IDENTITIES IN TWO-VALUED CALCULI

BY
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1. **Introduction.** Classical two-valued logic is ordinarily treated as a *deductive system*, with certain *propositions* (or propositional functions) given as axioms, from which all true propositions are derivable by substitution and *modus ponens*. Essentially the same calculus may be treated as an *algebraic system* (Boolean algebra), in which the axioms are now certain *equations* (or identities), from which all true equations are derivable by substitution and the usual rules for equality.

Following Post's enumeration(1) of all the two-valued logics, we obtain here a finite set of axioms for each of these, treated as an *algebraic system*. Algebraic axiomatization is ordinarily easier to establish than deductive axiomatization(2); but the algebraic concept is broader in its application—for example, to systems lacking a connective analogous to the conditional, for which the concept of deductive axiomatization is not clear.

We were led to the present investigation in connection with a more general problem(3): *Does every finite algebra* (in the sense of G. Birkhoff) *possess a finite set of identities from which all others are derivable*? It is perhaps surprising that this problem is not entirely trivial even for algebras that contain only two elements.

Our result can be interpreted directly as a theorem about Boolean algebras. Let $f_1, \cdots, f_n$ be any set of Boolean functions; then the algebraic theory having $f_1, \cdots, f_n$ as primitive operations possesses a finite complete set of axioms.

2. **Preliminaries.** In view of the elementary nature of our proofs, we permit ourselves a certain informality of expression.

By an *algebra*(4), A, we mean a certain set of *elements*, a_i, together with

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(1) Post [8]. Numbers in brackets refer to the bibliography.

(2) Deductive axiomatizability has been studied by Wajsberg [13]. Note that his example (p. 241) of a system not deductively axiomatizable is defined by algebraic axioms.

(3) Birkhoff [2] has given necessary and sufficient conditions for a family of algebras to be definable by a set of identities, but leaves open the question of when this set may be taken as finite. By a finite algebra we understand an algebra in which both the number of elements and the number of operations is finite. B. H. Neumann has raised this question for groups [7, p. 520]; Specht for rings [10, p. 575]; and an analogous problem for linear algebras has been posed by Amitsur and Levitzki [1, p. 462]. Somewhat different concepts of algebraic axiomatizability have been employed by Malcev [6] and by Lyndon [5].

(4) Our definition is an informal paraphrase of that of Birkhoff. The "identities" considered here may be interpreted rigorously as equations in a functionally free algebra of the family generated by A; see Tarski [12].

a set of *primitive functions* f_j (of various numbers of arguments) defined for all sets of arguments from A and assuming values in A. Any function compounded out of the primitive functions in the usual sense will be said to belong to A, and two such functions will be called equal if their values agree for all sets of arguments. It will be convenient further to identify a "reducible" function, that does not effectively depend upon certain of its arguments, with the corresponding function of the remaining arguments. Thus, in Boolean algebra, the reducible function $f(x)=x\vee x'$ of a single argument will be identified with the constant function 1.

A complete set of *axioms* for A is a set of (true) equations

$$\phi_1(x_1, \cdots, x_{n_1}) = \psi_1(x_1, \cdots, x_{m_1}), \cdots,$$
$$\phi_k(x_1, \cdots, x_{n_k}) = \psi_k(x_1, \cdots, x_{m_k}), \cdots,$$

where the x_i are indeterminates and the ϕ_i and ψ_i functions belonging to A, from which all identically true equations of the same form are derivable by means of the following rules:

(I) reflexivity and symmetry of equality;

(II) uniform substitution for a variable x_i in any established equation;

(III) given $\phi=\psi$, substitution of ϕ for ψ at any occurrence in an established equation.

An algebra is *axiomatizable* if it possesses a finite complete set of axioms.

Two algebras A and A' will be called *equivalent* if they possess the same elements and the same functions. The primitives of one must then be definable in terms of those of the other, whence we have the following theorem.

THEOREM 1. *If A and A' are equivalent, then A is axiomatizable if and only if A' is.*

For the present purpose it is thus not necessary to distinguish between equivalent algebras. Thus we avoid certain trivial complications. In particular, every algebra is equivalent to an algebra none of whose primitives is a reducible function, nor the identity function $f(x)=x$.

3. **Connection with deductive systems.** In an algebra A, let a certain element 1 be "designated" as "true," and let a certain function of two arguments be called the "conditional" and written multiplicatively as xy. Under certain restrictions, A, as a "logical matrix," will define a logical calculus $L(A)$[5], with the rule of inference:

If x and xy are theorems, then y is a theorem.

THEOREM 2. *Suppose that $L(A)$, as a deductive system, has a complete set of axioms: $\alpha_1, \cdots, \alpha_n$. Suppose further that in A the conditional function and the single designated element satisfy the following identities:*

(5) For these concepts see, for example, Wajsberg [13].

$$xx = 1, \tag{1}$$

$$1x = x, \tag{2}$$

$$(xy)y = (yx)x, \tag{3}$$

where x and y represent arbitrary elements of A. Then A, as an algebraic system, possesses a complete set of axioms consisting of (1), (2), *and* (3) *together with the equations* $\alpha_1 = 1, \cdots, \alpha_n = 1$.

Proof. Clearly the equations listed are true in A.

If f is an axiom of L, then $f=1$ is among the axioms listed for A.

If f is a theorem of L, and $f=1$ has been derived from the axioms for A by the rules I, II, and III, then $f'=1$ follows in A for any f' obtained from f by uniform substitution.

Suppose f and fg are theorems of L, and that $f=1$ and $fg=1$ have been derived from the axioms for A. Then substitution (in accordance with III) gives $1g=1$, whence from (2) it follows that $g=1$ is derivable from the axioms for A.—This proves, recursively, that if f is a theorem of L, then $f=1$ follows from the axioms for A.

Suppose now that $f=g$ is true in A. By (1) it follows that $fg=1$ and $gf=1$ are true equations, whence fg and gf must be theorems of L. Therefore $fg=1$ and $gf=1$ are derivable from the axioms for A. But (3) gives $(gf)f=(fg)g$, whence by substitution $1f=1g$, and applying (2) now gives $f=g$. Thus $f=g$ is derivable from the axioms for A.

Remark. That the three equations (1), (2), (3) in A cannot be replaced by the six corresponding conditionals in $L(A)$ is shown by the three-valued calculus with xy defined by the accompanying table; for in this calculus the six conditionals hold while equation (2) fails.

xy	1	2	3
1	1	3	3
2	1	1	1
3	1	1	1

Note that the class of "designated" elements and the selected "conditional" function play no special role in A as an algebra. Thus the correspondence between logic and algebra is not unique in either sense. This ambiguity may be exploited to deduce, from Rosser and Turquette's deductive axiomatization[6] of the Łukasiewicz-Tarski m-valued calculi (with an arbitrary number of designated values), the following result:

Corollary 2.1. *The algebra defined by the matrix for each of the Łukasiewicz-Tarski m-valued logics is axiomatizable.*

(6) Rosser and Turquette [9].

Henkin has shown(7) that every two-valued logic containing the classical conditional function is deductively axiomatizable. By Theorem 2 we conclude:

COROLLARY 2.2. *Every two-element algebra containing the classical conditional function is (algebraically) axiomatizable.*

4. **Post's iterative systems.** An algebra A containing just two elements, which we shall designate as 0 and 1, constitutes what Post has called a two-valued iterative system. Post has enumerated all such algebras, and we repeat below what is essentially his enumeration. However, in accordance with Theorem 1, we list only one out of each set of equivalent algebras. Also, we omit those systems with only constant functions, which are vacuously axiomatizable. Finally, we define the *dual* of a function f to be the function obtained from f under the interchange of the two elements 0 and 1 of A. The dual of an algebra A is the algebra whose functions are precisely the duals of those of A. Since an algebra is isomorphic to its dual, we include in our list only one out of each pair of duals.

A two-valued algebra is fully described by listing a set of primitive functions. For this purpose we employ the following notation:

0 and 1 for the two (dual) constant functions;

Nx for the self-dual function of complementation (or negation);

$x \vee y$ for the union (maximum) function, and $x \wedge y$, or simply xy, for the dual intersection (minimum) function;

$x \equiv y$ (equivalence) and its dual $x+y$ (symmetric difference);

$x \supset y$ (conditional) and its dual $x-y$ (set difference: xNy);

$x+y+z$, self-dual;

$$(x, y, z) = x(y \vee z), \ [x, y, z] = x(y \equiv z), \text{ and, for each } n \geqq 3,$$

$$d_n(x_1, \cdots, x_n) = x_2x_3 \cdots x_n \vee x_1x_3 \cdots x_n \vee \cdots \vee x_1 \cdots x_{n-2}x_{n-1};$$

we shall not require a notation for the duals of these functions. In listing the two-element algebras, we first give the name of the algebra (a capital letter with subscript) in Post's classification; next, a set $(f_1, \cdots, f_n)$ of primitive functions; and thirdly (in certain cases) a fuller equivalent set of primitive functions. For future convenience, we divide our list into five sections.

Ia. $O_4 = (N), \quad O_9 = (N, 0);$
$S_1(\vee), \quad S_4 = (\vee, 0), \quad S_3 = (\vee, 1), S_6 = (\vee, 0, 1);$
$A_4 = (\vee, \wedge), \quad A_2 = (\vee, \wedge, 0), \quad A_1 = (\vee, \wedge, 0, 1);$
$L_3 = (+) = (+, 0), \quad L_1 = (+, N) = (+, N, 0, 1);$
$C_3 = (-, \vee) = (+, \wedge, 0), \quad C_1 = (-, N) = (+, \wedge, 0, 1).$

Ib. $L_4 = (x + y + z), \quad L_5 = (x + y + z, N).$

II. $F_4 = (\supset)$, and $F_4^n = (\supset, d_n)$ for each $n \geqq 3.$

(7) Henkin [4]; the basic axiomatization of the conditional goes back to Tarski.

III. $F_6 = ((x, y, z)) = ((x, y, z), xy)$;
$F_7 = ((x, y, z), 0) = ((x, y, z), xy, 0)$;
$F_5 = ([x, y, z]) = ([x, y, z], (x, y, z), xy)$;
$C_4 = ([x, y, z], x \vee y) = ([x, y, z], (x, y, z), xy, x \vee y)$;
$F_6^n = ((x, y, z), d_n) = ((x, y, z), d_n, xy)$ for each $n \geqq 3$;
$F_7^n = ((x, y, z), d_n, 0) = ((x, y, z), d_n, 0, xy)$ for each $n \geqq 3$;
(Note that, for $n > 3$, $F_6^n = (d_n)$ and $F_7^n = (d_n, 0)$.)
$F_5^n = ([x, y, z], d_n = ([x, y, z], (x, y, z), d_n, xy)$ for each $n \geqq 3$.

IV. $D_2 = (d_3)$, $D_1 = (d_3, x + y + z)$, $D_3 = (d_3, x + y + z, N)$.

5. **Systems I.** For each of the systems Ia a complete set of axioms can be chosen by inspection from the various familiar sets of axioms for Boolean algebras and Boolean rings. Completeness can be proved by showing that the chosen set of axioms serves to reduce every expression to a prescribed normal form, and that distinct normal forms represent distinct functions.

The same method applies to systems Ib. For example, if we temporarily abbreviate $x+y+z$ to xyz, system L_5 has the following set of axioms:

$$NNx = x, \qquad N(xyz) = (Nx)yz, \qquad xyy = x,$$
$$xyz = yxz, \qquad xyz = xzy, \qquad xy(zuv) = (xyz)uv.$$

Completeness is established by reference to the normal forms

$$a, \; Na, \; ab(cd(\cdots(pqr)\cdots)), \; N(ab(cd(\cdots(pqr)\cdots))),$$

where $a, b, \cdots, p, q, r$ are distinct variables in alphabetical order.

6. **Systems II.** The axiomatizability of these systems, which all contain the conditional, follows from Corollary 2.2. Alternatively, for the dual systems, which contain $x-y$ and xy, a proof paralleling that for systems III can be given, in terms of representations by maximal dual ideals.

7. **Systems III.** Observe that all of the systems III contain the connectives (x, y, z) and xy.

THEOREM 3(8). *Let the algebra A with primitive connectives (x, y, z) and xy satisfy the axioms*

$$\mathfrak{A} \qquad \begin{array}{llll} xx = x, & xy = yx, & x(yz) = (xy)z, & (x, y, y) = xy, \\ (x, x, y) = x, & (x, y, z) = (x, z, y), & (x, y, z) = (x, xy, z), & \\ w(x, y, z) = (wx, y, z), & \textit{and} \;\; w(x, y, z) = (x, wy, wz). & & \end{array}$$

Then there exists a one-to-one mapping: $x \rightarrow \bar{x}$, of A into an algebra $\bar{A}$ of sets, such that $(x, y, z)^- = \bar{x}(\bar{y} \vee \bar{z})$ and $(xy)^- = \bar{x}\bar{y}$.

To prove this theorem, we first define $x \subset y$ to mean $xy = x$. It then follows that $x \subset x$; that $x \subset y$ and $y \subset x$ imply $x = y$; and that $x \subset y$ and $y \subset z$

(8) This representation theorem is patterned after that of Stone [11].

imply $x \subset z$.

An *ideal* in A is defined to be any subset $S \neq A$ satisfying

(1) if $x \subset y$ and x is in S, then y is in S,

(2) if x and y are in S, then xy is in S.

An *atom* in A is an ideal S with the further property

(3) if (x, y, z) is in S, then either xy or xz is in S.

LEMMA. *If $x \subset y$ does not hold, then there exists an atom containing x but not y.*

To prove the lemma, we first observe that the set S_0 of all z such that $x \subset z$ is an ideal containing x but not y. We shall show that every ideal with this property, if it is not already an atom, can be extended to a larger ideal with the same property. Since the union of an ascending chain of ideals with this property is clearly an ideal with the same property, it will follow by Zorn's lemma that there exists a maximal ideal with this property, which must therefore be an atom containing x but not y.

Let S be an ideal, but not an atom, containing x but not y. Then by definition S contains some (u, v, w) while neither uv nor uw is in S. Suppose there existed p and q in S such that $puv \subset y$ and $quw \subset y$. It would follow that $ruv \subset y$ and $ruw \subset y$, where $r = pq$ was in S. Hence

$$\begin{aligned} yr(u, v, w) &= (u, yrv, yrw) = (u, uyrv, uyrw) \\ &= (u, ruv, ruw) = (u, rv, rw) = r(u, v, w), \end{aligned}$$

that is, $r(u, v, w) \subset y$. Then, since r and (u, v, w) were in S it would follow that $r(u, v, w)$ was in S, and, since $r(u, v, w) \subset y$, that y was in S, contrary to hypothesis. Thus we may suppose, by symmetry, that $puv \subset y$ holds for no p in S. The set S' of all z such that $puv \subset z$ is clearly an ideal properly containing S, and hence x, but not y. This completes the proof of the lemma.

Define $\bar{x}$ to be the set of all atoms that contain x. From (1) it follows that $x \subset y$ implies $\bar{x} \subset \bar{y}$. The lemma shows that if not $x \subset y$, then not $\bar{x} \subset \bar{y}$. Since the mapping $x \rightarrow \bar{x}$ preserves inclusion, it is one-to-one.

That $(xy)^- = \bar{x}\bar{y}$ follows from (1) and (2), if one notes that $xy \subset x$ and $xy \subset y$. It remains to show that $(x, y, z)^- = \bar{x}(\bar{y} \vee \bar{z})$. First, let (x, y, z) be in S, an atom. Then

$$(x, y, z) = (x, y, z)(x, y, z) = ((x, y, z)x, y, z) = ((x, y, z), y, z)$$

whence by (3) either $(x, y, z)y = (xy, y, z) = (xy, xy, xyz) = xy$ is in S, or else $(x, y, z)z = xz$ is in S, and in either case S is in $(xy)^- \vee (xz)^- = \bar{x}\bar{y} \vee \bar{x}\bar{z} = \bar{x}(\bar{y} \vee \bar{z})$. Conversely, if S is in $\bar{x}(\bar{y} \vee \bar{z})$ we may suppose, by symmetry, that S is in $\bar{x}\bar{y} = (xy)^-$; then $xy(x, y, z) = (xy, xy, xyz) = xy$ implies that $xy \subset (x, y, z)$, so that xy in S implies that (x, y, z) is in S, that is, that S is in $(x, y, z)^-$. This completes the proof of the theorem.

THEOREM 4(9). *The axioms* $\mathfrak{A}$ *form a complete set for the algebra* F_6.

Proof. Let A be the free denumerably generated algebra with primitives xy and (x, y, z) subject to axioms $\mathfrak{A}$, and let $\overline{A}$ be the isomorphic algebra of sets. Every identity of the two-element algebra F_6 holds also in $\overline{A}$, as a subalgebra of a direct product of replicas of F_6. Thus every identity of F_6 holds in the free algebra A, and so is a consequence of the axioms $\mathfrak{A}$.

THEOREM 5. *Each of the algebras* F_7, F_5, C_4, F_6^n, F_7^n, *and* F_5^n *is axiomatizable.*

Proof. Each of these algebras can be obtained by adjoining further primitives to F_6. To extend the result obtained for F_6 it must be shown in each case that adjoining a finite number of new axioms to the set $\mathfrak{A}$ will ensure that the new primitives are properly represented in $\overline{A}$.

For the algebra F_7, with the additional primitive 0, it suffices to adjoin the single additional axiom $\mathfrak{A}_7$: $0x=0$. That $\overline{0}$ is indeed the empty set in $\overline{A}$ follows from the fact that 0 in S, for an atom S, would imply by (1) that all y were in S, contrary to the requirement $S\neq A$.

For F_5, with additional primitive $[x, y, z]$, we adjoin the additional axioms $\mathfrak{A}_5$:

$$[x, y, z] = [x, z, y], \qquad x[x, y, z] = [x, y, z], \qquad y[x, y, z] = xyz,$$
$$(x, (x, y, z), [x, y, z]) = x.$$

Suppose S is in $[x, y, z]^-$; then $[x, y, z]$ in S and $[x, y, z]\subset x$ implies x is in S. If neither y nor z is in S, then S is in $\bar{x}(\bar{y}\equiv\bar{z})$ as required. Otherwise we may suppose that y is in S, whence $y[x, y, z]=xyz$ is in S, so also z is in S, and again S is in $\bar{x}(\bar{y}\equiv\bar{z})$. For the converse, suppose that S is in $\bar{x}(\bar{y}\equiv\bar{z})$. If S is in $(xyz)^-$, it follows from $xyz[x, y, z]=xz(xyz)=xyz$ that $xyz\subset[x, y, z]$ and so $[x, y, z]$ is in S as required. Otherwise x is in S but neither y nor z is in S. By (3), that $x=(x, (x, y, z), [x, y, z])$ is in S implies that either $x(x, y, z)=(x, y, z)$ or $x[x, y, z]=[x, y, z]$ is in S. Since, by (3), (x, y, z) in S would imply that either y or z were in S, it must be that $[x, y, z]$ is in S.

For C_4, with additional primitive $x\vee y$, adjoin the further axioms $\mathfrak{A}_4$:

$$x \vee y = y \vee x, \qquad x(x \vee y) = x, \qquad (x \vee y, x, y) = x \vee y.$$

If $x\vee y=(x\vee y, x, y)$ is in S, it follows by (3) that either $(x\vee y)x=x$ or $(x\vee y)y=y$ is in S, so S is in $\bar{x}\vee\bar{y}$. Conversely, if either x or y is in S, it follows from $x\subset x\vee y$ and $y\subset x\vee y$ that $x\vee y$ is in S.

F_6^n, for $n\geqq 3$, contains the additional primitive d_n. Abbreviate

$$(x, y_1, \cdots, y_m) = (x, (x, \cdots (x, (x, y_1, y_2), y_3), \cdots, y_{m-1}), y_m)$$

(9) This result can also be established by the use of a "relative" normal form, in terms of the connectives $x\vee_a y=(a, x, y)$ and $x\wedge_a y=axy$, for a suitably chosen variable a. But this method, which bears a superficial connection to that of §8, is rather awkward to extend to the remaining systems III.

and write x^i for $x_1 \cdots x_{i-1}x_{i+1} \cdots x_n$, and d_n for $d_n(x_1 \cdots, x_n)$.

Adjoin the following finite set of further axioms:

$\mathfrak{S}_n$: axioms expressing that $d_n(x_1, \cdots, x_n)$ is invariant under any permutation of its arguments;

$\mathfrak{D}_n$: $d_n(x_1, \cdots, x_n) = (d_n(x_1, \cdots, x_n), x^1, \cdots, x^n)$,

$\mathfrak{D}_n'$: $x^1 d_n(x_1, \cdots, x_n) = x^1$.

From $\mathfrak{D}_n'$ with $\mathfrak{S}_n$ it follows that d_n is in S whenever any x^i is in S. For the converse, suppose that d_n is in S. Since $d_n = (d_n, x^1, \cdots, x^n)$, by (3) either $d_n(d_n, x^1, \cdots, x^{n-1}) = (d_n, x^1, \cdots, x^{n-1})$ is in S or else $d_n x^n = x^n$ is in S. If x^n is in S, then S is in $\bar{x}^1 \vee \cdots \vee \bar{x}^n$ as required. Otherwise from $(d_n, x^1, \cdots, x^{n-1})$ in S we conclude by (3) again that either $(d_n, x^1, \cdots, x^{n-2})$ or x^{n-1} is in S. Continuing thus, either some one of $x^n, \cdots, x^3$ is in S, or else (d_n, x^1, x^2) is in S, whence either $d_n x^1 = x^1$ or $d_n x^2 = x^2$ is in S. In any case, S is in $\bar{x}^1 \vee \cdots \vee \bar{x}^n$ as required.

Finally, for F_7^n it evidently suffices to adjoin the axiom $\mathfrak{A}_7$ to those for F_6^n; and for F_5^n to adjoin the axioms $\mathfrak{A}_5$ to those for F_6^n.

8. Systems IV.

THEOREM 6. *The algebra D_2 is axiomatizable.*

D_2 is defined by the single primitive(10) $d(x, y, z) = xy \vee xz \vee yz$. Let A be the free algebra on a denumerable set of generators $a, x, y, \cdots$ subject to the same set of identities as D_2. Fixing the generator a, introduce the definitions

$$(\Delta) \qquad x \wedge_a y = d(a, x, y), \qquad (x, y, z)_a = x \wedge_a d(x, y, z), \qquad 0_a = a.$$

Let A_a be the algebra with the same elements as A, but with primitive operations $d(x, y, z)$, $x \wedge_a y$, $(x, y, z)_a$, and 0_a. Let A_0 be the free algebra of type F_7^3, with primitives $d(x, y, z)$, $x \wedge y$, (x, y, z), and 0, on the generators $x, y, \cdots$. Then the mapping $x \rightarrow x \wedge Na$ of the underlying Boolean algebras clearly establishes an isomorphism of A_0 onto A_a.

Let $\mathfrak{A}_0$ be a finite set of axioms for F_7^3, and so for A_0. Let $\mathfrak{A}_a$ be the corresponding axioms for the isomorphic algebra A_a. Using (Δ) to eliminate defined operations, we obtain from $\mathfrak{A}_a$ a set of equations $\mathfrak{A}$ expressed in the variables $a, x, y, \cdots$ and the primitive d, of the algebra A.

If ϕ is any expression of A, substituting 0_a for a yields an expression ϕ_a in the notation of A_a. If ϕ_0 is the expression of A_0 corresponding to ϕ_a under the isomorphism of A_0 onto A_a, we see that formally ϕ_0 is obtained by substituting 0 for a in ϕ. In the full notation of Boolean algebra, let ϕ_1 be the dual of ϕ_0; since d is self-dual, we see that ϕ_1 is equivalent to the formal result of substituting 1 for a in ϕ, whence we have the identity

$$(\mathrm{H}) \qquad \phi = \phi_1 a \vee \phi_0 Na.$$

(10) This "median" operation has received considerable attention; see Birkhoff [3, p. 137], and further references given there. For the present argument, it is important that the operation d is self-dual.

Now suppose $\phi=\psi$ is one of the equations of $\mathfrak{A}$. This means that $\phi_a=\psi_a$ was one of the axioms $\mathfrak{A}_0$ of A_0, whence $\phi_0=\psi_0$, and its dual $\phi_1=\psi_1$, are Boolean identities. From (H) it follows that $\phi=\psi$ is a Boolean identity. This shows that all the equations $\mathfrak{A}$ are true in A.

For the converse, let $\phi=\psi$ be any true equation in the notation of A. Then, setting $a=0$, the equation $\phi_0=\psi_0$ is true in A_0, and hence a consequence of the axioms $\mathfrak{A}_0$ for A_0. Then $\phi_a=\psi_a$ is a consequence of the axioms $\mathfrak{A}_a$ in the isomorphic algebra A_a. Eliminating the defined operations by (Δ), it follows that $\phi=\psi$ is a consequence of the axioms $\mathfrak{A}$ for A.

This completes the proof that D_2 is axiomatizable. An obvious modification of this argument establishes the axiomatizability of the two remaining systems, D_1 and D_3.

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