THE BEHAVIOR AT INFINITY OF CERTAIN CONVOLUTION TRANSFORMS⁽¹⁾

BY

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1. Introduction. Let $b, \{a_k\}_1^\infty$ be real constants subject to the sole restriction that

$$(1) \quad \sum_{k=1}^{\infty} a_k^{-2} < \infty,$$

and let

$$(2) \quad E(s) = e^{bs} \prod_1^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k}.$$

We define

$$(3) \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{E(s)} ds \quad (-\infty < t < \infty).$$

It may be verified that

$$(4) \quad \int_{-\infty}^{\infty} G(t) e^{-st} dt = \frac{1}{E(s)},$$

the bilateral Laplace transform converging absolutely in the strip $(\alpha_1 < \sigma < \alpha_2)$ where

$$(5) \quad \alpha_2 = \min_{a_k > 0} a_k, \quad \alpha_1 = \max_{a_k < 0} a_k,$$

and $s = \sigma + i\tau$. The function $G(t)$ and the associated convolution transform

$$(6) \quad f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$$

have been studied in [2], [3], and [4]⁽²⁾.

A kernel $G(t)$ belongs to class II if the constants a_k are positive and if $\sum_1^{\infty} 1/a_k = \infty$. If $G(t) \in$ class II and if the convolution transform (6) converges for any one value of x then it converges for all larger values of x . Consequently there exists a number γ_c , the abscissa of convergence, such that the

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transform (6) converges for $x > \gamma_c$ and diverges for $x < \gamma_c$.

Our principal result is that if $f(x)$ is the generating function of a convolution transform with class II kernel $G(t)$ and determining function $\alpha(t)$, and if

$$(7) \quad f(x) = O[G(x - \rho)] \quad (x \rightarrow +\infty)$$

then $\alpha(t)$ is necessarily constant for $t > \rho$, so that

$$f(x) = \int_{-\infty}^{x+\epsilon} G(x-t) d\alpha(t)$$

for every $\epsilon > 0$. If (7) holds for arbitrarily large negative ρ then $f(x) \equiv 0$.

As an example we set

$$E(s) = [\Gamma(1-s)]^{-1}, \quad G(t) = e^{-s't}e^t.$$

If

$$f(x) = \int_{-\infty}^{\infty} e^{-ex-t} e^{x-t} \phi(t) dt \quad (x > \gamma_c),$$

$$f(x) = O[e^{-ex-\rho} e^{x-\rho}] \quad (x \rightarrow +\infty),$$

then $\phi(t) = 0$ for $(\rho < t < \infty)$. After an exponential change of variable this becomes a theorem concerning the Laplace transform. If

$$(8) \quad F(x) = \int_0^{\infty} e^{-x't} \Phi(t) dt \quad (x > g_c),$$

$$F(x) = O(e^{-rx}) \quad (x \rightarrow +\infty),$$

then $\Phi(t) = 0$ for $(0 < t < r)$. A proof of this special case has been given previously in [1].

A function $G(t)$ is said to belong to class Ia if there are both positive and negative a_k 's and if $\sum_1^{\infty} 1/|a_k| = \infty$. If $G(t) \in$ class Ia and if the convolution transform (6) converges for any one value of x , it converges for all x . With each kernel $G(t)$ class Ia we may associate a class II kernel $\bar{G}(t)$ defined by the equations

$$\bar{E}(s) = e^{bs} \prod_1^{\infty} \left(1 - \frac{s}{|a_k|}\right) e^{s/|a_k|},$$

$$\bar{G}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{\bar{E}(s)} ds \quad (-\infty < t < \infty).$$

We shall also prove that if $f(x)$ is a generating function with kernel $G(t) \in$ class Ia and if

$$f(x) = O[\bar{G}(x - \rho)] \quad (x \rightarrow +\infty)$$

for arbitrarily large negative ρ , or

$$f(x) = O[\bar{G}(\rho - x)] \quad (x \rightarrow -\infty)$$

for arbitrarily large positive ρ , then $f(x) \equiv 0$.

As an example of this result we may take

$$E(s) = \cos \pi s, \quad G(t) = \operatorname{sech} \frac{t}{2}.$$

After an exponential change of variables we obtain a theorem concerning the Stieltjes transform. If

$$F(x) = \int_{0+}^{\infty} \frac{\Phi(t)}{x+t} dt \quad (0 < x < \infty),$$

and if

$$F(x) = O(e^{-rx^{1/2}}) \quad (x \rightarrow +\infty)$$

or if

$$F(x) = O(e^{-rx^{-1/2}}) \quad (x \rightarrow 0+)$$

for arbitrarily large r , then $F(x) \equiv 0$.

2. A special case. In this section we shall prove our first theorem under very special assumptions. The most general case can, however, be reduced to this one by elementary transformations.

THEOREM 2. *If*

1. $G(t) \in \text{class II}$,
2. $\phi(t) \in L$ ($\rho \leq t \leq \infty$),
3. $h(x) = \int_{\rho}^{\infty} G(x-t)e^{ct}\phi(t)dt$ ($c < \alpha_2$), where α_2 is defined as in (5) §1,
4. $h(x) = O[G(x-\rho)]$ ($x \rightarrow +\infty$),

then

$$\phi(t) \equiv 0 \quad (\rho < t < \infty).$$

We define

$$(1) \quad \phi^*(s) = \int_{\rho}^{\infty} e^{ct}\phi(t)e^{-st}dt.$$

This bilateral Laplace transform converges absolutely for ($c \leq \sigma < \infty$). We know that

$$(2) \quad \frac{1}{E(s)} = \int_{-\infty}^{\infty} G(t)e^{-st}dt,$$

the integral converging absolutely for ($-\infty < \sigma/\alpha_2$).

Since $G(t)e^{-ct}$ is positive and bell-shaped, see [3; §10], we have for x sufficiently large and negative

$$\begin{aligned} |h(x)| &\leq e^{cx} \left[\max_{\rho \leq t \leq \infty} G(x-t)e^{-c(x-t)} \right] \int_{\rho}^{\infty} |\phi(t)| dt \\ &= O[G(x-\rho)] \quad (x \rightarrow -\infty). \end{aligned}$$

From this estimation and from assumption 4 it follows that

$$(3) \quad |h(x)| \leq O(1)G(x-\rho) \quad (-\infty < x < \infty).$$

This implies that the bilateral Laplace transform

$$(4) \quad h^*(s) = \int_{-\infty}^{\infty} h(x)e^{-sx}dx$$

converges absolutely for $(-\infty < \sigma < \alpha_2)$.

Since the integrals (1) and (2) have a common strip of absolute convergence $(c < \sigma < \alpha_2)$, the convolution theorem for the bilateral Laplace transform tells us that in this strip

$$h^*(s) = \frac{1}{E(s)} \phi^*(s),$$

or

$$(5) \quad \phi^*(s) = h^*(s)E(s).$$

Since $E(s)$ is an entire function, equation (5) provides a continuation of $\phi^*(s)$ into the half-plane $\sigma \leq c$, so that $\phi^*(s)$ is also an entire function.

By a change of variable in equation (1) we obtain

$$e^{\rho(s-c)}\phi^*(s) = \int_0^{\infty} e^{ct}\phi(t+\rho)e^{-st}dt.$$

It is clear that $e^{\rho s}\phi^*(s)$ is bounded in the half-plane $\sigma \geq c$. In particular $e^{\rho s}\phi^*(s)$ is bounded on the line $(\sigma=c, -\infty < \tau < \infty)$.

Using inequality (3) we have

$$|h^*(s)| = O(1)[E(\sigma)e^{\rho\sigma}]^{-1} \quad (-\infty < \sigma < \alpha_2)$$

from which we obtain

$$|e^{\rho s}\phi^*(s)| \leq O(1) \left| \frac{E(s)}{E(\sigma)} \right| \quad (-\infty < \sigma < \alpha_2).$$

It follows that $e^{\rho s}\phi^*(s)$ is bounded on the half-line $(-\infty < \sigma \leq c, \tau=0)$ and that in the half-plane $(-\infty < c \leq \sigma, \phi^*(s))$ is at most of order two minimal type.

We now know that $e^{\rho s}\phi^*(s)$ is bounded on the lines $(\sigma=c, -\infty < \tau < \infty)$ and $(-\infty < \sigma \leq c, \tau=0)$ and is at most of order two minimal type for $\sigma \leq c$.

Applying a familiar form of the Phragmén-Lindelöf principle to each of the two quadrants of this configuration we find that $e^{\rho s}\phi^*(s)$ is bounded for $\sigma \leq c$ and thus in the entire plane. By Liouville's theorem $e^{\rho s}\phi^*(s)$, being entire and bounded, is a constant. Since

$$\begin{aligned} e^{i\rho\tau}\phi^*(c+i\tau) &= \int_0^\infty \phi(t+\rho)e^{-it\tau}dt \\ &= o(1) \end{aligned} \quad (\tau \rightarrow \pm \infty)$$

it follows that $\phi^*(s) \equiv 0$. This implies that $\phi(t) = 0$ almost everywhere ($\rho \leq t < \infty$). q.e.d.

3. **Two lemmas.** We consider here two lemmas necessary to reduce the general case to the special case which we have just considered. Let

$$(1) \quad f(x) = \int_{-\infty}^\infty G(x-t)d\alpha(t)$$

have abscissa of convergence γ_c . We set

$$\begin{aligned} (2) \quad A_1(t) &= a_1 e^{a_1 t} \int_t^\infty e^{-a_1 u} d\alpha(u), \\ A_m(t) &= a_m e^{a_m t} \int_t^\infty e^{-a_m u} A_{m-1}(u) du \quad (m = 2, \dots, n) \end{aligned}$$

where n is chosen so large that $s - \alpha_2$ is not a zero of

$$\prod_{n+1}^\infty \left(1 - \frac{s}{a_k}\right) e^{s/a_k}.$$

We further define the auxiliary kernel

$$(3) \quad H(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(s(t - \sum_{k=1}^n a_k^{-1}))}{e^{bs} \prod_{n+1}^\infty (1 - (s/a_k)) e^{s/a_k}} ds \quad (-\infty < t < \infty).$$

LEMMA 3a. If $f(x)$ is given by (1) and if $A_n(t)$ and $H(t)$ are defined as in (2) and (3) then

$$A. \quad f(x) = \int_{-\infty}^\infty H(x-t)A_n(t)dt \quad (x > \gamma_c),$$

$$B. \quad A_n(t) = o(e^{a_2 t}) \quad (t \rightarrow +\infty).$$

The convergence of the integral (1) insures that $A_1(t)$ is defined, see [3; §15]. Writing

$$f(x) = \int_{-\infty}^\infty G(x-t)d\alpha(t) = \int_{-\infty}^\infty G(x-t)e^{a_1 t}e^{-a_1 t}d\alpha(t)$$

and integrating by parts we have

$$\begin{aligned} \int_{-\infty}^{\infty} G(x-t) d\alpha(t) &= \left[-G(x-t)e^{a_1 t} \int_t^{\infty} e^{-a_1 u} d\alpha(u) \right]_{-\infty}^{\infty} \\ &\quad + \int_{-\infty}^{\infty} A_1(t) \left[\frac{1}{a_1} e^{-a_1 t} \frac{d}{dt} e^{a_1 t} G(x-t) \right] dt. \end{aligned}$$

It is easily verified that for $x > \gamma_c$ the integrated term vanishes; see [3; §§16, 20]. Thus

$$(4) \quad f(x) = \int_{-\infty}^{\infty} \left\{ \left[1 + \frac{D}{a_1} \right] G(x-t) \right\} A_1(t) dt \quad (x > \gamma_c).$$

The convergence of the integral (4) insures that $A_2(t)$ is defined, and so forth. After n such steps we obtain

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \left\{ \prod_1^n \left(1 + \frac{D}{a_k} \right) G(x-t) \right\} A_n(t) dt \\ &= \int_{-\infty}^{\infty} H(x-t) A_n(t) dt, \end{aligned}$$

which is conclusion A.

We may establish conclusion B by induction. We know that $I(t) = \int_t^{\infty} e^{-a_2 u} d\alpha(u)$ converges. If $\alpha_2 = a_1$, conclusion B is obvious for $n=1$. If $\alpha_2 < a_1$, then

$$\begin{aligned} A_1(t) &= a_1 e^{a_1 t} \left\{ \left[-e^{-a_1 u + a_2 u} I(u) \right]_t^{\infty} + (\alpha_2 - a_1) \int_t^{\infty} e^{-a_1 u + a_2 u} I(u) du \right\}, \\ &= o(e^{a_2 t}) \quad (t \rightarrow +\infty). \end{aligned}$$

Suppose now that our result is proved for $m-1$. We will establish it for m . If $a_m = \alpha_2$ it is obvious. If $a_m > \alpha_2$, then using our induction assumption we see that

$$\begin{aligned} A_m &= o \left(e^{a_m t} \int_t^{\infty} e^{-a_m u} e^{a_2 u} du \right) \\ &= o(e^{a_2 t}) \quad (t \rightarrow +\infty). \end{aligned}$$

Conclusion B now follows.

LEMMA 3b. If

1. $G(t) \in \text{class II}$,

2. $f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ has abscissa of convergence γ_c ,

then

$$\int_{-\infty}^{\rho} G(x-t) d\alpha(t) = O[G(x-\rho)] \quad (x \rightarrow +\infty).$$

If δ is any number greater than γ_c then we know, see [3; §§16, 20], that there is a positive constant A such that

$$|\alpha(t)| \leq A e^{x(\delta-t)} \quad (-\infty < t \leq \rho),$$

where $\chi(t) = -\log G(t)$. Integrating by parts we find that for $x > \delta$

$$\int_{-\infty}^{\rho} G(x-t) d\alpha(t) = \int_{-\infty}^{\rho} \chi'(x-t) e^{-\chi(x-t)} \alpha(t) dt + O[G(x-\rho)].$$

For x sufficiently large $\chi'(x-t)$ is of constant (positive) sign in $(-\infty < t < \rho)$ therefore

$$\left| \int_{-\infty}^{\rho} \chi'(x-t) e^{-\chi(x-t)} \alpha(t) dt \right| \leq A \int_{-\infty}^{\rho} \chi'(x-t) e^{-\chi(x-t)+\chi(\delta-t)} dt.$$

Integrating by parts again,

$$\begin{aligned} \int_{-\infty}^{\rho} \chi'(x-t) e^{-\chi(x-t)+\chi(\delta-t)} dt \\ = \left[e^{-\chi(x-t)+\chi(\delta-t)} \right]_{-\infty}^{\rho} + \int_{-\infty}^{\rho} \chi'(\delta-t) e^{-\chi(x-t)+\chi(\delta-t)} dt. \end{aligned}$$

Now

$$\left[e^{-\chi(x-t)+\chi(\delta-t)} \right]_{-\infty}^{\rho} = G(x-\rho)/G(\delta-\rho).$$

Further if $(\theta > \delta > \gamma_c)$, then

$$\begin{aligned} \int_{-\infty}^{\rho} \chi'(\delta-t) e^{-\chi(x-t)+\chi(\delta-t)} dt \\ = \int_{-\infty}^{\rho} \chi'(\delta-t) e^{\chi(\delta-t)-\chi(\theta-t)} e^{\chi(\theta-t)-\chi(x-t)} dt \\ \leq \sup_{-\infty < t \leq \rho} [G(x-t)/G(\theta-t)] \int_{-\infty}^{\rho} \left| \chi'(\delta-t) \right| e^{\chi(\delta-t)-\chi(\theta-t)} dt. \end{aligned}$$

It may be verified that for x sufficiently large,

$$\sup_{-\infty < t \leq \rho} [G(x-t)/G(\theta-t)] = G(x-\rho)/G(\theta-\rho),$$

see [3; §20]. Thus

$$\int_{-\infty}^{\rho} \left| \chi'(\delta - t) \right| e^{\chi(\delta - t) - \chi(x - t)} dt = O[G(x - \rho)] \quad (x \rightarrow +\infty).$$

Combining our results we have as desired,

$$\int_{-\infty}^{\rho} G(x - t) d\alpha(t) = O[G(x - \rho)] \quad (x \rightarrow +\infty).$$

4. The general case, $G(t) \in \text{class II}$.

THEOREM 4. If

1. $G(t) \in \text{class II}$,
2. $f(x) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t)$ has abscissa of convergence γ_c ,
3. $f(x) = O[G(x - \rho)]$ ($x \rightarrow +\infty$),

then $\alpha(t)$ is constant for $\rho < t < \infty$.

By conclusion A of Lemma 3a we have

$$f(x) = \int_{-\infty}^{\infty} H(x - t) A_n(t) dt$$

where H and A_n are defined in §3. It is known [3; §§20, 21] that if $\epsilon > 0$

$$\lim_{x \rightarrow +\infty} G(x - \rho) / H(x - \rho - \epsilon) = 0.$$

Thus

$$(1) \quad f(x) = O[H(x - \rho - \epsilon)] \quad (x \rightarrow +\infty).$$

By Lemma 3b and equation (1)

$$\int_{\rho + \epsilon}^{\infty} H(x - t) A_n(t) dt = O[H(x - \rho - \epsilon)] \quad (x \rightarrow +\infty).$$

If we choose c ($\alpha_2 < c < \min_{k > n} a_k$), then by conclusion B of Lemma 3a

$$\int_{\rho + \epsilon}^{\infty} H(x - t) A_n(t) dt = \int_{\rho + \epsilon}^{\infty} H(x - t) e^{ct} [e^{-ct} A_n(t)] dt,$$

where $e^{-ct} A_n(t) \in L(\rho + \epsilon, \infty)$. We may apply Theorem 2 to conclude that $A_n(t) = 0$ ($\rho + \epsilon < t < \infty$). Since ϵ is arbitrary $A_n(t) = 0$ ($\rho < t < \infty$). Using the definition of $A_n(t)$ it is easily seen that $\alpha(t)$ is constant for ($\rho < t < \infty$). q.e.d.

5. $G(t) \in \text{class Ia}$. In order to illustrate the essential point of the argument which follows let us consider a very special case. Let $G(t) \in \text{class Ia}$ and let

$$(1) \quad f(x) = \int_{-\infty}^{\infty} G(x - t) \phi(t) dt$$

where $\phi(t) \in L^2(-\infty, \infty)$. We shall show that if

$$(2) \quad f(x) = O(\bar{G}(x - \rho)) \quad (x \rightarrow +\infty)$$

for arbitrarily large negative ρ then $f(x) \equiv 0$. Let $\mathfrak{F}$ denote the Fourier transformation

$$\mathfrak{F}_t \psi = \frac{1}{(2\pi)^{1/2}} \text{l.i.m.}_{T \rightarrow \infty}^{(2)} \int_{-T}^T \psi(u) e^{itu} du,$$

and let $\mathfrak{F}_t^{-1}$ denote its inverse

$$\mathfrak{F}_t^{-1} \psi = \frac{1}{(2\pi)^{1/2}} \text{l.i.m.}_{T \rightarrow \infty}^{(2)} \int_{-T}^T \psi(u) e^{-itu} du.$$

We assert that

$$(3) \quad f(x) = \int_{-\infty}^{\infty} \bar{G}(x-t) \left[\mathfrak{F}_t^{-1} \left(\frac{E(-iu)}{\bar{E}(-iu)} \mathfrak{F}_u \phi(t) \right) \right] dt.$$

From equation (1) we see that the Fourier transform of $f(x)$ satisfies the equation $\mathfrak{F}_u f = E(-iu) \mathfrak{F}_u \phi$. Computing the Fourier transform of the right-hand side of equation (3) we obtain

$$\bar{E}(-iu) \mathfrak{F}_u \mathfrak{F}_t^{-1} \left[\frac{E(-iu)}{\bar{E}(-iu)} \mathfrak{F}_u \phi \right] = E(-iu) \mathfrak{F}_u \phi = \mathfrak{F}_u f.$$

The validity of equation (3) follows. By equations (2) and (3) and Theorem 4 $f(x) \equiv 0$, which is what we wished to show. The transformation

$$\phi(t) \rightarrow \mathfrak{F}_t^{-1} \left[\frac{E(-iu)}{\bar{E}(-iu)} \mathfrak{F}_u \phi \right]$$

is a Watson transformation. Our procedure in the general case is an adaptation of this argument.

LEMMA 5a. *If*

1. $G(t) \in \text{class Ia}$,
 2. $|\beta_1|, |\beta_2| < \text{Min} [-\alpha_1, \alpha_2]$,
 3. $\phi(t) = O(e^{\beta_2 t})$ ($t \rightarrow +\infty$) $= O(e^{\beta_1 t})$ ($t \rightarrow -\infty$),
 4. $f(x) = \int_{-\infty}^{\infty} G(x-t) \phi(t) dt$,
 5. $f(x) = O[\bar{G}(x-\rho)]$ as $x \rightarrow +\infty$ for arbitrarily large negative ρ ,
- then $f(x) \equiv 0$.

Choose any constant c , $c \geq 2\alpha_2$. We define

$$(1) \quad F(s) = \exp \{ [b + (|a_1| + c)^{-1} + (|a_2| + c)^{-1}] s \} \cdot \prod_{n=3}^{\infty} \left(1 - \frac{s}{|a_n| + c} \right) e^{s/(|a_n| + c)},$$

$$(2) \quad K(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)} ds \quad (-\infty < t < \infty),$$

$$(3) \quad J(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{F(s)}{E(s)} e^{st} ds \quad (-\infty < t < \infty).$$

The function $K(t)$ is a class II kernel. Let $\eta > 0$ be so small that $\alpha_1 + \eta < \beta_1$ and $\alpha_2 - \eta > \beta_2$. By [3; §9]

$$(4) \quad \begin{aligned} K(t) &= O[e^{(\alpha_2 - \eta)t}] & (t \rightarrow -\infty) \\ &= O[e^{-\eta t}] & (t \rightarrow +\infty) \end{aligned}$$

for every positive K . Further,

$$(5) \quad \int_{-\infty}^{\infty} e^{-st} K(t) dt = \frac{1}{F(s)},$$

the bilateral Laplace transform converging absolutely for $(-\infty < \sigma < \alpha_2)$.

For c chosen as above and for any σ ($\alpha_1 < \sigma < \alpha_2$), it may be verified that

$$\left| 1 - \frac{\sigma + i\tau}{|a_n| + c} \right| \left| 1 - \frac{\sigma + i\tau}{a_n} \right|^{-1}$$

decreases as $|\tau|$ increases. Let $\alpha_1 < \sigma_1 \leq \sigma \leq \sigma_2 < \alpha_2$ be any proper subinterval of $(\alpha_1 < \sigma < \alpha_2)$, and let

$$M = \max_{\sigma_1 \leq \sigma \leq \sigma_2} |F(\sigma)/E(\sigma)|.$$

We have

$$\left| \frac{F(s)}{E(s)} \right| \leq \left| \left(1 - \frac{s}{a_1} \right) \left(1 - \frac{s}{a_2} \right) \right|^{-1} M \quad (\sigma_1 \leq \sigma \leq \sigma_2).$$

It follows that

$$(6) \quad \left| \frac{F(s)}{E(s)} \right| = O\left(\frac{1}{\tau^2}\right) \quad (\tau \rightarrow \pm \infty)$$

uniformly in σ for σ in any proper subinterval of $(\alpha_1 < \sigma < \alpha_2)$. Because of this order relation we may deform the line of integration of the integral (3) to $\sigma = \alpha_2 - \eta$ or to $\sigma = \alpha_1 + \eta$. We obtain

$$\begin{aligned} J(t) &= \frac{1}{2\pi i} \int_{\alpha_2 - \eta - i\infty}^{\alpha_2 - \eta + i\infty} \frac{F(s)}{E(s)} e^{st} ds \\ &= \frac{1}{2\pi i} \int_{\alpha_1 + \eta - i\infty}^{\alpha_1 + \eta + i\infty} \frac{F(s)}{E(s)} e^{st} ds, \end{aligned}$$

and these together with equation (6) imply that

$$(7) \quad \begin{aligned} J(t) &= O(e^{(\alpha_2 - \eta)t}) & (t \rightarrow -\infty) \\ &= O(e^{(\alpha_1 + \eta)t}) & (t \rightarrow +\infty). \end{aligned}$$

Further, a simple application of a theorem of Hamburger, see [6; pp. 265-266] shows that

$$(8) \quad \int_{-\infty}^{\infty} J(t) e^{-st} dt = \frac{F(s)}{E(s)},$$

the integral converging absolutely for $(\alpha_1 < \sigma < \alpha_2)$.

From equations (5) and (8) and the convolution theorem for the bilateral Laplace transform we have

$$(9) \quad \int_{-\infty}^{\infty} K(x-t) J(t) dt = G(x) \quad (-\infty < x < \infty).$$

Consequently we may write

$$(10) \quad f(x) = \int_{-\infty}^{\infty} G(x-t) \phi(t) dt = \int_{-\infty}^{\infty} \phi(t) dt \int_{-\infty}^{\infty} K(x-u) J(u-t) du.$$

Because of the order conditions (4) and (7) and assumption 3 we may invert the order of integration to obtain

$$(11) \quad f(x) = \int_{-\infty}^{\infty} K(x-u) \Phi(u) du,$$

where

$$\Phi(u) = \int_{-\infty}^{\infty} J(u-t) \phi(t) dt.$$

Consider the class II kernel $L(t)$,

$$L(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{e^{bs} \prod_1^{\infty} (1 - s/(|a_k| + c)) e^{s/(|a_k| + c)}} ds \quad (-\infty < t < \infty).$$

It may be shown, see [3; §20], that

$$\bar{G}(t) = \left[\prod_1^{\infty} \left(1 + \frac{c}{|a_k|} \right) e^{-c/|a_k|} \right]^{-1} e^{-ct} L \left(t - \sum_1^{\infty} \frac{c}{|a_k| (|a_k| + c)} \right).$$

Assumption 5 thus implies that $f(x) = O[L(x-\rho)]$ ($x \rightarrow +\infty$) for arbitrarily large negative ρ . It follows from [3; §§20, 21] that if $\epsilon > 0$ then

$$\lim_{t \rightarrow \infty} K(t)/L(t + \epsilon) = \infty.$$

Hence

$$(12) \quad f(x) = O[K(x - \rho)] \quad (x \rightarrow +\infty)$$

for arbitrarily large negative ρ . Equations (11) and (12) and Theorem 4 imply that

$$f(x) \equiv 0,$$

which is what we wished to prove.

THEOREM 5b. *If*

1. $G(t) \in \text{class Ia}$,

2. $f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$,

3. $f(x) = O[\bar{G}(x - \rho)] \quad (x \rightarrow +\infty)$ for arbitrarily large negative ρ where $\bar{G}$ is defined as in §1,
then

$$f(x) \equiv 0.$$

Choose n so large that $s - \alpha_1$ and $s - \alpha_2$ are no longer zeros of

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k},$$

and so large that if $\alpha_3 = \min |a_k|$, $k = n+1, n+2, \dots$, then $\alpha_3 > \max [-\alpha_1, \alpha_2]$. Let

$$H(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(s[t - \sum_{k=1}^n a_k^{-1}])}{e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k) e^{s/a_k}} ds,$$

$$\bar{H}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(s[t - \sum_{k=1}^n a_k^{-1}])}{e^{bs} \prod_{k=1}^{\infty} (1 - s/|a_k|) e^{s/|a_k|}} ds.$$

Further we set

$$A_1(t) = a_1 e^{a_1 t} \int_t^{\infty} e^{-a_1 u} d\alpha(u) \quad (\text{if } a_1 > 0)$$

$$= a_1 e^{a_1 t} \int_{-\infty}^t e^{-a_1 u} d\alpha(u) \quad (\text{if } a_1 < 0),$$

$$A_m(t) = a_m e^{a_m t} \int_t^{\infty} e^{-a_m u} A_{m-1}(u) du \quad (\text{if } a_m > 0)$$

$$= a_m e^{a_m t} \int_{-\infty}^t e^{-a_m u} A_{m-1}(u) du \quad (\text{if } a_m < 0),$$

for $m = 2, \dots, n$. Just as in Lemma 3 we may show that

$$f(x) = \int_{-\infty}^{\infty} H(x-t)A_n(t)dt$$

and that

$$\begin{aligned} A_n(t) &= o(e^{\alpha_2 t}) & (t \rightarrow +\infty) \\ &= o(e^{\alpha_1 t}) & (t \rightarrow -\infty). \end{aligned}$$

By our choice of n

$$H(t) = O(e^{-\alpha_3 |t|}) \quad (t \rightarrow +\infty)$$

where $\alpha_3 > \max[-\alpha_1, \alpha_2]$. It may be shown, see [3; §§20, 21] that if $\epsilon > 0$ then

$$\lim_{x \rightarrow +\infty} \bar{G}(t)/\bar{H}(t-\epsilon) = 0.$$

Hence assumption 3 implies that

$$f(x) = O(\bar{H}(x-\rho)) \quad (x \rightarrow +\infty)$$

for arbitrarily large negative ρ . We may now apply Lemma 5a to conclude that $f(x) \equiv 0$.

Similarly we may prove

THEOREM 5c. *If*

1. $G(t) \in \text{class Ia}$,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$,
3. $f(x) = \bar{G}(\rho-x)$ ($x \rightarrow -\infty$) for arbitrarily large positive ρ ,

then

$$f(x) \equiv 0.$$

6. Class III kernels. A kernel $G(t)$ is said to belong to class III if the corresponding product $E(s)$ has only positive zeros and if $\sum_1^{\infty} a_k^{-1} < \infty$. The following result may be proved exactly as Theorem 4 was proved. The analogue of Theorem 2 which is needed here reduces to a special case of a well known result, see [5; pp. 322-327].

THEOREM 6. *If*

1. $G(t) \in \text{class III}$,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$ is defined for $x > T + b + \sum_1^{\infty} a_n^{-1}$,
3. $f(x) = 0$ ($x > \rho + b + \sum_1^{\infty} a_n^{-1}$, $\rho \geq T$),

then $\alpha(t)$ is constant for $(\rho < t < \infty)$.

BIBLIOGRAPHY

1. I. I. Hirschman, Jr., *A new representation and inversion theory for the Laplace integral*, Duke Math. J. vol. 15 (1948) pp. 473-494.
2. I. I. Hirschman, Jr., and D. V. Widder, *An inversion and representation theory for convolu-*

tion transforms with totally positive kernels, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 152–156.

3. ———, *The inversion of a general class of convolution transforms*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201.

4. I. J. Schoenberg, *On totally positive functions, Laplace integrals, and entire functions of the Laguerre-Pólya-Schur type*, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 11–17.

5. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford University Press, 1937.

6. D. V. Widder, *The Laplace transform*, Princeton University Press, 1941.

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