j \qquad \text{for } i = m+1, \cdots, n; j = 1, \cdots, n,$$

where δ^i_j is the ordinary Kronecker δ-function.

The proof is divided into 3 parts.

Part 1. $(p^k_m:S:p^m_n:R) = (p^k_n:\overline{S}:R)$ for $R \in G_n$, $S \in G_m$.

Proof. For $x \in E^n$,

$$\begin{aligned}(p^k_m:S:p^m_n:R)(x) &= (p^k_m:S:p^m_n)\left(\sum_{i=1}^n x_i \cdot R^i\right)\\ &= p^k_m\left(\sum_{j=1}^m\left(\sum_{i=1}^n x_i \cdot R^i_j\right)S^j\right)\\ &= \sum_{h=1}^k\left(\sum_{j=1}^m\sum_{i=1}^n x_i R^i_j S^j_h\right){}^kI^h\\ &= \sum_{h=1}^k\left(\sum_{j=1}^n\sum_{i=1}^n x_i R^i_j \overline{S}^j_h\right){}^kI^h\\ &= (p^k_n:\overline{S}:R)(x).\end{aligned}$$

This proves Part 1.

Part 2. For $R \in G_n$,

$$\int_{E^m} N[(p^m_n:R), A, y]d\mathcal{J}^k_m y = \beta(m, k)^{-1}\int_{G_m}\int_{E^k} N[(p^k_n:\overline{S}:R), A, x]d\mathcal{L}_k x d\phi_m S.$$

Proof. Let

$$A_i = E^m \cap \underset{y}{E}\,[N[(p^m_n:R), A, y] = i] \quad \text{for } i = 1, 2, \cdots,$$
$$A_\infty = E^m \cap \underset{y}{E}\,[N[(p^m_n:R), A, y] = \infty].$$

Then A_∞ and A_i, for $i = 1, 2, \cdots$, are analytic and hence $\mathcal{J}^k_m$ measurable sets, and

$$A_i \cap A_j = \emptyset \qquad \text{for } i \neq j;\ 1 \leqq i, j \leqq \infty.$$

Therefore

$$\int_{E^m} N[(p^m_n:R), A, y]d\mathcal{J}^k_m y$$
$$= \sum_{i=1}^\infty i \cdot \mathcal{J}^k_m(A_i) + \infty \cdot \mathcal{J}^k_m(A_\infty)$$

$$= \sum_{i=1}^{\infty} i\cdot\beta(m, k)^{-1}\int_{G_m}\int_{E^k} N[(p_m^k:S), A_i, x]d\mathcal{L}_k x d\phi_m S$$

$$+ \infty\cdot\beta(m, k)^{-1}\int_{G_m}\int_{E^k} N[(p_m^k:S), A_\infty, x]d\mathcal{L}_k x d\phi_m S$$

$$= \beta(m, k)^{-1}\int_{G_m}\int_{E^k}\Big(\sum_{i=1}^{\infty} i\cdot N[(p_m^k:S), A_i, x]$$

$$+ \infty\cdot N[(p_m^k:S), A_\infty, x]\Big)d\mathcal{L}_k x d\phi_m S$$

$$= \beta(m, k)^{-1}\int_{G_m}\int_{E^k} N[(p_m^k:S:p_n^m:R), A, x]d\mathcal{L}_k x d\phi_m S$$

$$= \beta(m, k)^{-1}\int_{G_m}\int_{E^k} N[(p_n^k:\bar{S}:R), A, x]d\mathcal{L}_k x d\phi_m S.$$

This completes the proof of Part 2.

Part 3.

$$\mathcal{J}_n^k(A) = (\beta(m, k)/\beta(n, k))\int_{G_n}\int_{E^m} N[(p_n^m:R), A, y]d\mathcal{J}_m^k y d\phi_n R.$$

Proof. Applying Part 2 and 2.5,

$$\beta(m, k)\cdot\int_{G_n}\int_{E^m} N[(p_n^m:R), A, y]d\mathcal{J}_m^k y d\phi_n R$$

$$= \int_{G_n}\int_{G_m}\int_{E^k} N[(p_n^k:\bar{S}:R), A, x]d\mathcal{L}_k x d\phi_m S d\phi_n R$$

$$= \int_{G_m}\int_{G_n}\int_{E^k} N[(p_n^k:\bar{S}:R), A, x]d\mathcal{L}_k x d\phi_n R d\phi_m S$$

$$= \int_{G_m}\int_{G_n}\int_{E^k} N[(p_n^k:R), A, x]d\mathcal{L}_k x d\phi_n R d\phi_m S$$

$$= \int_{G_n}\int_{E^k} N[(p_n^k:R), A, x]d\mathcal{L}_k x d\phi_n R$$

$$= \beta(n, k)\cdot\mathcal{J}_n^k(A).$$

This completes the proof.

4. The integralgeometric (Favard) measure of cylinder sets. In this section we consider the case in which $\beta=k=1$.

4.1 LEMMA. *If*

(1) $$1 \leqq i \leqq n,$$

(2) $$S = (E^n \cap \underset{x}{E}\,[\,|x| = 1]),$$

(3) ψ *is the function on* $\underset{x}{E}\,[x \subset S]$ *defined by*

$$\psi(x) = \phi_n(G_n \cap \underset{R}{E}\,[R^i \in X]) \qquad \textit{for } X \subset S,$$

then

(4) $$\psi(X) = (\mathcal{H}_n^{n-1}(S))^{-1} \cdot \mathcal{H}_n^{n-1}(X) \quad \textit{whenever } X \subset S \textit{ is a Borel set,}$$

(5) $$\int_{G_n} g(R^i)d\phi_n R = (\mathcal{H}_n^{n-1}(S))^{-1} \cdot \int_S g(x)d\mathcal{H}_n^{n-1}x,$$

whenever g is a continuous function on S.

Proof. By straightforward methods it is checked that

ψ measures S,

Open sets in the relative topology on S are ψ measurable,

$\psi(S) = 1$,

$\psi(X) > 0$ whenever X is an open non-empty set in the relative topology on S,

$\psi[Q^*(X)] = \psi(X)$ whenever $Q \in G_n$, $X \subset S$.

Hence ψ is a Haar measure over S with respect to the transitive group G_n of isometries of S. Since

$$(\mathcal{H}_n^{n-1} \mid \underset{x}{E}\,[x \subset S])$$

is also a Haar measure over S with respect to G_n, we conclude by the uniqueness of Haar measures (on Borel sets) that (4) holds. (5) is now easily checked.

4.2 Lemma. *If* $1 \leqq i, j \leqq n$ *and* $-1 \leqq t \leqq 1$, *then*

$$\phi_n(G_n \cap \underset{R}{E}\,[R_j^i = t]) = 0.$$

Proof. By 4.1,

$$\phi_n(G_n \cap \underset{R}{E}\,[R_j^i = t]) = (\mathcal{H}_n^{n-1}(S))^{-1} \cdot \mathcal{H}_n^{n-1}(S \cap \underset{x}{E}\,[x_j = t]).$$

Since

$$\mathcal{H}_n^{n-1}(S \cap \underset{x}{E}\,[x_j = t]) = 0,$$

the proof is complete.

4.3 Lemma. *If n is a positive integer, and L is the set of those elements R of* G_n *for which* ${}^nI^n$ *is a linear combination of* $R^2, \cdots, R^n$, *then* $\phi_n(L) = 0$.

Proof. If $R \in L$, then $R^1 \cdot {}^nI^n = 0$, and therefore $R_n^1 = 0$. Apply 4.2.

4.4 LEMMA. *If* $0 \leqq m < n-1$, *then*

$$\phi_n\left(G_n \cap \underset{R}{E}\left[\sum_{i=m+2}^{n} |R_n^i|^2 = 1\right]\right) = 0.$$

Proof. If $R \in G_n$ is such that $\sum_{i=m+2}^{n} |R_n^i|^2 = 1$, then in particular, $R_u^1 = 0$. Apply 4.2.

4.5 THEOREM. *If*

(1) $$0 \leqq m \leqq n-1, \quad m \text{ and } n \text{ are integers},$$

(2) $$A = E^n \cap \underset{x}{E}\,[x_n = 0],$$

(3) $$B = E^n \cap \underset{x}{E}\,[x_1 = x_2 = \cdots = x_{n-1} = 0],$$

(4) $$S \subset A, \quad S \text{ is analytic}, \quad \mathcal{J}_n^m(S) < \infty,$$

(5) $$T \subset B, \quad T \text{ is analytic}, \quad \mathcal{J}_n^1(T) < \infty,$$

then

$$\mathcal{J}_n^{m+1}(S+T) = \mathcal{J}_n^1(T) \cdot \mathcal{J}_n^m(S).$$

Proof. If $m=0$, let $q = \mathcal{J}_n^0(S) < \infty$. Then S consists of q distinct points of A. Therefore $(S+T)$ consists of q distinct translates of T. Hence

$$\mathcal{J}_n^1(S+T) = q \cdot \mathcal{J}_n^1(T) = \mathcal{J}_n^0(S) \cdot \mathcal{J}_n^1(T).$$

If $m = n-1$, let $S' = (p_n^{n-1})^*(S)$. Since η_n^{n-1} is an isometric mapping of S' onto S, we use [F1, 4.5] to conclude that $\mathcal{J}_n^{n-1}(S) = \mathcal{L}_{n-1}(S')$. Similarly, if

$$T' = E^1 \cap \underset{x}{E}\,[(0, 0, \cdots, 0, x) \in T],$$

then $\mathcal{J}_n^1(T) = \mathcal{L}_1(T')$. Thus by Fubini's theorem,

$$\mathcal{J}_n^n(S+T) = \mathcal{L}_n(S+T) = \mathcal{L}_{n-1}(S') \cdot \mathcal{L}_1(T') = \mathcal{J}_n^{n-1}(S) \cdot \mathcal{J}_n^1(T).$$

Assume now that $0 < m < n-1$. Without loss of generality we may assume that $\mathcal{J}_n^1(T) > 0$. Let K be the set of those elements R of G_n for which

$$\sum_{i=m+2}^{n} |R_n^i|^2 < 1,$$

and for which $R^2, R^3, \cdots, R^n, {}^nI^n$ are linearly independent. Then by 4.3 and 4.4,

$$\phi_n(G_n - K) = 0.$$

Also it is easily seen that for $R \in K$,

$$\eta_n^{n-1}(p_n^{n-1}(R^2)),\ \eta_n^{n-1}(p_n^{n-1}(R^3)),\ \cdots,\ \eta_n^{n-1}(p_n^{n-1}(R^n)),$$

are linearly independent. Hence, for each $R\in K$, we may consider orthogonal transformations which leave ${}^nI^n$ fixed and which rotate ${}^nI^{m+1}, \cdots, {}^nI^{n-1}$ in such a way that they span the linear subspace generated by $\eta_n^{n-1}(p_n^{n-1}(R^{m+2}))$, $\cdots, \eta_n^{n-1}(p_n^{n-1}(R^n))$. For this purpose, we shall say that P satisfies condition I with respect to R if and only if $R\in K$, $P\in G_n$, and

$$P({}^nI^n) = {}^nI^n,$$

$$P({}^nI^{n-1}) = \eta_n^{n-1}(p_n^{n-1}(R^n))/\left|\eta_n^{n-1}(p_n^{n-1}(R^n))\right|,$$

$$(*)\quad P\left[\frac{-R_n^i\cdot R_n^n}{(1-|R_n^i|^2)^{1/2}(1-|R_n^n|^2)^{1/2}}\cdot {}^nI^{n-1} + \sum_{\alpha=0}^{n-i-2}\frac{-R_n^{i+1+\alpha}\cdot R_n^i\cdot {}^nI^{i+\alpha}}{(1-|R_n^i|^2)^{1/2}\left(1-\sum_{j=1}^{n-i-1-\alpha}|R_n^{i+1+\alpha+j}|^2\right)^{1/2}\left(1-\sum_{j=0}^{n-i-1-\alpha}|R_n^{i+1+\alpha+j}|^2\right)^{1/2}} + {}^nI^{i-1}\,\frac{\left(1-\sum_{j=0}^{n-i}|R_n^{i+j}|^2\right)^{1/2}}{(1-|R_n^i|^2)^{1/2}\left(1-\sum_{j=1}^{n-i}|R_n^{i+j}|^2\right)^{1/2}}\right] = \frac{\eta_n^{n-1}(p_n^{n-1}(R^i))}{\left|\eta_n^{n-1}(p_n^{n-1}(R^i))\right|},$$

for $i=m+2, \cdots, n-1$.

Further, we shall say that Q satisfies condition II with respect to R if and only if $R\in K$, $Q\in G_{n-1}$, and there is a $P\in G_n$ which satisfies condition I with respect to R and for which

$$Q_j^i = P_j^i \qquad \text{for } i, j = 1, \cdots, n-1.$$

In particular, we define the function f on K to G_{n-1} such that if $R\in K$, then

$$(f(R))_j^i = P_j^i \qquad \text{for } i, j = 1, \cdots, n-1,$$

where P satisfies condition I with respect to R and also P satisfies equation (*) for $i=2, \cdots, n-1$. It is obvious that f is a continuous mapping associating with each $R\in K$ a transformation $f(R)$ which satisfies condition II with respect to R.

Suppose $R\in K$ and P satisfies condition I with respect to R. We shall reduce P to a canonical form. Define $U_{P,R}=(P^{-1}:R)$. Then for $i=1, 2, \cdots, m$, we have

$$ {}^nI^i \in U^*_{P,R}(E^n \cap \underset{x}{E}\,[x_{m+2} = \cdots = x_n = 0]), $$

because $(U_{P,R})^j_i = 0$ for $j = m+2, \cdots, n$, $i = 1, \cdots, m$, and hence

$$ (U_{P,R})^j \cdot {}^nI^i = 0. $$

We may therefore choose $\tilde{U}_{P,R} \in G_{m+1}$ such that

$$ \tilde{U}_{P,R}((U^{-1}_{P,R}({}^nI^i))_1, \cdots, (U^{-1}_{P,R}({}^nI^i))_{m+1}) = {}^{m+1}I^i \qquad \text{for } i = 1, \cdots, m. $$

Define $V_{P,R} \in G_n$ by the formulas

$$ \begin{aligned} V_{P,R}({}^nI^i) &= ((\tilde{U}_{P,R})^i_1, \cdots, (\tilde{U}_{P,R})^i_{m+1}, 0, \cdots, 0) && \text{for } i = 1, \cdots, m+1, \\ V_{P,R}({}^nI^i) &= {}^nI^i && \text{for } i = m+2, \cdots, n. \end{aligned} $$

Finally, let $W_{P,R} = (U_{P,R} : V^{-1}_{P,R}) \in G_n$. Then it is easily checked that

$$ \begin{aligned} W_{P,R}({}^nI^i) &= {}^nI^i && \text{for } i = 1, \cdots, m, \\ W_{P,R}({}^nI^i) &= U_{P,R}({}^nI^i) = (P^{-1}:R)({}^nI^i) && \text{for } i = m+2, \cdots, n, \\ (W_{P,R})^i_n &= R^i_n && \text{for } i = m+2, \cdots, n. \end{aligned} $$

The remainder of the proof is divided into 8 parts.

Part 1. If P satisfies condition I with respect to R, then

$$ \int_{E^{m+1}} \gamma[(S+T) \cap \lambda_n^{n-m-1}(R, y)]\, d\mathcal{L}_{m+1}y $$
$$ = \int_{E^m} \int_{E^1} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (x_1, \cdots, x_m, y))]\, d\mathcal{L}_1 y\, d\mathcal{L}_m x. $$

Proof. Since $\mathcal{L}_{m+1}$ is invariant under transformations by elements of G_{m+1}, and applying Fubini's Theorem, we have

$$ \int_{E^{m+1}} \gamma[(S+T) \cap \lambda_n^{n-m-1}(R, y)]\, d\mathcal{L}_{m+1}y $$
$$ = \int_{E^{m+1}} \gamma[P^{-1*}((S+T) \cap \lambda_n^{n-m-1}(R, y))]\, d\mathcal{L}_{m+1}y $$
$$ = \int_{E^{m+1}} \gamma[P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(P^{-1}:R, y)]\, d\mathcal{L}_{m+1}y $$
$$ = \int_{E^{m+1}} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(U_{P,R}, \tilde{U}^{-1}_{P,R}(y))]\, d\mathcal{L}_{m+1}y $$
$$ = \int_{E^{m+1}} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}((U_{P,R} : V^{-1}_{P,R}), y)]\, d\mathcal{L}_{m+1}y $$

$$= \int_{E^m}\int_{E^1} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (x_1, \cdots, x_m, y))]d\mathcal{L}_1 y d\mathcal{L}_m x.$$

This completes Part 1.

Part 2. If P satisfies condition I with respect to R, then

$$|(W_{P,R})_n^{m+1}| = \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} > 0.$$

Proof. Since P and R are fixed, we omit the subscripts on $W_{P,R}$. Notice that

$$0 = W^{m+1}\cdot W^n = W_{n-1}^{m+1}(1 - |R_n^n|^2)^{1/2} + W_n^{m+1}R_n^n,$$

$$0 = W^{m+1}\cdot W^{n-1} = W_{n-2}^{m+1}\cdot\frac{(1 - |R_n^{n-1}|^2 - |R_n^n|^2)^{1/2}}{(1 - |R_n^n|^2)^{1/2}} - W_{n-1}^{m+1}\cdot\frac{R_n^{n-1}\cdot R_n^n}{(1 - |R_n^n|^2)^{1/2}}$$
$$+ W_n^{m+1}R_n^{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$0 = W^{m+1}\cdot W^{m+2} = W_{m+1}^{m+1}\frac{\left(1 - \sum_{j=m+2}^{n} |R_n^j|^2\right)^{1/2}}{\left(1 - \sum_{j=m+3}^{n} |R_n^j|^2\right)^{1/2}} - \sum_{\alpha=m+2}^{n-2} W_\alpha^{m+1}$$
$$\cdot\frac{R_n^{\alpha+1}\cdot R_n^{m+2}}{\left(1 - \sum_{j=1}^{(n-\alpha-1)} |R_n^{\alpha+1+j}|^2\right)^{1/2}\left(1 - \sum_{j=0}^{(n-\alpha-1)} |R_n^{\alpha+1+j}|^2\right)^{1/2}} - W_{n-1}^{m+1}$$
$$\cdot\frac{R_n^{m+2}\cdot R_n^n}{(1 - |R_n^n|^2)^{1/2}} + W_n^{m+1}R_n^{m+2},$$

$$1 = W^{m+1}\cdot W^{m+1} = \sum_{\alpha=m+1}^{n} |W_\alpha^{m+1}|^2.$$

Hence solving for $|W_n^{m+1}|$, we have

$$W_{n-1}^{m+1} = -\frac{R_n^n}{(1 - |R_n^n|^2)^{1/2}}\cdot W_n^{m+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$W_{m+1}^{m+1} = -\frac{R_n^{m+2}}{\left(1 - \sum_{j=m+3}^{n} |R_n^j|^2\right)^{1/2}\left(1 - \sum_{j=m+2}^{n} |R_n^j|^2\right)^{1/2}} W_n^{m+1},$$

and therefore

$$| W_n^{m+1} |^2 \cdot \left\{ 1 + \frac{| R_n^n |^2}{1 - | R_n^n |^2} + \frac{| R_n^{n-1} |^2}{(1 - | R_n^n |^2)(1 - | R_n^{n-1} |^2 - | R_n^n |^2)} + \cdots \right.$$

$$\left. + \frac{| R_n^{m+2} |^2}{\left(1 - \sum_{j=m+3}^{n} | R_n^j |^2\right)\left(1 - \sum_{j=m+2}^{n} | R_n^j |^2\right)} \right\}$$

$$= | W_n^{m+1} |^2 \cdot \left\{ \frac{1}{1 - \sum_{j=m+2}^{n} | R_n^j |^2} \right\} = 1,$$

$$| W_n^{m+1} | = \left(\sum_{j=1}^{m+1} | R_n^j |^2 \right)^{1/2} > 0.$$

The proof of Part 2 is complete.

Part 3. If P satisfies condition I with respect to R, then

$$\mathcal{L}_1(\underset{y}{E}\, [((\{\theta^n\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (0, \cdots, 0, y))) \neq \emptyset])$$
$$= \left(\sum_{j=1}^{m+1} | R_n^j |^2 \right)^{1/2} \cdot \mathcal{J}_n^1(T) > 0.$$

Proof. Let

$$T' = E^1 \cap \underset{x}{E}\, [(0, \cdots, 0, x) \in T].$$

Recalling that

$$(W_{P,R})^{m+1} \cdot (W_{P,R})^i = 0 \qquad \text{for } i \neq m + 1,$$
$$(W_{P,R})_j^{m+1} = 0 \qquad \text{for } j = 1, \cdots, m,$$

we have

$$[((\{\theta^n\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (0, \cdots, 0, y))) \neq \emptyset]$$

$$\rightleftarrows \left\{ \begin{array}{l} \text{There are numbers } \alpha_{m+2}, \cdots, \alpha_n \text{ such that} \\ y \cdot (W_{P,R})_j^{m+1} + \sum_{i=m+2}^{n} \alpha_i \cdot (W_{P,R})_j^i = 0 \qquad \text{for } j = m + 1, \cdots, n - 1, \\ y \cdot (W_{P,R})_n^{m+1} + \sum_{i=m+2}^{n} \alpha_i \cdot (W_{P,R})_n^i \in T', \end{array} \right.$$

which implies that

$$y\cdot\sum_{j=m+1}^{n}\left|(W_{P,R})_j^{m+1}\right|^2+\sum_{i=m+2}^{n}\alpha_i\cdot\sum_{j=m+1}^{n}(W_{P,R})_j^{m+1}\cdot(W_{P,R})_j^{i}=y\in(W_{P,R})_n^{m+1}\cdot T',$$

where $(W_{P,R})_n^{m+1}\cdot T'$ shall mean

$$(E^1\cap\underset{x}{E}\,[x=(W_{P,R})_n^{m+1}\cdot y\text{ for some }y\in T'].$$

Conversely, if $y\in(W_{P,R})_n^{m+1}\cdot T'$, let

$$\alpha_{m+2}=-\frac{y\cdot(W_{P,R})_{m+1}^{m+1}}{(W_{P,R})_{m+1}^{m+2}},$$

$$\alpha_{m+3}=-\frac{y\cdot(W_{P,R})_{m+2}^{m+1}+\alpha_{m+2}\cdot(W_{P,R})_{m+2}^{m+2}}{(W_{P,R})_{m+2}^{m+3}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\alpha_n=-\frac{y\cdot(W_{P,R})_{n-1}^{m+1}+\alpha_{m+2}\cdot(W_{P,R})_{n-1}^{m+2}+\cdots+\alpha_{n-1}(W_{P,R})_{n-1}^{n-1}}{(W_{P,R})_{n-1}^{n}}.$$

Then it easily follows that

$$y\cdot(W_{P,R})_n^{m+1}+\sum_{i=m+2}^{n}\alpha_i\cdot(W_{P,R})_n^{i}\in T'.$$

Hence, by Part 2 and the fact that $\mathcal{L}_1(T')=\mathcal{J}_n^1(T)>0$,

$$\mathcal{L}_1(\underset{y}{E}\,[((\{\theta^n\}+T)\cap\lambda_n^{n-m-1}(W_{P,R},(0,\cdots,0,y)))\neq\varnothing])$$

$$=\mathcal{L}_1((W_{P,R})_n^{m+1}\cdot T')=\left|(W_{P,R})_n^{m+1}\right|\cdot\mathcal{L}_1(T')=\left(\sum_{j=1}^{m+1}\left|R_n^j\right|^2\right)^{1/2}\cdot\mathcal{J}_n^1(T)>0.$$

This completes Part 3.

Part 4. If P satisfies condition I with respect to R, and $Q\in G_{n-1}$ is such that $Q_j^i=P_j^i$ for $i,j=1,\cdots,n-1$, then

$$\int_{E^1}\gamma[(P^{-1*}(S)+T)\cap\lambda_n^{n-m-1}(W_{P,R},(x_1,\cdots,x_m,y))]d\mathcal{L}_1y$$

$$=\gamma[(Q^{-1}:p_n^{n-1})^*(S)\cap\lambda_{n-1}^{n-m-1}({}^{(n-1)}I,x)]\cdot\left(\sum_{j=1}^{m+1}\left|R_n^j\right|^2\right)^{1/2}\cdot\mathcal{J}_n^1(T).$$

Proof. Assume that

$$q=\gamma[(Q^{-1}:p_n^{n-1})^*(S)\cap\lambda_{n-1}^{n-m-1}({}^{(n-1)}I,x)]<\infty.$$

Then

$$((Q^{-1}:p_n^{n-1})^*(S) \cap \lambda_{n-1}^{n-m-1}(^{(n-1)}I, x)) = \bigcup_{i=1}^{q} \{z^i\},$$

where $z^i \neq z^j$ for $i \neq j$ and $z^i \in E^{n-1}$ for $i=1, \cdots, q$. Furthermore

$$\int_{E^1} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (x_1, \cdots, x_m, y))] d\mathcal{L}_1 y$$

$$= \int_{E^1} \sum_{i=1}^{q} \gamma[(\{\eta_n^{n-1}(z^i)\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (x_1, \cdots, x_m, y))] d\mathcal{L}_1 y$$

$$= q \cdot \int_{E^1} \gamma[(\{(x_1, \cdots, x_m, 0, \cdots, 0)\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (x_1, \cdots, x_m, y))] d\mathcal{L}_1 y$$

$$= q \cdot \int_{E^1} \gamma[(\{\theta^n\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (0, \cdots, 0, y))] d\mathcal{L}_1 y.$$

Now if $(0, \cdots, 0, a)$ and $(0, \cdots, 0, b)$ are both in $\lambda_n^{m-n-1}(W_{P,R}, (0, \cdots, 0, y))$, then there exist numbers $\alpha_{m+2}, \cdots, \alpha_n$, and $\beta_{m+2}, \cdots, \beta_n$ such that

$$\begin{cases} y \cdot (W_{P,R})_n^{m+1} + \alpha_{m+2} \cdot (W_{P,R})_n^{m+2} + \cdots + \alpha_n \cdot (W_{P,R})_n^n = a, \\ y \cdot (W_{P,R})_n^{m+1} + \beta_{m+2} \cdot (W_{P,R})_n^{m+2} + \cdots + \beta_n \cdot (W_{P,R})_n^n = b, \end{cases}$$

$$\begin{cases} y \cdot (W_{P,R})_{m+1}^{m+1} + \alpha_{m+2} \cdot (W_{P,R})_{m+1}^{m+2} = 0, \\ y \cdot (W_{P,R})_{m+1}^{m+1} + \beta_{m+2} \cdot (W_{P,R})_{m+1}^{m+2} = 0, \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\begin{cases} y \cdot (W_{P,R})_{n-1}^{m+1} + \alpha_{m+2} \cdot (W_{P,R})_{n-1}^{m+2} + \cdots + \alpha_n \cdot (W_{P,R})_{n-1}^n = 0, \\ y \cdot (W_{P,R})_{n-1}^{m+1} + \beta_{m+2} \cdot (W_{P,R})_{n-1}^{m+2} + \cdots + \beta_n \cdot (W_{P,R})_{n-1}^n = 0. \end{cases}$$

From the definition of $W_{P,R}$ we see that

$$(W_{P,R})_{m+1+j}^{m+2+j} \neq 0 \quad \text{for } j = 0, \cdots, n-m-2,$$

hence

$$\alpha_{m+2} = \beta_{m+2}, \cdots, \alpha_n = \beta_n,$$

and therefore $(0, \cdots, 0, a) = (0, \cdots, 0, b)$. From this it follows that

$$\int_{E^1} \gamma[(\{\theta^n\} + T) \cap \lambda_n^{n-m-1}(W_{P,R}, (0, \cdots, 0, y))] d\mathcal{L}_1 y$$

$$= \mathcal{L}_1(\underset{y}{E}\,[((\{\theta^n\} + T) \cap \lambda_n^{n-m-1}(W_{P,R},\,(0, \cdots, 0, y))) \neq \emptyset])$$

$$= \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T) > 0.$$

Hence

$$\int_{E^1} \gamma[(P^{-1*}(S) + T) \cap \lambda_n^{n-m-1}(W_{P,R},\,(x_1, \cdots, x_m, y))]d\mathcal{L}_1 y$$

$$= q \cdot \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T).$$

The same method handles the case $q = \infty$, since

$$\left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T) > 0.$$

This completes the proof of Part 4.

Part 5. If Q satisfies condition II with respect to R, then

$$\int_{E^{m+1}} \gamma[(S + T) \cap \lambda_n^{n-m-1}(R, y)]d\mathcal{L}_{m+1} y$$

$$= \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T) \cdot \int_{E^m} \gamma[p_n^{n-1*}(S) \cap \lambda_{n-1}^{n-m-1}\,(Q, x)]d\mathcal{L}_m x.$$

Proof. Applying Parts 1 and 4, we have

$$\int_{E^{m+1}} \gamma[(S + T) \cap \lambda_n^{n-m-1}(R, y)]d\mathcal{L}_{m+1} y$$

$$= \int_{E^m} \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T) \cdot \gamma[(Q^{-1} : p_n^{n-1})^*(S) \cap \lambda_{n-1}^{n-m-1}({}^{(n-1)}I, x)]d\mathcal{L}_m x$$

$$= \left(\sum_{j=1}^{m+1} |R_n^j|^2\right)^{1/2} \cdot \mathcal{J}_n^1(T) \cdot \int_{E^m} \gamma[p_n^{n-1*}(S) \cap \lambda_{n-1}^{n-m-1}(Q, x)]d\mathcal{L}_m x.$$

The proof of Part 5 is complete.

For the remainder of this proof, we associate with each $P \in G_{n-1}$ the transformation $\tilde{P} \in G_n$ by the formulas

$$\tilde{P}^i = \eta_n^{n-1}(P^i) \qquad \text{for } i = 1, \cdots, n-1,$$
$$\tilde{P}^n = {}^nI^n.$$

Part 6. If Q satisfies condition II with respect to R, and $P \in G_{n-1}$, then $(P : Q)$ satisfies condition II with respect to $(\tilde{P} : R)$.

Proof. We have for $1 \leqq i \leqq n-1$, $1 \leqq k \leqq n$,

$$(\tilde{P}:R)_i^k = \sum_{j=1}^{n} R_j^k \cdot \tilde{P}_i^j = \sum_{j=1}^{n-1} R_j^k \cdot P_i^j,$$

and

$$(\tilde{P}:R)_n^k = \sum_{j=1}^{n} R_j^k \cdot \tilde{P}_n^j = R_n^k.$$

Hence

$$\begin{aligned}
(P:Q)({}^{(n-1)}I^{n-1}) &= P(p_n^{n-1}(R^n)/\,|\,p_n^{n-1}(R^n)\,|\,) \\
&= (1 - |\,R_n^n\,|^2)^{-1/2} \cdot P(R_1^n, \cdots, R_{n-1}^n) \\
&= (1 - |\,R_n^n\,|^2)^{-1/2} \cdot \sum_{j=1}^{n-1} R_j^n \cdot P^j \\
&= (1 - |\,(\tilde{P}:R)_n^n\,|^2)^{-1/2} \,((\tilde{P}:R)_1^n, \cdots, (\tilde{P}:R)_{n-1}^n) \\
&= p_n^{n-1}((\tilde{P}:R)^n)/\,|\,p_n^{n-1}((\tilde{P}:R)^n)\,|,
\end{aligned}$$

$$\begin{aligned}
(P:Q)\Bigg(&\frac{-(\tilde{P}:R)_n^i \cdot (\tilde{P}:R)_n^n}{(1 - |\,(\tilde{P}:R)_n^i\,|^2)^{1/2}(1 - |\,(\tilde{P}:R)_n^n\,|^2)^{1/2}} \cdot {}^{(n-1)}I^{n-1} \\
&+ \sum_{\alpha=0}^{n-i-2} \cdot \frac{-(\tilde{P}:R)_n^{i+1+\alpha} \cdot (\tilde{P}:R)_n^i \cdot {}^{(n-1)}I^{i+\alpha}}{(1 - |\,(\tilde{P}:R)_n^i\,|^2)^{1/2}\left(1 - \sum_{j=1}^{n-i-1-\alpha} |\,(\tilde{P}:R)_n^{i+1+\alpha+j}\,|^2\right)^{1/2}} \\
&\qquad \cdot \frac{1}{\left(1 - \sum_{j=0}^{n-i-1-\alpha} |\,(\tilde{P}:R)_n^{i+1+\alpha+j}\,|^2\right)^{1/2}} \\
&+ \frac{\left(1 - \sum_{j=0}^{n-i} |\,(\tilde{P}:R)_n^{i+j}\,|^2\right)^{1/2}}{(1 - |\,(\tilde{P}:R)_n^i\,|^2)^{1/2}\left(1 - \sum_{j=1}^{n-i} |\,(\tilde{P}:R)_n^{i+j}\,|^2\right)^{1/2}} \cdot {}^{(n-1)}I^{i-1}\Bigg) \\
&= P(p_n^{n-1}(R^i)/\,|\,p_n^{n-1}(R^i)\,|\,) \\
&= (1 - |\,R_n^i\,|^2)^{-1/2} \cdot \sum_{j=1}^{n-1} R_j^i P^j \\
&= (1 - |\,(\tilde{P}:R)_n^i\,|^2)^{-1/2}((\tilde{P}:R)_1^i, \cdots, (\tilde{P}:R)_{n-1}^i) \\
&= p_n^{n-1}((\tilde{P}:R)^i)/\,|\,p_n^{n-1}((\tilde{P}:R)^i)\,|,
\end{aligned}$$

for $i = m+2, \cdots, n-1$.

It easily follows now that $(P:Q)$ satisfies condition II with respect to $(\tilde{P}:R)$.

Part 7.

$$\mathcal{J}_n^{m+1}(S+T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^1(T)\cdot(\beta(n-1,m)/\beta(n,m+1)) \cdot\int_{G_n}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2}d\phi_n R.$$

Proof. Recalling the definition of f, and using Parts 5 and 6, we have for $P\in G_{n-1}$,

$$\begin{aligned}
\mathcal{J}_n^{m+1}&(S+T)\\
&= \beta(n,m+1)^{-1}\cdot\int_{G_n}\int_{E^{m+1}}\gamma[(S+T)\cap\lambda_n^{n-m-1}(R,y)]d\mathcal{L}_{m+1}y\,d\phi_n R\\
&= \beta(n,m+1)^{-1}\cdot\int_{K}\int_{E^{m+1}}\gamma[(S+T)\cap\lambda_n^{n-m-1}(R,y)]d\mathcal{L}_{m+1}y\,d\phi_n R\\
&= \beta(n,m+1)^{-1}\cdot\int_{K}\int_{E^{m+1}}\gamma[(S+T)\cap\lambda_n^{n-m-1}((\tilde{P}:R),y)]d\mathcal{L}_{m+1}y\,d\phi_n R\\
&= \beta(n,m+1)^{-1}\cdot\int_{K}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2}\cdot\mathcal{J}_n^1(T)\\
&\qquad\cdot\int_{E^m}\gamma[p_n^{n-1*}(S)\cap\lambda_{n-1}^{n-m-1}((P:f(R)),x)]d\mathcal{L}_m x\,d\phi_n R.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{J}_n^{m+1}&(S+T)\\
&= \mathcal{J}_n^1(T)\cdot(\beta(n,m+1))^{-1}\cdot\int_{G_{n-1}}\int_{K}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2}\\
&\qquad\cdot\int_{E^m}\gamma[p_n^{n-1*}(S)\ \cap\lambda_{n-1}^{n-m-1}((P:f(R)),x)]d\mathcal{L}_m x\,d\phi_n R\,d\phi_{n-1}P\\
&= \mathcal{J}_n^1(T)\cdot(\beta(n,m+1))^{-1}\cdot\int_{K}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2}\\
&\qquad\cdot\int_{G_{n-1}}\int_{E^m}\gamma[p_n^{n-1*}(S)\ \cap\lambda_{n-1}^{n-m-1}((P:f(R)),x)]d\mathcal{L}_m x\,d\phi_{n-1}P\,d\phi_n R\\
&= \beta(n-1,m)(\beta(n,m+1))^{-1}\cdot\mathcal{J}_n^1(T)\\
&\qquad\cdot\mathcal{J}_{n-1}^m(p_n^{n-1*}(S))\int_K\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2}d\phi_n R
\end{aligned}$$

$$= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^1(T)\cdot\beta(n-1, m)\cdot(\beta(n, m+1))^{-1}$$

$$\cdot \int_{G_n}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2} d\phi_n R.$$

The interchange of the order of integration in the above is justified by the following argument:

The function g defined on G_{n-1} by

$$g(P) = \int_{E^m}\gamma[p_n^{n-1*}(S)\cap\lambda_{n-1}^{n-m-1}(P, x)]d\mathcal{L}_m x \qquad \text{for } P\in G_{n-1},$$

is such that counter-images of open sets are analytic sets. Using the fact that counter-images of analytic sets by continuous functions are again analytic sets, we conclude that the function h on $(G_{n-1}\times K)$ is measurable, where

$$h(P, R) = \int_{E^m}\gamma[p_n^{n-1*}(S)\cap\lambda_{n-1}^{n-m-1}((P:f(R), x)]d\mathcal{L}_m x,$$

for $(P, R)\in(G_{n-1}\times K)$. The Fubini Theorem then applies immediately.

The proof of Part 7 is complete.

Part 8. $\mathcal{J}_n^{m+1}(S+T)=\mathcal{J}_n^m(S)\cdot\mathcal{J}_n^1(T)$.

Proof. To evaluate the factor

$$\beta(n-1, m)(\beta(n, m+1))^{-1}\int_{G_n}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2} d\phi_n R,$$

we take in particular

$$T_0 = E^n\cap \underset{x}{E}\,[x_1 = x_2 = \cdots = x_{n-1} = 0,\ 1\geqq x_n\geqq 0],$$

$$S_0 = E^n\cap \underset{x}{E}\,[0\leqq x_i\leqq 1 \text{ for } i = 1, \cdots, m, \text{ and } x_{m+1} = \cdots = x_n = 0].$$

Then

$$\mathcal{J}_n^{m+1}(S_0+T_0) = 1 = \mathcal{J}_n^1(T_0) = \mathcal{J}_n^m(S_0),$$

therefore

$$\beta(n, m+1)(\beta(n-1, m))^{-1} = \int_{G_n}\left(\sum_{j=1}^{m+1}|R_n^j|^2\right)^{1/2} d\phi_n R,$$

and we conclude that

$$\mathcal{J}_n^{m+1}(S+T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^1(T),$$

for all S, T satisfying the hypotheses of this theorem.

4.6 Corollary. *If $m+1 \leqq n$, then*

$$\beta(n, m+1)(\beta(n-1, m))^{-1} = \int_{G_n} \left(\sum_{j=1}^{m+1} | R_j^n |^2 \right)^{1/2} d\phi_n R.$$

Proof. Recall that for $R \in G_n$,

$$(R^{-1})_j^i = R_i^j \qquad \text{for } 1 \leqq i, j \leqq n.$$

Hence by 4.5 and 2.5,

$$\begin{aligned} \beta(n, m+1)(\beta(n-1, m))^{-1} &= \int_{G_n} \left(\sum_{j=1}^{m+1} | R_n^j |^2 \right)^{1/2} d\phi_n R \\ &= \int_{G_n} \left(\sum_{j=1}^{m+1} | (R^{-1})_j^n |^2 \right)^{1/2} d\phi_n R \\ &= \int_{G_n} \left(\sum_{j=1}^{m+1} | R_j^n |^2 \right)^{1/2} d\phi_n R. \end{aligned}$$

The proof is complete.

4.7 Remark. The preceding corollary affords a new method for the evaluation of $\beta(n, m)$ which differs from the method in [F1, 5]. Roughly speaking, by means of a "distribution measure," we can express

$$\int_{G_n} \left(\sum_{j=1}^{m+1} | R_j^n |^2 \right)^{1/2} d\phi_n R$$

as the integral of a continuous function over the $(n-1)$-dimensional surface of the unit sphere in E^n. Thus $\beta(n, m+1)(\beta(n-1, m))^{-1}$ can be effectively evaluated. Furthermore, it is quite easy to evaluate $\beta(n, 1)$ for all n. From this information, $\beta(n, m)$ can be immediately calculated. The details follow.

Let

$$S = E^n \cap \underset{x}{E} [| x | = 1].$$

Statement 1. For $n \geqq m > 1$,

$$\frac{\beta(n, m)}{\beta(n-1, m-1)} = \frac{\Gamma\left(\frac{n}{2}\right) \cdot \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{m}{2}\right)}.$$

Proof. We let ψ be the measure over S defined by

$$\psi(X) = \phi_n(G_n \cap \underset{R}{E} [R^n \in X]) \qquad \text{for } X \subset S.$$

Then by 4.6 and 4.1,

$$\frac{\beta(n, m)}{\beta(n-1, m-1)} = \int_{G_n} \left(\sum_{j=1}^{m} | R_j^n |^2 \right)^{1/2} d\phi_n R$$

$$= (\mathcal{H}_n^{n-1}(S))^{-1} \cdot \int_S \left(\sum_{j=1}^{m} | x_j |^2 \right)^{1/2} d\mathcal{H}_n^{n-1} x.$$

Using generalized spherical coordinates $(\rho, \theta_1, \cdots, \theta_{n-1})$ of a point $x \in E^n$, given by the relations

$$\begin{aligned} x_n &= \rho \cos \theta_{n-1}, \\ x_{n-1} &= \rho \sin \theta_{n-1} \cos \theta_{n-2}, \\ x_{n-2} &= \rho \sin \theta_{n-1} \sin \theta_{n-2} \cos \theta_{n-3}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ x_2 &= \rho \cdot \prod_{j=2}^{n-1} \sin \theta_j \cdot \cos \theta_1, \\ x_1 &= \rho \cdot \prod_{j=1}^{n-1} \sin \theta_j, \end{aligned}$$

we have

$$(\mathcal{H}_n^{n-1}(S))^{-1} \cdot \int_S \left(\sum_{j=1}^{m} | x_j |^2 \right)^{1/2} d\mathcal{H}_n^{n-1} x$$

$$= \frac{\begin{gathered}\int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \left(\prod_{j=1}^{n-1} \sin^2 \theta_j + \sum_{j=2}^{m} \left(\prod_{i=j}^{n-1} \sin^2 \theta_i \right) \cos^2 \theta_{j-1} \right)^{1/2} \\ \cdot \left(\prod_{j=1}^{n-2} \sin^j \theta_{j+1} \right) d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1}\end{gathered}}{\int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \left(\prod_{j=1}^{n-2} \sin^j \theta_{j+1} \right) d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1}}$$

$$= \frac{\left(\prod_{j=m}^{n-1} \int_0^\pi \sin^j \theta_j d\theta_j \right) \cdot \left(\prod_{j=2}^{m-1} \int_0^\pi \sin^{j-1} \theta_j d\theta_j \right) \cdot 2\pi}{\left(\prod_{j=2}^{n-1} \int_0^\pi \sin^{j-1} \theta_j d\theta_j \right) \cdot 2\pi}$$

$$= \frac{\int_0^\pi \sin^{n-1} x dx}{\int_0^\pi \sin^{m-1} x dx}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)^{-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)^{-1}}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}.$$

This completes the proof of Statement 1.

Statement 2. $\beta(n, 1) = \Gamma(n/2)\cdot\Gamma((n+1)/2)^{-1}\cdot\Gamma(1/2)^{-1}$.

Proof. By definition 2.6,

$$\beta(n, 1) = \int_{G_n} \Delta(R\,|_1^1)d\phi_n R = \int_{G_n} |\, R_1^1 \,|\, d\phi_n R.$$

By 4.1,

$$\int_{G_n} |\, R_1^1 \,|\, d\phi_n R = (\mathcal{H}_n^{n-1}(S))^{-1}\cdot \int_S |\, x_1 \,|\, d\mathcal{H}_n^{n-1} x$$

$$= \frac{\int_0^\pi\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{n-1}\theta_{n-1}\sin^{n-2}\theta_{n-2}\cdots\sin^2\theta_2 \cdot |\sin\theta_1|\, d\theta_1 d\theta_2\cdots d\theta_{n-2}d\theta_{n-1}}{\int_0^\pi\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{n-2}\theta_{n-1}\sin^{n-3}\theta_{n-2}\cdots\sin\theta_2 \cdot d\theta_1 d\theta_2\cdots d\theta_{n-2}d\theta_{n-1}}$$

$$= \frac{2\cdot \prod_{j=1}^{n-1}\left(\int_0^\pi \sin^j x dx\right)}{2\pi\cdot \prod_{j=1}^{n-2}\left(\int_0^\pi \sin^j x dx\right)}$$

$$= \pi^{-1}\cdot \int_0^\pi \sin^{n-1} x dx$$

$$= \pi^{-1}\cdot\Gamma(n/2)\cdot\Gamma(1/2)\cdot\Gamma((n+1)/2)^{-1}$$

$$= \Gamma(n/2)\cdot\Gamma((n+1)/2)^{-1}\cdot\Gamma(1/2)^{-1},$$

hence

$$\beta(n, 1) = \Gamma(n/2)\cdot\Gamma((n+1)/2)^{-1}\cdot\Gamma(1/2)^{-1}.$$

Statement 3.

$$\beta(n, m) = \frac{\Gamma\left(\frac{m+1}{2}\right)\cdot\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\cdot$$

Proof.

$$\begin{aligned}\beta(n, m) &= \frac{\beta(n, m)}{\beta(n-1, m-1)}\cdot\frac{\beta(n-1, m-1)}{\beta(n-2, m-2)}\cdot\cdots\\ &\quad\cdot\frac{\beta(n-m+2, 2)}{\beta(n-m+1, 1)}\cdot\beta(n-m+1, 1)\\ &= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}\cdot\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}\cdot\cdots\\ &\quad\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)}{\Gamma\left(\frac{n-m+3}{2}\right)\Gamma(1)}\cdot\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)}\\ &= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\cdot\end{aligned}$$

This completes the proof.

5. **The integralgeometric (Favard) measure of product sets.** In this section we consider the case in which $\beta = k$.

5.1 LEMMA. *If*

(1) $\quad 1 \leqq k \leqq n,\ 0 \leqq m \leqq n-k;$ *m, n, and k are integers,*

(2) $\quad A = E^n \cap \underset{x}{E}\,[x_{n-k+1} = \cdots = x_n = 0],$

(3) $\quad B = E^n \cap \underset{x}{E}\,[x_1 = \cdots = x_{n-k} = 0],$

(4) $\quad S \subset A$, *S is analytic,* $\mathcal{J}_n^m(S) < \infty,$

(5) $\quad T = B \cap \underset{x}{E}\,[x_{n-k+i} \in J_i$ *for* $i = 1, \cdots, k],$

where J_i is an analytic subset of E^1 of finite $\mathcal{L}_1$ measure, for $i = 1, \cdots, k$, then

$$\mathcal{J}_n^{m+k}(S+T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

Proof. Note first that (5) implies that $T\subset B$ is analytic.

The lemma will be proved by induction on k.

For $k=1$ and for arbitrary n, m with $0\leqq m\leqq n-1$, then 4.5 implies that $\mathcal{J}_n^{m+1}(S+T)=\mathcal{J}_n^m(S)\cdot\mathcal{J}_n^1(T)<\infty$, since $\mathcal{J}_n^1(T)=\mathcal{L}_1(J_1)<\infty$.

Suppose now that the lemma holds for all k such that $1\leqq k\leqq q$ and for all n, m, such that $0\leqq m\leqq n-k$. Let $k=q+1$ and let m, n be such integers that $0\leqq m\leqq n-k$. By the inductive hypothesis, we have

$$\mathcal{J}_{n-1}^{m+q}(p_n^{n-1*}(S)+p_n^{n-1*}(T)) = \mathcal{J}_{n-1}^m(p_n^{n-1*}(S))\cdot\mathcal{J}_{n-1}^q(p_n^{n-1*}(T)) < \infty,$$

since $\mathcal{J}_{n-1}^m(p_n^{n-1*}(S))=\mathcal{J}_n^m(S)<\infty$. Let

$$X = (\eta_n^{n-1}:p_n^{n-1})^*(S+T),$$
$$Y = E^n\cap \underset{x}{E}\,[x_1=\cdots=x_{n-1}=0,\ x_n\in J_k].$$

Now

$$\mathcal{J}_n^{m+q}(X) = \mathcal{J}_{n-1}^{m+q}(p_n^{n-1*}(S)+p_n^{n-1*}(T)),$$

and since we are essentially dealing with Lebesgue measures,

$$\mathcal{J}_n^1(Y)\cdot\mathcal{J}_{n-1}^q(p_n^{n-1*}(T)) = \mathcal{J}_n^k(T).$$

Also 4.5 implies that

$$\mathcal{J}_n^{m+k}(X+Y) = \mathcal{J}_n^{m+q}(X)\cdot\mathcal{J}_n^1(Y) < \infty,$$

therefore

$$\begin{aligned}\mathcal{J}_n^{m+k}(X+Y) &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_{n-1}^q(p_n^{n-1*}(T))\cdot\mathcal{J}_n^1(Y)\\ &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).\end{aligned}$$

But $X+Y=S+T$. Hence

$$\mathcal{J}_n^{m+k}(S+T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T) < \infty,$$

which completes the proof.

5.2 LEMMA. *If conditions* (1), (2), (3), (4) *of* 5.1 *are satisfied and*

(5′) $\quad T\subset B,\ \mathcal{J}_n^k(T)<\infty,\ (E^k\cap\underset{x}{E}\,[(0,\cdots,0,x_1,\cdots,x_k)\in T])$

is an open subset of E^k, *then*

$$\mathcal{J}_n^{m+k}(S+T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

Proof. Let

$$T' = E^k \cap \underset{x}{E}\,[(0, \cdots, 0, x_1, \cdots, x_k) \in T].$$

Then by standard methods, we can obtain a decomposition of T' into a countable number of parallelotopes (not necessarily open or closed) with "sides" parallel to the coordinate axes. More precisely, we may express

$$T' = \bigcup_{i=1}^{\infty} T_i',$$

such that

$$T_i' \cap T_j' = \emptyset \qquad \text{for } i \neq j,$$
$$T_i' = E^k \cap \underset{x}{E}\,[x_j \in J_j^i \text{ for } j = 1, \cdots, k],$$

where J_j^i is a bounded interval of E^1 for $i=1, 2, 3, \cdots; j=1, \cdots, k$. Let

$$T_i = E^n \cap \underset{x}{E}\,[x_1 = \cdots = x_{n-k} = 0 \text{ and } (x_{n-k+1}, \cdots, x_n) \in T_i'],$$

for $i=1, 2, 3, \cdots$. Then by 5.1, $\mathcal{J}_m^{m+k}(S + T_i) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T_i)$, for $i=1, 2, 3, \cdots$. Moreover,

$$T = \bigcup_{i=1}^{\infty} T_i,$$
$$T_i \cap T_j = \emptyset \qquad \text{for } i \neq j,$$
$$S + T = \bigcup_{i=1}^{\infty} (S + T_i),$$
$$(S + T_i) \cap (S + T_j) = \emptyset \qquad \text{for } i \neq j,$$

and therefore

$$\begin{aligned} \mathcal{J}_n^{m+k}(S + T) &= \sum_{i=1}^{\infty} \mathcal{J}_n^{m+k}(S + T_i) \\ &= \sum_{i=1}^{\infty} \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T_i) \\ &= \mathcal{J}_n^m(S)\cdot\sum_{i=1}^{\infty} \mathcal{J}_n^k(T_i) \\ &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T). \end{aligned}$$

The proof is complete.

5.3 THEOREM. *If conditions* (1), (2), (3), (4) *of* 5.1 *are satisfied and*

(5″) $T \subset B$, *T is analytic*, $\mathcal{J}_n^k(T) < \infty$,

then $\mathcal{J}_n^{m+k}(S+T) = \mathcal{J}_n^m(S) \cdot \mathcal{J}_n^k(T)$.

Proof. Let

$$T' = E^k \cap \underset{x}{E}\,[(0, \cdots, 0, x_1, \cdots, x_k) \in T].$$

Then T' is analytic, hence $\mathcal{L}_k$ measurable, and

$$\mathcal{L}_k(T') = \mathcal{J}_n^k(T) < \infty.$$

Let $\epsilon > 0$ and choose a set W_1, open in E^k, such that

$$T' \subset W_1 \quad \text{and} \quad \mathcal{L}_k(W_1) < \mathcal{L}_k(T') + \epsilon.$$

Now $(W_1 - T')$ is $\mathcal{L}_k$ measurable and

$$\mathcal{L}_k(W_1 - T') = \mathcal{L}_k(W_1) - \mathcal{L}_k(T') < \epsilon.$$

Hence we may choose W_2, open in E^k, such that

$$(W_1 - T') \subset W_2 \subset W_1,$$
$$\mathcal{L}_k(W_2) < \mathcal{L}_k(W_1 - T') + \epsilon < 2\epsilon.$$

For $i = 1, 2$, define

$$W_i' = E^n \cap \underset{x}{E}\,[x_1 = \cdots = x_{n-k} = 0 \text{ and } (x_{n-k+1}, \cdots, x_n) \in W_i].$$

Then by 5.2 we have, for $i = 1, 2$,

$$\mathcal{J}_n^{m+k}(S + W_i') = \mathcal{J}_n^m(S) \cdot \mathcal{J}_n^k(W_i') = \mathcal{J}_n^m(S) \cdot \mathcal{L}_k(W_i).$$

Since $S + (W_1' - W_2') \subset (S+T) \subset (S + W_1')$, it follows that

$$\mathcal{J}_n^{m+k}(S + (W_1' - W_2')) \leqq \mathcal{J}_n^{m+k}(S + T) \leqq \mathcal{J}_n^{m+k}(S + W_1').$$

Hence

$$\begin{aligned} \mathcal{J}_n^m(S) \cdot (\mathcal{L}_k(W_1) - 2\epsilon) &\leqq \mathcal{J}_n^m(S) \cdot (\mathcal{L}_k(W_1) - \mathcal{L}_k(W_2)) \\ &= \mathcal{J}_n^{m+k}(S + W_1') - \mathcal{J}_n^{m+k}(S + W_2') \\ &= \mathcal{J}_n^{m+k}(S + (W_1' - W_2')) \\ &\leqq \mathcal{J}_n^{m+k}(S + T) \\ &\leqq \mathcal{J}_n^{m+k}(S + W_1') \\ &= \mathcal{J}_n^m(S) \cdot \mathcal{L}_k(W_1). \end{aligned}$$

By the arbitrary nature of ϵ and because

$$\mathcal{L}_k(W_1) - 2\epsilon \leqq \mathcal{L}_k(W_1 - W_2) \leqq \mathcal{L}_k(T') = \mathcal{J}_n^k(T) = \mathcal{L}_k(T') \leqq \mathcal{L}_k(W_1),$$

we conclude that

$$\mathcal{J}_n^{m+k}(S + T) = \mathcal{J}_n^m(S) \cdot \mathcal{J}_n^k(T).$$

5.4 REMARK. Though the methods used in the proof of 4.5 can be generalized to prove 5.3, the above proof is simpler and more elegant.

5.5 THEOREM. *If*

(1) *A is a subspace of E^n of dimension $(n-k)$, $1 \leqq k < n$,*

(2) *B is the subspace of E^n of dimension k, orthogonal to A,*

(3) *$0 \leqq m \leqq n-k$,*

(4) *$S \subset A$, S is countably $\mathcal{J}_n^m$ measurable,*

(5) *$T \subset B$, T is countably $\mathcal{J}_n^k$ measurable,*

then

$$\mathcal{J}_n^{m+k}(S + T) = \mathcal{J}_n^m(S) \cdot \mathcal{J}_n^k(T).$$

Proof. By 2.10, we may just as well assume that

$$A = E^n \cap \underset{x}{E}\,[x_{n-k+1} = \cdots = x_n = 0],$$
$$B = E^n \cap \underset{x}{E}\,[x_1 = \cdots = x_{n-k} = 0].$$

Assume further that $\mathcal{J}_n^m(S) < \infty$, $\mathcal{J}_n^k(T) < \infty$. Then since S and T are respectively $\mathcal{J}_n^m$ and $\mathcal{J}_n^k$ measurable sets, we can find analytic sets U_1 and V_1 such that

$$S \subset U_1 \subset A, \qquad T \subset V_1 \subset B,$$
$$\mathcal{J}_n^m(S) = \mathcal{J}_n^m(U_1), \qquad \mathcal{J}_n^k(T) = \mathcal{J}_n^k(V_1).$$

Hence

$$\mathcal{J}_n^m(U_1 - S) = 0, \qquad \mathcal{J}_n^k(V_1 - T) = 0.$$

Choose analytic sets U_2 and V_2 such that

$$(U_1 - S) \subset U_2 \subset A, \quad (V_1 - T) \subset V_2 \subset B,$$
$$\mathcal{J}_n^m(U_2) = 0, \qquad \mathcal{J}_n^k(V_2) = 0.$$

Then

$$U_1 - U_2 \subset S, \qquad V_1 - V_2 \subset T,$$
$$\mathcal{J}_n^m(U_1 - U_2) = \mathcal{J}_n^m(U_1) = \mathcal{J}_n^m(S),$$
$$\mathcal{J}_n^k(V_1 - V_2) = \mathcal{J}_n^k(V_1) = \mathcal{J}_n^k(T).$$

By 5.3, we have

$$\mathcal{J}_n^{m+k}(U_1 + V_1) = \mathcal{J}_n^m(U_1)\cdot\mathcal{J}_n^k(V_1) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T),$$
$$\mathcal{J}_n^{m+k}((U_1 - U_2) + (V_1 - V_2)) = \mathcal{J}_n^m(U_1 - U_2)\cdot\mathcal{J}_n^k(V_1 - V_2)$$
$$= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

Finally,

$$\begin{aligned}\mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T) &= \mathcal{J}_n^{m+k}((U_1 - U_2) + (V_1 - V_2))\\ &\leqq \mathcal{J}_n^{m+k}(S + T)\\ &\leqq \mathcal{J}_n^{m+k}(U_1 + V_1)\\ &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T),\end{aligned}$$

therefore

$$\mathcal{J}_n^{m+k}(S + T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

If now S is countably $\mathcal{J}_n^m$ measurable, then we may express

$$S = \bigcup_{i=1}^{\infty} S_i, \qquad S_i \cap S_j = \emptyset \qquad \text{for } i \neq j,$$

S_i is $\mathcal{J}_n^m$ measurable and $\mathcal{J}_n^m(S_i) < \infty$ for $i = 1, 2, \cdots$. Similarly if T is countably $\mathcal{J}_n^k$ measurable, then we may write

$$T = \bigcup_{i=1}^{\infty} T_i, \qquad T_i \cap T_j = \emptyset \qquad \text{for } i \neq j,$$

T_i is $\mathcal{J}_n^k$ measurable and $\mathcal{J}_n^k(T_i) < \infty$ for $i = 1, 2, \cdots$. Then

$$\begin{aligned}\mathcal{J}_n^{m+k}(S + T) &= \mathcal{J}_n^{m+k}\left(\bigcup_{i=1}^{\infty} S_i + \bigcup_{i=1}^{\infty} T_i\right)\\ &= \mathcal{J}_n^{m+k}\left(\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}(S_i + T_j)\right)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{J}_n^{m+k}(S_i + T_j)\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{J}_n^m(S_i)\cdot\mathcal{J}_n^k(T_j)\\ &= \left(\sum_{i=1}^{\infty}\mathcal{J}_n^m(S_i)\right)\cdot\mathcal{J}_n^k(T)\\ &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).\end{aligned}$$

The proof is complete.

5.6 REMARK. If in the hypothesis of 5.5 we do not require S and T to be $\mathcal{J}_n^m$ and $\mathcal{J}_n^k$ measurable respectively, then it is obvious that

$$\mathcal{J}_n^{m+k}(S + T) \leqq \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

5.7 THEOREM. *If*

(1) *A is a subspace of E^n of dimension α,*

(2) *B is the subspace of E^n of dimension $\beta = n-\alpha$, orthogonal to A,*

(3) $0 \leqq m \leqq \alpha$, $0 \leqq k \leqq \beta$,

(4) *f is a Lipschitzian function,* dmn $f = X$, *where X is an $\mathcal{L}_m$ measurable subset of E^m, $f^*(X) = S \subset A$,*

(5) *g is a Lipschitzian function,* dmn $g = Y$, *where Y is an $\mathcal{L}_k$ measurable subset of E^k, $g^*(Y) = T \subset B$,*

then

$$\mathcal{J}_n^{m+k}(S + T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

Proof. Without loss of generality, we may assume that

$$A = E^n \cap \underset{x}{E}\,[x_{\alpha+1} = \cdots = x_n = 0],$$
$$B = E^n \cap \underset{x}{E}\,[x_1 = \cdots = x_\alpha = 0].$$

By [MF], we may further assume that both f and g are univalent.

Now for all functions u, v, such that dmn $u \subset E^m$, dmn $v \subset E^k$, rng $u \subset E^n$, rng $v \subset E^n$, we define the function $(u \otimes v)$ on (dmn $u \times$ dmn $v) \subset E^{m+k}$ (we shall make no distinction between $E^m \times E^k$ and E^{m+k}) to E^n by the formula

$$(u \otimes v)(x) = u(x_1, \cdots, x_m) + v(x_{m+1}, \cdots, x_{m+k})$$

for $x = (x_1, \cdots, x_{m+k}) \in$ dmn $(u \otimes v)$. Then

$(f \otimes g)$ is univalent and Lipschitzian,

$$(f \otimes g)^*(X \times Y) = (S + T).$$

Letting $Dh(x)$ denote the (total) differential of h at x, then by straightforward methods it is seen that for $\mathcal{L}_{m+k}$ almost all $z \in (X \times Y)$,

$$D(f \otimes g)(z) = Df(z_1, \cdots, z_m) \otimes Dg(z_{m+1}, \cdots, z_{m+k}),$$

and therefore $J(f \otimes g)(z) = Jf(z_1, \cdots, z_m)\cdot Jg(z_{m+1}, \cdots, z_{m+k})$. Hence by [F2, 5.13],

$$\mathcal{J}_n^{m+k}(S + T) = \int_{E^n} N[(f \otimes g), (X \times Y), z]\,d\mathcal{J}_n^{m+k}z$$

$$= \int_{X\times Y} J(f \otimes g)(z)\,d\mathcal{L}_{m+k}z$$

$$= \int_X Jf(x)d\mathcal{L}_m x \cdot \int_Y Jg(y)d\mathcal{L}_k y$$

$$= \int_{E^n} N[f, X, x]d\mathcal{J}_n^m x \cdot \int_{E^n} N[g, Y, y]d\mathcal{J}_n^k y$$

$$= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

The proof is complete.

5.8 THEOREM. *If*

(1) *A is a subspace of E^n of dimension α,*

(2) *B is the subspace of E^n of dimension $\beta = n - \alpha$, orthogonal to A,*

(3) $0 \leqq m \leqq \alpha$, $0 \leqq k \leqq \beta$,

(4) *$S \subset A$ is a countable union of m rectifiable, $\mathcal{J}_n^m$ measurable sets of finite $\mathcal{J}_n^m$ measure,*

(5) *$T \subset B$ is a countable union of k rectifiable, $\mathcal{J}_n^k$ measurable sets of finite $\mathcal{J}_n^k$ measure,*

then

$$\mathcal{J}_n^{m+k}(S + T) = \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T).$$

Proof. We may just as well assume that S is m rectifiable, $\mathcal{J}_n^m(S) < \infty$, T is k rectifiable, $\mathcal{J}_n^k(T) < \infty$.

Since the closure of a rectifiable set is rectifiable, we may choose analytic m rectifiable sets S' and S'' for which

$$S'' \subset S \subset S' \subset A, \qquad \mathcal{J}_n^m(S'') = \mathcal{J}_n^m(S) = \mathcal{J}_n^m(S').$$

By 2.11, choose a Lipschitzian function f such that dmn $f = E^m$, $f^*(E^m) \supset S' \supset S''$. By the analyticity of S' and S'' and the continuity of f,

$$X' = (E^m \cap \underset{x}{E}\,[f(x) \in S']) \text{ is an analytic set,}$$
$$X'' = (E^m \cap \underset{x}{E}\,[f(x) \in S'']) \text{ is an analytic set.}$$

Let $f_1 = (f|X')$, $f_2 = (f|X'')$,

Similarly choose Lipschitzian functions g_1, g_2, with $\mathcal{L}_k$ measurable domains and sets T', T'', such that

$$\text{rng } g_2 = T'' \subset T \subset T' = \text{rng } g_1 \subset B, \qquad \mathcal{J}_n^k(T'') = \mathcal{J}_n^k(T) = \mathcal{J}_n^k(T').$$

Then by 5.7,

$$\begin{aligned} \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T) &= \mathcal{J}_n^m(S'')\cdot\mathcal{J}_n^k(T'') = \mathcal{J}_n^{m+k}(S'' + T'') \\ &\leqq \mathcal{J}_n^{m+k}(S + T) \leqq \mathcal{J}_n^{m+k}(S' + T') = \mathcal{J}_n^m(S')\cdot\mathcal{J}_n^k(T') \\ &= \mathcal{J}_n^m(S)\cdot\mathcal{J}_n^k(T). \end{aligned}$$

The proof is complete.

6. **Hausdorff measure of product sets.** In this section we construct a subset A of E^2 such that the unit cylinder B with base A has larger 2-dimensional Hausdorff measure than the 1-dimensional Hausdorff measure of A. In doing this we use and extend some of the ideas of Besicovitch and Moran, who gave a rough outline of such a construction (see [BM]).

6.1 DEFINITION. If C is a subset of E^3, we shall say that

$$D \text{ is complete with respect to } C,$$

if and only if D is such a bounded subset of C that

$$(p \in C - D) \rightarrow (\operatorname{diam} (D \cup \{p\}) > \operatorname{diam} D).$$

Unless otherwise stated, we shall mean in this paper by a complete set, a set which is complete with respect to E^3.

We shall make use of the fact that a set is complete (in E^3) if and only if it has constant width.

6.2 REMARK. For the properties of convex sets and, in particular, complete sets, which we shall use, see [BF].

6.3 THEOREM. *Let* $a > 0$,

$$C = E^3 \cap \underset{z}{E} [z_1^2 + z_2^2 \leqq a^2/4],$$

$\infty > d > 0$. *Then of all subsets of C with diameter less than or equal to d, the set*

$$(C \cap \underset{z}{E} [|z| \leqq d/2])$$

has the largest volume.

Proof. Let $D \subset C$ be such that $\operatorname{diam} D = \delta \leqq d$. Choose a convex set $D_1 \subset C$ such that $D \subset D_1$ and $\operatorname{diam} D_1 = \delta$.

Let now H_1 be the supporting function of D_1 (see [BF, p. 23]). Then by the "central-symmetrization" of [BF, p. 73], we define the function H_2 on E^3 by $H_2(u) = (1/2)(H_1(u) + H_1(-u))$ for $u \in E^3$. H_2 is the supporting function of a convex set, call it D_2, with θ^3 as center and with the properties that $\operatorname{diam} D_2 = \operatorname{diam} D_1 = \delta$, $\mathcal{L}_3(D_2) \geqq \mathcal{L}_3(D_1)$. Moreover if $u \in E^3$, $|u| = 1$, $u_3 = 0$, then $|H_1(u)| \leqq a/2$, since $D_1 \subset C$. Hence $|H_2(u)| \leqq a/2$, which implies that $D_2 \subset C$.

Finally we shall show that

$$D_2 \subset (C \cap \underset{z}{E} [|z| \leqq d/2]).$$

Let $z \in D_2$. Since D_2 has θ^3 as center, $-z \in D_2$. Hence

$$2 \cdot |z| = |z - (-z)| \leqq \operatorname{diam} D_2 = \delta \leqq d, \qquad |z| \leqq d/2,$$

$$z \in C \cap \underset{z}{E}\,[\,|z| \leqq d/2].$$

Thus

$$\mathcal{L}_3(C \cap \underset{z}{E}\,[\,|z| \leqq d/2]) \geqq \mathcal{L}_3(D_2) \geqq \mathcal{L}_3(D),$$

and the proof is complete.

6.4 REMARK. The isodiametric inequality proved in 6.3 can obviously be generalized to any cylinder symmetric with respect to its axis. However we shall use only the result stated in 6.3.

6.5 DEFINITION. If c is a bounded subset of E^3, let g_c be the function on E^2 to E^1 such that for $x \in E^2$,

$$g_c(x) = \mathcal{H}_3^1(c \cap \underset{z}{E}\,[z = (x_1, x_2, t), 0 \leqq t \leqq 1]).$$

6.6 THEOREM. *If c is a complete set of diameter d, and $\delta > 0$, then*

$$(x \in E^2, y \in E^2 \text{ and } |x - y| \leqq \delta) \to (|g_c(x) - g_c(y)| \leqq 2^{3/2} \cdot (\delta d)^{1/2}).$$

Proof. Let $c_1 = p_3^{2*}(c)$ and let β be the boundary of c_1. The following fact is well known:

If h is a convex function on the interval

$$(E^1 \cap \underset{x}{E}\,[a \leqq x \leqq b])$$

to E_1 and $\lambda > 0$, then either

$$\sup_{|x-a| \leqq \lambda} |h(x) - h(a)| = \sup_{|x-y| \leqq \lambda} |h(x) - h(y)|,$$

or

$$\sup_{|x-b| \leqq \lambda} |h(x) - h(b)| = \sup_{|x-y| \leqq \lambda} |h(x) - h(y)|.$$

Suppose now that $x \in c_1$, $y \in c_1$, and $0 < |x - y| \leqq \delta$. Choose $a \in c_1$, $b \in c_1$ so that

$$\{a\} \cup \{b\} = \beta \cap \underset{z}{E}\,[z = y + t(x - y), -\infty < t < \infty].$$

Since

$$(g_c \mid \underset{z}{E}\,[z = a + t(b - a), 0 \leqq t \leqq 1])$$

is a convex function, we conclude that either

$$|g_c(x) - g_c(y)| \leqq \sup_{|z-a| \leqq \delta} |g_c(z) - g_c(a)|,$$

or

$$| g_c(x) - g_c(y) | \leqq \sup_{|z-b|\leqq\delta} | g_c(z) - g_c(b) |.$$

Hence it follows that

$$\sup_{x\in c_1,\, y\in c_1,\, |x-y|\leqq\delta} | g_c(x) - g_c(y) | = \sup_{x\in\beta,\, y\in c_1,\, |x-y|\leqq\delta} | g_c(x) - g_c(y) |.$$

Now let $x\in\beta$, $y\in c_1$, $0<|x-y|\leqq\delta$. By [BF, p. 127] there is a unique number t for which $(x_1, x_2, t)\in c$. Since x is a boundary point of c_1, there is a supporting line L of c_1 through x. Let L' be the line determined by $(x_1, x_2, 0)$ and $(x_1, x_2, 1)$. Then obviously L and L' determine a vertical supporting plane to c through (x_1, x_2, t). Since c is a set of constant width, we may choose $x'\in c_1$ such that $|x'-x|=d$, and such that $(x_1', x_2', t)\in c$. Therefore c is contained in the sphere of radius d and center (x_1', x_2', t). The intersection of this sphere with the plane determined by $(x_1, x_2, 0)$, $(y_1, y_2, 0)$, and $(y_1, y_2, 1)$ is then a disc of radius $r\leqq d$, whose circumference passes through (x_1, x_2, t). Hence since the length of the vertical chord (of this disc) passing through (y_1, y_2, t) is $2\cdot(r^2-(r-|x-y|)^2)^{1/2}=2\cdot(2r\cdot|x-y|-|x-y|^2)^{1/2}$, we conclude that

$$| g_c(x) - g_c(y) | = g_c(y) \leqq 2\cdot(2r\cdot | x - y | - | x - y |^2)^{1/2} \leqq 2\cdot(2d\delta)^{1/2}.$$

Hence $\sup_{|x-y|\leqq\delta} |g_c(x)-g_c(y)| \leqq 2^{3/2}\cdot(\delta d)^{1/2}$. This completes the proof.

6.7 REMARK. If for each bounded subset c of E^3 we let g_c' be the function on E^2 to E^1 such that

$$g_c'(x) = \mathcal{H}_3^1(c \cap \mathop{E}_z [z = (x_1, x_2, t), \ -\infty < t < \infty]),$$

for $x\in E^2$, then 6.6 remains true with g_c' substituted for g_c.

6.8 DEFINITION. Suppose n is a positive integer, B is a closed disc in E^2 of radius $a>0$ and with center $z\in E^2$. Let

$B_0 = (E^2 \cap \mathop{E}_x [| x | \leqq a])$,

$C = (E^2 \cap \mathop{E}_x [x_1 = (j/n)a,\ x_2 = (k/n)a$, where j and k are integers$])$,

$N = \gamma(C \cap B_0)$,

f be the function on E^2 such that $f(x) = (1 - 1/N)x$, for $x \in E^2$.

We shall say that

A is the n-uniform spread of B,

if and only if

$$A = \bigcup_{y\in f^*(C\cap B_0)} \{E^2 \cap \mathop{E}_x [| x - y - z | \leqq a/N]\}.$$

Since $N=\gamma(A)$ is obviously independent of the radius and center of B and depends only on n, we shall call it $N(n)$.

6.9 REMARK. Let B be a closed disc in E^2 of radius $a>0$, let A_n denote the n-uniform spread of B for $n=1, 2, 3, \cdots$, and let

$$D_n = \inf_{a_1\in A_n,\ a_2\in A_n,\ a_1\neq a_2} \left(\inf_{x\in a_1,\ y\in a_2} |x-y| \right),$$

for $n=1, 2, 3, \cdots$. Then the following statements are easily checked:

(1) $\sigma A_n \subset B$,

(2) $(a_1 \in A_n,\ a_2 \in A_n,\ a_1 \neq a_2) \to (a_1 \cap a_2 = \emptyset)$ for $n = 1, 2, 3, \cdots$,

(3) $\lim_{n\to\infty} N(n)/(\pi n^2) = 1$,

(4) $D_n = a/n - a/(n\cdot N(n)) - 2a/N(n)$,

(5) $\lim_{n\to\infty} D_n/(a/n) = 1$,

(6) $\lim_{n\to\infty} D_n^2 \cdot N(n)/(\pi a^2) = 1$.

6.10 FUNDAMENTAL CONSTRUCTION. Let $n_1=100$ and define the increasing sequence $n_1, n_2, \cdots$ inductively so that

$$N(n_{i+1}) \geqq M_i^4, \qquad \text{for } i = 1, 2, 3, \cdots,$$

where for brevity we shall let

$$M_i = \prod_{j=1}^{i} N(n_j), \qquad \text{for } i = 1, 2, 3, \cdots.$$

Let

$$A_0 = \{E^2 \cap \underset{x}{E}\,[\,|x| \leqq 1/2]\},$$

and define A_i inductively as follows:

Denoting the n_i-uniform spread of $a\in A_{i-1}$ by $u(a)$, we define

$$A_i = \bigcup_{a\in A_{i-1}} u(a) \qquad \text{for } i = 1, 2, 3, \cdots.$$

It is obvious that if $a\in A_i$, then a is a closed disc in E^2 of diameter M_i^{-1} and that $\gamma(A_i)=M_i$ for $i=1, 2, 3, \cdots$.

For the remainder of this paper we shall let

$$d_i = \inf_{a_1\in A_i,\ a_2\in A_i,\ a_1\neq a_2} \operatorname{dist}(a_1, a_2) \qquad \text{for } i = 1, 2, 3, \cdots.$$

Note that if in 6.9 we let B be some element of A_{i-1}, then

$$D_{n_i} = d_i.$$

We now define

$$A = \bigcap_{i=0}^{\infty} \sigma A_i.$$

Since $\sum_{a \in A_i} \text{diam } a = 1$ for $i = 1, 2, 3, \cdots$, it follows from the definition of $\mathcal{H}_2^1$ that $\mathcal{H}_2^1(A) \leqq 1$.

We define B_i for $i = 1, 2, 3, \cdots$, as follows:

$$B_i = \bigcup_{a \in A_i} \{a \times \underset{y}{E}\,[0 \leqq y \leqq 1]\}.$$

Further let

$$B = \bigcap_{i=0}^{\infty} \sigma B_i.$$

Then

$$B = A \times \underset{t}{E}\,[0 \leqq t \leqq 1].$$

A and B will be fixed for the remainder of this section. Theorem 6.18 will show that $\mathcal{H}_3^2(B) > 1 \geqq \mathcal{H}_2^1(A)$. But first we need some preliminary definitions and theorems.

6.11 THEOREM. *If c is a complete set, then*

$$0 \leqq \lim_{i \to \infty} \mathcal{L}_3(c \cap \sigma B_i)/\mathcal{L}_3(\sigma B_i) \leqq 1.$$

Proof. For $i = 1, 2, 3, \cdots$, define

$$\psi_i(S) = \mathcal{L}_2(S \cap \sigma A_i)/\mathcal{L}_2(\sigma A_i) \qquad \text{for } S \subset E^2.$$

Then ψ_i is a measure over E^2, $\psi_i(E^2) = 1$, and hence

$$\int f d\psi_i \leqq \|f\| = \sup_{x \in E^2} |f(x)|,$$

for any Baire function f on E^2 to E^1.

Let C_i, for $i = 1, 2, 3, \cdots$, be the set of all functions f on E^2 to E^1 such that $(f|a)$ is constant for all $a \in A_i$. The proof is divided into 4 parts.

Part 1. $(f \in C_i) \to (\lim_{k \to \infty} \int f d\psi_k$ exists).

Proof. Just notice that if $f \in C_i$, then $\int f d\psi_j = \int f d\psi_i$ for $j \geqq i$.

Part 2. If g is such a continuous function on E^2 that for some $i \geqq 1$ and for all $a \in A_i$,

$$(x \in a,\ y \in a) \to (|g(x) - g(y)| \leqq \epsilon),$$

then for $j \geqq i$, $k \geqq i$,

$$\left| \int g d\psi_j - \int g d\psi_k \right| \leqq \epsilon.$$

Proof. Define $f_1, f_2 \in C_i$ as follows:

$$\begin{aligned} f_1(x) &= \inf_{y \in a} g(y) && \text{if } x \in a \in A_i, \\ f_1(x) &= g(x) && \text{if } x \in E^2 - \sigma A_i, \\ f_2(x) &= \sup_{y \in a} g(y) && \text{if } x \in a \in A_i, \\ f_2(x) &= g(x) && \text{if } x \in E^2 - \sigma A_i. \end{aligned}$$

Then $f_1(x) \leqq g(x) \leqq f_2(x)$ for $x \in E^2$, $\|f_2 - f_1\| \leqq \epsilon$. Hence for $k \geqq i$,

$$\int f_1 d\psi_i = \int f_1 d\psi_k \leqq \int g d\psi_k \leqq \int f_2 d\psi_k = \int f_2 d\psi_i.$$

Since

$$\int f_2 d\psi_i - \int f_1 d\psi_i \leqq \|f_2 - f_1\| \leqq \epsilon,$$

we conclude that if $j \geqq i$, $k \geqq i$, then

$$\left| \int g d\psi_j - \int g d\psi_k \right| \leqq \epsilon.$$

This completes the proof of Part 2.

Part 3. If g is a continuous function on E^2, then

$$-\infty < \lim_{k \to \infty} \int g d\psi_k < \infty.$$

Proof. Note that g is uniformly continuous on the closed unit disc centered at the origin. Hence for any $\epsilon > 0$, we can choose $i \geqq 1$ so that for all $a \in A_i$,

$$(x \in a,\ y \in a) \to (|g(x) - g(y)| \leqq \epsilon).$$

Apply Part 2 to complete the proof.

Part 4. $0 \leqq \lim_{i \to \infty} \mathcal{L}_3(c \cap \sigma B_i)/\mathcal{L}_3(\sigma B_i) \leqq 1$.

Proof. Applying 6.6 and Part 3, we have

$$-\infty < \lim_{i \to \infty} \int g_c d\psi_i < \infty.$$

Since $0 \leqq \int g_c d\psi_i = \mathcal{L}_3(c \cap \sigma B_i)/\mathcal{L}_3(\sigma B_i) \leqq 1$ for $i = 1, 2, \cdots$, the proof is complete.

6.12 Corollary. *If c is such a complete set that for some $i \geqq 1$,*

$$(a \in A_i,\ x \in a,\ y \in a) \to (|g_c(x) - g_c(y)| \leqq \epsilon),$$

then

$$\left| \frac{\mathcal{L}_3(c \cap \sigma B_i)}{\mathcal{L}_3(\sigma B_i)} - \lim_{j\to\infty} \frac{\mathcal{L}_3(c \cap \sigma B_j)}{\mathcal{L}_3(\sigma B_j)} \right| \leqq \epsilon.$$

6.13 Definition. In view of 6.11, we define for c a complete set,

$$f(c) = \lim_{i\to\infty} \mathcal{L}_3(c \cap \sigma B_i)/\mathcal{L}_3(\sigma B_i).$$

6.14 Convention. We shall let C be the set of all complete sets c such that

$$f(c) \geqq \frac{1}{2} \cdot \sup_{x \in X} f(x), \quad \text{where} \quad X = \underset{x}{E}\,[x \text{ is complete and diam } x \leqq \text{diam } c].$$

6.15 Lemma. *Let $\lambda > 0$. Then there exists an integer K such that if $k \geqq K$, $c \in C$, and*

$$100 M_k^{-1} \geqq \text{diam } c \geqq d_{k+1}^2 \cdot M_{k+1}/(100\pi),$$

then

$$(1 - \lambda)\cdot f(c) \leqq \mathcal{L}_3(c \cap \sigma B_{k+1})/\mathcal{L}_3(\sigma B_{k+1}) \leqq (1 + \lambda)\cdot f(c).$$

Proof. By 6.9(6) choose K so large that for all $k \geqq K$,

$$(399/400)\cdot d_{k+1}^2 \leqq \pi\cdot(2M_k)^{-2}(N(n_{k+1}))^{-1} \leqq (401/400)\cdot d_{k+1}^2,$$

and

$$M_k \geqq 10^{13}/\lambda.$$

Assume $k \geqq K$ Then

$$(401 M_k)^{-1} \leqq (100\pi)^{-1}\cdot d_{k+1}^2 \cdot M_{k+1} \leqq (399 M_k)^{-1}.$$

Choose $b \in B_k$ with base $a \in A_k$ and let E be a right circular cylinder whose axis coincides with the axis of b, whose height is $(200\pi)^{-1}\cdot d_{k+1}^2 \cdot M_{k+1}$, the radius of whose base is $(400\pi)^{-1} d_{k+1}^2 \cdot M_{k+1}$, and which is completely contained in b. Then

$$\text{diam } E \leqq (100\pi)^{-1} d_{k+1}^2 \cdot M_{k+1},$$

and since there are more than $\pi((800\pi)^{-1}\cdot d_{k+1}\cdot M_{k+1})^2$ elements of A_{k+1} contained in the closed disc of radius $((400\pi)^{-1} d_{k+1}\cdot M_{k+1}) d_{k+1}$ and center coinciding with the center of a,

$$f(E) \geqq \frac{(\pi((800\pi)^{-1} d_{k+1} M_{k+1})^2)\cdot(\pi(2M_{k+1})^{-2})\cdot((200\pi)^{-1} d_{k+1}^2 M_{k+1})}{(M_{k+1})(\pi(2M_{k+1})^{-2})}$$

$$= (128\cdot 100^3\cdot \pi^2)^{-1}\cdot d_{k+1}^4 \cdot M_{k+1}^2.$$

Let now $c \in C$ be such that

$$100 M_k^{-1} \geqq \operatorname{diam} c \geqq (100\pi)^{-1} \cdot d_{k+1}^2 \cdot M_{k+1}.$$

Then

$$f(c) \geqq (256 \cdot 100^3 \cdot \pi^2)^{-1} d_{k+1}^4 \cdot M_{k+1}^2.$$

Also, if $a \in A_{k+1}$, $x \in a$, $y \in a$, then by **6.6**,

$$| g_c(x) - g_c(y) | \leqq 2^{3/2} \cdot (\operatorname{diam} c)^{1/2} (M_{k+1})^{-1/2} \leqq 2^{3/2} \cdot 10 \cdot M_k^{-1} \cdot N(n_{k+1})^{-1/2}.$$

Hence by 6.12,

$$\begin{aligned}
| (\mathcal{L}_3(c \cap \sigma B_{k+1})/\mathcal{L}_3(\sigma B_{k+1})) - f(c) | &\leqq 2^{3/2} \cdot 10 \cdot M_k^{-1} \cdot N(n_{k+1})^{-1/2} \\
&\leqq \frac{2^{3/2} \cdot 10 \cdot M_k^{-1} \cdot N(n_{k+1})^{-1/2}}{(256 \cdot 100^3 \cdot \pi^2)^{-1} \cdot d_{k+1}^4 \cdot M_{k+1}^2} \cdot f(c) \\
&= 2^{3/2} \cdot 256 \cdot 10^7 \cdot \pi^2 \cdot f(c) \cdot (d_{k+1}^8 \cdot M_k^6 \cdot N(n_{k+1})^5)^{-1/2} \\
&\leqq 10^{11} \cdot \pi^2 \cdot f(c) \cdot \left(\left(\frac{401}{100\pi} \right)^4 \cdot \frac{M_k^8 \cdot N(n_{k+1})^4}{M_k^6 \cdot N(n_{k+1})^5} \right)^{1/2} \\
&\leqq 10^{13} \cdot f(c) \cdot M_k \cdot N(n_{k+1})^{-1/2} \\
&\leqq 10^{13} \cdot f(c) \cdot M_k^{-1} \\
&\leqq \lambda \cdot f(c).
\end{aligned}$$

The proof is complete.

6.16 LEMMA. *Let $\lambda > 0$. There exists an integer K such that if $k \geqq K$, then*

(1) $100 \cdot M_k^{-1} \leqq d_k$,

(2) $(401 \cdot M_k)^{-1} \leqq (100\pi)^{-1} d_{k+1}^2 \cdot M_{k+1} \leqq (399 \cdot M_k)^{-1}$,

(3) *If c is a complete set which is also complete with respect to some $b \in B_k$,* diam $c =$ diam $(c \cap b)$, *and*

$$100\, M_k^{-1} \geqq \operatorname{diam} c \geqq (100\pi)^{-1} \cdot d_{k+1}^2 \cdot M_{k+1},$$

then

$$(1 - \lambda) \leqq \left(\frac{\mathcal{L}_3(c \cap \sigma B_{k+1})}{\mathcal{L}_3(\sigma B_{k+1})} \right) \cdot \left(\frac{\mathcal{L}_3(c \cap \sigma B_k)}{\mathcal{L}_3(\sigma B_k)} \right)^{-1} \leqq (1 + \lambda).$$

Proof. Let q be the infimum of the volumes of all complete sets of diameter 1. Then $q > 0$. By 6.9(6), (5), (3), choose K so large that for all $k \geqq K$, (1) and (2) of this lemma hold, and $M_k \geqq 10^{14} \cdot q^{-1} \cdot \lambda^{-1}$.

Let $k \geqq K$, and suppose c satisfies the hypotheses of (3). Then since diam $c \leqq d_k$, we see that c intersects exactly one element of B_k, namely b.

Because c is complete with respect to b, diam $(c\cap b)=$ diam $c\geqq(100\pi)^{-1}\cdot d_{k+1}^2\cdot M_{k+1}$, we have

$$\mathcal{L}_3(c\cap\sigma B_k)\geqq q((100\pi)^{-1}\cdot d_{k+1}^2\cdot M_{k+1})^3.$$

Let $a\in A_k$ be the base of b, and suppose $\alpha_1,\cdots,\alpha_{N(n_{k+1})}$ are the $N(n_{k+1})$ elements of A_{k+1} contained in a. Now associate to each α_i for $i=1,\cdots,N(n_{k+1})$, a Borel set $\alpha_i'\subset a$ such that

$$\alpha_i'\cap\alpha_j'=\emptyset\quad\text{for}\quad i\neq j,\qquad \alpha_i\subset\alpha_i',\qquad \operatorname{diam}\alpha_i'\leqq 2d_{k+1},$$

$$\bigcup_{i=1}^{N(n_{k+1})}\alpha_i'=a,\qquad \mathcal{L}_2(\alpha_i')=\pi\cdot(2M_k)^{-2}\cdot N(n_{k+1})^{-1}.$$

If $x\in\alpha_i'$, $y\in\alpha_i'$ for some $1\leqq i\leqq N(n_{k+1})$, then by 6.6,

$$|g_c(x)-g_c(y)|\leqq 2^{3/2}\cdot(2\cdot d_{k+1}\cdot\operatorname{diam} c)^{1/2}\leqq 40\cdot(d_{k+1}\cdot M_k^{-1})^{1/2},$$

and hence

$$\left|\int_{\alpha_i'}g_c(x)d\mathcal{L}_2x-N(n_{k+1})\cdot\int_{\alpha_i}g_c(x)d\mathcal{L}_2x\right|\leqq 40\cdot d_{k+1}^{1/2}\cdot M_k^{-1/2}\cdot\mathcal{L}_2(\alpha_i'),$$

$$|\mathcal{L}_3(c\cap\sigma B_k)-N(n_{k+1})\mathcal{L}_3(c\cap\sigma B_{k+1})|\leqq 40\cdot d_{k+1}^{1/2}\cdot M_k^{-1/2}\cdot\pi\cdot(2M_k)^{-2}.$$

Therefore,

$$\begin{aligned}
&\left|1-\left(\frac{\mathcal{L}_3(c\cap\sigma B_{k+1})}{\mathcal{L}_3(\sigma B_{k+1})}\right)\cdot\left(\frac{\mathcal{L}_3(c\cap\sigma B_k)}{\mathcal{L}_3(\sigma B_k)}\right)^{-1}\right|\\
&\qquad=\left|1-\frac{\mathcal{L}_3(c\cap\sigma B_{k+1})\cdot N(n_{k+1})}{\mathcal{L}_3(c\cap\sigma B_k)}\right|\\
&\qquad\leqq\frac{40\cdot d_{k+1}^{1/2}\cdot M_k^{-1/2}\cdot\pi\cdot(2M_k)^{-2}}{q\cdot[(100\pi)^{-1}\cdot d_{k+1}^2\cdot M_{k+1}]^3}\\
&\qquad=\frac{(100\pi)^3}{q}\cdot 10\pi\cdot(d_{k+1}^{11}\cdot M_k^{11}\cdot N(n_{k+1})^6)^{-1/2}\\
&\qquad\leqq\frac{10^{11}}{q}\left(\left(\frac{401}{100\pi}\right)^{11/2}\cdot\frac{M_k^{11}\cdot N(n_{k+1})^{11/2}}{M_k^{11}\cdot N(n_{k+1})^6}\right)^{1/2}\\
&\qquad\leqq\frac{10^{14}}{q}\cdot M_k^{-1}\leqq\lambda.
\end{aligned}$$

The proof is complete.

6.17 LEMMA. *Let $\lambda>1$. There exists an integer K such that if $k\geqq K$, then*

(1) $d_k \geqq 100 M_k^{-1}$,

(2) $(401 \cdot M_k)^{-1} \leqq (100\pi)^{-1} \cdot d_{k+1}^2 \cdot M_{k+1} \leqq (399 \cdot M_k)^{-1}$,

(3) *If* $c \in C$ *is complete with respect to some* $b \in B_k$, diam $(b \cap c) =$ diam c, *and*

$$100 M_k^{-1} \geqq \operatorname{diam} c \geqq (100\pi)^{-1} d_{k+1}^2 \cdot M_{k+1},$$

then

$$\lambda^{-1} \cdot f(c) \leqq \mathcal{L}_3(c \cap \sigma B_k)/\mathcal{L}_3(\sigma B_k) \leqq \lambda \cdot f(c).$$

Proof. Apply 6.15, 6.16.

6.18 Theorem. $\mathcal{H}_3^2(B) > 1 \geqq \mathcal{H}_2^1(A)$.

Proof. Note that in the definition of $\mathcal{H}_3^2$, we may restrict our coverings to consist of complete sets. We can do this because every set in E^3 is contained in a complete set of the same diameter.

Let $\lambda > 1$. Choose K as in 6.17. Then the theorem follows from Part 4 and the arbitrary nature of λ.

Part 1. If $k \geqq K$, c is such a complete set that

$$100 M_k^{-1} \leqq \operatorname{diam} c \leqq d_k, \qquad f(c) > 0,$$

then

$$(\operatorname{diam} c)^2/f(c) \geqq 100.$$

Proof. Since diam $c \leqq d_k$ and $f(c) > 0$, c intersects exactly one element of B_k, say b. Let

$$\tau = \inf \underset{t}{E} \left[(x_1, x_2, t) \in (b \cap c) \text{ for some } x_1, x_2\right].$$

Then

$$(b \cap c) \subset (b \cap \underset{x}{E} \left[\tau \leqq x_3 \leqq \tau + \operatorname{diam} c\right]),$$

and hence

$$f(c) \leqq \operatorname{diam} c \cdot M_k^{-1},$$
$$(\operatorname{diam} c)^2/f(c) \geqq M_k \cdot \operatorname{diam} c \geqq 100.$$

This completes the proof of Part 1.

Part 2. If $k \geqq K$, c is such a complete set that

$$d_{k+1} \leqq \operatorname{diam} c \leqq (100\pi)^{-1} \cdot d_{k+1}^2 \cdot M_{k+1}, \qquad f(c) > 0,$$

then

$$(\operatorname{diam} c)^2/f(c) \geqq 25.$$

Proof. The set c cannot intersect more than $\pi(2\cdot\text{diam } c/d_{k+1})^2$ elements of B_{k+1}, since $p_3^{2*}(c)$ is contained in a disc of radius equal to diam c. Hence, by a similar argument as was used in the proof of Part 1, namely that the intersection of c with an element of B_{k+1} is contained in a cylinder of height equal to diam c and base radius equal to $(2M_{k+1})^{-1}$, we conclude that

$$f(c) \leqq \pi(2\cdot\text{diam } c/d_{k+1})^2\cdot\text{diam } c\cdot M_{k+1}^{-1},$$

and therefore

$$(\text{diam } c)^2/f(c) \geqq d_{k+1}^2\cdot M_{k+1}/(4\pi\cdot\text{diam } c) \geqq 25.$$

This proves Part 2.

Part 3. If $k \geqq K$, c is such a complete set that

$$(100\pi)^{-1}\cdot d_{k+1}^2\cdot M_{k+1} \leqq \text{diam } c \leqq 100M_k^{-1}, \qquad f(c) > 0,$$

then

$$(\text{diam } c)^2/f(c) \geqq 2\cdot(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1}.$$

Proof. If $c \notin C$, then we can choose a set $c' \in C$ such that diam $c =$ diam c', and $f(c') > f(c)$. Hence

$$(\text{diam } c)^2/f(c) > (\text{diam } c')^2/f(c'),$$

so that we may just as well assume that $c \in C$.

Since diam $c \leqq d_k$, c intersects exactly one element of B_k, say b. Now there exists a complete set c' such that

$$(c \cap b) \subset (c' \cap b), \qquad \text{diam } (c' \cap b) = \text{diam } c = \text{diam } c',$$
$$c' \text{ is complete with respect to } b.$$

Hence $f(c') \geqq f(c)$, $(\text{diam } c)^2/f(c) \geqq (\text{diam } c')^2/f(c')$, and so we may assume that $c \in C$ is complete with respect to b and that diam $(b \cap c) =$ diam c.

Applying 6.17, we have

$$\lambda^{-1}\cdot f(c) \leqq \mathcal{L}_3(c \cap \sigma B_k)/\mathcal{L}_3(\sigma B_k) \leqq \lambda\cdot f(c),$$

and therefore

$$\lambda^{-1}\cdot(\text{diam } c)^2/f(c) \leqq (\text{diam } c)^2\cdot\mathcal{L}_3(\sigma B_k)/\mathcal{L}_3(c \cap \sigma B_k) \leqq \lambda\cdot(\text{diam } c)^2/f(c).$$

By 6.3, if s is a sphere of diameter equal to diam c, with center on the axis of b, and which does not meet either base of b, then

$$(\text{diam } c)^2\cdot\mathcal{L}_3(\sigma B_k)/\mathcal{L}_3(c \cap \sigma B_k) \geqq (\text{diam } s)^2\cdot\mathcal{L}_3(\sigma B_k)/\mathcal{L}_3(s \cap \sigma B_k).$$

Our problem is thus reduced to the following calculus problem:

Let

$$E = E^3 \cap \underset{x}{E}\,[x_1^2 + x_2^2 \leqq r^2];$$

then find the number s such that

$$r/401 \leqq s \leqq 100r, \quad \text{and} \quad 4s^2/\mathcal{L}_3(E \cap \underset{x}{E}\,[\,|x| \leqq s]) \text{ is least.}$$

It is easily checked that $s = r\cdot 3^{-1/4}\cdot 2^{1/2}$, and that

$$4s^2/\mathcal{L}_3(E \cap \underset{x}{E}\,[\,|x| \leqq s]) = 4(\pi r(108^{1/4} - 12^{1/4}))^{-1}.$$

Hence

$$\begin{aligned}(\text{diam } c)^2/f(c) &\geqq \lambda^{-1}\cdot(\text{diam } s)^2\cdot\mathcal{L}_3(\sigma B_k)/\mathcal{L}_3(s \cap \sigma B_k)\\ &\geqq \lambda^{-1}\cdot 4(\pi(108^{1/4} - 12^{1/4})/2\cdot M_k)^{-1}\cdot(\pi M_k/4M_k^2)\\ &= 2\cdot(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1}.\end{aligned}$$

This proves Part 3.

Part 4. $\mathcal{H}_3^2(B) \geqq \lambda^{-1}\cdot\pi\cdot(2(108^{1/4} - 12^{1/4}))^{-1} > (1.1)\cdot\lambda^{-1}$.

Proof. Recall that B is compact and hence we may assume that any covering of B by complete sets of positive diameter is finite. Let F be such a finite covering of B by complete sets whose diameters are less than d_K. Then by Parts 1, 2, and 3,

$$(c \in F \text{ and } f(c) \neq 0) \rightarrow ((\text{diam } c)^2/f(c) \geqq 2(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1}).$$

Also by the definition of f, it is obvious that

$$\sum_{c \in F} f(c) \geqq 1.$$

Hence

$$\begin{aligned}\sum_{c\in F}(\pi/4)(\text{diam } c)^2 &\geqq 2(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1}\cdot(\pi/4)\sum_{c\in F} f(c)\\ &\geqq \pi\cdot 2^{-1}\cdot(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1}\\ &> (1.1)\cdot\lambda^{-1}.\end{aligned}$$

By the definition of $\mathcal{H}_3^2$, we conclude that

$$\mathcal{H}_3^2(B) \geqq \pi\cdot 2^{-1}\cdot(108^{1/4} - 12^{1/4})^{-1}\cdot\lambda^{-1} > (1.1)\cdot\lambda^{-1}.$$

The proof is complete.

6.19 Corollary. $\mathcal{S}_3^2(B) > \mathcal{S}_2^1(A)$.

Proof. It is easy to see that in general,

$$\mathcal{S}_n^k(X) \geqq \mathcal{H}_n^k(X) \qquad \text{for } X \subset E^n,\ 0 \leqq k \leqq n.$$

Moreover it is obvious that $\mathcal{S}_2^1(A) \leqq 1$, hence

$$\mathcal{S}_3^2(B) \geqq \mathcal{H}_3^2(B) > 1 \geqq \mathcal{S}_2^1(A),$$

so that for sphere measure, it is likewise not true that the measure of a cylinder set is the product of the measure of the base by the height.

6.20 REMARK. Since $\mathcal{H}_3^2(B) > 0$, it follows that $\mathcal{H}_2^1(A) > 0$.

6.21 REMARK. By [R, 5], we have

$$C_3^2(B) \leqq C_2^1(A).$$

Since $C_2^1(A) \leqq 1$, we conclude that

$$C_3^2(B) \leqq 1 < \mathcal{H}_3^2(B),$$

thus proving that $C_3^2 \neq \mathcal{H}_3^2$.

6.22 REMARK. By means of the construction in 6.10, examples can immediately be constructed in E^n for $n > 3$, for which the Hausdorff measure of a product set is not equal to the product of the measures of the components.

BIBLIOGRAPHY

A. S. BESICOVITCH and P. A. P. MORAN

[BM] *The measure of product and cylinder sets*, J. London Math. Soc. vol. 20 (1945) pp. 110–120.

J. BONNESEN and W. FENCHEL

[BF] *Theorie der konvexen Körper*, Ergebnisse der Mathematik, vol. 3, part 1, 1934.

C. CARATHÉODORY

[C] *Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs*, Nachr. Ges. Wiss. Göttingen (1914) pp. 404–426.

H. FEDERER

[F1] *Coincidence functions and their integrals*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 441–466.

[F2] *The (ϕ, k) rectifiable subsets of n space*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 114–192.

[F3] *Dimension and measure*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 536–547.

F. HAUSDORFF

[H] *Dimension und aüsseres Mass*, Math. Ann. vol. 79 (1918) pp. 157–179.

L. H. LOOMIS

[L] *Abstract congruence and the uniqueness of Haar measure*, Ann. of Math. vol. 46 (1945) pp. 348–355.

A. P. MORSE and H. FEDERER

[MF] *Some properties of measurable functions*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 270–277.

A. P. MORSE and J. F. RANDOLPH

[MR] *Gillespie measure*, Duke Math. J. vol. 6 (1940) pp. 408–419.

H. RADEMACHER

[RAD] *Über partielle und totale Differenzierbarkeit*. I, Math. Ann. vol. 79 (1919) pp. 340–359.

J. F. Randolph
[R] *On generalizations of length and area*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 268–274.

S. Saks
[S] *Theory of the integral*, Warsaw, 1937.

A. Weil
[W] *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, vol. 869, Hermann, Paris, 1938.

Brown University,
Providence, R. I.