A THEORY OF POWER-ASSOCIATIVE COMMUTATIVE ALGEBRAS

BY

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1. **Introduction.** In any study of a class of linear algebras the main goal is usually that of determining the simple algebras. The author has recently made a number of such studies for classes of power-associative algebras defined by identities(1) or by the existence of a trace function(2), and the results have been somewhat surprising in that the commutative simple algebras have all been Jordan algebras.

In the present paper we shall derive the reason for this fact. Moreover we shall derive a structure theory which includes the structure theory for Jordan algebras of characteristic p.

We shall begin our study with a consideration of power-associative commutative rings $\mathfrak{A}$ under the customary hypotheses that the characteristic of $\mathfrak{A}$ is prime to 30 and that the equation $2x=a$ has a solution in $\mathfrak{A}$ for every a of $\mathfrak{A}$. We shall show that *if $\mathfrak{A}$ is simple and contains a pair of orthogonal idempotents whose sum is not the unity quantity of $\mathfrak{A}$, then $\mathfrak{A}$ is a Jordan ring.*

We shall apply the result just stated to commutative power-associative algebras $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic prime to 30. If such an algebra $\mathfrak{A}$ is simple we shall show that $\mathfrak{A}$ has a unity quantity e, and so there exists a scalar extension $\mathfrak{K}$ of the center $\mathfrak{F}$ of $\mathfrak{A}$ such that e is expressible as a sum of pairwise orthogonal absolutely primitive idempotents of $\mathfrak{A}_{\mathfrak{K}}$. We define the maximal number of such idempotents to be the *degree* of $\mathfrak{A}$ and use the result on rings to see that every simple algebra of degree $t>2$ is a Jordan algebra. Every Jordan algebra of degree $t\geqq 2$ is a classical Jordan algebra, that is, an algebra of one of the types obtained(3) for algebras of characteristic zero. We define the *radical* of a commutative power-associative algebra to be its *maximal nilideal* and show finally, that every semisimple algebra has a unity quantity and is expressible uniquely as a direct sum of simple algebras.

2. **Elementary properties.** Let $\mathfrak{A}$ be a commutative ring whose characteristic is prime to 30. Then it is known(4) that $\mathfrak{A}$ is power-associative if and only

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(1) For these studies see the author's *Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 522–593. We shall refer to this paper as PAR.

(2) See the paper, *A theory of trace-admissible algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 317–322.

(3) See Theorem 28 of *Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 524–555, and Theorem 17 of *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567. We shall refer to these papers as JA1 and JA2.

(4) *On the power-associativity of rings*, Summa Brasiliensis Mathematicae vol. 2 (1948) pp. 21–33.

if $x^2x^2=(x^2x)x$, an identity which is equivalent to the multilinear identity

$$\begin{aligned} 4[(xy)(zw) + (xz)(yw) + (xw)(yz)] &= x[y(zw) + z(wy) + w(yz)] \\ &+ y[x(zw) + z(wx) + w(xz)] + z[x(yw) + y(wx) + w(xy)] \\ &+ w[x(yz) + y(zx) + z(xy)]. \end{aligned} \tag{1}$$

If u is any idempotent of $\mathfrak{A}$ we may write $\mathfrak{A}=\mathfrak{A}_u(1)+\mathfrak{A}_u(1/2)+\mathfrak{A}_u(0)$ where $\mathfrak{A}_u(\lambda)$ is the subgroup of the additive group of $\mathfrak{A}$ consisting of all quantities $x_\lambda(\lambda)$ in $\mathfrak{A}$ such that $x_\lambda u=\lambda x_\lambda$. Moreover, every x of $\mathfrak{A}$ is uniquely expressible in the form $x=x_1+x_{1/2}+x_0$ with x_λ in $\mathfrak{A}_u(\lambda)$. The multiplicative relations between these modules have been determined(5) and may be expressed as the following formulas for $xy=z$:

$$\begin{gathered} x_1y_1 = z_1, \quad x_0y_0 = z_0, \quad x_1y_0 = 0, \quad x_{1/2}y_{1/2} = z_1 + z_0, \\ x_1y_{1/2} = z_{1/2} + z_0, \quad x_0y_{1/2} = z_{1/2} + z_1. \end{gathered} \tag{2}$$

The ring $\mathfrak{A}$ is said to be *u-stable* if $x_1y_{1/2}=z_{1/2}$, $x_0y_{1/2}=z_{1/2}$, that is, $\mathfrak{A}_u(\lambda)\mathfrak{A}_u(1/2)\subseteq\mathfrak{A}_u(1/2)$ for $\lambda=0,1$. It is known(6) that Jordan rings are u-stable for every idempotent u. We shall now give an example of a power-associative ring which is not u-stable. It provides a simple proof of the property that *power-associativity is not equivalent to the Jordan identity* as well as an example of an unstable power-associative algebra of any characteristic prime to 30.

Consider the *algebra* $\mathfrak{A}$ with a basis u, f, g, h over a field $\mathfrak{F}$ whose characteristic is not 2, 3, or 5, and let

$$u^2 = u, \quad f^2 = g^2 = h^2 = uh = fh = gh = 0, \quad uf = f, \quad ug = 1/2\, g, \quad fg = h$$

Then $\mathfrak{A}_u(1)=u\mathfrak{F}+f\mathfrak{F}$, $\mathfrak{A}_u(1/2)=g\mathfrak{F}$, $\mathfrak{A}_u(0)=h\mathfrak{F}$, $\mathfrak{A}_u(1)\mathfrak{A}_u(1/2)$ is not contained in $\mathfrak{A}_u(1/2)$, and $\mathfrak{A}$ is not stable. We need only prove then that $x^2x^2=(x^2x)x$ for every x of $\mathfrak{A}$. Let $x=\alpha u+\beta f+\gamma g+\delta h$. Then $x^2=\alpha^2u+2\alpha\beta f+\alpha\gamma g+2\beta\gamma h$ and so

$$x^2x^2 = \alpha^4u + 4\alpha^3\beta f + \alpha^3\gamma g + 4\alpha^2\beta\gamma h.$$

Also $x^2x=\alpha^3u+2\alpha^2\beta f+1/2\,\alpha^2\gamma g+\alpha^2\beta f+\alpha\beta\gamma h+1/2\,\alpha^2\gamma g+2\alpha\beta\gamma h=\alpha^3u+3\alpha^2\beta f+\alpha^2\gamma g+3\alpha\beta\gamma h$, $(x^2x)x=\alpha^4u+3\alpha^3\beta f+1/2\,\alpha^3\gamma g+\alpha^3\beta f+\alpha^2\beta\gamma h+1/2\,\alpha^3\gamma g+3\alpha^2\beta\gamma h=\alpha^4u+4\alpha^3\beta f+\alpha^3\gamma g+4\alpha^2\beta\gamma h=x^2x^2$ as desired. We have shown that $\mathfrak{A}$ is a power-associative algebra and is not stable.

3. **The basic machinery.** The relations in (2) imply that the mapping $a_{1/2}+a_0\rightarrow(a_{1/2}+a_0)x_1=a_{1/2}x_1$ is an endomorphism of the module $\mathfrak{A}_u(1/2)+\mathfrak{A}_u(0)$ which we may write as

(5) See Theorem 2 of PAR. The existence of the characteristic root one-half will cause our formulas to contain a number of fractions such as 1/2, 3/2, and 5/2. The reader should always read such formulas as 1/2 xyz as $(1/2)xyz$ and not as $(2xyz)^{-1}$.

(6) See Theorem 6 of JA2.

$$a_{1/2}x_1 = a_{1/2}S_{1/2}(x_1) + a_{1/2}S_0(x_1). \tag{3}$$

Here $a_{1/2}S_{1/2}(x_1)$ is in $\mathfrak{A}_u(1/2)$ so that $S_{1/2}(x_1)$ is an endomorphism of $\mathfrak{A}_u(1/2)$. Also $a_{1/2}S_0(x_1)$ is in $\mathfrak{A}_u(0)$ and so $S_0(x_1)$ is a homomorphism of $\mathfrak{A}_u(1/2)$ into $\mathfrak{A}_u(0)$. Note that when $\mathfrak{A}$ is a linear algebra over a field $\mathfrak{F}$ the mapping $S_{1/2}(x_1)$ is a linear transformation of $\mathfrak{A}_u(1/2)$, and $S_0(x_1)$ is a linear mapping of $\mathfrak{A}_u(1/2)$ into $\mathfrak{A}_u(0)$. While we shall state our proofs for rings they will hold, with only trivial alterations, for algebras.

The first of our tools will be a result of the substitution of $x=x_1$, $y=y_1$, $z=u$, $w=w_{1/2}$ in (1). We use the properties $u(w_{1/2}y_1)=u[w_{1/2}S_{1/2}(y_1)+w_{1/2}S_0(y_1)]=1/2\, w_{1/2}S_{1/2}(y_1)$, $x_1[u(y_1w_{1/2})]=1/2\, x_1[w_{1/2}S_{1/2}(y_1)]=1/2\, x_1(y_1w_{1/2})$, and (1) becomes

$$\begin{aligned} 2(x_1y_1)w_{1/2} + 4x_1(y_1w_{1/2}) + 4y_1(x_1w_{1/2}) &= 2x_1(y_1w_{1/2}) \\ &\quad + 2y_1(x_1w_{1/2}) + u[x_1(y_1w_{1/2}) + y_1(x_1w_{1/2}) + (x_1y_1)w_{1/2}] \\ &\quad + 3(x_1y_1)w_{1/2}. \end{aligned} \tag{4}$$

The component in $\mathfrak{A}_u(0)$ of the term of (4) in which u appears as an external factor must be zero and so (4) is equivalent to the relations

$$\begin{aligned} S_{1/2}(x_1y_1) &= S_{1/2}(x_1)S_{1/2}(y_1) + S_{1/2}(y_1)S_{1/2}(x_1), \\ 1/2\, S_0(x_1y_1) &= S_{1/2}(x_1)S_0(y_1) + S_{1/2}(y_1)S_0(x_1). \end{aligned} \tag{5}$$

The formulas of (2) also imply that

$$a_{1/2}x_0 = a_{1/2}T_{1/2}(x_0) + a_{1/2}T_1(x_0),$$

where $T_{1/2}(x_0)$ is an endomorphism of $\mathfrak{A}_u(1/2)$ and $T_1(x_0)$ maps $\mathfrak{A}_u(1/2)$ into $\mathfrak{A}_u(1)$. We substitute $x=x_0$, $y=y_0$, $z=u$, $w=w_{1/2}$ in (1), and obtain a formula which is readily seen to be equivalent to

$$\begin{aligned} T_{1/2}(x_0y_0) &= T_{1/2}(x_0)T_{1/2}(y_0) + T_{1/2}(y_0)T_{1/2}(x_0), \\ 1/2\, T_1(x_0y_0) &= T_{1/2}(x_0)T_1(y_0) + T_{1/2}(y_0)T_1(x_0). \end{aligned} \tag{6}$$

Let us finally obtain some relations between the mappings S_λ and T_λ. We substitute $x=x_0$, $y=y_1$, $z=u$, and $w+w_{1/2}$ in (1) to obtain $4(w_{1/2}x_0)y_1 = x_0[u(w_{1/2}y_1)+w_{1/2}y_1+1/2\,w_{1/2}y_1]+y_1[u(w_{1/2}x_0)+1/2\,w_{1/2}x_0]+u[(w_{1/2}x_0)y_1 + (w_{1/2}y_1)x_0]$. Then $4[w_{1/2}T_{1/2}(x_0)+w_{1/2}T_1(x_0)]y_1 = 4w_{1/2}T_{1/2}(x_0)S_{1/2}(y_1) + 4w_{1/2}T_{1/2}(x_0)S_0(y_1)+4[w_{1/2}T_1(x_0)]y_1 = 5/2\,w_{1/2}S_{1/2}(y_1)T_{1/2}(x_0)+3w_{1/2}S_{1/2}(y_1)T_1 \cdot(x_0)+3/2[w_{1/2}S_0(y_1)]x_0+3/2\,w_{1/2}T_{1/2}(x_0)S_{1/2}(y_1)+w_{1/2}T_{1/2}(x_0)S_0(y_1)+5/2\,w_{1/2} [T_1(x_0)]y_1$. Equating components in $\mathfrak{A}_u(1/2)$ we obtain $5/2\, w_{1/2}T_{1/2}(x_0)S_{1/2}(y_1) = 5/2\, w_{1/2}S_{1/2}(y_1)T_{1/2}(x_0)$. Our hypothesis that the characteristic of $\mathfrak{A}$ is prime to five implies that

$$S_{1/2}(y_1)T_{1/2}(x_0) = T_{1/2}(x_0)S_{1/2}(y_1). \tag{7}$$

We also equate components in $\mathfrak{A}_u(1)$ to obtain $3/2\,[w_{1/2}T_1(x_0)]y_1$

$=3w_{1/2}S_{1/2}(y_1)T_1(x_0)$, and equate components in $\mathfrak{A}_u(0)$ similarly. Use the fact that the characteristic of $\mathfrak{A}$ is prime to three to obtain the pair of results which we state as

$$
\begin{aligned}
[w_{1/2}T_1(x_0)]y_1 &= 2w_{1/2}S_{1/2}(y_1)T_1(x_0),\\
[w_{1/2}S_0(y_1)]x_0 &= 2w_{1/2}T_{1/2}(x_0)S_0(y_1).
\end{aligned}
\tag{8}
$$

The relations (5)–(8) will be used frequently in our proofs.

Our first result will be an application of (5). We consider the mapping $x_1 \rightarrow 2S_{1/2}(x_1)$. Define the operation $2A \cdot B = AB + BA$ for any endomorphisms A and B on $\mathfrak{A}_u(1/2)$, and see that $2S_{1/2}(x_1) \cdot 2S_{1/2}(y_1) = 2S_{1/2}(x_1y_1)$ by (5). Then we have the first part of the following lemma.

LEMMA 1. *The mapping $x_1 \rightarrow 2S_{1/2}(x_1)$ is a homomorphism of the ring $\mathfrak{A}_u(1)$ onto the special Jordan ring consisting of the endomorphisms $S_{1/2}(x_1)$. The kernel of this homomorphism is an ideal $\mathfrak{B}_u$ of $\mathfrak{A}_u(1)$ which contains the ideal $\mathfrak{C}_u$ of $\mathfrak{A}$ of all quantities x of $\mathfrak{A}_u(1)$ such that $xw_{1/2}=0$ for every $w_{1/2}$ of $\mathfrak{A}_u(1/2)$. Moreover $\mathfrak{B}_u^2 \subseteq \mathfrak{C}_u$.*

To complete our proof we note that $\mathfrak{B}_u$ is the set of all quantities x_1 of $\mathfrak{A}_u(1)$ such that $S_{1/2}(x_1)=0$ and that $\mathfrak{C}_u$ is the set of all x_1 of $\mathfrak{B}_u$ such that $S_0(x_1)=0$. Hence $\mathfrak{B}_u \supseteq \mathfrak{C}_u$. If c_1 is in $\mathfrak{C}_u$ and a is in $\mathfrak{A}$, then $a=a_1+a_{1/2}+a_0$, $ac_1=a_1c_1$. But by (5) we have $S_{1/2}(a_1c_1)=0$, $S_0(a_1c_1)=0$, and so $\mathfrak{C}_u$ is an ideal of $\mathfrak{A}$. Evidently if b_1 and c_1 are in $\mathfrak{B}_u$ then $S_0(b_1c_1)=0$, $\mathfrak{B}_u^2 \subseteq \mathfrak{C}_u$.

The relations of (6) are the result of interchanging the roles of $\lambda=1$ of (5) with $\lambda=0$. Thus (6) provides a lemma which is the counterpart of Lemma 1. Since we shall not use this result we shall not state it.

4. **Adjunction of a unity quantity.** Our study is concerned with nonassociative rings $\mathfrak{A}$ with the property that $1/2\,a$ is a unique element of $\mathfrak{A}$ for every a of $\mathfrak{A}$. Then the characteristic of $\mathfrak{A}$ is prime to two. We now imbed $\mathfrak{A}$ in a unique ring $\mathfrak{R}=\mathfrak{A}+[e]$ with a unity quantity e. If $\mathfrak{A}$ has finite characteristic m, the subring $[e]$ of $\mathfrak{R}$ is isomorphic to the ring of residue classes of the integers modulo m. Otherwise $[e]$ is isomorphic to the ring of all rational numbers whose denominator is a power of two. In every case every element of $\mathfrak{R}$ is uniquely expressible in the form $r=a+\alpha e$ for a in $\mathfrak{A}$ and αe in $[e]$. The ring $\mathfrak{R}$ has the same characteristic([7]) as $\mathfrak{A}$.

The identity (1) is linear in x, y, z, w and is satisfied identically whenever any one of x, y, z, w is a unity quantity. It follows that $\mathfrak{R}$ is power-associative if and only if $\mathfrak{A}$ is power-associative. Evidently $\mathfrak{R}$ is commutative if and only if $\mathfrak{A}$ is commutative. We shall now prove the following result.

LEMMA 2. *Let f be an idempotent of $\mathfrak{A}$ so that $g=e-f$ is an idempotent of the*

([7]) For a discussion of the characteristic of a nonassociative ring see the reference of footnote (4). The construction of $\mathfrak{R}$ is the well known construction as given, for example, in the author's *Modern higher algebra*, Theorem 2.5.

attached ring $\mathfrak{R}=\mathfrak{A}+[e]$ *and* $fg=0$. *Then* $\mathfrak{R}_f(1)=\mathfrak{R}_g(0)=\mathfrak{A}_f(1)$, $\mathfrak{R}_f(1/2)=\mathfrak{A}_f(1/2)$, $\mathfrak{R}_f(0)=\mathfrak{R}_g(1)=\mathfrak{A}_f(0)+[g]$. *If* $\mathfrak{A}$ *is also simple but has no unity quantity, every nonzero ideal of* $\mathfrak{R}$ *contains* $\mathfrak{A}$.

For if r is in $\mathfrak{R}_f(\lambda)$, then $rf=(a+\alpha e)f=af+\alpha f=\lambda r=\lambda a+\alpha\lambda e$. It follows that $\alpha\lambda e=0$ and so $\alpha e=0$ if $\lambda=1, 1/2$. But then r is in $\mathfrak{A}$, $\mathfrak{R}_f(\lambda)=\mathfrak{A}_f(\lambda)$ for $\lambda=1, 1/2$. When $\lambda=0$ then $a=a_1+a_{1/2}+a_0$, $af=-\alpha f=a_1+1/2a_{1/2}$, $a_{1/2}=0$, $a_1=-\alpha f$, $r=a_0+\alpha(e-f)$, $\mathfrak{R}_f(0)=\mathfrak{A}_f(0)+[g]$. Since $g=e-f$ we have $ga_1=ea_1-fa_1=0$, $ga_{1/2}=ea_{1/2}-fa_{1/2}=1/2\ a_{1/2}$, $g(a_0+\alpha g)=(e-f)a_0+\alpha g=a_0+\alpha g$ and the first part of our lemma has been proved.

Let us now suppose that $\mathfrak{D}$ is any nonzero ideal of $\mathfrak{R}$ and that $\mathfrak{A}$ is simple, f is an idempotent of $\mathfrak{A}$ but $\mathfrak{A}$ contains no unity quantity. The intersection $\mathfrak{D}_0$ of $\mathfrak{A}$ and $\mathfrak{D}$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ is an ideal of $\mathfrak{R}$ if $\mathfrak{D}_0=0$ then $\mathfrak{A}\mathfrak{D}=0$. If $d=d_1+d_{1/2}+d_0$ is a nonzero element of $\mathfrak{D}$, the quantity $fd=d_1+1/2\ d_{1/2}=0$, $d_1=d_{1/2}=0$, $d=d_0=a_0-\alpha g$ where a_0 is in $\mathfrak{A}_f(0)$. Evidently we may multiply by a power of two if necessary and so assume that α is an integer. If the set $\alpha\mathfrak{A}$ (of all sums αa for a in $\mathfrak{A}$) is the zero set, then every $\alpha a=0$, $\mathfrak{A}$ has characteristic a divisor m of α, $\alpha e=0$, $\alpha g=0$, $d=d_0=a_0=0$ which is contrary to hypothesis. Evidently $\alpha\mathfrak{A}$ is an ideal of $\mathfrak{A}$ and so $\alpha\mathfrak{A}=\mathfrak{A}$. Let $\mathfrak{S}$ be the set of all quantities s of $\mathfrak{A}$ such that $\alpha s=0$. By our proof $\mathfrak{S}\neq\mathfrak{A}$. But if $\alpha s=0$, then $\alpha(sa)=(\alpha s)a=0$ and so $\mathfrak{S}$ is an ideal of $\mathfrak{A}$, $\mathfrak{S}=0$, $\alpha a=0$ if and only if $a=0$. Since $\alpha\mathfrak{A}=\mathfrak{A}$ there exists a quantity c in $\mathfrak{A}$ such that $a_0=\alpha c$. Then $fa_0=\alpha(fc)=0$, $(fc)=0$, and c is in $\mathfrak{A}_f(0)$. We form $b_0d=b_0(a_0-\alpha g)=\alpha(b_0c-b_0)=0$, and we also form $b_{1/2}d=\alpha(b_{1/2}c-1/2\ b_{1/2})=0$ to obtain $b_0c=b_0$, $b_{1/2}c=1/2\ b_{1/2}$. Since c is in $\mathfrak{A}_f(0)$ we have $b_1c=0$. But then $b(f+c)=b$ for every b of $\mathfrak{A}$ contrary to our hypothesis that $\mathfrak{A}$ has no unity quantity. This completes our proof.

5. **Decomposition relative to a set of idempotents.** The decomposition of a ring relative to a set of pairwise orthogonal idempotents depends upon a result whose proof is rather trivial in the case of Jordan rings.

LEMMA 3. *Let* u *and* v *be orthogonal idempotents of a power-associative commutative ring* $\mathfrak{A}$. *Then* $(au)v=(av)u$ *for every* a *of* $\mathfrak{A}$.

The property of our lemma is the statement $R_uR_v=R_vR_u$ for the corresponding right multiplications. The identity (1) is equivalent to

$$\begin{aligned} R_{x(yz)}+R_{y(zx)}+R_{z(xy)} &= 4(R_xR_{yz}+R_yR_{zx}+R_zR_{xy}) \\ &\quad -(R_{yz}R_x+R_{zx}R_y+R_{xy}R_z)-[R_x(R_yR_z+R_zR_y) \\ &\quad +R_y(R_xR_z+R_zR_x)+R_z(R_xR_y+R_yR_x)]. \end{aligned}$$

Put $x=y=u$ and $z=v$ in this relation and use $u^2=u$, $vu=0$ to obtain

$$4R_vR_u-R_uR_v=2(R_u^2R_v+R_uR_vR_u+R_vR_u^2). \tag{9}$$

If we put $x=y=x=u$ in the relation above, we obtain

$$(10) \qquad 2R_u^3 = 3R_u^2 - R_u.$$

We now left multiply (9) by R_u and also right multiply by R_u to obtain $4R_vR_u^2 - R_uR_vR_u = 2(R_vR_u^3 + R_uR_vR_u^2 + R_u^2R_vR_u)$. By subtraction we obtain $4R_vR_u^2 + R_u^2R_v - 5R_uR_vR_u = 2(R_vR_u^3 - R_u^3R_v) = 3(R_vR_u^2 - R_u^2R_v) - (R_vR_u - R_uR_v)$. But then

$$(11) \qquad R_vR_u^2 - 5R_uR_vR_u + 4R_u^2R_v = R_uR_v - R_vR_u.$$

Equation (11) implies that $2R_vR_u^2 - 10R_uR_vR_u = 2R_uR_v - 2R_vR_u - 8R_u^2R_v$, and (9) implies that $2R_vR_u^2 + 2R_uR_vR_u = -R_uR_v + 4R_vR_u - 2R_u^2R_v$. Subtracting and deleting the factor 3 we obtain

$$(12) \qquad 4R_uR_vR_u = 2R_vR_u - R_uR_v + 2R_u^2R_v.$$

Substitute this result in (9) to obtain

$$(13) \qquad 4R_vR_u^2 = 6R_vR_u - R_uR_v - 6R_u^2R_v.$$

Our next manipulation begins with the right multiplication of (13) by R_u to obtain $6R_vR_u^2 - R_uR_vR_u - 4R_vR_u^3 = 6R_vR_u^2 - R_uR_vR_u - 2R_v(3R_u^2 - R_u) = 6R_u^2R_vR_u$, that is,

$$(14) \qquad 6R_u^2R_vR_u = 2R_vR_u - R_uR_vR_u.$$

We also left multiply (12) by $3R_u$ to obtain $12R_u^2R_vR_u = 6R_uR_vR_u - 3R_u^2R_v + 3(3R_u^2 - R_u)R_v = 6(R_u^2R_v + R_uR_vR_u) - 3R_uR_v$. By (14) we have $6R_u^2R_v + 6R_uR_vR_u = 3R_uR_v + 4R_vR_u - 2R_uR_vR_u$, that is,

$$(15) \qquad 8R_uR_vR_u = 4R_vR_u + 3R_uR_v - 6R_u^2R_v.$$

Combine this result with (12) to obtain $5R_uR_v = 10R_u^2R_v$ and so

$$(16) \qquad R_uR_v = 2R_u^2R_v, \qquad 2R_uR_vR_u = R_vR_u, \qquad 2R_vR_u^2 = 3R_vR_u - 2R_uR_v.$$

We have not used the fact that v is idempotent and we shall do so now. By symmetry we have

$$(17) \qquad R_vR_u = 2R_v^2R_u, \qquad 2R_vR_uR_v = R_uR_v, \qquad 2R_uR_v^2 = 3R_uR_v - 2R_vR_u.$$

We use (16) to write $4R_vR_u^2R_v = 2R_vR_uR_v = R_uR_v$ by (17). But also $4R_vR_u^2R_v = 6R_vR_uR_v - 4R_uR_v^2 = 3R_uR_v - 2(3R_uR_v - 2R_vR_u) = 4R_vR_u - 3R_uR_v$. Then $4R_vR_u - 3R_uR_v = R_uR_v$, $R_uR_v = R_vR_u$ as desired.

As a consequence of Lemma 3 we may now prove the following important decomposition theorem.

LEMMA 4. *Let u and v be orthogonal idempotents. Then the intersection of $\mathfrak{A}_u(1/2)$ and $\mathfrak{A}_v(1/2)$ contains $(au)v = (av)u$ for every a of $\mathfrak{A}$.*

For we write $a=a_u(1)+a_u(1/2)+a_u(0)$ with $a_u(\lambda)$ in $\mathfrak{A}_u(\lambda)$. Then $au=a_u(1)+1/2\ a_u(1/2)$ and, since v is in $a_u(0)$, $(au)v=1/2\ va_u(1/2)$, $[(au)v]u=1/2\ u[va_u(1/2)]=1/2\ v[ua_u(1/2)]=1/4\ va_u(1/2)$. Thus $(au)v$ is in $\mathfrak{A}_u(1/2)$ and is also in $\mathfrak{A}_v(1/2)$ by symmetry.

The following result implies that if u, v, and w are pairwise orthogonal the intersection of $\mathfrak{A}_u(1/2)$, $\mathfrak{A}_v(1/2)$, and $\mathfrak{A}_w(1/2)$ is zero.

LEMMA 5. *Let u, v, w be pairwise orthogonal idempotents of $\mathfrak{A}$. Then $[(au)v]w=0$ for every a of $\mathfrak{A}$.*

For $e=u+v$ is orthogonal to w, and if $b=(au)v$ then $ub=1/2\ b$, $vb=1/2\ b$ by Lemma 4, $eb=b$. But w is in $\mathfrak{A}_e(0)$, b is now in $\mathfrak{A}_e(1)$ and so $bw=0$ as desired.

We are now ready to obtain the decomposition of a power-associative ring relative to a set of pairwise orthogonal idempotents. We let $\mathfrak{R}$ be a commutative power-associative ring with a unity quantity e and suppose that $e=e_1+\cdots+e_t$ for pairwise orthogonal idempotents e_i of $\mathfrak{R}$. Every quantity a of $\mathfrak{R}$ has the form

$$a = (2ae)e - ae = a(2R_e^2 - R_e) = \sum_{i\leq j} a_{ij}$$

where

$$a_{ii} = (2ae_i)e_i - ae_i, \qquad a_{ij} = a_{ji} = 4(ae_i)e_j. \tag{18}$$

This expression is unique(8) and we have written $\mathfrak{R}$ as the sum of its modules $\mathfrak{R}_{ij}=\mathfrak{R}_{ji}$ where $\mathfrak{R}_{ii}=\mathfrak{R}_{e_i}(1)$ and $\mathfrak{R}_{ij}$ is the intersection of $\mathfrak{R}_{e_i}(1/2)$ and $\mathfrak{R}_{e_j}(1/2)$ for all distinct i and j. If $g=e_i+e_j$, then $\mathfrak{R}_g(1)=\mathfrak{C}=\mathfrak{R}_{ii}+\mathfrak{R}_{ij}+\mathfrak{R}_{ij}$ is a subring of $\mathfrak{R}$ such that $\mathfrak{C}_{e_i}(1)=\mathfrak{R}_{ii}=\mathfrak{C}_{e_j}(0)$, $\mathfrak{C}_{e_i}(1/2)=\mathfrak{C}_{e_j}(1/2)=\mathfrak{R}_{ij}$, $\mathfrak{C}_{e_i}(0)=\mathfrak{R}_{jj}$. But then (2) implies that

$$\mathfrak{R}_{ii}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ij} + \mathfrak{R}_{jj} \qquad (i \neq j). \tag{19}$$

If $p\neq i, j$ and $q\neq i, j$, then $\mathfrak{R}_{ij}\subseteq\mathfrak{R}_g(1)$, $\mathfrak{R}_{pq}\subseteq\mathfrak{R}_g(0)$ where $g=e_i+e_j$ or e_i according as $i\neq j$ or $i=j$. It follows that

$$\mathfrak{R}_{ij}\mathfrak{R}_{pq} = 0 \qquad (p \neq i, j;\ q \neq i, j). \tag{20}$$

The properties of (19) and (20) are quite trivial but this is not true of the important property

$$\mathfrak{R}_{ij}\mathfrak{R}_{jk} \subseteq \mathfrak{R}_{ik} \qquad (i \neq j, k;\ j \neq k). \tag{21}$$

To prove this result we let $w=e_i+e_j+e_k$ and see that $\mathfrak{S}=\mathfrak{R}_w(1)=\mathfrak{R}_{ij}+\mathfrak{R}_{jj}+\mathfrak{R}_{kk}+\mathfrak{R}_{ij}+\mathfrak{R}_{ik}+\mathfrak{R}_{jk}$. Put $g=e_i+e_j$ and see that $\mathfrak{R}_{ij}\subseteq\mathfrak{S}_g(1)$, $\mathfrak{R}_{jk}\subseteq\mathfrak{S}_g(1/2)$, and so (2) implies that $\mathfrak{R}_{ij}\mathfrak{R}_{jk}\subseteq\mathfrak{S}_g(1/2)+\mathfrak{S}_g(0)=\mathfrak{R}_{ik}+\mathfrak{R}_{jk}+\mathfrak{R}_{kk}$. Put $h=e_j+e_k$ and see that $\mathfrak{R}_{ij}\subseteq\mathfrak{S}_h(1/2)$, $\mathfrak{R}_{jk}\subseteq\mathfrak{S}_h(1)$. By (2) we have $\mathfrak{R}_{ij}\mathfrak{R}_{jk}\subseteq\mathfrak{S}_h(1/2)$

(8) Cf. §14 of JA2.

$+\mathfrak{S}_h(0)=\mathfrak{R}_{ij}+\mathfrak{R}_{ik}+\mathfrak{R}_{ii}$. Since $\mathfrak{R}_{ik}$ is the intersection of $\mathfrak{S}_g(1/2)+\mathfrak{S}_g(0)$ and $\mathfrak{S}_h(1/2)+\mathfrak{S}_h(0)$ we have (21).

6. **The ring theorem.** Let $\mathfrak{R}$ be a power-associative commutative ring whose characteristic is prime to 30. We assume first that $\mathfrak{R}$ has a unity quantity $e=e_1+e_2+e_3$ for pairwise orthogonal idempotents e_i. We shall also assume that either $\mathfrak{R}=\mathfrak{A}$ is simple or that $\mathfrak{R}$ is the result of adjoining the unity quantity e to a simple ring $\mathfrak{A}$ which contains the idempotents e_1 and e_2 but *does not have a unity quantity*. By Lemma 2 every nonzero ideal $\mathfrak{C}$ of $\mathfrak{R}$ contains $\mathfrak{A}$.

Let g be any one of the idempotents $f=e_1+e_2$, $h=e_1+e_3$,and $k=e_1+e_3$ and define the ideals $\mathfrak{B}_g$ and $\mathfrak{C}_g$ of Lemma 1. Then $\mathfrak{C}_g$ is an ideal of $\mathfrak{R}$ contained in $\mathfrak{R}_g(1)$, and if $\mathfrak{C}_g\neq 0$ the subring $\mathfrak{R}_g(1)$ contains $\mathfrak{A}$. However e_1 and e_2 are both contained in $\mathfrak{R}_g(1)$ except when $g=f$, and in this case we can conclude that $\mathfrak{R}_g(1)=\mathfrak{A}_f(1)$ contains $\mathfrak{A}$. This implies that f is the unity quantity of $\mathfrak{A}$, which is contrary to our hypothesis that either e is the unity quantity of $\mathfrak{A}$ or $\mathfrak{A}$ has no unity quantity. It follows that $\mathfrak{C}_g=0$.

By §5 we may write $\mathfrak{R}=\mathfrak{R}_{11}+\mathfrak{R}_{22}+\mathfrak{R}_{33}+\mathfrak{R}_{12}+\mathfrak{R}_{13}+\mathfrak{R}_{23}$. The ideal $\mathfrak{B}_g$ of $\mathfrak{R}_g(1)$ is defined by the property $\mathfrak{B}_g\mathfrak{R}_g(1/2)\subseteq\mathfrak{R}_g(0)$. Now (21) implies that $\mathfrak{R}_{ij}\mathfrak{R}_g(1/2)=\mathfrak{R}_g(1/2)$. If $\mathfrak{D}_g$ is the intersection of $\mathfrak{R}_{ij}$ and $\mathfrak{B}_g$, then $\mathfrak{D}_g\mathfrak{R}_g(1/2)$ is contained in both $\mathfrak{R}_g(1/2)$ and $\mathfrak{R}_g(0)$ and is zero, $\mathfrak{D}_g\subseteq\mathfrak{C}_g$, $\mathfrak{D}_g=0$. Since $\mathfrak{B}_g$ is an ideal of $\mathfrak{R}_g(1)$, the components of any $b=b_{ii}+b_{ij}+b_{jj}$ are in $\mathfrak{B}_g$ and so the component $b_{ij}=0$. Thus $\mathfrak{B}_g=\mathfrak{B}_{gi}\oplus\mathfrak{B}_{gj}$ where $\mathfrak{B}_{gi}$ is the intersection of $\mathfrak{B}_g$ and $\mathfrak{R}_{ii}$, $\mathfrak{B}_{gj}$ is the intersection of $\mathfrak{B}_g$ and $\mathfrak{R}_{jj}$.

Define the submodule $\mathfrak{B}=\mathfrak{B}_f+\mathfrak{B}_h+\mathfrak{B}_k$. By Lemma 1 the subring $\mathfrak{R}_h(1)=\mathfrak{R}_{11}+\mathfrak{R}_{13}+\mathfrak{R}_{33}$ has the property that $\mathfrak{R}_h(1)-\mathfrak{B}_h$ is a Jordan ring. Since a Jordan ring is stable, it follows that $\mathfrak{R}_{11}\mathfrak{R}_{13}\subseteq\mathfrak{R}_{13}+\mathfrak{B}_h$. But $\mathfrak{B}_{f1}\mathfrak{R}_{13}\subseteq\mathfrak{R}_f(0)=\mathfrak{R}_{33}$ and so $\mathfrak{B}_{f1}\mathfrak{R}_{13}\subseteq\mathfrak{B}_{k3}$. Evidently $\mathfrak{B}_{f1}\mathfrak{R}_{23}=0$. By symmetry $\mathfrak{B}_{f2}\mathfrak{R}_{23}\subseteq\mathfrak{B}_{k3}$ and so $\mathfrak{B}_f\mathfrak{R}_f(1/2)\subseteq\mathfrak{B}$. Since $\mathfrak{B}_f\mathfrak{R}_f(0)=0$, and $\mathfrak{B}_f\mathfrak{R}_f(1)\subseteq\mathfrak{B}_f$ since $\mathfrak{B}_f$ is an ideal of $\mathfrak{B}_f(1)$, we know that $\mathfrak{B}_f\mathfrak{R}\subseteq\mathfrak{B}$. By symmetry we see that $\mathfrak{B}_h\mathfrak{R}\subseteq\mathfrak{B}$, $\mathfrak{B}_k\mathfrak{R}\subseteq\mathfrak{B}$, $\mathfrak{B}\mathfrak{R}\subseteq\mathfrak{B}$. Since $\mathfrak{B}\subseteq\mathfrak{R}_{11}\oplus\mathfrak{R}_{22}\oplus\mathfrak{R}_{33}$, our hypothesis about the ideals of $\mathfrak{R}$ implies that $\mathfrak{B}$ cannot be a nonzero ideal of $\mathfrak{R}$. Hence $\mathfrak{B}=0$. We have proved the following result.

LEMMA 6. *The subrings* $\mathfrak{R}_{11}+\mathfrak{R}_{12}+\mathfrak{R}_{22}$, $\mathfrak{R}_{11}+\mathfrak{R}_{13}+\mathfrak{R}_{23}$, $\mathfrak{R}_{22}+\mathfrak{R}_{23}+\mathfrak{R}_{33}$ *are Jordan rings, and so* $\mathfrak{R}_{ii}\mathfrak{R}_{ij}\subseteq\mathfrak{R}_{ij}$ *for* $i\neq j$ *and* $i, j=1, 2, 3$.

Let us now write $\mathfrak{L}=\mathfrak{R}_{ii}+\mathfrak{R}_{ij}+\mathfrak{R}_{jj}$, $\mathfrak{N}=\mathfrak{R}_{ik}+\mathfrak{R}_{jk}$, $\mathfrak{M}=\mathfrak{R}_{kk}$. Then $\mathfrak{L}=\mathfrak{R}_g(1)$, $\mathfrak{M}=\mathfrak{R}_g(0)$, $\mathfrak{N}=\mathfrak{R}_g(1/2)$ where g is one of the idempotents f, h, k. But $\mathfrak{R}_g(1)\mathfrak{R}_g(1/2)=\mathfrak{R}_{ii}\mathfrak{R}_{ik}+\mathfrak{R}_{jj}\mathfrak{R}_{jk}+\mathfrak{R}_{ij}(\mathfrak{R}_{ik}+\mathfrak{R}_{jk})\subseteq\mathfrak{R}_{ik}+\mathfrak{R}_{jk}=\mathfrak{N}$, $\mathfrak{R}_g(0)\mathfrak{R}_g(1/2)=\mathfrak{R}_{kk}(\mathfrak{R}_{ik}+\mathfrak{R}_{jk})\subseteq\mathfrak{R}_{ik}+\mathfrak{R}_{jk}$ by Lemma 6 and we have proved that

$$\mathfrak{L}\mathfrak{N}\subseteq\mathfrak{N},\qquad \mathfrak{M}\mathfrak{N}\subseteq\mathfrak{N}. \tag{22}$$

We may also use (5) and (6) for the idempotent g where (22) implies that $S_0(x_1)=T_1(x_0)=0$. This yields

$$w_{1/2}(x_\lambda y_\lambda) = (w_{1/2}x_\lambda)y_\lambda + (w_{1/2}y_\lambda)x_\lambda \qquad (\lambda = 0, 1), \tag{23}$$

where we have written z_λ for the general element of $\mathfrak{R}_e(\lambda)$. By (7) we have

$$(w_{1/2}x_\lambda)y_{1-\lambda} = (w_{1/2}y_{1-\lambda})x_\lambda \qquad (\lambda = 0, 1). \tag{24}$$

We shall also prove the following result which will be seen later to imply that $\mathfrak{R}$ is a Jordan ring.

Lemma 7. *Let x_λ be in $\mathfrak{R}_e(\lambda)$ for $\lambda = 0, 1$. Then*

$$(w_{1/2}y_{1/2})x_\lambda = [(x_\lambda y_{1/2})w_{1/2} + (x_\lambda w_{1/2})y_{1/2}]_\lambda \tag{25}$$

and

$$[w_{1/2}(y_{1/2}x_\lambda)]_{1-\lambda} = [y_{1/2}(w_{1/2}x_\lambda)]_{1-\lambda}. \tag{26}$$

For proof we apply the Jordan identity

$$(wx)(yz) + (wy)(xz) + (wz)(xy) = [w(yz)]x + [w(xz)]y + [w(xy)]z \tag{27}$$

in the ring $\mathfrak{L}$. Put $w = w_{ij}$, $x = x_{ii}$, $y = y_{ij}$, $z = e_i$ in (27) to obtain $1/2\,(w_{ij}x_{ii})y_{ij} + (w_{ij}y_{ij})x_{ii} + 1/2\, w_{ij}(x_{ii}y_{ij}) = 1/2\, x_{ii}(y_{ij}w_{ij}) + y_{ij}(w_{ij}x_{ii}) + e_i[w_{ij}(x_{ii}y_{ij})]$. Then $x_{ii}(w_{ij}y_{ij}) + w(xy) = y_{ij}(x_{ii}w_{ij}) + 2e_i[w_{ij}(x_{ii}y_{ij})]$ and so we have

$$x_{ii}(w_{ij}y_{ij}) = e_i[y_{ij}(x_{ii}w_{ij}) + w_{ij}(y_{ij}x_{ii})], \tag{28}$$

$$e_j[(x_{ii}y_{ij})w_{ij}] = e_j[(x_{ii}w_{ij})y_{ij}]. \tag{29}$$

These results are special cases of (25) and (26) and we pass on to the general case.

Let us assume first that x_1 is in $\mathfrak{R}_{ii}$. If $y_{1/2}$ and $w_{1/2}$ are in $\mathfrak{R}_{jk}$, all products in (25) and (26) vanish and so the relation is valid. If $y_{1/2}$ and $w_{1/2}$ are in $\mathfrak{R}_{ik}$ we use (28) and (29) for the Jordan algebra $\mathfrak{R}_{ii} + \mathfrak{R}_{ik} + \mathfrak{R}_{kk}$ and obtain (25) and (26) immediately. There remains the case where $y_{1/2} = y_{ik}$ and $w_{1/2} = w_{jk}$. We apply (23) for the idempotent $e_i + e_k$ and have $w_{jk}(x_{ii}y_{ik}) = (w_{jk}x_{ii})y_{ik} + (w_{jk}y_{ik})x_{ii} = (w_{jk}y_{ik})x_{ii}$ since $x_{ii}w_{jk} = 0$. But this yields (25) and (26).

Assume next that $x_1 = x_{ij}$. If $y = y_{ik}$ and $w = w_{ik}$ then $x_{ij}(y_{ik}w_{ik}) = (x_{ij}y_{jk})w_{ik} + (x_{ij}w_{ik})y_{ik}$ by (23). This yields (25) immediately. The components in $\mathfrak{R}_{kk} = \mathfrak{R}_e(0)$ of all terms are zero and so (26) holds. There remains the case $y = y_{ik}$, $w = w_{jk}$. We substitute these values with $z = e_j$ in (1) to obtain $2x_{ij}(y_{ik}w_{jk}) + 2w_{jk}(x_{ij}y_{ik}) = x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik}) + y_{ik}(x_{ij}w_{jk}) + e_j[x_{ij}(y_{ik}w_{jk}) + y_{ik}(x_{ij}w_{jk}) + w_{jk}(x_{ij}y_{ik})]$. However $e_j[y_{ik}(x_{ij}w_{jk})] = 0$ and so we have $x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik}) = y_{ik}(x_{ij}w_{jk}) + e_j[x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik})]$. Since $x_{ij}(y_{ik}w_{jk})$ is in $\mathfrak{R}_{ii} + \mathfrak{R}_{jj}$, $w_{jk}(x_{ij}y_{ik})$ is in $\mathfrak{R}_{jj} + \mathfrak{R}_{kk}$, and $y_{ik}(x_{ij}w_{jk})$ is in $\mathfrak{R}_{ii} + \mathfrak{R}_{kk}$, we may equate components and find first that $e_k[(x_{ij}y_{ik})w_{jk}] = e_k[y_{ik}(x_{ij}w_{jk})]$ which is (26). We also see that $e_i[x_{ij}(y_{ik}w_{jk})] = e_i[y_{ik}(x_{ij}w_{jk})]$. Interchange i and j and y and w to obtain $e_j[x_{ij}(y_{ik}w_{jk})] = e_j[w_{jk}(x_{ij}y_{ik})]$, add to get $x_{ij}(y_{ik}w_{jk})$

$=e_i[y_{ik}(x_{ij}w_{jk})]+e_j[w_{jk}(x_{ij}y_{ik})]$, and (25) is proved. This completes our proof in the case $\lambda=1$.

In the case $\lambda=0$ we have $x=x_{kk}$. As above (23) implies that $w_{jk}(x_{kk}y_{ik}) =(w_{jk}x_{kk})y_{ik}+(w_{jk}y_{ik})x_{kk}=(w_{jk}x_{kk})y_{ik}$. This yields (26). Equation (25) is automatically satisfied since $(w_{jk}y_{ik})x_{kk}=0$, $e_k[(x_{kk}y_{ik})w_{jk}]=e_k[(x_{kk}w_{jk})y_{ik}]=0$. The remaining case is that where $y=y_{ik}$, $w=w_{ik}$ and (25) and (26) follow from (28) and (29).

The results just obtained are sufficient to prove our theorem on power-associative commutative rings.

THEOREM 1. *Let $\mathfrak{A}$ be a simple commutative power-associative ring whose characteristic is prime to* 30, *and let $\mathfrak{A}$ contain a pair of orthogonal idempotents u and v such that $u+v$ is not the unity quantity of $\mathfrak{A}$. Then $\mathfrak{A}$ is a Jordan ring.*

For proof we note that Lemma 2 implies that $\mathfrak{A}$ can be imbedded in a ring $\mathfrak{R}$ with a unity quantity $e=e_1+e_2+e_3$ for pairwise orthogonal idempotents $e_1=u$, $e_2=v$, e_3 and that $\mathfrak{R}=\mathfrak{R}_{11}+\mathfrak{R}_{22}+\mathfrak{R}_{33}+\mathfrak{R}_{12}+\mathfrak{R}_{13}+\mathfrak{R}_{23}$ where Lemma 6 holds. Moreover we have (22), (23), (24), (25), (26).

The Jordan identity (27) is linear in x, y, z, and w and so it suffices to prove this identity for the variables in component submodules $\mathfrak{R}_{ij}$. If all four of the variables are in $\mathfrak{L}=\mathfrak{R}_{ii}+\mathfrak{R}_{ij}+\mathfrak{R}_{jj}$, then the Jordan identity holds by Lemma 6. If three factors are in $\mathfrak{L}$ and one in $\mathfrak{N}=\mathfrak{R}_{kk}$, then all products in (27) are zero and (27) holds trivially. Hence let three factors be in $\mathfrak{L}$ and one in $\mathfrak{M}=\mathfrak{R}_{ik}+\mathfrak{R}_{jk}$. The Jordan identity (27) is an identity $J(w;\, x,\, y,\, z)=0$ which is symmetric in x, y, z but not in w. The power-associative identity (1) is the identity

$$J(w;\, x,\, y,\, z) + J(x;\, y,\, z,\, w) + J(y;\, z,\, w,\, x) + J(z;\, w,\, x,\, y) = 0,$$

and if $J(x;\, y,\, z,\, w)=J(y;\, z,\, w,\, x)=J(z;\, w,\, x,\, y)=0$ then also $J(w;\, x,\, y,\, z)=0$. Hence we need only consider the case of (27) in which $w=w_\lambda$, $x=x_{1/2}$, $y=y_\lambda$, $z=z_\lambda$. We then use (23) to write

$$\begin{aligned} H &= (x_{1/2}y_\lambda)(z_\lambda w_\lambda) + (x_{1/2}z_\lambda)(y_\lambda w_\lambda) + (x_{1/2}w_\lambda)(y_\lambda z_\lambda) \\ &= (x_{1/2}w_\lambda)(y_\lambda z_\lambda) + [(x_{1/2}y_\lambda)w_\lambda]z_\lambda + [(x_{1/2}z_\lambda)w_\lambda]y_\lambda \\ &\quad + [(x_{1/2}y_\lambda)z_\lambda + (x_{1/2}z_\lambda)y_\lambda]w_\lambda. \end{aligned}$$

But $[(x_{1/2}y_\lambda)z_\lambda+(x_{1/2}z_\lambda)y_\lambda]w_\lambda=[x_{1/2}(y_\lambda z_\lambda)]w_\lambda=x_{1/2}[w_\lambda(y_\lambda z_\lambda)]-(x_{1/2}w_\lambda)(y_\lambda z_\lambda)$ and so $H=[(x_{1/2}y_\lambda)w_\lambda]z_\lambda+[(x_{1/2}z_\lambda)w_\lambda]y_\lambda+[w_\lambda(z_\lambda y_\lambda)]x_{1/2}$ as desired.

We next assume that only two factors are in $\mathfrak{L}$. If one of the remaining factors is in $\mathfrak{M}$ and one in $\mathfrak{N}$ we may write $w=w_{1-\lambda}$ or $w=w_\lambda$ or $w=w_{1/2}$. In the first case we take $w=w_{1-\lambda}$, $x=x_{1/2}$, $y=y_\lambda$, $z=z_\lambda$ and have $wy=wz=0$, $H=(w_{1-\lambda}x_{1/2})(y_\lambda z_\lambda)$, $K=[w_{1-\lambda}(x_{1/2}y_\lambda)]z_\lambda+[w_{1-\lambda}(x_{1/2}z_\lambda)]y_\lambda=w_{1-\lambda}[(x_{1/2}y_\lambda)z_\lambda +(x_{1/2}z_\lambda)y_\lambda]=w_{1-\lambda}[x_{1/2}(y_\lambda z_\lambda)]=(x_{1/2}w_{1-\lambda})(y_\lambda z_\lambda)$ by (23) and (24) so that we have the Jordan identity $H=K$. Next write $w=w_\lambda$, $x=x_{1/2}$, $y=y_\lambda$, $z=z_{1-\lambda}$

and have $H=(w_\lambda y_\lambda)(x_{1/2}z_{1-\lambda})$, $K=[w_\lambda(x_{1/2}z_{1-\lambda})]y_\lambda+[w_\lambda(x_{1/2}y_\lambda)]z_{1-\lambda}$ and $H=K$ by (23) and (24). Thus $J(w_\lambda;\ x_\lambda,\ y_{1/2},\ z_{1-\lambda})=J(x_\lambda;\ w_\lambda,\ y_{1/2},\ z_{1-\lambda})=J(z_{1-\lambda};\ w_\lambda,\ x_\lambda,\ y_{1/2})=0$ from which $J(y_{1/2};\ x_\lambda,\ w_\lambda,\ z_{1-\lambda})=0$, and the identity is proved.

There remains the more difficult case where two of w, x, y, z are in $\mathfrak{L}$ and two are in $\mathfrak{M}$. We first write $w=w_{1/2}$, $x=x_\lambda$, $y=y_\lambda$, $z=z_{1/2}$ and have $H=(w_{1/2}x_\lambda)(z_{1/2}y_\lambda)+(w_{1/2}y_\lambda)(x_\lambda z_{1/2})+(w_{1/2}z_{1/2})(x_\lambda y_\lambda)$, $K=[w_{1/2}(x_\lambda y_\lambda)]z_{1/2}+y_\lambda[w_{1/2}(x_\lambda z_{1/2})]+[w_{1/2}(y_\lambda)z_{1/2}]$. We compute

$$x_\lambda[w_{1/2}(y_\lambda z_{1/2})] = [(x_\lambda w_{1/2})(y_\lambda z_{1/2})]_\lambda + \{w_{1/2}[x_\lambda(y_\lambda z_{1/2})]\}_\lambda, \tag{30}$$

and

$$y_\lambda[w_{1/2}(x_\lambda z_{1/2})] = [(y_\lambda w_{1/2})(x_\lambda z_{1/2})]_\lambda + \{w_{1/2}[y_\lambda(x_\lambda z_{1/2})]\}_\lambda \tag{31}$$

by the use of (25). Also (25) implies that

$$(x_\lambda y_\lambda)(w_{1/2}z_{1/2}) = \{[(x_\lambda y_\lambda)w_{1/2}]z_{1/2}\}_\lambda + \{[(x_\lambda y_\lambda)z_{1/2}]w_{1/2}\}_\lambda.$$

It follows that

$$\begin{aligned} K_\lambda &= x_\lambda[w_{1/2}(y_\lambda z_{1/2})] + y_\lambda[w_{1/2}(x_\lambda z_{1/2})] \\ &\quad + \{[(x_\lambda y_\lambda)w_{1/2}]z_{1/2}\}_\lambda = [(x_\lambda w_{1/2})(y_\lambda z_{1/2})]_\lambda \\ &\quad + [(y_\lambda w_{1/2})(x_\lambda z_{1/2})]_\lambda + \{w_{1/2}[(x_\lambda y_\lambda)z_{1/2}]\}_\lambda \\ &\quad + (x_\lambda y_\lambda)(w_{1/2}z_{1/2}) - \{w_{1/2}[(x_\lambda y_\lambda)z_{1/2}]\}_\lambda = H_\lambda. \end{aligned}$$

Also $K_{1-\lambda}=\{[(x_\lambda y_\lambda)w_{1/2}]z_{1/2}\}_{1-\lambda}$ and $H_{1-\lambda}=[(x_\lambda w_{1/2})(y_\lambda z_{1/2})]_{1-\lambda}+[(x_\lambda z_{1/2})\cdot(y_\lambda w_{1/2})]_{1-\lambda}=\{z_{1/2}[(w_{1/2}x_\lambda)y_\lambda+(w_{1/2}y_\lambda)x_\lambda]\}_{1-\lambda}$ by (26). Hence (23) implies that $K_{1-\lambda}=H_{1-\lambda}$, $H=K$.

We next write $w=w_\lambda$, $x=x_{1/2}$, $y=y_\lambda$, $z=z_{1/2}$. Then we know that $H=(w_\lambda x_{1/2})(y_\lambda z_{1/2})+(w_\lambda y_\lambda)(x_{1/2}z_{1/2})+(w_\lambda z_{1/2})(x_{1/2}y_\lambda)=x_{1/2}[z_{1/2}(w_\lambda y_\lambda)]+y_\lambda\cdot[z_{1/2}(w_\lambda x_{1/2})]+w_\lambda[z_{1/2}(x_{1/2}y_\lambda)]$ by the result above. Since $K=x_{1/2}[w_\lambda\cdot(y_\lambda z_{1/2})]+y_\lambda[w_\lambda(x_{1/2}z_{1/2})]+z_{1/2}[w_\lambda(x_{1/2}y_\lambda)]$, the Jordan identity is true if and only if

$$\begin{aligned} &x_{1/2}[w_\lambda(y_\lambda z_{1/2}) - z_{1/2}(w_\lambda y_\lambda)] + y_\lambda[w_\lambda(x_{1/2}z_{1/2}) - z_{1/2}(w_\lambda x_{1/2})] \\ &\qquad = w_\lambda[z_{1/2}(x_{1/2}y_\lambda)] - z_{1/2}[w_\lambda(x_{1/2}y_\lambda)]. \end{aligned} \tag{32}$$

However $z_{1/2}(w_\lambda y_\lambda)=(z_{1/2}w_\lambda)y_\lambda+w_\lambda(y_\lambda z_{1/2})$ by (23), and $w_\lambda(x_{1/2}z_{1/2})=[(w_\lambda x_{1/2})z_{1/2}+(w_\lambda z_{1/2})x_{1/2}]$ by (25), so that $y_\lambda[w_\lambda(x_{1/2}z_{1/2})]=y_\lambda[(w_\lambda x_{1/2})z_{1/2}]+y_\lambda[(w_\lambda z_{1/2})x_{1/2}]$. It follows that (32) is equivalent to

$$y_\lambda[(w_\lambda z_{1/2})x_{1/2}] + z_{1/2}[w_\lambda(x_{1/2}y_\lambda)] = w_\lambda[z_{1/2}(x_{1/2}y_\lambda)] + x_{1/2}[(z_{1/2}w_\lambda)y_\lambda]. \tag{33}$$

We now compute

$$y_\lambda[(w_\lambda z_{1/2})x_{1/2}] = [(y_\lambda x_{1/2})(w_\lambda z_{1/2})]_\lambda + \{x_{1/2}[y_\lambda(w_\lambda z_{1/2})]\}_\lambda, \tag{34}$$

$$\{z_{1/2}[w_\lambda(x_{1/2}y_\lambda)]\}_{1-\lambda} = \{(z_{1/2}w_\lambda)(x_{1/2}y_\lambda)\}_{1-\lambda}, \tag{35}$$

$$(36)\qquad \{x_{1/2}[y_\lambda(w_\lambda z_{1/2})]\}_{1-\lambda} = [(x_{1/2}y_\lambda)(z_{1/2}w_\lambda)]_{1-\lambda},$$

$$(37)\qquad w_\lambda[z_{1/2}(y_\lambda x_{1/2})] = [(wz_{1/2})(y_\lambda x_{1/2})]_\lambda + \{[w_\lambda(y_\lambda x_{1/2})]z_{1/2}\}_\lambda.$$

If we label the terms in (33) as a, b, c, d and so need $a+b=c+d$, then (34) states that $a=k_\lambda+d_\lambda$, (35) states that $b_{1-\lambda}=k_{1-\lambda}$, (36) states that $k_{1-\lambda}=d_{1-\lambda}$ and hence that $b_{1-\lambda}=d_{1-\lambda}$, (37) states that $c=k_\lambda+b_\lambda$. Then $a+b=k_\lambda+d_\lambda+b_{1-\lambda}+b_\lambda$, $c+d=k_\lambda+b_\lambda+d_\lambda+b_{1-\lambda}$ and so (33) holds. This proves the Jordan identity when two factors are in $\mathfrak{L}$ and two in $\mathfrak{M}$.

The only remaining case is that where at most one factor is in $\mathfrak{L}$. Then at least three factors are in $\mathfrak{M}+\mathfrak{N}=\mathfrak{N}_{ik}+\mathfrak{N}_{jk}+\mathfrak{N}_{kk}$ and so at least two factors are in $\mathfrak{L}'=\mathfrak{N}_{ii}+\mathfrak{N}_{ik}+\mathfrak{N}_{kk}$ or in $\mathfrak{L}''=\mathfrak{N}_{jj}+\mathfrak{N}_{jk}+\mathfrak{N}_{kk}$. This completes our proof.

The theorem just proved holds for algebras (not necessarily finite-dimensional) as well as for rings. The proof goes through with practically no change. However it holds as corollary of the ring result since an algebra without proper algebra ideals is easily seen to have no ring ideals. Indeed let $\mathfrak{A}$ be a simple algebra not a zero algebra of order one over $\mathfrak{F}$. Then $\mathfrak{A}=\mathfrak{A}^2$. If $\mathfrak{M}$ is any nonzero ring ideal of $\mathfrak{A}$, the set $\mathfrak{M}_{\mathfrak{F}}$ of all finite sums $\sum\lambda_i z_i$ with λ_i in $\mathfrak{F}$ and z_i in $\mathfrak{M}$ is an algebra ideal, $\mathfrak{M}_{\mathfrak{F}}\neq 0$, $\mathfrak{M}_{\mathfrak{F}}=\mathfrak{A}$, every x of $\mathfrak{A}$ has the form $x=\sum\lambda_i z_i$. But then $xy=\sum z_i(\lambda_i y)$, $z_i(\lambda_i y)$ is in $\mathfrak{M}$, $\mathfrak{A}=\mathfrak{A}^2\subseteq\mathfrak{M}$, $\mathfrak{M}=\mathfrak{A}$, $\mathfrak{A}$ is a simple ring.

7. **A theorem on special Jordan algebras.** An algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic not two is called a *special* Jordan algebra if $\mathfrak{A}$ has a faithful representation $x\rightarrow T_x$ as a vector space $\mathfrak{A}_0$ of linear transformations T_x on a vector space such that $T_{xy}=1/2\ (T_xT_y+T_yT_x)$. We shall prove the following result for algebras over a field $\mathfrak{F}$ of characteristic not two.

THEOREM 2. *Let $\mathfrak{A}$ be a special Jordan algebra with a unity quantity e which is an absolutely primitive idempotent of $\mathfrak{A}$. Then there exists a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}=e\mathfrak{K}+\mathfrak{N}$ where $\mathfrak{N}$ is the radical of $\mathfrak{A}_{\mathfrak{K}}$.*

By a well known argument, if $\mathfrak{A}_{\mathfrak{K}}=e\mathfrak{K}+\mathfrak{N}$ where $\mathfrak{K}$ is the algebraic closure of $\mathfrak{F}$, then there exists a subfield $\mathfrak{L}$ *of finite degree* over $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{L}}=e\mathfrak{L}+\mathfrak{N}_0$. It is therefore sufficient to consider the case where $\mathfrak{F}$ is algebraically closed and so every quantity x of $\mathfrak{A}$ has the form $x=\alpha e+y$ for α in $\mathfrak{F}$ and y nilpotent. Since x is a linear transformation, x is singular if and only if x is nilpotent.

We may identify $\mathfrak{A}$ with its representation by linear transformations and shall use $x\cdot y$ for the product $1/2\ (xy+yx)$ of $\mathfrak{A}$ where xy is the associative product. It is also known([9]) that the representation may always be selected so that e is the identity transformation.

([9]) For if $ex+xe=2x$ where e is an idempotent linear transformation, then $ex+exe=2ex$, $xe+exe=2xe$ and so $ex-xe=2(ex-xe)$, $ex=xe$, x is in the space of linear transformations for which e is the unity quantity. These linear transformations are then transformations on a subspace for which e is the identity transformation.

If a is any quantity of $\mathfrak{A}$, the right multiplications R_a, R_{aa} generate an associative algebra $\mathfrak{A}_a$ which is known(10) to be nilpotent when a is nilpotent. If $\mathfrak{B}$ is any nilpotent subalgebra of $\mathfrak{A}$, the right multiplications R_b defined for b in $\mathfrak{B}$ generate(10) an associative algebra $\mathfrak{B}^*$ of linear transformations on $\mathfrak{A}$, and $\mathfrak{B}^*$ is nilpotent. Since $\mathfrak{B}$ is a nilpotent Jordan algebra of linear transformations, the enveloping associative algebra $\mathfrak{B}_0$ is nilpotent(11).

Let $\mathfrak{B}$ be a maximal nilpotent subalgebra of $\mathfrak{A}$, and let $\mathfrak{C}=\mathfrak{B}+e\mathfrak{F}$. If $\mathfrak{C}=\mathfrak{A}$ there is nothing to prove. Hence we assume that $\mathfrak{C}\neq\mathfrak{A}$ and that there exists an x in $\mathfrak{A}$ and not in $\mathfrak{C}$. We now form $x\mathfrak{B}^{*k}=0$ for k sufficiently large. But then there exists an x not in $\mathfrak{C}$ such that $xb+bx$ is in $\mathfrak{C}$ for every b of $\mathfrak{B}$. Since $x=\alpha e+y$ where y is not in $\mathfrak{C}$ and is nilpotent, we know that $yb+by=xb+bx-2\alpha b$ is in $\mathfrak{C}$. Hence we may assume that x is nilpotent. We now prove

LEMMA 8. *The quantities* bxb, xb^2+b^2x, $x^2b^2+b^2x^2$, xb^2x, bx^2b, $bxc+cxb$, $xbc+cbx$, $x(bc+cb)x$ *are in* $\mathfrak{B}$ *for every* b *and* c *of* $\mathfrak{B}$, *and* $cxvx+xbxc$ *is in* $\mathfrak{C}$.

For $xb+bx=\beta e+b'$ with β in $\mathfrak{F}$ and b' in $\mathfrak{B}$. Then $b(xb+bx)+(xb+bx)b=b^2x+xb^2+2bxb=2\beta b+bb'+b'b$ is in $\mathfrak{B}$. By hypothesis $b^2x+xb^2=\beta_2e+b_2'$ for β_2 in $\mathfrak{F}$ and b_2 in $\mathfrak{B}$, $2bxb=-\beta_2e+b''$ with b'' in $\mathfrak{B}$. But β_2e+b'' is singular only when $\beta_2=0$. Hence bxb and b^2x+xb^2 are in $\mathfrak{B}$. Now $(b+c)x(b+c)=bxc+cxb+bxb+cxc$ is in $\mathfrak{B}$ and so $bxc+cxb$ is in $\mathfrak{B}$.

We now see that $(bxb)x+x(bxb)=\gamma e+b_3$ for γ in $\mathfrak{F}$ and b_3 in $\mathfrak{B}$. Form $(\beta e+b')^2=\beta^2e+2\beta b'+b'^2=(xb+bx)^2=x(bxb)+(bxb)x+xb^2x+bx^2b$. Since bxb is in $\mathfrak{B}$ the quantity $x(bxb)+(bxb)x$ is in $\mathfrak{C}$. It follows that $bx^2b+xb^2x=\delta e+b_4$ for δ in $\mathfrak{F}$ and b_4 in $\mathfrak{B}$. We also form $x(xb+bx)+(xb+bx)x=x^2b+bx^2+2xbx=2\beta x+(xb'+b'x)=2\beta x+\delta e+b_5$. Then $2bx^2b+b^2x^2+x^2b^2+2(xbxb+bxbx)=2\beta(\beta e+b')+2\delta b+bb_5+b_5b$. It follows that $b^2x^2+x^2b^2+2bx^2b=\epsilon e+b_6$ for ϵ in $\mathfrak{F}$ and b_6 in $\mathfrak{B}$. However $xb^2+b^2x=b_2'$ is in $\mathfrak{B}$, $x^2b^2+b^2x^2+2xb^2x=xb'+b'x=\lambda e+b_7$, bx^2b-xb^2x is in $\mathfrak{C}$. Since bx^2b+xb^2x is in $\mathfrak{C}$ so are bx^2b and xb^2x. But both quantities are singular and so must be in $\mathfrak{B}$.

The quantity $(xb)^2+(bx)^2=\beta^2e+2\beta b'+b'^2-xb^2x-bx^2b$ and so $b^2x^2+x^2b^2=2\beta^2e+2\beta b'+2\delta b+bb_5+b_5b-2bx^2b-2[(xb)^2+(bx)^2]$ is in $\mathfrak{B}$. The quantity $x(b+c)^2x=xb^2x+xc^2x+x(bc+cb)x$ is in $\mathfrak{B}$ and so $x(bc+cb)x$ is in $\mathfrak{B}$. We now form $cbx+xbc+c(\beta e+b'-xb)+(\beta e+b'-bx)c=2\beta c+cb'+b'c-(cxb+bxc)$ which is in $\mathfrak{B}$. Now $cxbx+xbxc=(\gamma e+c'-xc)bx+xb(\gamma e+c'-cx)=\gamma(bx+xb)+c'bx+xbc'-x(cb+bc)x$ which is in $\mathfrak{C}$ by the results already proved.

The quantity $2xbx=x(xb+bx)+(xb+bx)x-(x^2b+bx^2)$ is in the algebra $\mathfrak{A}$. If xbx is in $\mathfrak{C}$ for every b of $\mathfrak{B}$, then xbx is singular and is in $\mathfrak{B}$. But then $(xbx)b+b(xbx)$ is in $\mathfrak{B}$, $(\beta e+b')^2+(xbx)b+b(xbx)+bx^2b+xb^2x$ is in $\mathfrak{B}$ and is nilpotent, b' is nilpotent and so $\beta=0$. We have proved that in this case $xb+bx$ is in $\mathfrak{B}$ for every b of $\mathfrak{B}$.

(10) See Theorem 1 of JA2.

(11) See Theorem 8 of JA1.

Assume now that there exists a quantity d_1 in $\mathfrak{B}$ such that $y_1=xd_1x$ is not in $\mathfrak{B}$. Evidently y_1 is nilpotent. Then $y_2=y_1^2=x(d_1x^2d_1)x=xd_2x$ where d_2 is in $\mathfrak{B}$. If y_2 is not in $\mathfrak{B}$ we continue the process and obtain a quantity $y_3=y_1^4=xd_3x$ where $d_3=d_2x^2d_2$ is in $\mathfrak{B}$. But the nilpotency of y_1 implies that this process must yield a quantity y such that y^2 is in $\mathfrak{B}$ while $y=xdx$ is not in $\mathfrak{C}$ and d is in $\mathfrak{B}$. Evidently $yb+by=xdxb+bxdx$ is in $\mathfrak{C}$ by Lemma 14. We now let $yb+by=\beta e+b'$ and compute $y(\beta e+b')+(\beta e+b')y=2\beta y+(yb'+b'y)=y^2b+by^2+2yby$. If $yb'+b'y=\beta'e+b''$, then $\beta'e+b''-(y^2b+by^2)=2(yby-\beta y)$ is singular and $b''-(y^2b+by^2)$ is in $\mathfrak{B}$. Then $\beta'=0$.

Assume that $\mathfrak{B}_1$ is the set of all quantities b_1 of $\mathfrak{B}$ such that yb_1+b_1y is in $\mathfrak{B}$. By Lemma 8 we see that $\mathfrak{B}_1\supseteq\mathfrak{B}^2$. The result above shows that

$$yb+by=\beta e+b_1$$

where b_1 is in $\mathfrak{B}_1$ for every b of $\mathfrak{B}$. Also $yy^2+y^2y=2y^3$ is nilpotent and so y^2 is in $\mathfrak{B}_1$. It follows that $\mathfrak{D}=y\mathfrak{F}+\mathfrak{C}$ is an algebra and that $\mathfrak{B}_1$ is an ideal of $\mathfrak{D}$. Let $yb+by=\beta e+b$, where $\beta\neq 0$, for some b. We may take $\beta=1$ without loss of generality. The homomorphism $w\rightarrow w_0$ of $\mathfrak{D}$ onto $\mathfrak{D}-\mathfrak{B}$ maps e onto the unity quantity e_0 of $\mathfrak{D}$ and y onto y_0, b onto b_0 such that $y_0b_0+b_0y_0=e_0$. Also $y_0^2=(y^2)_0=b_0^2=(b^2)_0=0$. But then the fact that y is not in $\mathfrak{C}$ implies that $u_0=1/2\ (y_0+b_0+e_0)\neq e_0$, $u_0^2=1/2\ (e_0+2y_0+2b_0+b_0y_0+y_0b_0)=u_0$. The class u_0 must then contain an idempotent $u\neq e$ contrary to the hypothesis that e is primitive. It follows that $yb+by$ is in $\mathfrak{B}$ for every b of $\mathfrak{B}$, $\mathfrak{B}_1=\mathfrak{B}$, $\mathfrak{C}=y\mathfrak{F}+\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$ as an ideal $\mathfrak{C}-\mathfrak{B}=y\mathfrak{F}$, $\mathfrak{C}-\mathfrak{B}$ is a nilalgebra contrary to our hypothesis that $\mathfrak{B}$ is maximal. We have proved that $\mathfrak{A}$ always contains a nilpotent quantity x not in $\mathfrak{B}$ such that $xb+bx$ and xbx are in $\mathfrak{B}$ for every b of $\mathfrak{B}$.

If $xb+bx=b'$ then $xb'+b'x=b''=x^2b+bc^2+2xbx$ and so x^2b+bx^2 is in $\mathfrak{B}$. Assume now that $x^kb+bx^k=b_k$ is in $\mathfrak{B}$. Then $x^{k+1}b+bx^{k+1}=x(b_k-bx^k)+(b_k-x^kb)x=xb_k+b_kx-[(xbx)b^{k-1}+b^{k-1}(xbx)]$ which is in $\mathfrak{B}$ since xbx is in $\mathfrak{B}$. It follows that $\mathfrak{C}=\mathfrak{F}[x]+\mathfrak{B}$ is an algebra and that $\mathfrak{C}-\mathfrak{B}=\mathfrak{F}[x]$ is a nilalgebra, $\mathfrak{C}\neq\mathfrak{B}$, $\mathfrak{C}$ is nilpotent. This contradicts our hypothesis that $\mathfrak{B}$ is maximal and implies that $\mathfrak{A}=e\mathfrak{F}+\mathfrak{B}$ as desired.

8. **Principal idempotents.** Consider an idempotent e of $\mathfrak{A}$ and write $\mathfrak{A}=\mathfrak{A}_e(1)+\mathfrak{A}_e(1/2)+\mathfrak{A}_e(0)$. If x is in $\mathfrak{A}$ we use (1) with $w=e$, $x=y=z$ to obtain $4(ex)x^2=ex^3+(ex^2)x+2[(ex)x]x$. When x is in $\mathfrak{A}_e(1/2)$ this relation becomes $x^3=ex^3+(ex^2)x$. But if $x^2=w_1+w_0$, then $x^3=x(w_0+w_1)=x[S_{1/2}(w_1)+T_{1/2}(w_0)]+xS_0(w_1)+xT_1(w_0)=ex^3+w_1x=1/2\,x[S_{1/2}(w_1)+T_{1/2}(w_0)]+xT_1(w_0)+xS_{1/2}(w_1)+xS_0(w_1)$. We have proved the relation

$$x_{1/2}S_{1/2}(w_1)=x_{1/2}T_{1/2}(w_0),\qquad w_1+w_0=x_{1/2}^2. \tag{38}$$

By (5) we see that $S_{1/2}(w_1^k)=2^{k-1}[S_{1/2}(w_1)]^k$ and similarly $T_{1/2}(w_0^k)=2^{k-1}[T_{1/2}(w_0)]^k$. But if $x[S_{1/2}(w_1)]^k=x[T_{1/2}(w_0)]^k$, then $x[S_{1/2}(w_1)]^{k+1}=x[T_{1/2}(w_0)]^k S_{1/2}(w_1)=xS_{1/2}(w_1)[T_{1/2}(w_0)]^k$ by (7). Hence $x[S_{1/2}(w_1)]^k=x[T_{1/2}(w_0)]^k$ for all positive integers k and so $xS_{1/2}(w_1^k)=xT_{1/2}(w_0^k)$ for every k.

An idempotent e of $\mathfrak{A}$ is said to be *principal* if there is no idempotent orthogonal to e. Then $\mathfrak{A}_e(0)$ *is a nilalgebra* and we may now obtain a proof of a result previously proved only for stable algebras, and which will be completed in Theorem 7.

LEMMA 9. *Let e be a principal idempotent of a commutative power-associative algebra $\mathfrak{A}$ whose characteristic is prime to* 30. *Then the quantities of $\mathfrak{A}_e(1/2)$ are nilpotent.*

For if e is principal we have $w_0^k=0$ for some k since w_0 is in $\mathfrak{A}_e(0)$. Then $x_{1/2}S_{1/2}(w_1^k)=0$. Put $z=x^{2k+1}=xx^{2k}=xw^k=x(w_1^k+w_0^k)=xw_0^k=xS_0(w_1^k)$. It follows that z is in $\mathfrak{A}_e(0)$ and is nilpotent, x is nilpotent.

9. **The first property of simple algebras.** Let $\mathfrak{A}$ be a commutative power-associative algebra with a unity quantity e over a field $\mathfrak{F}$ of characteristic prime to 30 and suppose that $e=u+v$ where u and v are orthogonal idempotents. Then we may write $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_2+\mathfrak{A}_{12}$ where $\mathfrak{A}_1=\mathfrak{A}_u(1)=\mathfrak{A}_v(0)$, $\mathfrak{A}_2=\mathfrak{A}_u(0)=\mathfrak{A}_v(1)$, $\mathfrak{A}_{12}=\mathfrak{A}_u(1/2)=\mathfrak{A}_v(1/2)$. We adopt the corresponding notation $x=x_1+x_{12}+x_2$ for the quantities of $\mathfrak{A}$ and will use (6), (7), (8) with the subscript zero replaced by two. Thus $x_1y_{12}=y_{12}S_{1/2}(x_1)+y_{12}S_2(x_1)$, $x_2y_{12}=y_{12}T_{1/2}(x_2)+y_{12}T_1(x_2)$. We now prove

LEMMA 10. *Let $\mathfrak{A}_1=u\mathfrak{F}+\mathfrak{G}_1$, $\mathfrak{A}_2=v\mathfrak{F}+\mathfrak{G}_2$ where $\mathfrak{G}=\mathfrak{G}_1\oplus\mathfrak{G}_2$ is a nilalgebra. Then $xy=\alpha e+g$ for every x and y of $\mathfrak{A}_{12}$ where α is in $\mathfrak{F}$ and g is in $\mathfrak{G}$.*

We begin our proof by observing that the mapping $1/2\ x_1\rightarrow S_{1/2}(x_1)$ is a homomorphism of $\mathfrak{A}_1$ onto a special Jordan algebra and its maps the nilalgebra $\mathfrak{G}_1$ onto a nilpotent Jordan algebra $\mathfrak{P}_1$ of linear transformations $P_1=2S_{1/2}(g_1)$ on $\mathfrak{A}_{12}$. It is known[11] that the enveloping algebra of $\mathfrak{P}_1$ is nilpotent and so $\mathfrak{P}_1$ is nilpotent. Similarly the mapping $x_2\rightarrow 2T_{1/2}(x_2)$ maps every g_2 of $\mathfrak{G}_2$ on $P_2=2T_{1/2}(g_2)$ where P_2 is nilpotent. By (7) we have $P_1P_2=P_2P_1$ and hence P_1+P_2 is nilpotent. It follows that $\mathfrak{P}=\mathfrak{P}_1+\mathfrak{P}_2$ is a vector space of nilpotent linear transformations $P=S_{1/2}(g_1)+T_{1/2}(g_2)$ and that the enveloping algebra of $\mathfrak{P}$ is nilpotent.

Suppose now that x is any nonzero quantity of $\mathfrak{A}_{12}$ and so write $x^2=w_1+w_2$, $w_1=\alpha u+g_1$, $w_2=\beta v+g_2$ for g_i in $\mathfrak{G}_i$, α and β in $\mathfrak{F}$. Apply (38) and compute $xS_{1/2}(w_1)=1/2\ \alpha x+xS_{1/2}(g_1)=xT_{1/2}(w_2)=1/2\ \beta x+xT_{1/2}(g_2)$. Hence $(\alpha-\beta)x=xP$ where the linear transformation $P=2T_{1/2}(g_2)-2S_{1/2}(g_1)$ is nilpotent. Since $x\neq 0$ we must have $\alpha=\beta$. We now let x and y be in $\mathfrak{A}_{12}$ and write $x^2=\alpha_0e+g'$, $y^2=\beta e+g''$, $(x+y)^2=\gamma e+g'''$. This yields $2xy=(x+y)^2-x^2-y^2=(\gamma-\alpha_0-\beta)e+(g'''-g'-g'')$, $xy=\alpha e+g$ as desired.

LEMMA 11. *Let $y_{12}S_2(x_1)$ be in $\mathfrak{G}_2$ for every x_1 of $\mathfrak{A}_1$ and y_{12} of $\mathfrak{A}_{12}$. Then $y_{12}T_1(x_2)$ is in $\mathfrak{G}_1$ for every x_2 of $\mathfrak{A}_2$ and y_{12} of $\mathfrak{A}_{12}$.*

For $T_1(v)=0$, and we need only prove that $y_{12}T_1(g_2)$ is in $\mathfrak{G}_1$ for every g_2 of

$\mathfrak{G}_2$. Suppose that there is a quantity p_{12} in $\mathfrak{A}_{12}$ and a quantity g_2 in $\mathfrak{G}_2$ such that $f_1 = p_{12}T_1(g_2)$ is not in $\mathfrak{G}_1$. Then f_1 has an inverse in $\mathfrak{A}_1$ and $f_1f_1^{-1} = u = 2p_{12}S_{1/2}(f_1^{-1})T_1(g_2) = y_{12}T_1(g_2)$ by (8) where $y_{12} = 2p_{12}S_{1/2}(f_1^{-1})$ is in $\mathfrak{A}_{12}$. Then we have

$$y_{12}g_2 = b_{12} + u, \qquad b_{12} = y_{12}T_{1/2}(g_2). \tag{39}$$

We next write

$$b_{12}g_2 = c_{12} + c_1, \qquad c_{12} = b_{12}T_{1/2}(g_2), \qquad c_1 = b_{12}T_1(g_2). \tag{40}$$

By (6) we have $y_{12}g_2^2 = y_{12}T_{1/2}(g_2^2) + y_{12}T_1(g_2^2) = 2[y_{12}T_{1/2}(g_2)]T_{1/2}(g_2) + 4[y_{12}T_{1/2}(g_2)]T_1(g_2)$ and so

$$y_{12}g_2^2 = 2c_{12} + 4c_1. \tag{41}$$

Let us substitute $y = z = y_{12}$, $x = w = g_2$ in (1) and obtain

$$2[2(y_{12}g_2)^2 + y_{12}^2g_2^2] = g_2[g_2y_{12}^2 + 2(g_2y_{12})y_{12}] + y_{12}[y_{12}g_2^2 + 2(y_{12}g_2)g_2]. \tag{42}$$

This becomes $4(b_{12}+u)^2 + 2y_{12}^2g_2^2 = g_2(g_2y_{12}^2) + 2g_2(b_{12}y_{12} + 1/2\ y_{12}) + y_{12}(2c_{12}+4c_1) + 2y_{12}(b_{12}g_2) = g_2(g_2y_{12}^2) + 2g_2(b_{12}y_{12}) + b_{12} + u + 2y_{12}c_{12} + 4y_{12}c_1 + y_{12}c_{12} + 2y_{12}c_1 = 4b_{12}^2 + 4b_{12} + 4u + 2y_{12}^2g_2^2$. Computing the components in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ we obtain

$$3u + 4u(b_{12}^2) = 4u(y_{12}c_{12}), \tag{43}$$

$$4vb_{12}^2 - 4v(y_{12}c_{12}) = 6b_2 + g_2(g_2y_{12}^2) + 2g_2(b_{12}y_{12}) - 2y_{12}^2g_2^2, \tag{44}$$

where $b_2 = y_{12}S_2(c_1)$ is in $\mathfrak{G}_2$ by the hypothesis of our theorem and all other terms of the right member of (44) are also in $\mathfrak{G}_2$.

We now substitute $y = z = y_{12}$, $w = u$, $x = h_2$ in (1) and use the fact that $u(h_2y_{12}^2) = h_2(y_{12}^2u) = 0$ to obtain

$$4y_{12}(y_{12}h_2) = y_{12}(y_{12}h_2) + 2y_{12}[u(y_{12}h_2)] + h_2y_{12}^2 + 2u[y_{12}(y_{12}h_2)]. \tag{45}$$

Write

$$y_{12}h_2 = d_{12} + d_1, \qquad d_1 = y_{12}T_1(h_2), \qquad d_{12} = y_{12}T_{1/2}(h_2) \tag{46}$$

and obtain $3y_{12}(d_{12}+d_1) = 2u(y_{12}d_1 + y_{12}d_{12}) + h_2y_{12}^2 + 2y_{12}d_1 + y_{12}d_{12}$. This yields $y_{12}d_1 + 2y_{12}d_{12} = 2(y_{12}d_1)u + 2u(y_{12}d_{12}) + h_2y_{12}^2$ and the component in $\mathfrak{A}_2$ is

$$2(y_{12}d_{12})v + y_{12}S_0(d_1) = h_2y_{12}^2. \tag{47}$$

In the particular case where $h_2 = g_2^2$ we may use (41) and have $d_{12} = 2c_{12}$, $d_1 = 4c_1$ so that

$$4(y_{12}c_{12})v + 4b_2 = g_2^2y_2^2. \tag{48}$$

Since b_2 is in $\mathfrak{G}_2$ so is $v(y_{12}c_{12})$. By (44) the quantity $4vb_{12}^2$ is in $\mathfrak{G}_2$ and by

Lemma 10 both $y_{12}c_{12}$ and b_{12}^2 are in $\mathfrak{G}$. But then $(y_{12}c_{12})u$ and b_{12}^2u are in $\mathfrak{G}$ and this contradicts (43).

Let us now return to the study of simple power-associative commutative algebras $\mathfrak{A}$ over a center $\mathfrak{F}$ of characteristic not two, three, or five. We let $\mathfrak{K}$ be the algebraic closure of $\mathfrak{F}$ so that $\mathfrak{A}_{\mathfrak{K}}$ is simple and every primitive idempotent of $\mathfrak{A}_{\mathfrak{K}}$ is absolutely primitive. Define the *degree* of $\mathfrak{A}$ *to be the maximum number of elements in all sets* $e_1, \cdots, e_t$ *of pairwise orthogonal idempotents* e_i *of* $\mathfrak{A}_{\mathfrak{K}}$. The sum $f=e_1+\cdots+e_t$ will then be a principal idempotent of $\mathfrak{A}_{\mathfrak{K}}$ and the e_i will be primitive idempotents of $\mathfrak{A}_{\mathfrak{K}}$.

If $t>2$ or $t=2$ but f is not the unity quantity of $\mathfrak{A}$ we apply Theorem 1 to see that $\mathfrak{A}_{\mathfrak{K}}$ is a Jordan algebra. But then $\mathfrak{A}$ is a Jordan algebra. Thus when $t\geqq 2$ the algebra $\mathfrak{A}$ is either a Jordan algebra or has a unity quantity.

Let $t=1$ and thus assume that $\mathfrak{A}$ is a simple algebra containing a primitive idempotent quantity u which is the only idempotent of $\mathfrak{A}_{\mathfrak{K}}$. There is no loss of generality if we take $\mathfrak{K}=\mathfrak{F}$. Adjoin a unity quantity to $\mathfrak{A}$ as in §4 and obtain an algebra $\mathfrak{R}=\mathfrak{A}+v\mathfrak{F}$ with a unity quantity $e=u+v$ and which is such that all nonzero ideals of $\mathfrak{R}$ contain $\mathfrak{A}$. If $\mathfrak{A}_u(1/2)=0$ then $\mathfrak{A}=\mathfrak{A}_u(1)\oplus\mathfrak{A}_u(0)$ contrary to our hypothesis that $\mathfrak{A}$ is simple. Thus $\mathfrak{R}_u(1/2)=\mathfrak{A}_u(1/2)\neq 0$. Since u is a principal idempotent, the algebra $\mathfrak{A}_u(0)=\mathfrak{G}_2$ is a nilalgebra.

By Lemma 1 the algebra $\mathfrak{A}_u(1)$ contains an ideal $\mathfrak{B}_u$ which is a zero algebra and is such that $\mathfrak{A}_u(1)-\mathfrak{B}_u$ is a special Jordan algebra $\mathfrak{J}$. The unity quantity of $\mathfrak{J}$ is the image u' of u and must be absolutely primitive since every idempotent element of $\mathfrak{J}$ is the image of an idempotent of $\mathfrak{A}_u(1)$ when $\mathfrak{A}$ is power-associative. By Theorem 2 we have $\mathfrak{J}=u'\mathfrak{F}+\mathfrak{H}$ where $\mathfrak{H}$ is a nilpotent Jordan algebra. But then $\mathfrak{A}_u(1)=u\mathfrak{F}+\mathfrak{G}_1$ where $\mathfrak{G}_1-\mathfrak{B}_u=\mathfrak{H}$. It follows that $\mathfrak{G}_1$ is a nilalgebra and that we have the hypotheses of Lemma 10 for the algebra $\mathfrak{R}$.

If x and y are in $\mathfrak{A}_u(1/2)$ then $xy=\alpha e+g$ where α is in $\mathfrak{F}$ and g is in $\mathfrak{G}_1+\mathfrak{G}_2$ by Lemma 10. But x and y are in $\mathfrak{A}$ and so xy is in $\mathfrak{A}$, $xy=g$, $\mathfrak{A}_u(1/2)\mathfrak{A}_u(1/2)\subseteq\mathfrak{G}$. If y is in $\mathfrak{A}_u(1/2)$ and x is in $\mathfrak{A}_u(1)$, then xy is in $\mathfrak{A}_u(1/2)+\mathfrak{A}_u(0)$. But then $yS_2(x)$ is in $\mathfrak{G}_2=\mathfrak{A}_u(0)$ and the hypothesis of Lemma 11 is satisfied, yz is in $\mathfrak{A}_u(1/2)+\mathfrak{G}_1$ for every y of $\mathfrak{A}_u(1/2)$ and z of $\mathfrak{R}_u(0)$. It follows that $\mathfrak{B}=\mathfrak{A}_u(1/2)+\mathfrak{G}_1+\mathfrak{G}_2$ is an ideal of $\mathfrak{R}$. However $\mathfrak{B}$ does not contain u and so $\mathfrak{B}=0$, $\mathfrak{A}_u(0)=\mathfrak{A}_u(1/2)=0$, a contradiction. We have proved the following property.

THEOREM 3. *Every simple commutative power-associative algebra of characteristic prime to* 30 *is either an algebra of degree one or two with a unity quantity or is a Jordan algebra.*

10. **Jordan algebras of degree two.**

Let $\mathfrak{A}$ be a Jordan algebra with a unity quantity $e=u+v$ where u and v are orthogonal idempotents, and write $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_{12}+\mathfrak{A}_2$ as in §9. Since $\mathfrak{A}$ is a Jordan algebra we have the properties $\mathfrak{A}_1\mathfrak{A}_{12}\subseteq\mathfrak{A}_{12}$, $\mathfrak{A}_2\mathfrak{A}_{12}\subseteq\mathfrak{A}_{12}$. Thus $u(g_iy_{12})=1/2\, g_iy_{12}$ for every g_i of $\mathfrak{A}_i$ and y_{12} of $\mathfrak{A}_{12}$.

THEOREM 4. *Let* $\mathfrak{A}_1 = u\mathfrak{F} + \mathfrak{G}_1$, $\mathfrak{A}_2 = v\mathfrak{F} + \mathfrak{G}_2$ *where* $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ *is a nilalgebra. Then* $\mathfrak{B} = \mathfrak{G} + \mathfrak{G}\mathfrak{A}_{12}$ *is a nilideal of* $\mathfrak{A}$.

We begin by interchanging w and x in the Jordan identity (27). The left members are unaltered and so the right members must be equal, that is, we have the identity

$$(49) \quad [w(yz)]x + [w(xz)]y + [w(xy)]z = [x(yz)]w + [x(wz)]y + [x(wy)]z.$$

Put $y = y_{12}$, $z = g_2$, $w = w_{12}$, $x = u$ and obtain

$$[w_{12}(y_{12}g_2)]u + 1/2\ (w_{12}y_{12})g_2 = 1/2\ (y_{12}g_2)w_{12} + 1/2\ (w_{12}g_2)y_{12},$$

where we have used the property $xz = 0$ and $[u(y_{12}w_{12})]g_2 = 0$. Write $w_{12}(y_{12}g_2) = \alpha e + h_1 + h_2$, $(w_{12}g_2)y_{12} = \beta e + k_1 + k_2$ for α and β in $\mathfrak{F}$, h_1 and k_1 in $\mathfrak{G}_1$, h_2 and k_2 in $\mathfrak{G}_2$, where we have used Lemma 10 and the fact that $y_{12}g_2$ and $w_{12}g_2$ are in $\mathfrak{A}_{12}$. We substitute these expressions in the formula above and have

$$\alpha u + h_1 + 1/2\ (w_{12}y_{12})g_2 = 1/2\ (\alpha e + h_1 + h_2) + 1/2\ (\beta e + k_1 + k_2).$$

Since $(w_{12}y_{12})g_2$ is in $\mathfrak{G}_2$ we have $\alpha u = 1/2\ \alpha u + 1/2\ \beta u$, $1/2\ (\alpha v + \beta v) = 0$. Thus $\alpha = \beta$, $\alpha = -\beta$, $\alpha = \beta = 0$. This proves that $\mathfrak{A}_{12}(\mathfrak{A}_{12}\mathfrak{G}) \subseteq \mathfrak{G}$. By symmetry $\mathfrak{A}_{12}(\mathfrak{A}_{12}\mathfrak{G}_1) \subseteq \mathfrak{G}$ and so $\mathfrak{A}_{12}(\mathfrak{A}_{12}\mathfrak{G}) \subseteq \mathfrak{G} \subseteq \mathfrak{B}$. But then $\mathfrak{A}\mathfrak{B} = (\mathfrak{A}_{12} + \mathfrak{A}_1 + \mathfrak{A}_2) \cdot (\mathfrak{A}_{12}\mathfrak{G} + \mathfrak{G}) = \mathfrak{A}_{12}(\mathfrak{A}_{12}\mathfrak{G}) + (\mathfrak{A}_1 + \mathfrak{A}_2)(\mathfrak{A}_{12}\mathfrak{G}) + (\mathfrak{A}_1 + \mathfrak{A}_2)\mathfrak{G} + \mathfrak{A}_{12}\mathfrak{G} \subseteq \mathfrak{A}_{12}\mathfrak{G} + \mathfrak{A}_{12}\mathfrak{G} + \mathfrak{G} + \mathfrak{A}_{12}\mathfrak{G} \subseteq \mathfrak{B}$ where we have used the fact that $[y_{12}S_{1/2}(g_1)]a_2 = [y_{12}T_{1/2}(a_2)] \cdot S_{1/2}(g_1)$, $[y_{12}S_{1/2}(g_1)]a_1 = y_{12}S_{1/2}(g_1)S_{1/2}(a_1) = y_{12}S_{1/2}(g_1a_1) - [y_{12}S_{1/2}(a_1)]S_{1/2}(g_1)$ to prove that $(\mathfrak{A}_1 + \mathfrak{A}_2)(\mathfrak{A}_{12}\mathfrak{G}) \subseteq \mathfrak{A}_{12}\mathfrak{G}$. This proves that $\mathfrak{B}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{B}$ were not a nilideal, it would contain an idempotent $w = w_1 + w_{1/2} + w_2 = w^2 = w_1^2 + w_2^2 + w_{1/2}^2 + 2w_{1/2}(w_1 + w_2)$. But then $w_{1/2} = 2w_{1/2}(w_1 + w_2) = w_{1/2}P$ where $P = 2S_{1/2}(w_1) + 2T_{1/2}(w_2)$ is nilpotent since w_i is in $\mathfrak{G}_i$. It follows that $w_{1/2} = 0$ and that the idempotent w is in the nilalgebra $\mathfrak{G}$ which is impossible.

We now have the following consequence:

COROLLARY I. *Let* $\mathfrak{A}$ *be a simple Jordan algebra of degree two over a center whose characteristic is prime to* 30. *Then* $\mathfrak{A}$ *is a classical Jordan algebra of degree two.*

For the ideal $\mathfrak{B}$ of Theorem 4 must be zero and every quantity of the scalar extension $\mathfrak{A}_{\mathfrak{K}}$ of $\mathfrak{A}$ has the form $x = \alpha u + \beta v + x_{1/2}$ where $x_{1/2}^2 = f(x_{1/2}) \cdot e$ with $f(x_{1/2})$ a quadratic form in the coordinates of $x_{1/2}$. This is the classical Jordan algebra of degree two.

11. **Classification of simple Jordan algebras of characteristic p.** We shall proceed to extend the result of Theorem 4 to the case $t > 2$. We let $\mathfrak{A}$ be a Jordan algebra whose unity quantity e is the sum $e = e_1 + \cdots + e_t$ of pairwise orthogonal idempotents e_i and so write $\mathfrak{A}$ as the sum of subspaces $\mathfrak{A}_{ij} = \mathfrak{A}_{ji}$ for $i \leq j$ and $i, j = 1, \cdots, t$. Here $\mathfrak{A}_{ii} = \mathfrak{A}_{e_i}(1)$ and $\mathfrak{A}_{ij}$ is the intersection of $\mathfrak{A}_{e_i}(1/2)$ and $\mathfrak{A}_{e_j}(1/2)$. Theorem 4 may now be extended as follows:

THEOREM 5. *Let* $\mathfrak{A}_{ii}=e_i\mathfrak{F}+\mathfrak{G}_i$ *where* $\mathfrak{G}_i$ *is a nilalgebra, and define* $\mathfrak{G}_{ij}=\mathfrak{A}_{ij}(\mathfrak{G}_i+\mathfrak{G}_j)$ *for* $i\neq j$, $\mathfrak{G}_{ijk}=\mathfrak{A}_{ij}(\mathfrak{G}_j\mathfrak{A}_{jk})$ *for* i, j, k *all distinct. Then the sum* $\mathfrak{G}$ *of the spaces* $\mathfrak{G}_i$, $\mathfrak{G}_{ij}$, *and* $\mathfrak{G}_{ijk}$ *is an ideal of* $\mathfrak{A}$ *which contains none of the idempotents* e_i.

We note first that $\mathfrak{G}_{ij}=\mathfrak{G}_{ji}=\mathfrak{A}_{ij}$. We also substitute $w=w_{jk}$, $x=e_i$, $y=y_{ij}$, $z=g_j$ in the Jordan identity (27) and obtain

$$(w_{jk}g_j)y_{ij} = w_{jk}(y_{ij}g_j). \tag{50}$$

We also substitute $x=e_i$, $y=y_{ij}$, $z=g_k$, $w=w_{jk}$ in (27) and obtain

$$y_{ij}(w_{jk}g_k) = g_k(y_{ij}w_{jk}). \tag{51}$$

We have thus proved the relations

$$\mathfrak{G}_{ijk} = \mathfrak{G}_{kji} \subseteq \mathfrak{A}_{ik}, \qquad \mathfrak{A}_{ij}(\mathfrak{A}_{jk}G_k) \subseteq \mathfrak{A}_{ik}\mathfrak{G}_k. \tag{52}$$

Theorem 4 is applicable and yields the relations $\mathfrak{A}_{ii}\mathfrak{G}_{ij}\subseteq\mathfrak{G}_{ij}$, $\mathfrak{A}_{ij}\mathfrak{G}_{ij}\subseteq\mathfrak{G}_i+\mathfrak{G}_j$. Since $\mathfrak{A}_{pq}\mathfrak{G}_{ij}=0$ if p and q are distinct from i and j and since (52) implies that $\mathfrak{A}_{ki}\mathfrak{G}_{ij}=\mathfrak{G}_{kij}+\mathfrak{G}_{jk}$, we have proved that $\mathfrak{A}\mathfrak{G}_{ij}\subseteq\mathfrak{G}$. Evidently our definitions have been so constructed that $\mathfrak{A}\mathfrak{G}_i\subseteq\mathfrak{G}$.

Let us now pass to the more difficult task of proving that $\mathfrak{A}\mathfrak{G}_{ijk}=\mathfrak{G}$. We put $x=x_i$, $w=w_{ij}$, $y=y_{jk}$, $z=g_j$ in (27) and obtain $(x_iw_{ij})(y_{jk}g_j)=x_i[w_{ij}(y_{jk}g_j)]$. Since x_iw_{ij} is in $\mathfrak{A}_{ij}$ we have proved that

$$\mathfrak{A}_{ii}\mathfrak{G}_{ijk} \subseteq \mathfrak{G}_{ijk}. \tag{53}$$

By symmetry $\mathfrak{A}_{kk}\mathfrak{G}_{ijk}=\mathfrak{G}_{ijk}$. But $\mathfrak{G}_{ijk}=\mathfrak{A}_{ik}$ and so $\mathfrak{A}_{pp}\mathfrak{G}_{ijk}=0$ for every $p\neq i, k$. Hence $(\sum_{p=1}^{t}\mathfrak{A}_{pp})\mathfrak{G}_{ijk}\subseteq\mathfrak{G}_{ijk}$.

We next write $x=x_{ij}$, $w=w_{ij}$, $y=y_{jk}$, $z=g_j$ in (27) and obtain $(x_{ij}w_{ij})(x_{jk}g_j)+(w_{ij}g_j)(x_{ij}y_{jk})+(w_{ij}y_{jk})(x_{ij}g_j)=x_{ij}[w_{ij}(y_{jk}g_j)]+y_{jk}[w_{ij}(x_{ij}g_j)]+g_j[w_{ij}(y_{jk}x_{ij})]$. Evidently $g_j[w_{ij}(y_{jk}x_{ij})]$ is in $\mathfrak{G}_{jk}$. By Theorem 4 the product $w_{ij}(x_{ij}g_j)$ is in $\mathfrak{G}_i+\mathfrak{G}_j$ and so $y_{jk}[w_{ij}(x_{ij}g_j)]$ is in $\mathfrak{G}_{jk}$. The products $(w_{ij}y_{jk})(x_{ij}g_j)$ and $(x_{ij}y_{jk})\cdot(w_{ij}g_j)$ are clearly in $\mathfrak{A}_{ik}(\mathfrak{A}_{ij}\mathfrak{G}_j)\subseteq\mathfrak{A}_{kj}G_j=\mathfrak{G}_{jk}$ and $(x_{ij}w_{ij})(y_{jk}g_j)$ is in $(\mathfrak{A}_i+\mathfrak{A}_j)\mathfrak{G}_{jk}=\mathfrak{A}_j\mathfrak{G}_{jk}\subseteq\mathfrak{G}_{jk}$. We have then shown that $x_{ij}[w_{ij}(y_{jk}g_j)]$ is in $\mathfrak{G}_{jk}$, that is,

$$\mathfrak{A}_{ij}\mathfrak{G}_{ijk} = \mathfrak{G}_{jk}. \tag{54}$$

Substitute $x=x_{ik}$, $y=y_{jk}$, $w=w_{ij}$, $z=g_j$ in (27) and obtain $(w_{ij}x_{ik})(g_jy_{jk})+(w_{ij}g_j)(x_{ik}y_{jk})=[w_{ij}(g_jy_{jk})]x_{ik}+[w_{ij}(x_{ij}y_{jk})]g_j$. The term $(w_{ij}x_{ik})(g_jy_{jk})$ is in $\mathfrak{G}_j+\mathfrak{G}_k$ by Theorem 4, $(x_{ik}y_{jk})(w_{ij}g_j)$ is in $\mathfrak{G}_i+\mathfrak{G}_j$ and $[w_{ij}(x_{ik}y_{jk})]g_j$ is in $\mathfrak{G}_j$. Hence $x_{ik}[w_{ij}(g_jy_{jk})]$ is in $\mathfrak{G}_i+\mathfrak{G}_k$, that is,

$$\mathfrak{A}_{ik}\mathfrak{G}_{ijk} \subseteq \mathfrak{G}_i + \mathfrak{G}_k. \tag{55}$$

There remains the possibility that $t>3$ and so that there exists a subscript $p\neq i, j, k$. Evidently $\mathfrak{A}_{pq}\mathfrak{G}_{ijk}=0$ unless $q=i$ or k. Put $x=x_{ik}$, $y=y_{jk}$, $w=w_{ij}$, $z=g_j$ in (27) and obtain $(w_{ij}x_{ip})(y_{jk}g_j)=x_{ip}[w_{ij}(y_{jk}g_j)]$, a result which may be written as

(56) $$\mathfrak{A}_{ip}\mathfrak{S}_{ijk} \subseteq \mathfrak{S}_{pjk}.$$

This completes our proof that $\mathfrak{A}\mathfrak{S}_{ijk}\subseteq\mathfrak{S}$ and that $\mathfrak{S}$ is an ideal of $\mathfrak{A}$. Since the intersection of $\mathfrak{S}$ and $\mathfrak{A}_{ii}$ is $\mathfrak{S}_{ii}$, no e_i is in $\mathfrak{S}$.

Let us now assume that $\mathfrak{R}$ is a Jordan algebra over an algebraically closed field $\mathfrak{F}$ and that $\mathfrak{R}$ has a unity quantity $e=e_1+\cdots+e_t$ for absolutely primitive idempotents e_i. We first apply Theorem 6 in the case where $\mathfrak{R}$ is simple. Then some $\mathfrak{R}_{ij}\neq 0$ and we may apply Lemma 1 to see $\mathfrak{R}_{ii}$ has an ideal $\mathfrak{B}_i$ which is a zero algebra and is such that $\mathfrak{A}_{ii}-\mathfrak{B}_i$ is a special Jordan algebra. As in the application of Theorem 4 we see that $\mathfrak{R}_{ii}=e_i\mathfrak{F}+\mathfrak{S}_i$ where $\mathfrak{S}_i$ is nilpotent. By Theorem 5, $\mathfrak{S}_i$ is contained in a proper ideal $\mathfrak{S}$ of $\mathfrak{R}$, $\mathfrak{S}_i=0$, $\mathfrak{R}_{ii}=e_i\mathfrak{F}$.

We next assume that u is a principal idempotent of a simple Jordan algebra $\mathfrak{A}$ over an algebraically closed field $\mathfrak{F}$ and adjoin a unity quantity e to $\mathfrak{A}$ to obtain an algebra $\mathfrak{R}=\mathfrak{A}+e\mathfrak{F}=\mathfrak{A}+v\mathfrak{F}$ where $v=e-u$. Clearly $\mathfrak{A}_u(1/2)\neq 0$ and $\mathfrak{R}_u(0)=\mathfrak{A}_u(0)+v\mathfrak{F}$ where $\mathfrak{A}_u(0)$ is a nilpotent Jordan algebra. Since every nonzero ideal of $\mathfrak{R}$ contains $\mathfrak{A}$ we may again apply the argument above to see that some $\mathfrak{R}_{ij}\neq 0$ for every $i=1, \cdots, t-1$ where we have written $u=e_1+\cdots+e_{t-1}$, $v=e_t$. Then $\mathfrak{R}_{ii}=e_i\mathfrak{F}+\mathfrak{S}_i$. We again apply Theorem 5 and see that every $\mathfrak{S}_i=0$, $\mathfrak{R}_{ii}=e_i\mathfrak{F}$. Note that in this case $\mathfrak{A}_u(0)=0$.

By Lemma 9 the quantities of $\mathfrak{A}_u(1/2)$ are all nilpotent. Then if $x=x_{it}$, we have $x^2=\alpha(e_i+e_t)$ by Lemma 10. But then the nilpotency of x implies that $\alpha=0$, $x^2=0$, $2xy=(x+y)^2-x^2-y^2=0$ for every x and y of $\mathfrak{R}_{it}$ where $i=1, \cdots, t-1$.

We now observe that the quantities of $\mathfrak{R}$ are uniquely expressible in the form $x=\xi_1e_1+\cdots+\xi_te_t+\sum_{i<j}x_{ij}$ in both of our cases. Define the function $\tau(x)=\xi_1+\cdots+\xi_t$ and $\tau(x, y)=\tau(xy)$. Since $\mathfrak{R}$ is commutative, we have $\tau(x, y)=\tau(y, x)$. The function $\tau(x, y)$ is a bilinear function and the property $\tau(x, yz)=\tau(xy, z)$ will follow if proved for components in subspaces $\mathfrak{R}_{ij}$. Since $\mathfrak{R}_{ij}\mathfrak{R}_{jk}=\mathfrak{R}_{ik}$ and $\tau(x_{ik})=0$ for $i\neq k$, it is evident that $\tau(x, yz)=\tau(xy, z)=0$ unless all three of x, y, z are scalar multiples of e_i or $x=x_{ii}$, $y=y_{ij}$, $z=z_{ij}$ or $x=x_{ij}$, $y=y_{jk}$, $z=z_{ki}$. The first case is trivial. If $x=x_{ii}$, then $x=\xi e_i$, $x(yz)=\xi e_i(y_{ij}z_{ij})=\xi e_i(\gamma e_i+\gamma e_j)=\gamma\xi e_i$ and $(xy)z=1/2\ \xi y_{ij}z_{ij}=1/2\ \gamma\xi(e_i+e_j)$ so that $\tau(xy, z)=\gamma\xi=\tau(x, yz)$. Also $x_{ij}y_{jk}=w_{ik}$, $(x_{ij}y_{jk})z_{ki}=w_{ik}z_{ki}=\alpha(e_i+e_k)$. Define $y_{jk}z_{ki}=a_{ji}$, $x_{ij}a_{ji}=\beta(e_i+e_j)$. Apply (26) for the idempotent $g=e_j+e_k$ and $y=y_{jk}=y_\lambda$, $x=x_{ij}=x_{1/2}$, $z=z_{ki}=z_{1/2}$ to obtain $e_i[(x_{ij}y_{jk})z_{ki}]=\alpha e_i=e_i[x_{ij}(y_{jk}z_{ki})]=\beta e_i$. Then $\tau(xy, z)=2\alpha=2\beta=\tau(x, yz)$.

The properties just obtained imply that *the set of all quantities x of $\mathfrak{R}$ such that $\tau(xy)=0$ for every y of $\mathfrak{R}$ is an ideal of* $\mathfrak{R}$. Since $\tau(e_ie_i)=\tau(e_i)=1$, this ideal contains no one of the idempotents e_i. But in both of our cases every ideal of $\mathfrak{R}$ contains $e_1+\cdots+e_{t-1}$. We state our result as

LEMMA 12. *Let $\mathfrak{N}$ be the ideal of all quantities x of $\mathfrak{R}$ such that $\tau(xy)=0$ for every y of $\mathfrak{R}$. Then $\mathfrak{N}=0$.*

We shall now apply the result just obtained in the case where $\mathfrak{A}$ was assumed to have no unity quantity. We have already seen that $\mathfrak{R}_{it}\mathfrak{R}_{it}=0$ and so $\tau(x_{it}y_{it})=0$ for every y_{it} of $\mathfrak{R}_{it}$ where x_{it} is in $\mathfrak{R}_{it}$. Now $\mathfrak{R}_{it}\mathfrak{R}_{jt}\subseteq\mathfrak{R}_{ij}$ and so $\tau(x_{it}y_{jt})=0$. But then $\tau(xy)=0$ for every x and y of $\mathfrak{A}_u(1/2)$. Since $\mathfrak{R}_u(1/2)=\mathfrak{A}_u(1/2)$ and $\mathfrak{R}_u(1)\mathfrak{R}_u(1/2)\subseteq\mathfrak{R}_u(1/2)$, $\mathfrak{R}_u(0)\mathfrak{R}_u(1/2)\subseteq\mathfrak{R}_u(1/2)$, we see that $\tau(xy)=0$ for every y of $\mathfrak{R}$ if x is in $\mathfrak{R}_u(1/2)$. By Lemma 12 we have $\mathfrak{A}_u(1/2)=0$, a contradiction.

We have now proved that *every simple Jordan algebra $\mathfrak{A}$ has a unity quantity e*, and that there exists a scalar extension $\mathfrak{K}$ of the center of $\mathfrak{A}$ such that $e=e_1+\cdots+e_t$ for pairwise orthogonal idempotents e_i of $\mathfrak{B}=\mathfrak{A}_{\mathfrak{K}}$ such that $\mathfrak{B}_{e_i}(1)=e_i\mathfrak{K}$. Assume now that $\mathfrak{K}=\mathfrak{F}$ and so $\mathfrak{B}=\mathfrak{A}$. We then prove

LEMMA 13. *The subalgebras $\mathfrak{D}_{ij}=\mathfrak{A}_{ii}+\mathfrak{A}_{ij}+\mathfrak{A}_{jj}$ are all simple algebras of the same order $s+2$.*

For let $\mathfrak{G}$ be an ideal of $\mathfrak{D}_{ij}$. If $g=e_i+e_j+g_{ij}$ is a nonzero quantity of $\mathfrak{G}$, the component g_{ij} is also in $\mathfrak{G}$. Suppose first that there exists a quantity a_{ij} in $\mathfrak{A}_{ij}$ such that $g_{ij}a_{ij}\neq 0$. Then $g_{ij}a_{ij}=\gamma(e_i+e_j)$ and $\mathfrak{G}$ contains e_i+e_j and so $\mathfrak{G}=\mathfrak{D}_{ij}$. Otherwise $g_{ij}a_{ij}=0$ for every a_{ij} of $\mathfrak{A}_{ij}$, $\tau(g_{ij}a_{ij})=0$ and it is easily seen that $\tau(g_{ij}a)=0$ for every a of $\mathfrak{A}$. By Lemma 12 we see that $g_{ij}=0$. It follows that $g=\alpha e_i+\beta e_j$ and that $\mathfrak{G}$ contains $e_ig=\alpha e_i$ and $e_jg=\beta e_j$. In either case $\mathfrak{G}$ contains $a_{ij}=2e_ia_{ij}$, $\mathfrak{G}\supseteq\mathfrak{A}_{ij}$, $\mathfrak{A}_{ij}=0$. If every $\mathfrak{A}_{ik}=0$, then $\mathfrak{A}_{e_i}(1/2)=0$, $\mathfrak{A}=\mathfrak{A}_{e_i}(1)\oplus\mathfrak{A}_{e_i}(0)$ contrary to our hypothesis that $\mathfrak{A}$ is simple. It follows that there exists an integer $k\neq i, j$ such that $\mathfrak{A}_{ik}\neq 0$. If $a_{ik}^2=0$ for every a_{ik} of $\mathfrak{A}_{ik}$, then $a_{ik}b_{ik}=0$ for every a_{ik} and b_{ik} of $\mathfrak{A}_{ik}$ and it is again true that $\tau(a_{ik}a)=0$ for every a of $\mathfrak{A}$, $\mathfrak{A}_{ik}=0$ by Lemma 12. Hence there exists a quantity e_{ik} in $\mathfrak{A}_{ik}$ such that $e_{ik}^2\neq 0$ and we may assume that $e_{ik}^2=e_i+e_k$.

We now let g be any integer for which $\mathfrak{A}_{ig}=0$. We substitute $x=y=e_{ik}$, $z=a_{gk}$, $w=e_k$ in (27) to obtain $1/2\, e_{ik}^2a_{gk}+e_{ik}(e_{ik}a_{gk})=1/2\, a_{gk}+2[e_k(e_{ik}a_{gk})]e_{ik}$. But $e_{ik}a_{gk}$ is in $\mathfrak{A}_{ig}=0$ and so $a_{gk}=0$, that is $\mathfrak{A}_{gk}=0$. We define v to be the sum of all idempotents e_g for which $\mathfrak{A}_{ig}=0$ and u to be the sum of all the idempotents e_k for which $\mathfrak{A}_{ik}\neq 0$. Clearly e_i is one of the components of u and e_j is one of the components of v. Moreover every $\mathfrak{A}_{gk}=0$. But then $\mathfrak{A}_u(1/2)=0$, $\mathfrak{A}=\mathfrak{A}_u(1)\oplus\mathfrak{A}_u(0)$, a contradiction. Hence $\mathfrak{A}_{ij}\neq 0$, $\mathfrak{G}=\mathfrak{D}_{ij}$ is simple. We have actually proved that if $\mathfrak{A}$ is simple, every $\mathfrak{A}_{ij}\neq 0$ and it is known[12] that the spaces $\mathfrak{A}_{ij}$ all have the same dimension s.

The result above is now adequate for a classification of simple Jordan algebras of degree $t>2$ as in the case of algebras of characteristic zero and

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[12] This is the result of Lemma 18 of JA2. The basic properties are actually contained in the proof of the present lemma.

without a word of change([13]). We call the type of algebras so obtained the *classical* Jordan algebras and collect our results as follows:

THEOREM 6. *Every simple Jordan algebra of degree $t \geqq 2$ over a center $\mathfrak{F}$ of characteristic not two, three, or five*([14]) *is a classical Jordan algebra. Every simple power-associative commutative algebra over $\mathfrak{F}$, which is not a nilalgebra, has a unity element and is either a classical Jordan algebra or has degree $t=1$ or* 2.

It is not yet known whether there exist commutative power-associative algebras which are nilalgebras but are not either solvable or nilpotent. It is also not known whether Theorem 2 can be extended to arbitrary power-associative commutative algebras, but we shall do so for algebras of characteristic zero. It also seems to be exceedingly difficult to determine the structure of simple power-associative algebras of degree two.

12. **The general structure theory.** Let $\mathfrak{A}$ be any power-associative commutative algebras over $\mathfrak{F}$ of characteristic prime to 30. If $\mathfrak{B}$ and $\mathfrak{C}$ are nilideals of $\mathfrak{A}$, their sum $\mathfrak{B}+\mathfrak{C}$ is also a nilideal of $\mathfrak{A}$. It follows that $\mathfrak{A}$ has a maximal nilideal $\mathfrak{N}$ which we call the *radical* of $\mathfrak{A}$. We call $\mathfrak{A}$ *semisimple* if $\mathfrak{N}=0$ and it is easy to prove that $\mathfrak{A}-\mathfrak{N}$ is semisimple.

Let us note that if $\mathfrak{A}$ has a unity quantity e, there is no principal idempotent of $\mathfrak{A}$ except e. Indeed if f is an idempotent, so is $e-f$ and $f(e-f)=0$, $e \neq f$, implies that f is not principal.

THEOREM 7. *Let e be a principal idempotent of a commutative power-associative algebra $\mathfrak{A}$ of characteristic not* 2, 3, *or* 5. *Then $\mathfrak{A}_e(1/2)+\mathfrak{A}_e(0)$ is contained in the radical of $\mathfrak{A}$.*

The theorem is trivial for algebras of order $m=1$. Assume it true for all algebras $\mathfrak{B}$ of order $m<n$ where n is the order of $\mathfrak{A}$. If $\mathfrak{A}$ is not semisimple, we form $\mathfrak{B}=\mathfrak{A}-\mathfrak{N}$ and see that $\mathfrak{B}$ has order $m<n$. The homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ sends e onto an idempotent u of $\mathfrak{B}$ and $\mathfrak{A}_e(\lambda)$ onto $\mathfrak{B}_u(\lambda)$. Hence u is principal in $\mathfrak{B}$. But $\mathfrak{B}$ is semisimple and the hypothesis of our induction implies that $\mathfrak{B}_u(1/2)+\mathfrak{B}_u(0)=0$, $\mathfrak{A}_e(1/2)+\mathfrak{A}_e(0)\subseteq\mathfrak{N}$ as desired. There remains the case where $\mathfrak{A}$ is semisimple.

If $\mathfrak{A}$ is simple it has a unity quantity e by Theorem 6. Since e is the only principal idempotent of $\mathfrak{A}$ we have $\mathfrak{A}_e(1/2)=\mathfrak{A}_e(0)=0$ and our result is true. Assume then that $\mathfrak{A}$ contains a nonzero ideal $\mathfrak{D}\neq\mathfrak{A}$. Since $\mathfrak{A}$ is semisimple $\mathfrak{D}$ is not a nilideal of $\mathfrak{A}$ and must contain a principal idempotent e. Write $\mathfrak{D}$

([13]) All of the material on pages 562–567 of JA2 is valid without change.

([14]) The theorems of this paper are not true for algebras of characteristic two but are probably true for algebras of characteristic three and five. The proofs will involve the use of the associativity of fifth and sixth powers in addition to the associativity of fourth powers (which is all that has been utilized here). The question is being studied as the subject of a Ph.D. dissertation at the University of Chicago.

$=\mathfrak{D}_e(1)+\mathfrak{D}_e(1/2)+\mathfrak{D}_e(0)$ and let $\mathfrak{M}$ be the radical of $\mathfrak{D}$. By the hypothesis of our induction

$$\mathfrak{D}_e(1/2) + \mathfrak{D}_e(0) \subseteq \mathfrak{M}.$$

We may also write $\mathfrak{A}=\mathfrak{A}_e(1)+\mathfrak{A}_e(1/2)+\mathfrak{A}_e(0)$ and it should be evident that $\mathfrak{D}_e(\lambda)$ is the intersection of $\mathfrak{D}$ and $\mathfrak{A}_e(\lambda)$. However $xe=\lambda x$ is in $\mathfrak{D}$ for every x of $\mathfrak{A}$ and so $\mathfrak{A}_e(1)+\mathfrak{A}_e(1/2)\subseteq\mathfrak{D}$,

$$\mathfrak{A} = \mathfrak{D}_e(1) + \mathfrak{D}_e(1/2) + \mathfrak{A}_e(0).$$

Moreover $\mathfrak{D}_e(0)\mathfrak{A}_e(0)\subseteq\mathfrak{D}$ as well as in $\mathfrak{A}_e(0)$ since $\mathfrak{A}_e(0)$ is an algebra. It follows that

$$\mathfrak{D}_e(0)\mathfrak{A}_e(0) \subseteq \mathfrak{D}_e(0).$$

We suppose now that x_0 is in $\mathfrak{A}_e(0)$ and that $y_{1/2}$ is in $\mathfrak{D}_e(1/2)$. Use (1) with $x=z=x_0$, $y=w=y_{1/2}$ to obtain

$$\begin{aligned} 4(x_0y_{1/2})^2 + 2x_0^2y_{1/2}^2 = {} & x_0[x_0y_{1/2}^2 + 2(x_0y_{1/2})y_{1/2}] \\ & + y_{1/2}[y_{1/2}x_0^2 + 2x_0(x_0y_{1/2})]. \end{aligned}$$

Since $y_{1/2}$ is in $\mathfrak{M}$ so is $y_{1/2}^2=z_0+z_1$. Thus z_0 is in $\mathfrak{D}_e(0)$ and so $2x_0^2y_{1/2}^2=2x_0^2z_0$ is in $\mathfrak{M}$. Also $y_{1/2}x_0^2$ is in $\mathfrak{A}_e(1)+\mathfrak{A}_e(1/2)=\mathfrak{D}$, and since $y_{1/2}$ is in $\mathfrak{M}$ the product $y_{1/2}(y_{1/2}x_0^2)$ is in $\mathfrak{M}$. Write

$$x_0y_{1/2} = b_1 + b_{1/2}, \qquad b_1 = y_{1/2}T_1(x_0).$$

Then $x_0(x_0y_{1/2})=x_0b_{1/2}=c_{1/2}+c_1$ is in $\mathfrak{D}$ and $y_{1/2}[x_0(x_0y_{1/2})]=y_{1/2}(c_{1/2}+c_1)$ is in $\mathfrak{M}$. We now compute $(x_0y_{1/2})y_{1/2}=(b_1+b_{1/2})y_{1/2}=b_{1/2}y_{1/2}+y_{1/2}S_{1/2}(b_1)+y_{1/2}S_0(b_1)$. Since $y_{1/2}$ is in $\mathfrak{M}$ we know that $y_{1/2}S_0(b_1)$ is in $\mathfrak{D}_e(0)$, $b_{1/2}y_{1/2}=g_1+g_0$ where g_0 is in $D_e(0)$. Thus $[b_{1/2}y_{1/2}+y_{1/2}S_0(b)]x_0$ is in $\mathfrak{M}$. We have proved that

$$w = 4(x_0y_{1/2})^2 - 2x_0[y_{1/2}S_{1/2}(b_1)]$$

is in $\mathfrak{M}$. However $y_{1/2}S_{1/2}(b_1)T_{1/2}(x_0)$ is in $\mathfrak{D}_e(1/2)$ and is in $\mathfrak{M}$. By (8) we know that $2y_{1/2}S_{1/2}(b_1)T_1(x_0)=[y_{1/2}T_1(x_0)]b_1=b_1^2$ and $(x_0y_{1/2})^2=b_1^2+2b_1b_{1/2}+b_{1/2}^2$ where we already know that $2b_1b_{1/2}+b_{1/2}^2$ is in $\mathfrak{M}$. Thus $4b_1^2-b_1^2$ is in $\mathfrak{M}$ and so b_1^2 is in $\mathfrak{M}$.

Define $\mathfrak{S}(x_0)$ to be the set of all quantities of $\mathfrak{D}_e(1)$ of the form $z_{1/2}T_1(x_0)$ for any fixed x_0 of $\mathfrak{A}_e(0)$ and $z_{1/2}$ ranging over all quantities of $\mathfrak{D}_e(1/2)$. By (8) we know that $[z_{1/2}T_1(x_0)]y_1=2[z_{1/2}S_{1/2}(y_1)]T_1(x_0)$ and so $\mathfrak{S}(x_0)$ is an ideal of $\mathfrak{D}_e(1)$. Form $\mathfrak{D}-\mathfrak{M}$ and see that the homomorphic mapping $d\rightarrow d'$ of $\mathfrak{D}$ onto the semisimple algebra $\mathfrak{D}'=\mathfrak{D}-\mathfrak{M}$ is such that every d' is the image of an element d_1 of $\mathfrak{D}_e(1)$. Then the image of $\mathfrak{S}(x_0)$ is an ideal of $\mathfrak{S}'(x_0)$ of $\mathfrak{D}'$. But we have shown that if b_1 is in $\mathfrak{S}(x_0)$ then b_1^2 is in $\mathfrak{M}$, $(b_1')^2=0$ and so $\mathfrak{S}'(x_0)=0$, $\mathfrak{S}(x_0)\subseteq\mathfrak{M}$. Thus $x_0y_{1/2}$ is in $\mathfrak{M}$, that is. $\mathfrak{A}_e(0)\mathfrak{D}_e(1/2)=\mathfrak{M}$.

We already know that $\mathfrak{M}\mathfrak{A}_e(1)=\mathfrak{M}\mathfrak{D}_e(1)\subseteq\mathfrak{M}$, $\mathfrak{M}\mathfrak{A}_e(1/2)=\mathfrak{M}\mathfrak{D}_e(1/2)\subseteq\mathfrak{M}$. Also $\mathfrak{M}=\mathfrak{M}_1+\mathfrak{D}_e(1/2)+\mathfrak{D}_e(0)$ where $\mathfrak{M}_1$ is the intersection of $\mathfrak{M}$ and $\mathfrak{D}_e(1)$, $\mathfrak{M}\mathfrak{A}_e(0)=\mathfrak{D}_e(1/2)\mathfrak{A}_e(0)+\mathfrak{D}_e(0)\mathfrak{A}_e(0)\subseteq\mathfrak{M}$ by the proof above. We have proved that $\mathfrak{M}$ is a nilideal of $\mathfrak{A}$ contrary to the hypothesis that $\mathfrak{A}$ is semisimple. It follows that $\mathfrak{M}=0$, $\mathfrak{D}=\mathfrak{D}_e(1)$ is a semisimple algebra with the unity quantity e, $\mathfrak{D}_e(1/2)=0$, $\mathfrak{A}=\mathfrak{D}\oplus\mathfrak{A}_e(0)$. But then $\mathfrak{A}_e(0)$ is an ideal of $\mathfrak{A}$ and must have a unity quantity u by the proof just completed. The idempotent $f=e+u$ is the unity quantity of $\mathfrak{A}$ and is the only principal idempotent of $\mathfrak{A}$, $\mathfrak{A}_f(1/2)\neq\mathfrak{A}_f(0)=0$ is contained in the radical of $\mathfrak{A}$. This completes our induction and proves the theorem.

The theorem just proved implies that if $\mathfrak{A}$ is semisimple, it has a unity element. Moreover the proof above implies that if $\mathfrak{D}$ is an ideal of $\mathfrak{A}$, it is semisimple and has a unity quantity e, $\mathfrak{A}=\mathfrak{D}\oplus\mathfrak{A}_e(0)$. We thus have the decomposition of the following theorem. The uniqueness is well known.

THEOREM 8. *Every semisimple power-associative commutative algebra of characteristic not two, three, or five has a unity quantity and is uniquely expressible as a direct sum of simple algebras.*

13. Simple algebras of degree one and characteristic zero. We shall close our discussion with a proof of the following generalization of Theorem 5 for the characteristic zero case.

THEOREM 9. *Let $\mathfrak{A}$ be a simple power-associative commutative algebra of degree one over a center $\mathfrak{F}$ of characteristic zero. Then $\mathfrak{A}=e\mathfrak{F}$ where e is the unity quantity of $\mathfrak{A}$.*

We shall first prove the following elementary property.

LEMMA 14. *Let e be a primitive idempotent of a commutative power-associative algebra $\mathfrak{A}$ over an infinite field $\mathfrak{F}$ and let $\mathfrak{K}=\mathfrak{F}(\xi_1,\cdots,\xi_r)$ for independent indeterminates $\xi_1,\cdots,\xi_r$ over $\mathfrak{F}$. Then e is a primitive idempotent of $\mathfrak{A}_{\mathfrak{K}}$.*

It is clearly sufficient to prove the lemma for $r=1$ and $\mathfrak{K}=\mathfrak{F}(\xi)$. Let $\mathfrak{J}=\mathfrak{F}[\xi]$ be the ring of polynomials in ξ and the unity quantity e of $\mathfrak{A}$ and let $\mathfrak{B}=\mathfrak{A}_e(1)$. If $\mathfrak{B}_{\mathfrak{K}}$ contains an idempotent $u\neq e$ we may write $u=\phi^{-1}v$ for ϕ in $\mathfrak{J}$ and v in $\mathfrak{B}_{\mathfrak{J}}$. Let $u_1,\cdots,u_q$ form a basis of $\mathfrak{B}$ over $\mathfrak{F}$ such that $u_1=e$ and so write $v=\zeta_1u_1+\cdots+\zeta_mu_m$ for ζ_i in $\mathfrak{J}$. At least one $\zeta_i\neq0$ for $i>1$ since otherwise $u=\phi^{-1}\zeta_1e=\phi^{-2}\zeta_1^2e$ and $(\phi^{-1}\zeta_1)^2=(\phi^{-1}\zeta_1)$, $\phi^{-1}\zeta_1\neq0$, $\phi^{-1}\zeta_1=1$, $u=e$, a contradiction. Then there exists a quantity ξ_0 in $\mathfrak{F}$ such that $\phi(\xi_0)\zeta_i(\xi_0)\neq0$. But the quantity $u_0=[\phi(\xi_0)]^{-1}v(\xi_0)\neq0$ and u_0 is evidently an idempotent of $\mathfrak{B}$. This contradicts our hypothesis that e is a primitive idempotent of $\mathfrak{A}$.

We now assume that $\mathfrak{A}$ is simple and of degree one. By Theorem 6, $\mathfrak{A}$ has a unity quantity e and the hypothesis that $\mathfrak{A}$ has degree one implies that e is absolutely primitive over the center $\mathfrak{F}$ of $\mathfrak{A}$. We may assume that $\mathfrak{F}$ is absolutely closed without loss of generality. The algebra $\mathfrak{F}[a]$ of polynomials in a (but not in e) is an associative algebra and so $\mathfrak{F}[a]=\mathfrak{C}+\mathfrak{N}_a$ where $\mathfrak{N}_a$ is the radical of $\mathfrak{F}[a]$ and $\mathfrak{C}$ is semisimple. Since $\mathfrak{A}$ is commutative $\mathfrak{C}=e\mathfrak{F}$, $a=\alpha e+b$ where α is in $\mathfrak{F}$ and b is nilpotent.

Let $u_1=e, u_2, \cdots, u_n$ be a basis of $\mathfrak{A}$ over $\mathfrak{F}$ and select the u_i to be nilpotent in the case $n>1$. Let $\xi_1, \cdots, \xi_n$ be independent indeterminates over $\mathfrak{F}$, $\mathfrak{K}=\mathfrak{F}(\xi_1, \cdots, \xi_n)$, $\mathfrak{J}=\mathfrak{F}[\xi_1, \cdots, \xi_n]$. By Lemma 14 we see that e is a primitive idempotent of $\mathfrak{A}_{\mathfrak{K}}$. Write $x=\xi_1u_1+\cdots+\xi_nu_n$ and see that $\mathfrak{K}[x]=\mathfrak{D}+\mathfrak{N}_x$ where $\mathfrak{N}_x$ is the radical of $\mathfrak{K}[x]$ and $\mathfrak{D}$ is semisimple. Since e is primitive, $\mathfrak{D}$ is a field, $\mathfrak{D}=\mathfrak{K}(y)$ where we may assume that y is in $\mathfrak{J}[x]$. If $\mathfrak{D}$ has degree $s>1$ over $\mathfrak{K}$ the quantity y is a root of an irreducible polynomial $\phi(\lambda)$ with coefficients in $\mathfrak{J}$, leading coefficient unity, and degree s. Write $y=\zeta_1u_1+\cdots+\zeta_nu_n$ with ζ_i in $\mathfrak{J}$. Evidently some $\zeta_i\neq 0$ and we may assume that $\zeta_2\neq 0$. The discriminant of $\phi(\lambda)$ is a polynomial $\Delta(\xi_1, \cdots, \xi_n)\neq 0$ since $\phi(\lambda)$ is irreducible and there exist values $\xi_{10}, \cdots, \xi_{n0}$ in $\mathfrak{F}$ such that $\pi(\xi_{10}, \cdots, \xi_{n0})\neq 0$ where $\pi=\zeta_2\Delta$. The corresponding element $y_0=y(\xi_{10}, \cdots, \xi_{n0})$ is not in $e\mathfrak{F}$ and is a root of $\phi_0(\lambda)=\phi(\lambda; \xi_{10}, \cdots, \xi_{n0})=0$. Evidently $\phi_0(\lambda)$ has discriminant $\Delta(\xi_{10}, \cdots, \xi_{n0})\neq 0$ and distinct roots. But $y_0-\beta e$ is nilpotent for some β in $\mathfrak{F}$, the minimum function of y_0 divides both $(\lambda-\beta)^n$ and $\phi_0(\lambda)$ which is possible only if $y_0=\beta e$, a contradiction. Hence $s=1$, $\mathfrak{D}=e\mathfrak{K}$, $x-e\psi(\xi_1, \cdots, \xi_n)$ is nilpotent for some $\psi(\xi_1, \cdots, \xi_n)$ in $\mathfrak{K}$.

The quantity x of $\mathfrak{A}_{\mathfrak{K}}$ is a root of $f(\lambda)=|\lambda I-R_x|=0$ where R_x is the linear transformation $a\rightarrow ax$. The polynomial $f(\lambda)$ is homogeneous of degree n in $\xi_1, \cdots, \xi_n$, and is divisible by the minimum function $g(\lambda)$ of x. But then $g(\lambda)=g(\lambda; \xi_1, \cdots, \xi_n)$ is a polynomial in λ with leading coefficient unity and which is homogeneous in $\lambda, \xi_1, \cdots, \xi_n$. However $g(\lambda)$ divides $[\lambda-\psi(\xi_1, \cdots, \xi_n)]^n$ and so $\lambda-\psi(\xi_1, \cdots, \xi_n)$ divides $g(\lambda)$, $\psi(\xi_1, \cdots, \xi_n)$ is in $\mathfrak{J}$, ψ must be a linear homogeneous function in $\xi_1, \cdots, \xi_n$. Indeed $g(\lambda)=(\lambda-\psi)^r$ where r is the degree of $g(\lambda)$, ψ^r has degree r. Hence we may write $\psi=\gamma_1\xi_1+\cdots+\gamma_n\xi_n$ for γ_i in $\mathfrak{F}$, and write $\psi(a)=\gamma_1\xi_{10}+\cdots+\gamma_n\xi_{n0}$ where $a=\xi_{10}u_1+\cdots+\xi_{n0}u_n$. Then $u_i-e\psi(u_i)$ is nilpotent and so is u_i for $i>1$. It follows that $\psi(u_i)=\gamma_i=0$ for $i>1$, $\psi(u_1)=\gamma_1=1$, $\psi(\xi_1, \cdots, \xi_n)=\xi_1$. Then all quantities of $\mathfrak{N}=u_2\mathfrak{F}+\cdots+u_n\mathfrak{F}$ are nilpotent and every nilpotent quantity of $\mathfrak{A}$ is in $\mathfrak{N}$. If b and c are in $\mathfrak{N}$ so are $b+c$, b^2, c^2, $(b+c)^2$, $2bc=(b+c)^2-b^2-c^2$, $\mathfrak{N}$ is a subalgebra of $\mathfrak{A}$, $\mathfrak{N}$ is a proper ideal of $\mathfrak{A}$. But $\mathfrak{A}$ was assumed to be simple and so $\mathfrak{N}=0$, that is, $n=1$, $\mathfrak{A}=e\mathfrak{F}$ as desired.

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