A GEOMETRIC APPROACH TO THE REPRESENTATIONS OF THE FULL LINEAR GROUP OVER A GALOIS FIELD

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1. Introduction. In this paper, methods are given for obtaining a large number of representations of $G \equiv GL(n, q)$, the group of all nondegenerate linear transformations of $S(n, q)$, the n -dimensional vector space over the Galois field $GF(q)$ of $q = p^r$ elements; this group will be considered equivalently as the group of all nondegenerate n by n matrices over the Galois field.

In §2, through a favorable comparison of G and H , the symmetric group on n symbols, we obtain a basic set of $p(n)$ irreducible⁽¹⁾ characters of G closely related to those of H . Among the characters obtained is one of degree $q^{n(n-1)/2}$ which is of particular interest from the group theoretical and the modular representations [1]⁽²⁾ points of view since $q^{n(n-1)/2}$ is the highest power of a prime p dividing the order g of G . In §3, the characters of this representation are computed explicitly.

In §4, by making use of linear characters of suitably chosen subgroups of G , a large number of irreducible characters of G is obtained.

The methods used involve the elementary properties of finite group representations⁽³⁾ and characters, especially of permutation representations, and the Frobenius formula for induced characters which enables one to find a character of a group if he knows one for a subgroup [2].

2. We shall first determine $p(n)$ irreducible representations of G by considering simple geometric properties of $S(n, q)$ or, briefly, of $S(n)$. Corresponding to a fixed partition of n , $n = \nu_1 + \nu_2 + \cdots + \nu_n \equiv (\nu)$, say, with $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$, let $e(\nu)$ be a sequence of subspaces $S(\nu_1), S(\nu_1 + \nu_2), \cdots, S(\nu_1 + \nu_2 + \cdots + \nu_{n-1}), S(n)$ which are such that each subspace is contained in the following one. The $e(\nu)$'s are permuted by the elements of G , and hence we get a permutation representation $C(\nu)$ of G of degree

$$(2.1) \quad c(\nu) = \{n\} / \{\nu_1\} \{\nu_2\} \cdots \{\nu_n\},$$

the number of $e(\nu)$'s, where the notation used is

$$(2.2) \quad [r] = q^{r-1} + q^{r-2} + \cdots + q + 1, \quad \{r\} = \prod_{i=1}^r [i].$$

Presented to the Society, February 26, 1949; received by the editors December 18, 1948 and November 23, 1949 and, in revised form, February 3, 1951.

⁽¹⁾ $p(n)$ denotes the number of ways of dividing n into non-negative integers.

⁽²⁾ Numbers in brackets refer to the references at the end of the paper.

⁽³⁾ An account of these properties and methods can be found in [3].

We shall find it convenient to define $[r]$ for all integers by

$$(2.3) \quad [-r] = -q^{-r}[r].$$

Then, for any two integers r and s , we have

$$(2.4) \quad [r] - [s] = q^s[r - s].$$

Corresponding to the $p(n)$ partitions of n , we thus get $p(n)$ representations of G .

Now, let $\lambda_i = \nu_i + (i-1)$. Then, the λ 's satisfy the relations

$$(2.5) \quad \begin{aligned} 0 &\leq \lambda_1 < \lambda_2 < \cdots < \lambda_n, \\ \sum_{i=1}^n \lambda_i &= n(n+1)/2. \end{aligned}$$

Also, let us define $\text{sgn}(\kappa_1, \kappa_2, \dots, \kappa_n)$ to be 0 if two of the κ 's are equal and otherwise 1 or -1 according as $\kappa_1, \kappa_2, \dots, \kappa_n$ form an even or an odd permutation of the κ 's written in ascending order of magnitude.

Then our main result of this section is the following theorem.

THEOREM 2.1. $\Gamma(\nu) = \sum_{\kappa} \text{sgn}(\kappa_1, \kappa_2, \dots, \kappa_n) C(\lambda_1 - \kappa_1, \lambda_2 - \kappa_2, \dots, \lambda_n - \kappa_n) = \sum_{\kappa} \text{sgn}(\kappa) C(\lambda - \kappa)$ is an irreducible representation of G . (The summation is made as $\kappa_1, \kappa_2, \dots, \kappa_n$ run through the $n!$ permutations of the numbers $0, 1, 2, \dots, n-1$; and $C(\lambda - \kappa)$ is defined to be 0 if any $\lambda_i - \kappa_i$ is negative.) Moreover, the $p(n)$ representations $\Gamma(\nu)$ so obtained are all distinct.

We shall first prove two lemmas of a geometric nature on which the proof of Theorem 2.1 depends.

LEMMA 2.1. If $e^{(1)}$ and $e^{(2)}$ are two⁽⁴⁾ $e(1^n)$'s, there exist n vectors $V_1, V_2, \dots, V_n$ which, when taken in the order $V_1 V_2 \cdots V_n$, span $e^{(1)}$, and, taken in some other order $V_{p_1} V_{p_2} \cdots V_{p_n}$, span $e^{(2)}$.

We say that the ordered set of vectors $V_1 V_2 \cdots V_n$ spans $e^{(1)} = S^{(1)}(1), S^{(1)}(2), \dots, S^{(1)}(n)$ if $S^{(1)}(i) = \{V_1, V_2, \dots, V_i\}$ for $i = 1, 2, \dots, n$.

Proof of Lemma 2.1. The proof will be by induction, and, since the lemma is trivially true for $n = 1$ or 2 , we shall assume $n \geq 3$, and that the lemma is true for $n-1$.

Now $e^{(1)}$ and $e^{(2)}$ intersect $S^{(1)}(n-1)$ in two $e(1^{n-1})$'s which we shall denote by $e^{(1)'}$ and $e^{(2)'}$. By the induction assumption, we can choose $n-1$ vectors $V_1, V_2, \dots, V_{n-1}$ such that $V_1 V_2 \cdots V_{n-1}$ and $V_{p_1} V_{p_2} \cdots V_{p_{n-1}}$ span $e^{(1)'}$ and $e^{(2)'}$ respectively. If $S^{(1)}(n-1)$ contains $S^{(2)}(i)$ but not $S^{(2)}(i+1)$, choose V_n to be any vector in $S^{(2)}(i+1)$ but not in $S^{(2)}(i)$. Then $V_1 V_2 \cdots V_n$ and $V_{p_1} V_{p_2} \cdots V_{p_i} V_n V_{p_{i+1}} \cdots V_{p_{n-1}}$ span $e^{(1)}$ and $e^{(2)}$ respectively, and the lemma is proved.

(4) (1^n) denotes the partition of n into n ones.

We next note an analogy between G and H , the symmetric group on n symbols, and make use of this analogy. Corresponding to the general partition (ν) of n , we define an $s(\nu)$ to be an entity consisting of the symbols 1 to n in any order, the first ν_1 symbols being bracketed together, the next ν_2 being bracketed together, and so forth. Two $s(\nu)$'s are considered to be the same if one can be obtained from the other by permuting the symbols in their separate brackets; for example, a typical $s(2^2)$ is 12, $34 \equiv 21$, $34 \equiv 12$, $43 \equiv 21$, 43. The $s(\nu)$'s are permuted by the elements of H , and hence furnish a permutation representation $D(\nu)$ of H of degree

$$(2.6) \quad d(\nu) = n!/\nu_1!\nu_2! \cdots \nu_n!,$$

the number of $s(\nu)$'s (cf. (2.1)).

It should be noted here that $D(\nu)$ is the representation of H induced by the unit representation of the subgroup $H(\nu) = H\nu_1 \times H\nu_2 \times \cdots \times H\nu_n$, where H_i is a symmetric group on i symbols, and that $C(\nu)$ is the representation of G induced by the unit representation of the subgroup $G(\nu)$ consisting of matrices with square blocks of degrees $\nu_1, \nu_2, \cdots, \nu_n$ (in this order) down the main diagonal and zeros above them.

LEMMA 2.2. *The number of classes of $(e(\nu), e(\mu))$'s is equal to the number of classes of $(s(\nu), s(\mu))$'s.*

Here, we mean that $(e^{(1)}(\nu), e^{(1)}(\mu))$ and $(e^{(2)}(\nu), e^{(2)}(\mu))$, for example, belong to the same class if there is an element of G taking $e^{(1)}(\nu)$ into $e^{(2)}(\nu)$ and $e^{(1)}(\mu)$ into $e^{(2)}(\mu)$.

Proof of Lemma 2.2. For any $(e(\nu), e(\mu))$, we supply the missing subspaces, if necessary, and then by Lemma 2.1 we choose n vectors $V_1, V_2, \cdots, V_n$ such that $V_1 V_2 \cdots V_n$ and $V_{p_1} V_{p_2} \cdots V_{p_n}$ span $e(\nu)$ and $e(\mu)$ respectively. We then associate the class containing $(e(\nu), e(\mu))$ with the class containing the $(s(\nu), s(\mu))$ given by

$$(1 \ 2 \ \cdots \ \nu_1, \nu_1 + 1 \ \cdots \ \nu_1 + \nu_2, \cdots, n; \\ p_1 \ p_2 \ \cdots \ p_{\mu_1}, p_{\mu_1+1} \ \cdots \ p_{\mu_1+\mu_2}, \cdots \ p_n).$$

Since there is a nondegenerate linear transformation taking any n vectors which span $S(n)$ into any other n vectors which span $S(n)$, we see that two $(e(\nu), e(\mu))$'s belong to the same class if and only if the corresponding $(s(\nu), s(\mu))$'s do, and this proves the lemma.

Geometrically, we have the following result.

COROLLARY. *There is a nondegenerate linear transformation taking any $(e(\nu), e(\mu))$ into any other $(e(\nu), e(\mu))$ with the same degrees of intersection of all corresponding pairs of subspaces.*

Now, let $\psi(\nu)$, $\phi(\nu)$, and $\chi(\nu)$ be the characters of $C(\nu)$, $D(\nu)$, and $\Gamma(\nu)$

respectively. Then, the permutation representations of the $(e(\nu), e(\mu))$'s and the $(s(\nu), s(\mu))$'s are given by the Kronecker products $C(\nu) \times C(\mu)$ and $D(\nu) \times D(\mu)$, and thus their characters by $\psi(\nu)\psi(\mu)$ and $\phi(\nu)\phi(\mu)$ respectively. But, for any permutation representation of character χ of a group of order g , $g^{-1} \sum \chi(x)$ is the number of times χ contains the unit character, that is, the number of classes of transitivity. (The summation is made over all elements x of G .) This remark together with Lemma 2.2 proves the following lemma.

LEMMA⁽⁵⁾ 2.3. $g^{-1} \sum_{x \in G} \psi(\nu, x)\psi(\mu, x) = h^{-1} \sum_{y \in H} \phi(\nu, y)\phi(\mu, y)$.

Proof of Theorem 2.1. Let $\lambda_i = \nu_i + i - 1$ and $\sigma_i = \mu_i + i - 1$, and let $\kappa_1, \kappa_2, \dots, \kappa_n$ and $\rho_1, \rho_2, \dots, \rho_n$ be permutations of the numbers $0, 1, 2, \dots, n-1$.

Then ⁽⁶⁾,

$$\begin{aligned} g^{-1} \sum_{x \in G} \chi(\nu, x)\chi(\mu, x) &= g^{-1} \sum_{x \in G} \sum_{\kappa} \text{sgn}(\kappa) \psi(\lambda - \kappa, x) \sum_{\rho} \text{sgn}(\rho) \psi^*(\sigma - \rho, x) \\ &= h^{-1} \sum_{y \in H} \sum_{\kappa} \text{sgn}(\kappa) \phi(\lambda - \kappa, y) \sum_{\rho} \text{sgn}(\rho) \phi^*(\sigma - \rho, y), \end{aligned}$$

by the definitions and Lemma 2.3 respectively.

But, $\sum_{\kappa} \text{sgn}(\kappa) \phi(\lambda - \kappa)$ is the character of $\Delta(\nu) = \sum_{\kappa} \text{sgn}(\kappa) D(\lambda - \kappa)$, and it was proved by Frobenius [3] that the $p(n)$ representations $\Delta(\nu)$ of H are irreducible and distinct. Thus

$$(2.7) \quad g^{-1} \sum_{x \in G} \chi(\nu, x)\chi^*(\mu, x) = \delta_{(\nu)(\mu)}.$$

Thus, if in addition $\chi(\nu, E) > 0$, where E is the identity element of G , $\chi(\nu)$, being an integral linear combination of characters, will be an irreducible character.

Now,

$$\begin{aligned} \chi(\nu, E) &= \sum_{\kappa} \text{sgn}(\kappa) \{n\} / \{\lambda_1 - \kappa_1\} \{\lambda_2 - \kappa_2\} \cdots \{\lambda_n - \kappa_n\} \\ (2.8) \quad &= \{n\} \det \left| \{\lambda_i - (j-1)\}^{-1} \right| \quad (\text{where } \{r\}^{-1} \equiv 0 \text{ if } r < 0) \\ &= \frac{\{n\} \det \left| [\lambda_i] [\lambda_i - 1] \cdots [\lambda_i - (j-2)] \right|}{\{\lambda_1\} \{\lambda_2\} \cdots \{\lambda_n\}}. \end{aligned}$$

To evaluate this, let us consider the determinant

$$\left| X_i \left(\frac{X_i - [1]}{q} \right) \left(\frac{X_i - [2]}{q^2} \right) \cdots \left(\frac{X_i - [j-2]}{q^{j-2}} \right) \right|.$$

It is of degree $n(n-1)/2$ in the n indeterminates $X_1, X_2, \dots, X_n$ and van-

⁽⁵⁾ $\psi(\nu, x)$ denotes the character in $\psi(\nu)$ of the element x of G .

⁽⁶⁾ * denotes conjugate complex.

ishes if $X_i = X_j$, $i \neq j$. Hence, it is a constant⁽⁷⁾ times $\Delta(X_1, X_2, \dots, X_n) = \prod_{i < j} (X_j - X_i)$, and a perusal of the coefficient of $X_2 X_3^2 \dots X_n^{n-1}$ shows this constant to be $q^{-n(n-1)(n-2)/3}$. Since this determinant reduces to the one in (2.8) if $X_i = [\lambda_i]$, we finally get

$$(2.9) \quad \begin{aligned} \gamma(\nu) &= \chi(\nu, E) \\ &= q^{-n(n-1)(n-2)/3} \{n\} \Delta([\lambda_1], [\lambda_2], \dots, [\lambda_n]) / \{\lambda_1\} \{\lambda_2\} \dots \{\lambda_n\}. \end{aligned}$$

This is positive since $\lambda_i < \lambda_j$ for $i < j$. This completes the proof of Theorem 2.1, and the equation (2.9) furnishes us with the degree $\gamma(\nu)$ of $\Gamma(\nu)$.

As an immediate consequence of Theorem 2.1 and Lemma 2.3, we get the following corollaries.

COROLLARY 1. *$C(\nu)$ and $D(\nu)$ split into irreducible representations in exactly the same manner; that is, if $C(\nu) = \sum_{(\mu)} k(\nu, \mu) \Gamma(\mu)$, then $D(\nu) = \sum_{(\mu)} k(\nu, \mu) \Delta(\mu)$.*

COROLLARY 2. *Let $(\nu_1), (\nu_2), \dots, (\nu_n)$ be further partitions of $\nu_1, \nu_2, \dots, \nu_n$, let C be the representation of G induced by the representation $\Gamma(\nu_1) \times \Gamma(\nu_2) \times \dots \times \Gamma(\nu_n)$ of $G(\nu)$, and let D be the corresponding representation of H . Then C and D split into irreducible representations in exactly the same manner.*

We can give an alternate representation to that given by (2.5) for the partitions of n , and this will lead to a simpler formula for the degree $\gamma(\nu)$ of $\Gamma(\nu)$. Among the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ there are a certain number, say r , which are not less than n . Denote these by $n+b_1, n+b_2, \dots, n+b_r$. Denote the remaining λ 's by $n-1-a_{r+1}, n-1-a_{r+2}, \dots, n-1-a_n$. Let $n-1-a_1, n-1-a_2, \dots, n-1-a_r$ be the rest of the numbers $0, 1, \dots, n-1$ so that the a 's are a permutation of the numbers $0, 1, \dots, n-1$. Then, we can represent the partition by the two sets of integers $a_1, a_2, \dots, a_r$ and $b_1, b_2, \dots, b_r$. These a 's and b 's can be ordered so that

$$(2.10) \quad 0 \leq a_1 < a_2 < \dots < a_r, \quad 0 \leq b_1 < b_2 < \dots < b_r.$$

Because of (2.5), we have also

$$(2.11) \quad \sum_{i=1}^r a_i + \sum_{i=1}^r b_i = n - r.$$

Conversely, it can be shown that two sets of r integers satisfying the relations (2.10) and (2.11) correspond to a unique partition of n . The partition obtained from a given one by interchanging the a 's and b 's we shall call the conjugate partition⁽⁸⁾.

⁽⁷⁾ Hereafter we shall call this the Vandermonde determinant.

⁽⁸⁾ One can represent a partition diagrammatically by the lattice points (i, j) which satisfy $1 \leq j \leq \nu_{n-i}$ and alternately define the conjugate partition to be the one represented by the points (j, i) . These two definitions are equivalent.

Then, by a computation similar to that used by Frobenius [3], and making use of (2.4), we can show that

$$\gamma(\nu) = q^{r(r-1)/2 + \sum_{i=1}^r (a_i+1)a_i/2} \cdot \frac{\{n\} \Delta([a_1], [a_2], \dots, [a_r]) \Delta([b_1], [b_2], \dots, [b_r])}{\{a_1\} \{a_2\} \cdots \{a_r\} \{b_1\} \{b_2\} \cdots \{b_r\} \prod_{\alpha, \beta=1}^r [a_\alpha + b_\beta + 1]}.$$

From this a direct computation shows that the degree $\gamma(\nu)$, $\gamma(\nu')$ of representations corresponding to conjugate partitions (ν) , (ν') are polynomials in q related by the following formula:

$$(2.13) \quad \gamma(\nu', q) = q^{n(n-1)/2} \gamma(\nu, q^{-1}).$$

As an important special case, if $(\nu) = (0^{n-1}n)$ and $(\nu') = (1^n)$, then $\gamma(\nu) = 1$ (since $\Gamma(\nu)$ in this case is the unit representation) and $\gamma(\nu') = q^{n(n-1)/2}$. As previously stated, this last representation is of group-theoretical importance and in the next section we shall compute its characters explicitly.

In closing this section, a remark on the analogy between G and H seems to be in order. Instead of considering G as a group of linear transformations of a vector space, we could consider G as a collineation group of a finite $(n-1)$ -dimensional geometry. If $q=1$, the vector space fails to exist but the finite geometry does exist and, in fact, reduces to the n vertices of a simplex with a collineation group isomorphic to H . Hence, if we put $q=1$, $[r]=r$, and $\{r\}=r!$ in (2.1), (2.9), (2.12), and (2.13), we get corresponding results for H .

3. In this section, the characters of the representation of degree $q^{n(n-1)/2}$ of G mentioned in the previous section will be found. These characters are given by Theorem 3.1 at the end of this section.

The determination will depend on a number of lemmas which we proceed to state and prove.

By the generating function of an element x of G , we shall mean the symmetric polynomial

$$f(q, t) = \sum f(\nu) t^{(\nu)},$$

where the sum is taken over all partitions (ν) of n , $f(\nu)$ is the number of $e(\nu)$'s left fixed by x , and $t^{(\nu)} = t_1^{\nu_1} t_2^{\nu_2} \cdots t_n^{\nu_n}$.

LEMMA 3.1. *Let A be an irreducible matrix of degree m and En the unit matrix of degree n (over $GF(q)$). Then, if the generating function of En is $f(q, t)$, that of $En \times A$ is $f(q^m, t^m)$.*

Proof. Let the field $GF(q)$ be extended to $GF(q^m)$. In this field A can be transformed to a diagonal matrix R whose main-diagonal elements are

conjugate relative to $GF(q)$. Then, after a further simple transformation, we get $En \times A$ similar to $R \times En$. The latter matrix leaves fixed all of the vectors of m disjoint conjugate (relative to $GF(q)$) $S(n)$'s and all subspaces spanned by such vectors. If we take m conjugate vectors, one from each $S(n)$, we get a fixed real $S(m)$. Similarly, the only fixed real $S(rm)$'s are spanned by conjugate $S(r)$'s. Since each $S(n)$ is a $GF(q^m)$ vector space, the lemma is proved.

LEMMA 3.2. *Let f_1, f_2, f be the generating functions for $A_1, A_2, A_1 \dot{+} A_2$, where A_1 and A_2 are matrices which, when written in canonical form, have no common constituents. Then $f = f_1 f_2$.*

This lemma follows from the definitions and from the fact that any fixed $S(i)$ of $A_1 \dot{+} A_2$ is obtained as the join of a fixed $S(j)$ in the space of A_1 and a fixed $S(i-j)$ in the space of A_2 , this representation being unique.

LEMMA 3.3. *Let f_1, f_2 be homogeneous polynomials of degree m, n respectively, symmetric in the $m+n$ variables t_i , and let $\Delta_1, \Delta_2, \Delta$ be the Vandermonde determinants in $t_1, t_2, \dots, t_m; t_{m+1}, t_{m+2}, \dots, t_{m+n}; t_1, t_2, \dots, t_{m+n}$ respectively. Let c_1, c_2, c be the coefficients of $t_1 t_2^2 \dots t_m^m, t_{m+1} t_{m+2}^2 \dots t_{m+n}^n, t_1 t_2^2 \dots t_{m+n}^{m+n}$ in $f_1 \Delta_1, f_2 \Delta_2, f \Delta$ respectively. Then $c_1 c_2 = c$.*

Proof. If we consider $f_2 \Delta$, it is evident that any term with a repeated index will vanish, and, since we are interested only in those terms in which each t_i occurs to a power of at most $m+n$, we need consider only terms of the sort $t_2 t_3^2 \dots t_m^{m-1} t_{m+1}^{m+1} \dots t_{m+n}^{m+n}$. The coefficient of this term is c_2 . If we multiply by f_1 , we readily see that the terms involving $t_{m+1}^{m+1} \dots t_{m+n}^{m+n}$ are $c_2 f_1 \Delta_1 t_{m+1}^{m+1} \dots t_{m+n}^{m+n}$. Hence the lemma is established.

LEMMA 3.4. *Let $f(t)$ be a homogeneous polynomial of degree n , symmetric in the n variables $t_1, \dots, t_n$, and let Δ_1, Δ be the Vandermonde determinants in $t_1, t_2, \dots, t_n$ and $t_1, t_2, \dots, t_{mn}$ respectively. Then the coefficient of $t_1 t_2^2 \dots t_n^n$ in $f(t) \Delta_1$ is equal to $(-1)^{(m-1)n}$ times the coefficient of $t_1 t_2^2 \dots t_{mn}^{mn}$ in $f(t^m) \Delta$.*

This lemma is evident if one considers how the $t_1 t_2^2 \dots t_{mn}^{mn}$ term is obtained when $f(t^m)$ is multiplied by Δ .

LEMMA 3.5. *For the matrix En , the coefficient of $t_1 t_2^2 \dots t_n^n$ in $f \Delta$ is $q^{n(n-1)/2}$.*

This follows from §2 and the following statement.

This coefficient, in the case of En , gives the degree of the representation under consideration, and, in general, the character of the element in this representation.

LEMMA 3.6. *For a matrix which cannot be reduced to diagonal form in any extension of $GF(q)$, the coefficient of $t_1 t_2 \dots t_n$ in $f \Delta$ is 0.*

This follows from a theorem due to R. Brauer and C. Nesbitt [1] to the

effect that the character of any p -singular element⁽⁹⁾ is 0 in an irreducible representation of order divisible by p^r , where p^r is the highest power of a prime p which divides the order of the group.

From these lemmas and the statement in Lemma 3.5, we get the following theorem.

THEOREM 3.1. *For the element $En \times A$, where A is irreducible and of degree m , the character is $(-1)^{(m-1)n} q^{mn(n-1)/2}$. For any element x of G which is made up of blocks of the type $En \times A$, the character is obtained by multiplying the $(-1)^{(m-1)n} q^{mn(n-1)/2^s}$ for the various blocks. For any p -singular element, the character is 0.*

4. We can make use of the $p(n)$ basic irreducible representations $\Gamma(\nu)$ to determine a large set of irreducible representations of G .

We first note that there are $q-1$ linear representations of G corresponding to the powers of the determinants of the elements x of G . Thus if ρ is a primitive $(q-1)$ th root of unity in $GF(q)$ and ϵ is one in the field of complex numbers, and if $\det x = \rho^\alpha$, these representations are given by $x \rightarrow \epsilon^{\alpha u}$, $u = 1, 2, \dots, q-1$.

If we multiply $\Gamma(\nu)$ by each of these representations, we get $q-1$ irreducible representations which we shall denote by $\Gamma_u(\nu)$.

Let us now return to our general partition $n = \nu_1 + \nu_2 + \dots + \nu_n$ and partition each ν_i further: $\nu_i = \sum \nu_{ij}$. We shall denote these partitions by (ν_i) . Let the partition of ν_i have k_{i1} ones, k_{i2} twos, and so forth, and let $k = \sum k_{ij}$. Corresponding to these partitions, we can take the Kronecker product of representations of the type $\Gamma_u(\nu)$ of the constituents of the subgroup $G(\nu)$ of G and get a representation of G which we can write as $\times \prod_{i=1}^n \Gamma_{u_i}(\nu_i)$. This representation of $G(\nu)$ induces one in G which we shall call Γ .

THEOREM 4.1. *If no two u 's are equal, Γ , is irreducible. Moreover, the number of representations so obtained (by varying the u 's and keeping the ν 's fixed) is $(q-1)(q-2) \dots (q-k) / \prod k_{ij}!$. The degree of each is*

$$(4.1) \quad \gamma = \frac{\gamma(\nu_1)\gamma(\nu_2) \dots \gamma(\nu_n) \{n\}}{\{\nu_1\} \{\nu_2\} \dots \{\nu_n\}}$$

Due to the complexities of notation, the proof will not be given here. It depends on a computation similar to that used by Frobenius [3] in his determination of the characters of the symmetric group.

A combinatorial argument shows that the number of different degrees of characters obtained by this method has as generating function $h(t) = \prod_{i=1}^{\infty} (1-t^i)^{-p(i)}$. I conjecture that the total number of different degrees has as generating function $\prod_{i=1}^{\infty} h(t^i)$. For $n = 1, 2, 3, 4$, this method gives

(9) A p -singular element is one of order divisible by p .

characters of 1, 3, 6, 14 distinct degrees, and the total⁽¹⁰⁾ number of distinct degrees is 1, 4, 8, 22 respectively.

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⁽¹⁰⁾ All of the characters of G for $n=1, 2, 3, 4$ have been determined [5].