ON SOME HIGH INDICES THEOREMS

BY

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1. A series

$$\sum_{n=1}^{\infty} c_n \tag{1.1}$$

is said to be lacunary if all its terms are zero, except perhaps for a set of indices

$$0 < n_1 < n_2 < \cdots$$

which satisfy the condition

$$n_{i+1}/n_i \geqq q > 1, \qquad i = 1, 2, \cdots.$$

The Hardy-Littlewood "high indices" theorem [2](1) asserts that for a lacunary series Abel summability implies convergence. Several proofs of this theorem have since been given, and of these the proof of Ingham [1] is of particular interest to us.

Instead of considering the power series

$$\phi(\rho) = \sum_{n=1}^{\infty} c_n \rho^n, \qquad 0 \leqq \rho < 1, \tag{1.2}$$

it is useful to make the substitution

$$\rho = e^{-s}$$

and, letting $a_i = c_{n_i}$, transform the series (1.2) into a Dirichlet series

$$\phi(e^{-s}) = \sum_{i=1}^{\infty} a_i e^{-n_i s}.$$

A more general Dirichlet series will be considered here, namely

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{1.3}$$

where the λ_n's satisfy the condition

$$\lambda_{n+1}/\lambda_n \geqq q > 1, \qquad 0 < \lambda_1 < \lambda_2 < \cdots. \tag{1.4}$$

The sequence $\{\lambda_n\}$ need not be a sequence of integers.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

In the case where $c_n \to 0$ the convergence of (1.1) is implied by its Abel summability as a special case of a well known theorem of Landau [3]. The principal difficulty in the proof of the Hardy-Littlewood theorem is in showing that for a lacunary series (1.1), Abel summability implies $c_n \to 0$. This is an easily obtainable consequence of a result stated by Ingham, namely:

If $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ *and the* λ_n*'s satisfy* (1.4),

then

$$| a_n | \leqq A_q \sup | f(s) |$$

for all n, A_q *being a function of* q *only, finite for all* $q > 1$ *and bounded as* $q \to \infty$.

More recently Zygmund [5] has shown that if we consider a series (1.1) such that the infinite series (1.2) converges for $\rho \in [0, 1]$ and $\phi(\rho)$ is of bounded variation over $[0, 1]$, then lacunarity of this series would imply its absolute convergence.

To demonstrate this theorem the inequality

$$\sum_{n=1}^{\infty} | c_n | \leqq A_q V(\phi; 0, 1) \tag{1.5}$$

was proved, where $V(\phi; 0, 1)$ denotes the variation of $\phi(\rho)$ over $[0, 1]$. The inequality which was actually obtained by Zygmund was

$$\sum_{n=1}^{\infty} | a_n | \leqq A_q \int_0^{\infty} | f'(s) | \, ds. \tag{1.6}$$

Clearly the result of Ingham may be put in the form

$$| a_n | \cdot \lambda_n \leqq A_q \sup | f'(s) |. \tag{1.7}$$

Then the inequalities (1.6) and (1.7) are seen to be the extreme cases, corresponding to $m = 1$ and $m = \infty$, of the following inequality

$$\sum_{n=1}^{\infty} | a_n |^m \lambda_n^{m-1} \leqq A_{q,m} \int_0^{\infty} | f'(s) |^m ds. \tag{1.8}$$

We shall establish this inequality.

THEOREM 1. *If the series* (1.3) *is lacunary, the sequence* $\{\lambda_n\}$ *satisfying* (1.4), $m > 1$, *and*

$$\int_0^{\infty} | f'(s) |^m ds < \infty, \tag{1.9}$$

then

$$\sum_{n=1}^{\infty} | a_n |^m \lambda_n^{m-1} \leqq A_{q,m} \int_0^{\infty} | f'(s) |^m ds \tag{1.10}$$

where $A_{q,m}$ *is a constant depending on* q *and* m *alone.*

The proof of this theorem utilizes the ideas of Ingham's proof of the "high indices" theorem and of Zygmund's proof of his theorem on absolute Abel summability.

We first prove (1.10) for Dirichlet polynomials.

LEMMA. *If* $0<\lambda_1<\lambda_2<\cdots$, $\lambda_{k+1}/\lambda_k \geqq q>1$ *and* $m>1$, *then there exists* $A_{q,m}$ *depending only on* q *and* m *such that if*

$$f(s) = \sum_{k=1}^{N} a_k e^{-\lambda_k s},$$

then

$$\sum_{k=1}^{N} |a_k|^m \lambda_k^{m-1} \leqq A_{q,m} \int_0^\infty |f'(s)|^m ds. \tag{1.11}$$

If

$$P(s) = \sum_l p_l e^{-\mu_l s}, \qquad 0<\mu_1<\mu_2<\cdots, \tag{1.12}$$

is a finite sum, then

$$F(s) \equiv \sum_l p_l f(\mu_l s) = \sum_{k=1}^{N} a_k P(\lambda_k s). \tag{1.13}$$

We set

$$p(s) = e^{-\alpha s} - e^{-\beta s}, \qquad 0<\alpha<\beta,$$

and

$$P(s) = \{[p(\sigma s)/p(\sigma)]^r\}^R$$

where

$$\sigma = \sigma(\alpha, \beta) = \frac{1}{\beta-\alpha} \log \beta/\alpha$$

is the value of s for which $p(s)$ attains its maximum, and r and R are positive integers to be determined later. We note that $p(0)=p(\infty)=0$ and that $p(s)$ decreases steadily as we go either to the right or the left from $s=\sigma$. Also for s outside any neighbourhood of $s=1$, $|P'(s)|<1$ if r is chosen sufficiently large. Clearly

$$\sum_l |p_l| = 2^{Rr}/p^{Rr}(\sigma) \tag{1.14}$$

and

$$\max_l \mu_l = R r\sigma\beta. \tag{1.15}$$

Let I_k denote the interval $(q^{-1/2}\lambda_k^{-1},\ q^{1/2}\lambda_k^{-1})$. These intervals are disjunct. Also for r sufficiently large and $n\neq k$,

$$\lambda_n \int_{I_k} |P'(\lambda_n s)|^m ds = \int_{\lambda_n q^{-1/2}\lambda_k^{-1}}^{\lambda_n q^{1/2}\lambda_k^{-1}} |P'(s)|^m ds \leqq \int_{\lambda_n q^{-1/2}\lambda_k^{-1}}^{\lambda_n q^{1/2}\lambda_k^{-1}} |P'(s)|\, ds \tag{1.16}$$
$$\leqq P(\lambda_n q^{\pm 1/2}\lambda_k^{-1}),$$

the sign of the exponent of q being plus or minus according as k is greater than or less than n, for the variation of $P(s)$ over the interval $(\lambda_n q^{-1/2}\lambda_k^{-1}$, $\lambda_n q^{1/2}\lambda_k^{-1})$ is less than the largest value of the function in this interval.

By applying Minkowski's inequality to (1.12), we have

$$\left\{\int_0^\infty |F'(s)|^m ds\right\}^{1/m} \leqq \sum_l \left\{\int_0^\infty |p_l f'(\mu_l s)\mu_l|^m ds\right\}^{1/m}$$
$$= \sum_l |p_l|\, \mu_l^{1-1/m} \left\{\int_0^\infty |f'(s)|^m ds\right\}^{1/m}.$$

Thus

$$\int_0^\infty |F'(s)|^m ds \leqq \left\{\sum_l |p_l|\, \mu_l^{1-1/m}\right\}^m \int_0^\infty |f'(s)|^m ds$$
$$< \left\{\sum_l |p_l|\right\}^m \max_l\, (\mu_l)^{m-1} \int_0^\infty |f'(s)|^m ds \tag{1.17}$$
$$= 2^{mRr} p^{-mRr}(\sigma)(Rr\sigma\beta)^{m-1} \int_0^\infty |f'(s)|^m ds.$$

We shall introduce the notation

$$J = \int_{q^{-1/2}}^{q^{1/2}} |P'(s)|^m ds \tag{1.18}$$

and

$$J_{n,k} = \int_{\lambda_n q^{-1/2}\lambda_k^{-1}}^{\lambda_n q^{1/2}\lambda_k^{-1}} |P'(s)|^m ds. \tag{1.19}$$

By again applying Minkowski's inequality to (1.12), integrating now over I_k, we obtain

$$\left\{\int_{I_k} |F'(s)|^m ds\right\}^{1/m} \geqq |a_k|\, \lambda_k^{1-1/m} J^{1/m} - \sum_{n\neq k} |a_n|\, \lambda_n^{1-1/m} J_{n,k}^{1/m}$$

or

$$\sum_{n\neq k}|a_n|\lambda_n^{1-1/m}J_{n,k}^{1/m}+\left\{\int_{I_k}|F'(s)|^m ds\right\}^{1/m}\geqq |a_k|\lambda_k^{1-1/m}J^{1/m}. \tag{1.20}$$

From (1.20) and the use of Jensen's inequality in the form

$$(a+b)^m\leqq 2^{m-1}(a^m+b^m)$$

we have

$$\left[\sum_{n\neq k}|a_n|\lambda_n^{1-1/m}J_{n,k}^{1/m}\right]^m+\int_{I_k}|F'(s)|^m ds\geqq 1/2^{m-1}|a_k|^m\lambda_k^{m-1}J. \tag{1.21}$$

Hölder's inequality gives us

$$\left[\sum_{n\neq k}|a_n|\lambda_n^{1-1/m}J_{n,k}^{1/m}\right]^m\leqq\left[\sum_{n\neq k}J_{n,k}^{1/m}\right]^{m-1}\left[\sum_{n\neq k}|a_n|^m\lambda_n^{m-1}J_{n,k}^{1/m}\right]. \tag{1.22}$$

Combining (1.21) and (1.22) and summing with respect to k we have

$$\begin{aligned}\int_0^\infty|F'(s)|^m ds\geqq\sum_{k=1}^N|a_k|^m\lambda_k^{m-1}\\ \cdot\left\{1/2^{m-1}J-\sum_{j\neq k}\left\{J_{k,j}^{1/m}\left[\sum_{n\neq j}J_{n,j}^{1/m}\right]^{m-1}\right\}\right\}.\end{aligned} \tag{1.23}$$

By choosing r sufficiently large we can obtain

$$\begin{aligned}P(s)&<s^R, & s&<q^{-1/2},\\ P(s)&<(1/s)^R, & s&>q^{1/2}.\end{aligned} \tag{1.24}$$

Using this with (1.16) and the lacunarity property in (1.23) we have

$$\begin{aligned}\int_0^\infty|F'(s)|^m ds\geqq\sum_{k=1}^N|a_k|^m\lambda_k^{m-1}\Biggl\{1/2^{m-1}J\\ -\sum_{j\neq k}\left[\left(\frac{q^{1/2}}{q^{\pm(j-k)}}\right)^{R/m}\left[\sum_{n\neq j}\left(\frac{q^{1/2}}{q^{\pm(n-j)}}\right)^{R/m}\right]^{m-1}\right]\Biggr\}\end{aligned} \tag{1.25}$$

where the signs are chosen so that $\pm(j-k)>0$, $\pm(n-j)>0$.

But

$$\begin{aligned}&\sum_{j\neq k}\left(\frac{q^{1/2}}{q^{\pm(j-k)}}\right)^{R/m}\left[\sum_{n\neq j}\left(\frac{q^{1/2}}{q^{\pm(n-j)}}\right)^{R/m}\right]^{m-1}\\ &<\left[2\sum_{i=1}^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]\left[2\sum_{i=1}^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^{m-1}=\left[2\sum_{i=1}^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^m,\end{aligned}$$

and so (1.25) becomes

$$(1.26)\qquad \int_0^\infty |F'(s)|^m ds \geqq \sum_{k=1}^N |a_k|^m \lambda_k^{m-1}\left\{\frac{1}{2^{m-1}}J - \left[2\sum_{i=1}^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^m\right\}.$$

The infinite series

$$\sum_1^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}$$

is essentially a geometric series with the term independent of the ratio removed. It behaves then as does its first term and is small when R is large. We may choose R so large that the variation of $P(s)$ is greater than one on the set of points $(q^{-1/2}, q^{+1/2})$ where $|P'(s)|>1$, and thus

$$J = \int_{q^{-1/2}}^{q^{1/2}} |P'(s)|\, ds > 1.$$

Choose R so large that

$$\left[2\sum_{i=1}^\infty\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^m < \frac{1}{2^{m-2}}\cdot$$

This in conjunction with (1.26) and (1.17) yields the lemma with

$$A_{q,m} = 2^{m-2+Rrp-mRr}(\sigma)(Rr\sigma\beta)^{m-1}.$$

We note that (1.26) could have been written with $\int_0^\infty|F'(s)|^m ds$ replaced by $\int_0^{q^{1/2}\lambda_1^{-1}}|F'(s)|^m ds$.

In order to prove Theorem 1, let I_δ denote the interval $[\delta, q^{1/2}\lambda_1^{-1}]$. Then the infinite series (1.3) differentiated term by term converges uniformly in I_δ to $f'(s)$. Therefore if we denote the Nth partial sum of (1.3) by $f_N(s)$,

$$(1.27)\qquad \int_{I_\delta}|f_N'(s)|^m ds \to \int_{I_\delta}|f'(s)|^m ds,\qquad N\to\infty,$$

and so

$$(1.28)\qquad \begin{aligned}\int_{I_\delta}|f_N'(s)|^m ds &\leqq \int_{I_\delta}|f'(s)|^m ds + \epsilon\\ &\leqq \int_0^\infty|f'(s)|^m ds + \epsilon\end{aligned}$$

for $N\geqq N_0(\epsilon, \delta, m)$.

Then by the lemma

$$(1.29)\qquad \sum_{n=1}^N |a_n e^{-\lambda_n\delta}|^m\lambda_n^{m-1} \leqq A_{q,m}\int_0^\infty |f'(s)|^m ds + \epsilon.$$

Letting $N\to\infty$, $\delta\to 0$, $\epsilon\to 0$ in that order we obtain (1.10).

2. The absolute Abel summability of Zygmund is equivalent to the finiteness of the integral

$$\int_0^1 |\phi'(\rho)|\, d\rho \tag{2.1}$$

which represents the total variation of $\phi(\rho)$ over the interval $[0, 1]$.

This integral is connected with a function which has been studied by Littlewood and Paley [4] and Zygmund [5], namely,

$$g_m(\theta, \phi) = \left\{\int_0^1 (1-\rho)^{m-1} |\phi'(\rho e^{i\theta})|^m d\rho\right\}^{1/m} \tag{2.2}$$

where ϕ is any function regular in $|z| < 1$, for clearly (2.1) is $g_1(0, \phi)$.

We shall consider the integral

$$\gamma_m(\theta, \phi) = \int_0^1 (1-\rho)^{m-1}\rho^{m-1} |\phi'(\rho e^{i\theta})|^m d\rho, \qquad m > 1, \tag{2.3}$$

where

$$\phi(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \qquad |z| < 1, \tag{2.4}$$

and the n_k's satisfy the condition $n_{k+1}/n_k \geqq q > 1$, $k = 1, 2, \cdots$. The introduction of the factor ρ^{m-1} is unimportant, for it only moderates the integrand near $z = 0$, a point at which its was already sufficiently well behaved. Its purpose is to facilitate the transformation to a Dirichlet series.

We shall prove the following theorem, a theorem about $\gamma_m(0, \phi)$, $m > 1$.

THEOREM 2. *If the series* (1.3) *is lacunary*, $m > 1$, *the sequence* $\{\lambda_n\}$ *satisfies the condition*

$$0 < h < \lambda_1 < \lambda_2 < \cdots, \qquad \lambda_{n+1}/\lambda_n \geqq q > 1, \qquad n = 1, 2, \cdots, \tag{2.5}$$

and

$$\int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds < \infty. \tag{2.6}$$

Then

$$\sum_{n=1}^{\infty} |a_n|^m \leqq A_{h,q,m} \int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds. \tag{2.7}$$

The proof of this theorem is much like that of Theorem 1. We begin by proving it for Dirichlet polynomials.

LEMMA. *If* $0 < h < \lambda_1 < \lambda_2 < \cdots$, $\lambda_{k+1}/\lambda_k \geqq q > 1$, *and* $m > 1$, *then there* *exist* $A_{h,q,m}$ *depending on h, q, and m such that if*

$$f(s) = \sum_{k=1}^{N} a_k e^{-\lambda_k s}$$

then

$$\sum_{k=1}^{N} |a_k|^m \leqq A_{h,q,m} \int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds. \tag{2.8}$$

Suppose that $P(s)$ and $F(s)$ are defined as in (1.12) and (1.13). Then by Minkowski's inequality

$$\begin{aligned}\left\{\int_0^\infty (1 - e^{-s})^{m-1} |F'(s)|^m ds\right\}^{1/m} &\leqq \sum_l \left\{\int_0^\infty |p_l^m \mu_l^m (1 - e^{-s})^{m-1} |f'(s)|^m ds\right\}^{1/m} \\ &= \sum_l |p_l| \mu_l^{1-1/m} \left\{\int_0^\infty (1 - e^{-s/\mu_l})^{m-1} |f'(s)|^m ds\right\}^{1/m} \\ &\leqq \sum_l |p_l| \mu_l^{1-1/m} \left\{\int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds\right\}^{1/m}\end{aligned}$$

since $\min_l \mu_l = Rr\sigma\alpha > 1$ for r sufficiently large. Thus

$$\begin{aligned}\int_0^\infty &(1 - e^{-s})^{m-1} |F'(s)|^m ds \\ &\leqq \left\{\sum_l |p_l| \mu_l^{1-1/m}\right\}^m \int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds \\ &\leqq \left\{\sum_l |p_l|\right\}^m \max (\mu_l)^{m-1} \int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds \\ &= 2^{mRr} p^{-mRr}(\sigma)(Rr\sigma\beta)^{m-1} \int_0^\infty (1 - e^{-s})^{m-1} |f'(s)|^m ds.\end{aligned} \tag{2.9}$$

By again applying Minkowski's inequality we also have

$$\begin{aligned}\Big\{\int_{I_k} &(1 - e^{-s})^{m-1} |F'(s)|^m ds\Big\}^{1/m} \\ &\geqq |a_k| \lambda_k^{1-1/m} \int_{q^{-1/2}}^{q^{1/2}} |P'(s)|^m (1 - e^{-s/\lambda_k})^{m-1} ds \\ &\quad - \sum_{n \neq k} |a_n| \lambda_n^{1-1/m} \left\{\int_{\lambda_n q^{-1/2}\lambda_k^{-1}}^{\lambda_n q^{1/2}\lambda_k^{-1}} |P'(s)|^m (1 - e^{s/\lambda_n})^{m-1} ds\right\}^{1/m}.\end{aligned} \tag{2.10}$$

But

$$(2.11)\qquad (1-e^{-1/\lambda_n q^{1/2}}) = -\left(-\frac{1}{\lambda_n q^{1/2}}+\frac{1}{2!}\left(\frac{1}{\lambda_n q^{1/2}}\right)^2-\cdots\right) > \frac{C_{q,h}}{\lambda_n}$$

since $\lambda_1 > h > 0$ and

$$(2.12)\qquad (1-e^{-q^{1/2}/\lambda_n}) = -\left(-\frac{q^{1/2}}{\lambda_n}+\frac{1}{2!}\left(\frac{q^{1/2}}{\lambda_n}\right)^2-\cdots\right) < \frac{q^{1/2}}{\lambda_n}.$$

Using (2.11) and (2.12) in (2.10) we have

$$(2.13)\qquad \begin{aligned}&\left\{\int_{I_k}(1-e^{-s})^{m-1}|F'(s)|^m ds\right\}^{1/m}\\ &\qquad \geqq |a_k|\, C_{q,h}^{1-1/m} J^{1/m} - \sum_{n\neq k}|a_n|\left(\frac{\lambda_n q^{1/2}}{\lambda_k}\right)^{1-1/m} J_{n,k}^{1/m}.\end{aligned}$$

By applying Jensen's inequality to (2.13) we obtain

$$(2.14)\qquad \begin{aligned}&\left[\sum_{n\neq k}|a_n|\left(\frac{\lambda_n q^{1/2}}{\lambda_k}\right)^{1-1/m} J_{n,k}^{1/m}\right]^m + \int_{I_k}(1-e^{-s})^{m-1}|F'(s)|^m ds\\ &\qquad \geqq \frac{1}{2^{m-1}}|a_k|^m C_{q,h}^{m-1} J.\end{aligned}$$

Hölder's inequality gives us

$$(2.15)\qquad \begin{aligned}&\left[\sum_{n\neq k} a_n\left[\frac{\lambda_n q^{1/2}}{\lambda_k}\right]^{1-1/m} J_{n,k}^{1/m}\right]^m\\ &\qquad \leqq \left[\sum_{n\neq k}|a_n|^m\left(\frac{\lambda_n q^{1/2}}{\lambda_k}\right)^{m-1} J_{n,k}^{1/m}\right]\left[\sum_{n\neq k} J_{n,k}^{1/m}\right]^{m-1}.\end{aligned}$$

Combining this with (2.14) and summing with respect to k, we have

$$(2.16)\qquad \begin{aligned}\int_0^\infty (1-e^{-s})^{m-1}|F'(s)|^m ds \geqq \sum_{k=1}^N |a_k|^m &\left\{\frac{1}{2^{m-1}} C_{q,h}^{m-1} J\right.\\ &\left. - \sum_{j\neq k}\left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{m-1} J_{k,j}\left(\sum_{n\neq j} J_{n,j}^{1/m}\right)^{m-1}\right\}.\end{aligned}$$

By the lacunarity property and (1.16) and (1.24) we have for $k>j$

$$(2.17)\qquad \begin{aligned}\left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{m-1} J_{k,j}^{1/m} &\leqq \left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{m-1}\left[\frac{\lambda_j q^{1/2}}{\lambda_k}\right]^{R/m}\\ &\leqq \frac{q^{1/2(R/m+m-1)}}{q^{(k-j)(R/m-(m-1))}} \leqq 1/q^{(k-j)(R/2m-3/(m-1)/2)}.\end{aligned}$$

For $k<j$ we have

$$\left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{m-1} J_{k,j}^{1/m} \leqq \left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{m-1}\left[\frac{\lambda_k q^{1/2}}{\lambda_j}\right]^{R/m}$$

$$\leqq \frac{q^{1/2(R/m+m-1)}}{q^{(j-k)(R/m+m-1)}} \tag{2.18}$$

$$< 1/q^{(j-k)(R/2m-3(m-1)/2)}.$$

Substituting these results in (2.16) and applying (1.16) and (1.24), we have

$$\int_0^\infty (1-e^{-s})^{m-1}|f'(s)|^m ds$$

$$\geqq \sum_{k=1}^{N}|a_k|^m\left\{\frac{1}{2^{m-1}}C_{q,h}J - \sum_{j\neq k}\left[\frac{1}{q^{\pm(j-k)}}\right]^{R/m-3(m-1)/2}\left[2\sum_{i=1}^{\infty}\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^{m-1}\right\} \tag{2.19}$$

$$\geqq \sum_{k=1}^{N}|a_k|^m\left\{\frac{1}{2^{m-1}}C_{q,h}J - \left[\sum_{i=1}^{\infty}\left(\frac{1}{q^i}\right)^{R/2m-3(m-1)/2}\right]\left[2\sum_{i=1}^{\infty}\left(\frac{q^{1/2}}{q^i}\right)^{R/m}\right]^{m-1}\right\}.$$

The product of the infinite series on the right of (2.19) can be made as small as we wish and J can be made greater than one by choosing R large enough. This in conjunction with (2.9) yields the lemma.

Theorem 2 may be deduced from this lemma by exactly the same argument as was used for Theorem 1.

A slight generalization of Theorem 2 can be proved and we state this as a theorem.

THEOREM 3. *If the series* (1.3) *is lacunary,* $m>1$, *the sequence* $\{\lambda_n\}$ *satisfies* (2.5), $1\leqq\beta\leqq m$, *and*

$$\int_0^\infty (1-e^{-s})^{\beta-1}|f'(s)|^m ds < \infty, \tag{2.20}$$

then

$$\sum_{n=1}^{\infty}|a_n|^m\lambda_n^{m-\beta} \leqq A_{q,m,h,\beta}\int_0^\infty (1-e^{-s})^{\beta-1}|f'(s)|^m ds. \tag{2.21}$$

The proof of this theorem is almost exactly the same as that of Theorem 2 and so need not be given here.

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