MEANS FOR THE BOUNDED FUNCTIONS AND ERGODICITY OF THE BOUNDED REPRESENTATIONS OF SEMI-GROUPS[1]

BY

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1. A mean on a semi-group Σ is a positive linear functional of norm one on the space $m(\Sigma)$ of bounded, real-valued functions on Σ. A bounded semi-group $\mathfrak{S}$ of linear operators from a Banach space B to itself is called ergodic if there exists a system $\mathcal{A}$ of averages A such that for every S in $\mathfrak{S}$ $\lim_A (AS-A)=\lim_A (SA-A)=0$; we have three strengths of ergodicity of $\mathfrak{S}$ according as uniform, strong, or weak convergence is used in the operator algebra.

The first part of this paper deals with the relationship between existence of invariant means and ergodicity of bounded representations. In Theorem 2 it is shown that weak ergodicity of every bounded representation of Σ is equivalent to weak ergodicity of the right and left representations of Σ by right and left translations on $m(\Sigma)$, and equivalent to the existence of a mean on $m(\Sigma)$ invariant under right and left translations. These conditions, in turn, are equivalent to existence of a directed system of finite means on $m(\Sigma)$ converging weakly to two-sided invariance under all right and left translations of $m(\Sigma)$. Uniform ergodicity is similarly related to existence of finite means converging in the norm of $m(\Sigma)^*$ to two-sided invariance (Theorem 4).

The second part of this paper gives some sufficient conditions for existence of invariant means or of finite means converging in norm to invariance. For the former, the Markoff method of proof by fixed-point arguments is applied (§5) to "solvable" semi-groups and groups, and to semi-groups which are the union of expanding directed systems of sub-semi-groups with means. (For example, a group G such that every finite subset generates a finite subgroup has an invariant mean.) It is also proved by a direct construction (Theorem 6) that if G is a normal subgroup of a group H and if G and H/G have two-sided invariant means, so has H. In §6 a parallel result is proved for finite means converging in norm to invariance. These results greatly increase the family of groups known to have invariant means. Solvable groups formed the only such class previously known; §6 now shows that a solvable group satisfies a stronger property; it has a system of finite means converging in norm to invariance.

Presented to the Society, September 1, 1949 and September 2, 1949 under the titles *Means on semigroups and groups* and *Means and ergodicity of semigroups*; received by the editors July 5, 1949.

(1) Written while the author was on sabbatical leave and partially supported by a stipend from the Office of Naval Research.

The last section shows that if G has an invariant mean, then every bounded representation of G on a Hilbert space is equivalent to a unitary representation, and also shows that the free group on two generators has no invariant mean.

It is to be emphasized that the ergodic and invariance properties of major interest here are properties of abstract semi-groups or groups; in this sense they are "algebraic" properties of a semi-group though the definitions go outside the semi-group to bounded representations or to function spaces.

2. The definition of ergodicity below is adapted from that of Eberlein[2] to the special case of bounded semi-groups. In what follows B will be a Banach space[3] and $L(B, C)$ the Banach space of all linear operators from B into C. $\mathfrak{S}$ will be a bounded[4] semi-group contained in the Banach algebra $L(B, B)$; that is[5] lub $\{\|S\| \mid S \in \mathfrak{S}\} = N < \infty$, and for each S, S' in $\mathfrak{S}$, SS' is also in $\mathfrak{S}$. For each b in B let $\mathfrak{S}(b) = \{Sb \mid S \in \mathfrak{S}\}$ and $K(b)$ = closed convex hull of $\mathfrak{S}(b)$. Let $K = \{T \mid T \in L(B, B)$ and $Tb \in K(b)$ for every b in $B\}$; that is, if $\mathfrak{P}_{b \in B} K(b)$ is the cartesian product of the sets $K(b)$, then $K = L(B, B) \cap \mathfrak{P}_{b \in B} K(b)$. The symbol $\mathcal{A}$ will represent a directed system[6] each element of which is an element of $L(B, B)$.

Definition 1. $\mathfrak{S}$ is called *ergodic under the system of averages* $\mathcal{A}$ when all the following conditions are satisfied:

(a) $\mathcal{A} \subseteq K$.

(b) For each S in $\mathfrak{S}$, $\lim_A A(S-I) = 0$, where I is the identity operator in $L(B, B)$.

(c) For each S in $\mathfrak{S}$, $\lim_A (S-I)A = 0$.

The strength of the assumption of ergodicity of $\mathfrak{S}$ depends on the topology of $L(B, B)$ used in (b) and (c). Corresponding to the uniform (or norm) s^*- and w^*-topologies[7] in $L(B, B)$, we have uniform s^*- or w^*-ergodicity of $\mathfrak{S}$ under $\mathcal{A}$. $\mathfrak{S}$ is called ergodic in a given sense if and only if there exists a directed system $\mathcal{A}$ of averages under which S is ergodic in the given sense.

s^*-ergodicity of $\mathfrak{S}$ is the property which carries the burden of the F. Riesz proof[8] of the mean ergodic theorem. It may be noted that if in the proofs of most such results the s^*-limit is used in (b), it suffices to use the w^*-limit in (c) since its only use is to get a certain point into the manifold of common fixed points of $\mathfrak{S}$.

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(2) W. F. Eberlein, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 43–47, and Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240.

(3) S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

(4) For brevity we keep to the bounded case even in this section where the results can be adapted to the more general definition of ergodicity used by Eberlein.

(5) $\{p \mid Q\}$ is the set of elements p with the property Q.

(6) SS' is the element of $L(B, B)$ such that $SS'(b) = S(S'(b))$ for all b in B; that is, multiplication in $L(B, B)$ is functional composition.

(7) M. M. Day, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 583–608.

(8) F. Riesz, J. London Math. Soc. vol. 13 (1938) pp. 274–278.

Let us begin with a simple characterization of w^*-ergodicity of $\mathfrak{S}$. Assume[9] that B is embedded in B^{**}, the second conjugate of B, by the canonical mapping $Qb(\beta)=\beta(b)$ for all β in B^*; then $L(B, B)$ can be regarded as a subspace of $L(B, B^{**})$. To the uniform, s^*-, and w^*-topologies of $L(B, B^{**})$ we must add a fourth, the w_*-topology; this is the relative topology imposed on $L(B, B^{**})$ by the product space topology (also called the point-open topology) of the space of all functions from B into B^{**} when the w^*-topology is used in B^{**}. $w_*\text{-}\lim_A A=A_0$ if and only if $\lim_A Ab(\beta)=A_0b(\beta)$ for every b in B and β in B^*. In this w_*-topology every sphere in $L(B, B^{**})$ is compact[10].

Let $K_*=w_*$-closure of K in $L(B, B^{**})$.

Lemma 1. *$\mathfrak{S}$ is w^*-ergodic if and only if there exists A_* in $L(B, B^{**})$ such that* (a_*) $A_*\in K_*$; (b_*) $A_*(S-I)=0$, *and* (c_*) $(S-I)^{**}A_*=0$[11].

K is bounded so K_* is bounded and w_*-compact. If $\mathfrak{S}$ is ergodic under $\mathcal{A}$, let A_* be a w_*-cluster point of $\mathcal{A}$; that is, for every A in $\mathcal{A}$ and every w_*-neighborhood U of A_* there exists $A'\geqq A$ such that $A'\in U$. Clearly $A_*\in K_*$; the proofs of (b_*) and (c_*) are so similar that we display only that of (c_*). Given S in $\mathfrak{S}$, b in B, β in B^*, and $\epsilon>0$, there exists A_ϵ in $\mathcal{A}$ with $|\beta[(S-I)Ab]|<\epsilon$ if $A\geqq A_\epsilon$; then there exists $A\geqq A_\epsilon$ such that $|(A_*b)[(S-I)^*\beta]-[(S-I)^*\beta](Ab)|<\epsilon$. Hence $|[(S-I)^{**}A_*b](\beta_2)=|(A_*b)[(S-I)^*\beta]|<|[(S-I)^*\beta](Ab)|+\epsilon=|\beta[(S-I)Ab]|+\epsilon<2\epsilon$. Using in order the freedom of ϵ, β, and b gives (c_*).

If A_* in K_* satisfies (a_*) to (c_*), each w_*-neighborhood U of A_* meets K; for each U let A_U belong to $K\cap U$. Ordering neighborhoods by inclusion then gives a directed system $\mathcal{A}\subseteq K$ such that $w_*\text{-}\lim_A A=A_*$. To prove, for example, (c) take S in $\mathfrak{S}$, b in B, and β in B^*; then

$$\begin{aligned}\lim_A \beta(S-I)Ab &= \lim_A\, [(S-I)^*\beta](Ab)\\ &= (A_*b)[(S-I)^*\beta] = [(S-I)^{**}A_*b](\beta) = 0,\end{aligned}$$

so $0=w_*-\lim_A (S-I)A$.

Theorem 1. *If $K_*=K$, in particular, if $\mathfrak{S}(b)$ is w-conditionally compact or w-sequentially conditionally compact for each b, or if B is reflexive*[12], *then $\mathfrak{S}$ is w^*-ergodic if and only if $\mathfrak{S}$ is uniformly ergodic.*

Under either compactness hypothesis Eberlein[13] has shown that $K(b)$ is

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(9) As in M. M. Day, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 399–412.

(10) Compact is used here in the sense once denoted by bicompact. To prove compactness, C, the unit sphere in $L(B, B^{**})$, is closed in $P_{b\in B}\, C(b)$, where $C(b)$ is the sphere of radius $\|b\|$ about 0 in B^{**}. Each $C(b)$ is w^*-compact, so Tychonoff's theorem asserts C is compact.

(11) As usual the adjoint S^* of an S in $L(B, B)$ is that element of $L(B^*, B^*)$ defined by $S^*\beta(b)=\beta(Sb)$ for all b in B, β in B^*. $S^{**}=(S^*)^*$ is, therefore, in $L(B^{**}, B^{**})$.

(12) B reflexive means $Q(B)=B^{**}$.

(13) W. F. Eberlein, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 51–53.

w^*-closed in B^{**}; hence K is w_*-closed and $K_*=K$. Under the reflexivity assumption, $L(B, B^{**})=L(B, B)$, so $K_*=K$. Then $A_*\in K$ and is invariant.

It will be necessary to recall that attached to each semi-group $\mathfrak{S}$ in $L(B, B)$ there are two closed linear manifolds $M_0(\mathfrak{S})=\{b \mid (S-I)b=0 \text{ for all } S \text{ in } \mathfrak{S}\}$, and $M_1(\mathfrak{S})=$closed linear hull of $\{(S-I)b \mid S\in\mathfrak{S}, b\in B\}$. Let $M(\mathfrak{S})=M_0(\mathfrak{S})+M_1(\mathfrak{S})=\{x+y \mid x\in M_0(\mathfrak{S}),\ y\in M_1(\mathfrak{S})\}$. The gist of the usual ergodic theorems is in the following collection of conclusions; these and further references can be found in the references of footnotes 2, 8, and 9.

If $\mathfrak{S}$ is s^*-ergodic under $\mathcal{A}$, then

(1) $\lim_A AB=\tau b$ *exists in the norm topology if and only if* $b\in M(\mathfrak{S})$.

(2) $M(\mathfrak{S})$ *is closed in* B.

(3) $\tau b=b$ *if and only if* $b\in M_0(\mathfrak{S})$; $\tau b=0$ *if and only if* $b\in M_1(\mathfrak{S})$; *and for all S in $\mathfrak{S}$, $\tau S=S\tau=\tau\tau=\tau$; hence τ is the projection of $M(\mathfrak{S})$ along $M_1(\mathfrak{S})$ onto $M_0(\mathfrak{S})$.*

(4) $\|\tau\|\leqq N=\text{lub }\{\|S\| \mid S\in\mathfrak{S}\}$.

(5) τb *is the unique point in* $K(b)\cap M_0(\mathfrak{S})$.

(6) $b\in M(\mathfrak{S})$ *if and only if* Ab, $A\in\mathcal{A}$ *have a weak cluster point* b_0 *in* B; *in this case* $b_0=\tau b$.

If $\mathfrak{S}$ is only w^*-ergodic under $\mathcal{A}$, the conclusions change only in that the first limit is taken in the weak rather than the norm topology of B.

Using (6) we see: *If $\mathfrak{S}$ is w^*-ergodic under a system $\mathcal{A}$ such that Ab has a weak cluster point in B for every b in B, then $\mathfrak{S}$ is uniformly ergodic with τ as a single invariant average.* This gives another proof of Theorem 1; (1) identifies A_* as τ. It may be noted that this could also be stated as: *$\mathfrak{S}$ is ergodic under a single invariant average A in K if and only if $\mathfrak{S}$ is w^*-ergodic and $M(\mathfrak{S})=B$ (then $A=\tau$).*

3. In this section we consider bounded representations of a fixed abstract semi-group. Let Σ be a semi-group with elements σ; that is, in Σ is defined a binary multiplication which satisfies the associative law. A *right* (*left*) *representation of* Σ is a function F from Σ to an $L(B, B)$ such that $F_{\sigma\sigma'}=F_\sigma F_{\sigma'}$ $(F_{\sigma\sigma'}=F_{\sigma'}F_\sigma)$. We wish to find conditions under which every bounded right and left representation of Σ is ergodic in some sense. To this end we employ certain regular representations of Σ.

Let $m(\Sigma)$ be the Banach space of all bounded, real-valued functions x on Σ. For each σ in Σ define $R=r(\sigma)=r_\sigma$ in $L(m(\Sigma), m(\Sigma))$ by $Rx(\sigma')=x(\sigma'\sigma)$ for all σ' in Σ and x in $m(\Sigma)$. The function r is *the right representation* of Σ, although we shall also apply this name to the semi-group $\mathfrak{R}=r(\Sigma)$. Similarly we define *the left representation* l by $L=l(\sigma)=l_\sigma$ if $Lx(\sigma')=x(\sigma\sigma')$ for all σ' in Σ and x in $m(\Sigma)$; let $\mathfrak{L}=l(\Sigma)$.

It is clear that elements of $\mathfrak{R}$ commute with elements of $\mathfrak{L}$, and that the elements of $\mathfrak{R}$ and $\mathfrak{L}$ are of norm not greater than 1, so $\mathfrak{R}$ and $\mathfrak{L}$ are bounded representations of Σ.

Let e be the unit function, $e(\sigma)\equiv 1$, in $m(\Sigma)$. A *mean* μ over Σ is an element of $m(\Sigma)^*$ such that $\mu(e)=\|\mu\|=1$; it is well known that for an element of $m(\Sigma)^*\mu(e)=\|\mu\|$ is equivalent to $\mu(x)\geqq 0$ if $x(\sigma)\geqq 0$ for all σ.

THEOREM 2. *The following conditions on a semi-group Σ are equivalent* ([14]).

(a) *Every bounded right or left representation is w^*-ergodic.*

(b) *The regular representations $\mathfrak{R}$ and $\mathfrak{L}$ are w^*-ergodic.*

(c) *There exists a mean μ in $m(\Sigma)^*$ invariant under $\mathfrak{R}^*$ and $\mathfrak{L}^*$ (where $\mathfrak{R}^*=\{R^*\mid R\in\mathfrak{R}\}$).*

(d) *e is at distance one from $M_1(\mathfrak{R})+M_1(\mathfrak{L})$.*

(e) *For every bounded representation F of Σ and $\mathfrak{S}=F(\Sigma)$, there exists A_* satisfying* (a$_*$), (b$_*$), *and* (c$_*$); *in fact, A_* is in the w_*-closure of the convex hull of $\mathfrak{S}$.*

Obviously (e)→(a)→(b); the rest of the argument depends on the following propositions:

(A) If $\mathfrak{R}$ and $\mathfrak{L}$ are w^*-ergodic, then $M_0(\mathfrak{R})=M_0(\mathfrak{L})=\{te\mid t \text{ real}\}$.

If $x\in M_0(\mathfrak{R})$, $r_\sigma(x)=x$ for all σ or $r_\sigma x(\sigma')=x(\sigma'\sigma)=x(\sigma')$ for all σ, σ'. Hence $l_{\sigma'}x(\sigma)=x(\sigma')$ for all σ'; that is, $l_{\sigma'}x=x(\sigma')e$ if $\sigma'\in\Sigma$. Clearly scalar multiples of e are in $M_0(\mathfrak{R})$ and in $M_0(\mathfrak{L})$ so $x=(x-l_{\sigma'}x)+l_{\sigma'}x\in M_1(\mathfrak{L})+M_0(\mathfrak{L})=M(\mathfrak{L})$. By ergodicity of $\mathfrak{L}$ (items 1 and 5) $l_{\sigma'}x=l_{\sigma''}x$ for every σ', σ'' in Σ, so $x(\sigma')=x(\sigma'')$ for all σ', σ'' in Σ.

Let $\mathfrak{P}$ = smallest semi-group in $L(m(\Sigma), m(\Sigma))$ containing $\mathfrak{R}$ and $\mathfrak{L}$; by commutativity, $\mathfrak{P}=\{RL\mid R\in\mathfrak{R},\ L\in\mathfrak{L}\}$. We also recall the definition of partial order and convergence in a semi-group given by Alaoglu and Birkhoff([15]). We shall use the symbols ρ, λ, π for elements of the convex hulls of $\mathfrak{R}$, $\mathfrak{L}$, and $\mathfrak{P}$, respectively. Ordering these by multiplication, we say $\rho\geqq\rho'$ if there exists ρ'' such that $\rho=\rho''\rho'$. Alaoglu and Birkhoff showed that, for given x, $\lim_\rho \rho x$ exists (in norm) if and only if (5) of the conclusions of ergodicity holds for every $\rho' x$. Applying this to a w^*-ergodic $\mathfrak{R}$ we find that $\lim_\rho \|\rho x-\tau_R x\|=0$ for every x in $M(\mathfrak{R})$, where τ_R is related to $\mathfrak{R}$ as τ was related to $\mathfrak{S}$.

(B) If $\mathfrak{R}$ and $\mathfrak{L}$ are w^*-ergodic and $x\in M(\mathfrak{R})$, then $\lim_\pi \|\tau_R x-\pi x\|=0$.

This argument depends on the commutativity of $\mathfrak{R}$ with $\mathfrak{L}$; compare also Eberlein's Theorem 8.2.

Since a similar result holds with $\mathfrak{R}$ replaced by $\mathfrak{L}$ we obtain:

(C) If $\mathfrak{R}$ and $\mathfrak{L}$ are w^*-ergodic and $x\in M(\mathfrak{R})+M(\mathfrak{L})$, then $\lim_\pi \pi x$ exists in the norm topology. $\lim_\pi \pi x=0$ if $x\in M_1(\mathfrak{R})+M_1(\mathfrak{L})$; $\lim_\pi \pi x=x$ if $x(=te)\in M_0(\mathfrak{R})=M_0(\mathfrak{L})$.

Since every π is of norm not greater than 1, $\|\lim_\pi \pi x\|\leqq\|x\|$. It follows that if $y\in M_1(\mathfrak{R})+M_1(\mathfrak{L})$, $\|y+e\|\geqq\|e\|\geqq 1$. This proves that (b) implies (d).

([14]) Alaoglu and Birkhoff, Proc. Nat. Acad. Sci. U.S.A. vol. 12 (1939) pp. 628–630, have observed that for groups (c) implies some form of ergodicity.

([15]) Alaoglu and Birkhoff, Ann. of Math. (2) vol. 42 (1940) pp. 293–309.

By the Hahn-Banach theorem and (d) there is a mean μ such that $\mu(x)=0$ for x in $M_1(\mathcal{R})+M_1(\mathcal{L})$. But $\mu=0$ on $M_1(\mathcal{R})$ means $\mu(Rx-x)=0$ for all x or $R^*\mu=\mu$ for all R. Similarly, μ is invariant under $\mathcal{P}^*$, so (d) implies (c).

Now let F be a bounded representation of Σ in $L(B, B)$ and define A_* from μ, as in footnote 9, as follows: For each b in B, $F_\sigma b$ is a bounded function of $\sigma, f_b(\sigma)$; for each β in B^*, $\beta:f_b$, defined by $\beta:f_b(\sigma)=\beta(f_b(\sigma))$, is in $m(\Sigma)$. Define $A_*b(\beta)=\mu(\beta:f_b)$ for all β in B^*. It is easily verified that $A_*\in L(B, B^{**})$; (b_*) and (c_*) can be checked as in footnote 9. For (a_*) recall that each mean μ is a w^*-limit of finite means μ_γ, where γ is a function on Σ such that $\gamma(\sigma)\geqq 0$, $\sum_\sigma\gamma(\sigma)=1$, $\gamma(\sigma)=0$ except at a finite number of points, and $\mu_\gamma(x)=\sum_\sigma\gamma(\sigma)x(\sigma)$. It is easily verified that if $\mu=w^*\text{-}\lim_\gamma\mu_\gamma$, then $A_*=w_*\text{-}\lim_\gamma\sum_\sigma\gamma(\sigma)F(\sigma)$, so $A_*\in K_*$.

COROLLARY 1. *If every bounded right and left representation of Σ is w^*-ergodic, then every bounded representation F over a reflexive space B has the invariant τ (of Theorem 1 and the conclusions from s^*-ergodicity) given by $\beta(\tau b)=\mu(\beta:f_b)$ for every β in B^*, b in B, where μ is an arbitrary mean invariant under $\mathcal{R}^*$ and $\mathcal{L}^*$.*

In some semi-groups ergodicity of one regular representation is enough to enforce ergodicity of all right and left representations. The hypothesis of the next corollary is satisfied if on the appropriate side Σ has either a unit or unique cancellation; a similar result holds with left and right interchanged.

COROLLARY 2. *Let Σ be a semi-group such that for every x in $m(\Sigma)$ there exists σ'' in Σ and x' in $m(\Sigma)$ such that $l_{\sigma''}x'=x$. Let ρ_* be an average for $\mathcal{R}$ defined, as at the end of the preceding proof, from a mean μ. Then $\mathcal{R}$ is ergodic under ρ_* if and only if μ is invariant under $\mathcal{R}^*$ and $\mathcal{L}^*$.*

All that need be proved is invariance of μ; we sketch one case. By hypothesis

$$0=[\rho_*(R-I)x](\alpha)=\mu(\alpha:f_{(R-I)x})$$

for all $\alpha\in m(\Sigma)^*$, $x\in m(\Sigma)$, $R\in\mathcal{R}$. Applying this to an α defined by $\alpha(x)=x(\sigma'')$ for all $x\in m(\Sigma)$, we can show that $\alpha:f_{(R-I)x}=(R-I)l_{\sigma''}x$ for all x, so that

$$0=[(R-I)^*\mu](l_{\sigma''}x) \qquad \text{for all } \sigma'', x.$$

Under the hypothesis, $R^*\mu=\mu$ for all R in $\mathcal{R}$. A similar proof disposes of $L^*\mu$ and of the cases with right and left interchanged.

We remark here that if both cancellation laws hold in Σ, then it can be shown that if an element μ of $m(\Sigma)^*$ is invariant under $\mathcal{R}^*$ and $\mathcal{L}^*$, the same is true of the positive part of μ. This gives the following corollary.

COROLLARY 2'. *If both cancellation laws hold in Σ (for example, if Σ is a group), then the following conditions are equivalent to those of Theorem 2:*

(c′) *There exists* $\mu \neq 0$ *in* $m(\Sigma)^*$ *invariant under* $\mathfrak{R}^*$ *and* $\mathfrak{L}^*$.

(d′) $M_1(\mathfrak{R})+M_1(\mathfrak{L})$ *is not dense (in norm) in* $m(\Sigma)$.

Invariance of μ under $\mathfrak{R}^*$ and $\mathfrak{L}^*$ is equivalent to the vanishing of μ on $M_1(\mathfrak{R})$ and $M_1(\mathfrak{L})$, so (c′) and (d′) are equivalent. If (c') holds, $\|\mu^+\|+\|\mu^-\| =\|\mu\| \neq 0$, so at least one of $\mu^+/\|\mu^+\|$ and $\mu^-/\|\mu^-\|$ is defined; that one will satisfy (c).

4. In this section[16] we discuss a restricted form of ergodicity in which the averages A used are actually finite averages of the elements of the semi-group $\mathfrak{S}$. Let $\mathfrak{T}$ be the convex hull in $L(B, B)$ of the semi-group $\mathfrak{S}$. It was shown in footnote 15 that $\mathfrak{T}$ is also a semi-group in $L(B, B)$; clearly $M_0(\mathfrak{T}) =M_0(\mathfrak{S})$ and $M_1(\mathfrak{T})=M_1(\mathfrak{S})$. For further notation let $\mathfrak{K}=w^*$-closure of $\mathfrak{T}$ in $L(B, B)$ and let $\mathfrak{K}_* =w_*$-closure of $\mathfrak{T}$ in $L(B, B^{**})$ (so $K=K_*\cap L(B, B)$).

Lemma 2. *Lemma* 1 *and Theorem* 1 *remain valid if K and K_* are replaced by $\mathfrak{K}$ and $\mathfrak{K}_*$, through all of* §2.

Definition 2. $\mathfrak{S}$ is *restrictedly ergodic* in a given topology of $L(B, B)$ if it is ergodic in that topology under a directed system of averages $\mathcal{A}\subseteq\mathfrak{T}$.

Lemma 3. *In $L(B, B)$ a convex set is w^*-closed if and only if it is s^*-closed; hence $\mathfrak{K}=s^*$-closure of $\mathfrak{T}$.*

If $T_0 \notin s^*$-closure of C, a convex set, there exist $\epsilon>0$ and $b_1, \cdots, b_n$ in B with $U=\{T \mid \|Tb_i-T_0b_i\|<\epsilon\}$ disjoint from C. Let $B'=B\times B\times \cdots \times B$ (using n factors) and let $\|(b_1, \cdots, b_n)\|=\max_{i\leqq n} \|b_i\|$. Let $C'=\{T' =(Tb_1, \cdots, Tb_n) \mid T\in C\}$ and let $T_0'=(T_0b_1, \cdots, T_0b_n)$. Then C' is convex and $\|T'-T_0'\|\geqq \epsilon$ if $T'\in C'$. By Mazur's theorem[17] there exists $\beta'\in B'^*$ such that $\beta'(T_0)>\text{lub}_{T'\in C'} \beta'(T')+\epsilon$. Since $\beta'(Tb_1, \cdots, Tb_n)= \sum_{i\leqq n}\beta_i (Tb_i)$, where $\beta_i\in B^*$, the w^*-neighborhood $V=\{T \mid |\beta_i(T-T_0)b_i|<\epsilon/n\}$ is disjoint from C; hence T_0 is not in the w^*-closure of C if it is not in the s^*-closure of C.

Paralleling Theorem 1, this lemma now yields the following theorem.

Theorem 3. *Under the hypotheses of Theorem* 1, $\mathfrak{S}$ *is restrictedly w^*-ergodic if and only if it is restrictedly s^*-ergodic.*

In this case $\mathfrak{K}_*\subseteq K_* =K\subseteq B$, so $\mathfrak{K}_* =\mathfrak{K}$; hence $\mathfrak{K}$ is w^*-compact. Hence S w^*-ergodic under $\mathcal{A}\subseteq\mathfrak{K}$ implies $\mathcal{A}$ has a w^*-cluster point A_0; by Lemma 3, this A_0 is the s^*-limit of a directed system $\mathcal{A}'\subseteq\mathfrak{T}$. It is easily verified that $\mathfrak{S}$ is s^*-ergodic under $\mathcal{A}'$ and we have the additional property that s^*-$\lim_{A'}A'$ exists; by the remarks at the end of §2 we know $A_0=s^*$-$\lim_{A'} A'$ is τ, the pro-

[16] The amount of space used here in §4 on the special case of restricted uniform ergodicity and in §6 on existence of finite means converging in norm to invariance is justifiable on the grounds of ignorance; every known example of a group with an invariant mean (including the new cases introduced by the theorems of §5) has, as I have since shown, a system of finite means converging in norm to invariance.

[17] S. Mazur, Studia Mathematica vol. 4 (1933) pp. 70–84.

jection associated with $\mathfrak{S}$.

Combining this with (b) implies (e) of Theorem 2 we derive:

COROLLARY 3. *When the regular representations* $\mathfrak{R}$ *and* $\mathfrak{L}$ *of a semi-group* Σ *are* w^**-ergodic, then every bounded right or left representation* $\mathfrak{S}=F(\Sigma)$ *is restrictedly* w^**-ergodic and every bounded representation of* Σ *over a reflexive* B *is restrictedly* s^**-ergodic.*

To strengthen this somewhat, let $\mathfrak{C}$ be the set of finite means over Σ; that is, $\gamma\in\mathfrak{C}$ means $\gamma(\sigma)\geqq 0$ for all σ, $\gamma(\sigma)=0$ except on a finite subset of Σ, and $\sum_{\sigma\in\Sigma}\gamma(\sigma)=1$. As we mentioned before, each mean μ is a w^*-limit of a directed system $\Gamma\subseteq\mathfrak{C}$. Applying this to the μ of Theorem 2, (c), we have the following corollary.

COROLLARY 4. *If* $\mathfrak{R}$ *and* $\mathfrak{L}$ *are* w^**-ergodic, then there exists a directed system* $\Gamma\subseteq\mathfrak{C}$ *such that each bounded representation* F *of* Σ *is restrictedly* w^**-ergodic under the directed system* $\{\phi_\gamma \mid \gamma\in\Gamma\}$ *of finite means, where* $\phi_\gamma=\sum_{\sigma\in\Sigma}\gamma(\sigma)F(\sigma)$.

A similar result holds for uniform ergodicity. To prove it we give the following lemma.

LEMMA 4. *If there exists a directed system* $\Gamma\subseteq\mathfrak{C}$ *such that, for each* σ' *in* Σ, $\lim_\gamma$ $[\text{lub } \{ | \sum_\sigma\gamma(\sigma)(x(\sigma\sigma')-x(\sigma)) | \mid \|x\|\leqq 1\}=0$, *then, for* ϕ_γ *defined as in Corollary* 4, $\lim_\gamma \|\phi_\gamma F_{\sigma'}-\phi_\gamma\|=0$ *for each right representation* F *of* Σ.

For $\|\phi_\gamma F_{\sigma'}-\phi_\gamma\|=\text{lub } \{ | \sum_\sigma\gamma(\sigma)\beta[F(\sigma)(F(\sigma')-I)b] | \mid \|\beta\|\leqq 1,\ \|b\|\leqq 1\}$ $\leqq\text{lub } \{ | \sum_\sigma\gamma(\sigma)(x(\sigma\sigma')-x(\sigma)) | \mid \|x\|\leqq \text{lub}_\sigma \|F_\sigma\|\}$.

It should be pointed out that if Q maps $l_1(\Sigma)$ into $m(\Sigma)^*$ by $Q_\gamma(x)=\sum_{\sigma\in\Sigma}\gamma(\sigma)x(\sigma)$, then the hypothesis of Lemma 4 becomes

$$\lim_\gamma \|r^*_{\sigma'}Q_\gamma - Q_\gamma\| = 0.$$

Using this and the corresponding results for $\mathfrak{L}$ and for left representations gives the following theorem.

THEOREM 4. *If there exists a directed system* $\Gamma\subseteq\mathfrak{C}$ *which converges strongly to 2-sided invariance in the sense that* $\lim_\gamma \|(r_\sigma-I)^*Q_\gamma\|=0$ *and* $\lim_\gamma\|(l_\sigma-I)^*Q_\gamma\|=0$ *for every* σ, *then every bounded right or left representation of* Σ *is restrictedly uniformly ergodic; a suitable system of averages is* $\{\phi_\gamma\mid\gamma\in\Gamma\}$, *where* $\phi_\gamma=\sum_\sigma\gamma(\sigma)F(\sigma)$. *Conversely, if* Σ *is a semi-group such that for every* x *in* $m(\Sigma)$ *there exists* σ'' *in* Σ *and* x' *in* $M(\Sigma)$ *such that* $l_{\sigma''}x'=x$ *and* $\|x'\|\leqq\|x\|$, *then uniform ergodicity of* $\mathfrak{R}$ *under a directed system* $\{\rho_\gamma\mid\gamma\in\Gamma\}$ *of finite means implies that the system* Γ *converges strongly to 2-sided invariance.* (*Right and left can be interchanged here.*)

For one typical case of the converse, we have by assumption $\|r_\sigma\rho_\gamma-\rho_\gamma\|\to 0$. But

$$\begin{aligned}
\|r_\sigma\rho_\gamma - \rho_\gamma\| &= \Big\| \sum_{\sigma'} \gamma(\sigma')(r_{\sigma\sigma'} - r_{\sigma'}) \Big\| \\
&= \operatorname*{lub}_{\|x\|\leqq 1, \sigma''\in\Sigma} \Big| \sum_{\sigma'} \gamma(\sigma') x(\sigma''\sigma\sigma') - x(\sigma''\sigma') \Big| \\
&= \operatorname*{lub}_{\|x\|\leqq 1, \sigma''\in\Sigma} \Big| \sum_{\sigma'} \gamma(\sigma') [l_\sigma l_{\sigma''} x(\sigma') - l_{\sigma''} x(\sigma')] \Big| \\
&= \operatorname*{lub}_{\|x\|\leqq 1, \sigma''\in\Sigma} | Q_\gamma[(l_\sigma - I) l_{\sigma''} x] | \\
&= \operatorname*{lub}_{\|x\|\leqq 1, \sigma''\in\Sigma} | [(l_\sigma - I)^* Q_\gamma](l_{\sigma''} x) | .
\end{aligned}$$

This will be

$$\geqq \operatorname*{lub}_{\|x\|\leqq 1} | [(l_\sigma - I)^* Q_\gamma](x) | = \|(l_\sigma - I)^* Q_\gamma\|$$

under the hypothesis of the converse.

5. The last sections showed the relationship between restricted ergodicity of all bounded representations and the existence of finite means approaching two-sided invariance, and (in the w^*-case) the existence of a single two-sidedly invariant mean. We present in this section some existence proofs for such means.

For semi-groups we can give (Theorem 5) an invariant μ if the semi-group is solvable in a sense to be made precise below. For groups we show that existence of a left-invariant λ suffices for existence of a two-sided invariant μ and then (Theorem 6) show that if G and H/G have such means, so has H. This, of course, applies to arbitrary solvable groups and, with slight modification, to show existence of a μ in $C(H)^*$ if H is an extension of a compact G by a solvable H.

For groups with finite means converging in norm to invariance we have parallel results (see §6); we can reduce to the left-sided approach, and we can prove H has such means if G and G/H have.

LEMMA 5. *Let $\mathfrak{S}$ be a semi-group of distributive operators in a linear space B, let p be a positive-homogeneous sub-additive functional on B, and suppose there is a constant N with $p(Sb) \leqq Np(b)$ for all b, S. Let E_0 be a linear subspace of B invariant under $\mathfrak{S}$ and let ϕ_0 be a distributive functional on E_0 invariant under $\mathfrak{S}$ and dominated by p (that is, $\phi_0(Sb) = \phi_0(b)$ for all $b \in E_0$, $S \in \mathfrak{S}$, and $\phi_0(b) \leqq p(b)$ for $b \in E_0$). If there exists a distributive extension β_0 of ϕ_0 to all B such that $\beta_0(b) \leqq p(b)$ for all b and $\beta_0(S_1S_2b) = \beta_0(S_2S_1b)$ for all $b \in B$, $S_i \in \mathfrak{S}$, then there is a distributive extension β of ϕ_0 to all B such that $\beta(Sb) = \beta(b)$ for all S, b, and $\beta(b) \leqq Np(b)$.*

The assumption on β_0, that $S_1^*S_2^*\beta_0 = S_2^*S_1^*\beta_0$ for all S_i in $\mathfrak{S}$, provides just enough commutativity to make the Markoff proof(18) (that there is a fixed point common to all S^* in the w^*-closed convex hull of $\{S^*\beta_0 | S \in \mathfrak{S}\}$) effective. Morse and Agnew(19) have also proved such a theorem by an adaptation of Banach's proof of existence of a Banach limit for bounded sequences.

If $\mathfrak{S}$ is a semi-group, call $\mathfrak{S}_1$ a *commutator set* for $\mathfrak{S}$ if for each pair S, S' in $\mathfrak{S}$ there exists S_1 in $\mathfrak{S}_1$ such that $SS' = S_1S'S$ or $S'S = S_1SS'$, and there exists S_2 in $\mathfrak{S}_1$ such that $SS' = S'SS_2$ or $S'S = SS'S_2$.

LEMMA 6. *Let $\mathfrak{S}$, p, ϕ_0, and E_0 be as in Lemma 5. If there exists a distributive extension β_0 of ϕ_0 to all B such that $S_1^*\beta_0 = \beta_0$ for all S_1^* in a commutator set $\mathfrak{S}_1^*$ of $\mathfrak{S}^*$, and $\beta_0(b) \leqq p(b)$ for all b, then there exists an extension β of ϕ_0 to all B such that $S^*\beta = \beta$ for all S in S and $\beta(b) \leqq Np(b)$ for all b in B.*

If S, S' are given in $\mathfrak{S}$, arrange them in the proper order so that $S^*S'^* = S'^*S^*S_2^*$. Then $S^*S'^*\beta_0 = S'^*S^*S_2^*\beta_0 = S'^*S^*\beta_0$, so Lemma 5 applies.

Call Σ *solvable* if there exists a chain of sub-semi-groups $\Sigma = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_n$, with Σ_{i+1} a commutator set for Σ_i and Σ_n commutative.

THEOREM 5. *Let Σ be a solvable semi-group. Then there exists a mean μ invariant under $\mathcal{R}^*$ and $\mathcal{L}^*$ (so the other conditions of Theorem 2 also hold).*

Let $\mathcal{R}_i = r(\Sigma_i)$ and $\mathcal{L}_i = l(\Sigma_i)$ and let $\mathcal{P}_i$ be the semi-group generated by $\mathcal{R}_i$ and $\mathcal{L}_i$; then $\mathcal{R}_n$ and $\mathcal{L}_n$ are abelian and commute. The Hahn-Banach theorem gives a μ_0 such that $\mu_0(e) = \|\mu_0\| = 1$; since $\mathcal{P}_n$ is commutative, Lemma 5 gives a μ_1 invariant under $\mathcal{P}_n^*$. Since $\mathcal{R}_i$ and $\mathcal{L}_i$ commute, it is easily verified that $\mathcal{P}_{i+1}$ is a commutator sub-semi-group for $\mathcal{P}_i$. Lemma 6 and induction complete the proof.

This applies, of course, to solvable groups. Other notable simplifications of proof appear, however, for groups, due to the presence of the cancellation law and the inverse operation. In what follows G will be a group and T will be the element of $L(m(G), m(G))$ defined by $Tx(g) = x(g^{-1})$ for all g in G, x in $m(G)$. The formulas $r_gT = Tl_{g^{-1}}$, $l_gT = Tr_{g^{-1}}$, $T^2 = I$, $Te = e$, and $\|Tx\| = \|x\|$ for all x are easily established.

LEMMA 7. *If there exists a mean λ in $m(\Sigma)^*$ invariant under $\mathcal{L}^*$, then there exists a mean μ in $m(\Sigma)^*$ invariant under both $\mathcal{R}^*$ and $\mathcal{L}^*$.*

To define first a right-invariant mean ρ let $\rho = T^*\lambda$. Then from $l_g^*\lambda = \lambda$ for all g, we see that $r_g^*\rho = r_g^*T^*\lambda = T^*l_g^{*-1}\lambda = T^*\lambda = \rho$. $\lambda(e) = 1$ implies $\rho(e) = T^*\lambda(e) = \lambda(Te) = \lambda(e) = 1$. $\|T^*\| = 1$ and T^{*2} is the identity, so $\|\rho\| = \|\lambda\|$ $(=1)$.

Given right- and left-invariant means ρ and λ define a two-sided invariant mean μ as follows: for $x \in m(G)$ and $g \in G$, let $\bar{x}(g) = \rho(l_gx)$, and let $\mu(x) = \lambda(\bar{x})$. This μ can be shown to be an invariant mean.

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(18) A. Markoff, C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 10 (1936) pp. 299–301.

(19) R. P. Agnew and A. P. Morse, Ann. of Math. (2) vol. 38 (1938) pp. 20–30.

The argument above can be applied to any subspace of $m(G)$ closed under the operations in question. In particular if G is a topological group and $U(G)$ the space of uniformly continuous, bounded functions on G, then $U(G)$ is closed under right and left translations and under the formation of $\bar{x}$. If G is the extension of a compact group by a discrete group, this space $U(G)$ coincides with the space $C(G)$ of all bounded continuous functions, and is also closed under T.

COROLLARY 5. *If Σ is a group, then the conditions of Theorem 2 and Corollaries 2, 2′, 3, 4 are also equivalent to*

(c″) *there exists a mean μ or a $\mu \neq 0$ in $m(\Sigma)^*$ invariant under $\mathcal{R}^*$ or $\mathcal{L}^*$,*

or

(d″) *$M_1(\mathcal{R})$ or $M_1(\mathcal{L})$ is not dense in $m(\Sigma)$,*

or

(f″) there exists a directed system Γ of finite means converging weakly to one-sided invariance.

THEOREM 6. *Let H be a group and G a normal sub-group of H. If there exist invariant means over $m(G)$ and over $m(H/G)$, then there is an invariant mean over $m(H)$.*

By Lemma 7 it suffices to find a left-invariant mean over H. Suppose that α is left-invariant over $m(G)$ and β left-invariant over $m(\overline{H})$, where $\overline{H}=H/G$. For fixed h in H and x in $m(H)$, define x_h in $m(G)$ by $x_h(g)=x(hg)$; then let $\bar{x}(h)=\alpha(x_h)$. We prove that if h and h' are in the same coset $\bar{h}$, then $\bar{x}(h)=\bar{x}(h')$; for $h=h'g'$ so

$$x_h(g) = x(hg) = x(h'g'g) = x_{h'}(g'g) = (l_{g'}x_{h'})(g),$$

so $x_h=l_{g'}x_{h'}$, and $\alpha(x_h)=\alpha(x_{h'})$. Hence $\bar{x}(h)$ depends only on the coset $\bar{h}$; we define $\bar{x}$ on $\overline{H}$ by $\bar{x}(\bar{h})=\bar{x}(h)$.

Now define γ on $m(H)$ by $\gamma(x)=\beta(\bar{x})$. This γ can be shown to be a left-invariant mean, and Lemma 7 completes the proof.

Again this proof is valid for any subspace $E(H)$ for which the constructed functions x_h lie in an $E(G)$ with left mean and the $\bar{x}$ lie in an $E(\overline{H})$ with a left mean. As before it may be noted that $U(G)$, $U(H)$, and $U(H/G)$ would satisfy these conditions.

COROLLARY 6. *If the chain of commutator sub-groups of a group G ends at the identity, then there is in $m(G)^*$ a two-sided invariant mean γ_1 so all the conditions of Theorem 2 hold.*

This follows by induction from Theorem 6.

COROLLARY 7. *Let H be a topological group and G a compact normal subgroup of H such that H/G satisfies the hypotheses of Corollary 6; then there is in $C(H)^*$ a two-sided invariant mean.*

The Haar measure in G is a left-invariant α on $C(G)=U(G)$. Corollary 5 provides a left-invariant β on $m(H/G)\supseteq C(H/G)\supseteq U(H/G)$. Hence the remarks after Theorem 6 apply to give in $C(H)^*$ an invariant mean γ.

An elementary argument using w^*-compactness proves the following corollary.

COROLLARY 8. *Let Σ be a semi-group with a system Δ of sub-semi-groups δ, directed by $\supseteq$, with $\bigcup_{\delta\in\Delta}\delta=\Sigma$, such that for each δ there is in $m(\Sigma)^*$ a mean μ_δ invariant under $r(\delta)^*$ and $l(\delta)^*$. Then there is a mean μ invariant under all $r(\Sigma)^*$ and $l(\Sigma)^*$.*

As an aid in applying this we use the following lemma.

LEMMA 8. *If Σ' is a sub-semi-group of Σ, and if there exists a mean μ' in $m(\Sigma')^*$ invariant under $r'(\Sigma')^*$ and $l'(\Sigma')^*$, then there exists a mean μ in $m(\Sigma)^*$ invariant under $r(\Sigma')^*$ and $l(\Sigma')^*$.*

Letting $Px=x'$ mean $x'(\sigma')=x(\sigma')$ for all σ' in Σ', the definition $\mu=P^*\mu'$ gives an appropriate mean on $m(\Sigma)$.

From these, Theorem 6, and the obvious mean on a finite group follow (a) if every finitely generated sub-semi-group of Σ has a two-sided invariant mean, so has Σ; (b) if G is a group such that every finite subset generates a finite sub-group, then G has an invariant mean; and (c) if $G=\bigcup_{i<\alpha}H_i$, where H_1 and H_{i+1}/H_i are finite or solvable for all ordinals $i<\alpha$, and if $H_\lambda=\bigcup_{i<\lambda}H_i$ for every limit ordinal $\lambda<\alpha$, then G has an invariant mean.

6. The existence of finite means over $m(G)$ converging in norm to two-sided invariance is a stronger restriction on G than is the existence of a single invariant mean μ in $m(G)^*$, for this is equivalent to existence of finite means converging weakly to invariance. For this stronger restriction we have, however, almost the same properties (except for the analogues of Corollary 7 to the end of §5).

LEMMA 9. *If on a group G there exists a directed system Λ of finite means λ such that $\lim_\lambda \|l_g\lambda-\lambda\|=0$ for every g in G, then there exists a directed system Φ of finite means ϕ such that $\lim_\phi \|l_g\phi-\phi\|=0$ and $\lim_\phi \|r_g\phi-\phi\|=0$ for every g in G.*

As in Lemma 7, let

$$\|r_g\rho-\rho\| = \sum_{g'}|\rho(g'g)-\rho(g')| = \sum_{g'}|\lambda(g^{-1}g'^{-1})-\lambda(g'^{-1})|$$
$$= \|l_{g^{-1}}\lambda-\lambda\|,$$

so $\lim_\rho \|r_g\rho-\rho\|=0$. To construct a two-sided mean ϕ from one left mean λ and one right mean ρ, define $\phi=\phi(\lambda,\rho)$ by

$$\phi(g) = \sum_{h\in G}\lambda(h)\rho(h^{-1}g) = \sum_{f\in G}\lambda(gf)\rho(f^{-1}).$$

Then

$$\begin{aligned}\|r_{g_0}\phi - \phi\| &= \sum_g |\phi(gg_0) - \phi(g)| \\ &= \sum_g \Big|\sum_h \lambda(h)[\rho(h^{-1}gg_0) - \rho(h^{-1}g)]\Big| \\ &\leqq \sum_g \sum_h \lambda(h)\,|r_{g_0}\rho(h^{-1}g) - \rho(h^{-1}g)| \\ &= \sum_h \lambda(h) \sum_{h'} |r_{g_0}\rho(h') - \rho(h')| = \|r_{g_0}\rho - \rho\|.\end{aligned}$$

Similarly

$$\begin{aligned}\|l_{g_0}\phi - \phi\| &= \sum_g |\phi(g_0g) - \phi(g)| \\ &= \sum_g \Big|\sum_f [\lambda(g_0gf) - \lambda(gf)]\rho(f^{-1})\Big| \\ &\leqq \sum_f \rho(f^{-1}) \sum_g |l_{g_0}\lambda(gf) - \lambda(gf)| \\ &= \sum_f \rho(f^{-1}) \sum_{g'} |l_{g_0}\lambda(g') - \lambda(g')| \\ &= \|l_{g_0}\lambda - \lambda\|.\end{aligned}$$

Let $\Phi = \{\phi(\lambda,\ T^*\lambda) \mid \lambda \in \Lambda\}$ and order Φ as Λ was ordered; that is, $\phi(\lambda,\ T^*\lambda) \geqq \phi(\lambda',\ T^*\lambda')$ if and only if $\lambda \geqq \lambda'$. Then

$$\lim_\phi \|r_g\phi - \phi\| = \lim_\phi \|l_g\phi - \phi\| = 0$$

for all g in G.

Now as with invariant means we have the following theorem.

THEOREM 7. *Let H be a group, G a normal sub-group of H, and $H^* = G/H$. If there exist on G and on H^* directed systems of finite means converging in norm to (either-sided) invariance, then there exists on H a directed system of means converging in norm to two-sided invariance.*

By Lemma 9 we can consider only left invariance, so suppose that $\lim_\alpha \|l_g\alpha - \alpha\| = 0$ for each g in G and $\lim_\beta \|l_{h^*}\beta - \beta\| = 0$ for each h^* in H^*. Let F be a system of representatives in H of the group H/G; then each h in H is uniquely representable as $h = fg$, $f \in F$, $g \in G$. Then $(fg)(f'g') = (ff')(g^{f'}g')$, where $g^h = h^{-1}gh \in G$ also.

Now for α and β given define $\mu = \mu(\alpha,\ \beta)$ by $\mu(fg) = \alpha(g)\beta(f^*)$, where h^* = coset containing h. Then, if $f_0f = \phi g'$, $\phi \in F$, $g' \in G$, and $g'g_0^f = g_1$,

$$l_{f_0g_0}\mu(fg) = \mu(f_0g_0fg) = \beta((f_0f)^*)\alpha(g'g_0^fg) = \beta((f_0f)^*)\alpha(g_1g).$$

Hence

$$\begin{aligned}
\|l_{f_0g_0}\mu - \mu\| &= \sum_{f\in F, g\in G} |\beta((f_0f)^*)\alpha(g_1g) - \beta(f^*)\alpha(g)| \\
&\leqq \sum_{f\in F, g\in G} \beta((f_0f)^*)\,|\alpha(g_1g) - \alpha(g)| \\
&\quad + \sum_{f\in F, g\in G} |\beta((f_0f)^*) - \beta(f^*)|\,\alpha(g) \\
&\leqq \sum_f \beta((f_0f)^*)\|l_{g_1}\alpha - \alpha\| + \|l_{f_0^*}\beta - \beta\| \\
&\leqq \|l_{f_0^*}\beta - \beta\| + \operatorname*{lub}_{\beta((f_0f)^*)\neq 0} \|l_{g_1}\alpha - \alpha\|.
\end{aligned}$$

If F and G are finite, the theorem is evident from the beginning. If F or G is infinite, the new directed system of means will be ordered by triples (β, δ, π), where β is in the given system of means on H^*, δ is a finite subset of G, and π is a finite subset of F; $(\beta, \delta, \pi) \geqq (\beta', \delta', \pi')$ means that $\beta \geqq \beta'$, $\delta \supseteq \delta'$, and $\pi \supseteq \pi'$. Define $|\delta|$ to mean the number of elements in δ. For given f_0, f, and g_0, as above, let $f_0f = \phi' g'$, and $g_1 = g' g_0^f$. For given (β, δ, π) the set of such g_1 obtained for $f_0 \in \pi$, $g_0 \in \delta$, and $\beta(f^*) \neq 0$ is finite. Hence there exists $\alpha = \alpha(\beta, \delta, \pi)$ such that $\|l_{g_1}\alpha - \alpha\| < 1/|\delta| \cdot |\pi|$ for each such g_1.

Let $\mu(\beta, \delta, \pi) = \mu(\alpha(\beta, \delta, \pi), \beta)$; then

$$\lim_{(\beta,\delta,\pi)} \|l_{f_0g_0}\mu - \mu\| = 0.$$

This and Lemma 9 complete our proof.

COROLLARY 9. *If the chain of commutator sub-groups of G ends at the identity in a finite number of steps, then there exists a directed system of finite means converging in norm to two-sided invariance.*

$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$, abelian, and G_i/G_{i+1} is abelian. To verify the hypotheses of Theorem 7 for an induction argument requires only the following lemma.

LEMMA 10. *If G is an abelian group, then the set Γ of all finite means on G is an abelian semi-group under the operation of convolution, $(\gamma \circ \gamma')(g) = \sum_{h\in G} \gamma(h^{-1})\gamma'(gh)$. Ordering Γ by $\gamma \geqq \gamma'$ if there exists γ'' with $\gamma = \gamma'' \circ \gamma'$, Γ becomes a directed system and $\lim_\gamma \|l_g\gamma - \gamma\| = 0$ for all g.*

The proof is essentially that of Eberlein's remark that every abelian semi-group of linear operators is uniformly ergodic under its semi-group of finite means, or can be derived from that fact by Theorem 4.

Returning for a moment to Corollary 9, it can be seen from the proofs of Theorem 7 and Lemma 9 that a directed system Φ of means on G is isomorphic to $\Gamma \times \Delta \times \Pi$, where Γ is the system approaching invariance on G/G_1, Δ is the system of finite subsets of G_1, and Π is the system of finite subsets of G/G_1.

COROLLARY 10. *If G satisfies the hypotheses of Corollary* 9, *then every bounded representation of G is restrictedly uniformly ergodic.*

Since $r_g^* Q\gamma - Q\gamma = Q(r_{g^{-1}}\gamma - \gamma)$ and Q is norm-preserving, the conclusion of Corollary 9 gives the hypothesis of Theorem 4.

Cohen[20] gave, as a sufficient condition for (s^*-) ergodicity of every bounded representation of the positive integers under a sequence of means defined by a regular Toeplitz matrix $\{a_{ik}\}$, essentially the condition $\lim_i \sum_k |a_{ik+1} - a_{ik}| = 0$. The discussion here and around Theorem 4 can be modified to prove this condition necessary and sufficient for uniform ergodicity under the given sequence of means.

Lorentz[21] uses this same condition in a paper on Banach limits. His work shows that an element x lies in the ergodic subspace $M(\mathfrak{S})$ of the regular representation of the semi-group $\mathfrak{S}$ of integers if and only if $\mu(x)$ has the same value for every Banach limit (=invariant mean). His proof can easily be adapted to each abelian semi-group, but it is not yet determined for how general a class of groups the result is valid.

7. We conclude with counter-examples and applications.

To show that there need not be a mean even one-sided invariant on the space of all bounded uniformly continuous functions on a locally compact group, let G be the discrete free group on two generators; the space in question then reduces to $m(G)$. If each element of G is written in reduced form, then it begins with either a, b, a^{-1}, or b^{-1} (unless $g=u$, the identity of G). Let A, B, A^{-1}, and B^{-1} be the sets of this form. Then $A, bA, b^2A, \cdots, b^nA$ are all disjoint. If, for example, $x(g)=0$ for g not in A, then $l_bx, l_{b^2}x, \cdots, l_{b^n}x$ are all disjoint so $\mu(\sum_{i\leqq n} l_{b^i}x) = n\mu(x)$ if μ is left-invariant, while $\|\sum_{i\leqq n} l_{b^i}x\| = \|x\|$, so $n|\mu(x)| \leqq \|x\|$ for all n, and $\mu(x)=0$. But if $x_A(g)=x(g)$ if $g\in A$, $x_A(g)=0$ if $g\notin A$, we have $\mu(x)=\mu(x_A)+\mu(x_B)+\mu(x_{A^{-1}})+\mu(x_{B^{-1}}) +\mu(x_u)=0$.

I. Kaplansky suggested the following generalization of a theorem proved by Sz. Nagy[22] for G = the integers or the real line.

THEOREM 8. *Let G be a group with a two-sided invariant mean μ in $m(G)^*$. Then every bounded representation F of G in an inner product space B is equivalent to a unitary representation in the sense that there exists a new inner product defining the same topology in B, and such that every F_g is unitary with respect to this new inner product.*

As in the paper cited in footnote 22, for $\phi, \psi \in B$ let (ϕ, ψ) be the original inner-product in the space B. Define a new inner-product stepwise, by $x(g) = (F_g\phi, F_g\psi)$, $\langle\phi, \psi\rangle = \mu(x)$. Then $\langle\phi, \psi\rangle$ is an inner product in H, and, if $\|F_g\|$

[20] L. W. Cohen, Ann. of Math. (2) vol. 41 (1940) pp. 505–509.

[21] G. G. Lorentz, Acta Math. vol. 80 (1948) pp. 167–190.

[22] B. de Sz. Nagy, Acta Univ. Szeged Sect. Sci. Math. vol. 11 (1947) pp. 152–157.

$\leqq N$ for all g, then $(\phi, \phi)/N \leqq \langle\phi, \phi\rangle \leqq N(\phi, \phi)$. Also, defining $|\phi|^2 = \langle\phi, \phi\rangle$, $|F_g\phi| = |\phi|$ for all ϕ, g.

[It is known that if an inner product with these properties is given, there exists a positive definite P such that $\langle\phi, \psi\rangle = (P^{-1}\phi, P^{-1}\psi)$.]

In case G is a topological group and F_g is continuous in w^*-, s^*-, or u-topologies, it suffices that μ be invariant over $C(G)$.

COROLLARY 11. *If G satisfies the hypotheses of Corollary 6, every bounded representation on a Hilbert space is equivalent to a unitary representation. If G satisfies the hypotheses of Corollary 7, then every continuous bounded representation is equivalent to a continuous unitary representation.*

It should be mentioned that very simple (even finite) semi-groups need have no invariant mean. If Σ is an arbitrary set and $\sigma\sigma' = \sigma'$ for all σ, σ', then Σ has no right-invariant mean if it has more than one element. $l_\sigma = I$ for every σ and $r_\sigma x = x(\sigma)e$ for every σ, and $M_1(\mathfrak{R}) = m(\Sigma)$.

That there is no invariant mean does not (apparently) imply that there is a bounded representation on a Hilbert space not equivalent to a unitary representation. Thus far I have not succeeded in settling this more difficult question even for the free group G on two generators.

Another problem of interest is this: Suppose every bounded representation of a group or semi-group is s^*-ergodic. Is it restrictedly uniformly ergodic?

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