

A LINEAR OPERATOR METHOD ON ARBITRARY RIEMANN SURFACES

BY
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1. Introduction. This investigation is concerned with an operator method, which is of utility for a unified treatment of certain mapping, existence, uniqueness, and boundary value problems on Riemann surfaces.

Arbitrary Riemann surfaces R , closed or open, of finite or infinite genus are considered. On a compact or noncompact domain $S \subset R$ normal linear operators L will be defined associating to every harmonic function v in a neighborhood of the relative boundary of S a uniquely determined harmonic function Lv on S . Given on R a finite collection l of closed curves, we select on $S = R - l$ a "singular function" s , that is, a real function, harmonic near l and permitting a harmonic continuation across l . If a simple condition concerning s is fulfilled, then there exists on R a real function p , harmonic in a neighborhood of l and such that

$$p - s = L(p - s)$$

on S . The function p is uniquely determined and will be called the *principal function* for a given s and L . If different initial functions s and different operators L are selected, the corresponding principal function $p(z; s, L)$ furnishes the solution to different mapping and existence problems on R .

In §1 the operator method is presented for single-valued functions. An estimation lemma is given, the above theorem is proved, and a special operator L_0 with the corresponding principal function p_0 introduced. The nature of the function p_0 is illustrated by its connection to classical slit mappings. §2 is devoted to the existence problem of regular harmonic and analytic functions. Necessary and sufficient conditions, in terms of the normal linear operator L_0 , are given. Applications to certain uniqueness questions are made. A theorem is presented for harmonic functions without existence restrictions. §3 contains the operator method for multiple-valued functions. By way of preparation the solution of the first and second boundary value problem on abstract Riemann surfaces is given in terms of the principal function. A theorem on analytic functions without existence restrictions is proved.

In many cases the linear operator method allows, as it seems to us, a simpler approach than the Dirichlet principle or Hilbert space methods. Since all proofs are purely constructive, a numerical computation of the functions in question is possible. What is assumed a priori is the solution of the

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first boundary value problem for Riemann domains with the boundary consisting of a finite number of analytic Jordan curves. As is well known this can be done simply by the Schwarz alternating method [18] which was the starting point of this investigation.

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§1. OPERATOR METHOD FOR SINGLE-VALUED FUNCTIONS

2. An estimation lemma. The Riemann surfaces to be considered may be given either in an abstract way by parameter discs or by covering or embedding properties.

The convergence proofs will be based on the following simple statement.

LEMMA 1. *Let R be an arbitrary Riemann surface and K a closed point set on R . Consider on R all single-valued harmonic functions $\{u\}$, normalized, by adding a suitable constant, so that*

$$(1) \quad \int_K u d\mu = 0,$$

where μ is a positive mass distribution on K . There exists a constant $0 < q < 1$, independent of u , such that

$$(2) \quad \max_K |u| \leq q \cdot \sup_R |u|.$$

Proof. If u is constant ($=0$), the inequality (2) is evident. The same is the case if u is not bounded. Hence we can assume $\sup_R |u| = 1$.

If (2) were not true, there would exist a sequence $\{u_n\}$ of functions u such that

$$\lim_{n \rightarrow \infty} \max_K |u_n| = 1.$$

A subsequence, say again $\{u_n\}$, would converge towards a function u , harmonic and bounded, $|u| \leq 1$, on R . The points where u_n takes its maximum on K accumulate at at least one point z_0 on K . It follows from the continuity of u and the uniform convergence of $\{u_n\}$ on K that $|u(z_0)| = 1$. But u cannot be constant, since (1) obviously is valid for it. This contradicts the maximum principle. The lemma follows.

3. Normal linear operators. By the *relative boundary* l of a subdomain $D \subset R$ is meant the set of (inner) points of R which are boundary points of D .

⁽¹⁾ The notes [8], [9], [10], where the generalization of the alternating method was given and the problem of integrals without assumption of a nullboundary was attacked, were presented to Académie des Sciences on November 28, and December 5, 1949.

For the sake of brevity, D is said in this paper to have an *analytic relative boundary* if l consists of a finite number of disjoint analytic Jordan curves and if the complement of the closure $\bar{D} = D + l$ of D has the same boundary.

A function u is said to be harmonic on a curve on R if it is harmonic in a neighborhood of the curve. Correspondingly, u is said to be harmonic on \bar{D} if it is harmonic on an open set containing \bar{D} . The conjugate function of a harmonic function u is denoted by \bar{u} . For the local parameter (uniformizer) of R the letter z is used.

Consider linear operators with the following properties. Given a domain D with an analytic relative boundary l and any single-valued harmonic function v on l , the operator L defines a single-valued and unique function Lv on D ; the operator is assumed to have the following properties:

$$(3) \quad Lv = v \text{ on } l, \quad \int_i d\bar{L}v = 0.$$

In addition the functional L is supposed to satisfy the conditions

$$(4) \quad \begin{aligned} L1 &\equiv 1, & Lv &\geq 0 \text{ if } v \geq 0 \text{ on } l, \\ L(c_1v_1 + c_2v_2) &= c_1Lv_1 + c_2Lv_2. \end{aligned}$$

An operator Lv satisfying (3) and (4) will be termed a *normal linear operator*.

4. Principal functions. Given a finite collection of disjoint Jordan curves l on a Riemann surface R , the complement $S = R - l$ consists of a finite number of regions S_i . If in each S_i there is given a single-valued function $s = s_i$, harmonic near l and with a harmonic continuation across l , then s is in general two-valued on l . The operation L in S will denote, in the following, the operation L applied, in each S_i , to the branch $s = s_i$ defined in S_i . The two edges of l are denoted by l^+ and l^- , each traced in the positive direction with respect to the domains S_i bounded by it. When integrating functions s along l^+ or l^- the branch of s will be used which is defined in the domain S_i bounded by this edge.

THEOREM 1. *Let R be an arbitrary Riemann surface, closed or open, of finite or infinite genus. Consider on R a finite collection l of closed curves, which is the analytic relative boundary of a compact or noncompact subregion of R . In $S = R - l$, let L be a normal linear operator and s a single-valued real function, harmonic near l and such that both branches permit a harmonic continuation across l . The condition*

$$(5) \quad \int_{l^+ + l^-} d\bar{s} = 0$$

is necessary and sufficient for the existence of a single-valued real function p on R , harmonic on l and such that in S

$$(6) \quad p - s = L(p - s).$$

The function $p = p(z; s, L)$ is unique up to an additive constant and will be called the principal function for given s, L on R .

p is not constant if and only if $s \neq Ls$.

Proof. By (6) and (3), it is evident that condition (5) is necessary. In order to prove the sufficiency, let S_0 be the region mentioned above of which l is the relative boundary. Denote the complement $R - \bar{S}_0$ by S_1 . Any closed curve $l_i \subset l$ can be mapped onto a circle C_i in such a way that a neighborhood T_i of l_i is mapped conformally and biuniquely onto a neighborhood T'_i of C_i . Select the domains T_i disjoint, such that s is harmonic in \bar{T}_i and such that T'_i is a concentric circular ring divided by C_i into two conformally equivalent rings. Let k be the boundary of $T = \cup T_i$ and let $k_0 = k \cap S_0$, $k_1 = k \cap S_1$.

Begin with the function

$$s_0 = s - Ls$$

in S and denote by s' the harmonic function in T with boundary values s_0 on k . The harmonic function Ls' in S is a linear functional Ks_0 of s_0 on k . Write K^i for i successive operations K and consider the sequence

$$(7) \quad s_n = \sum_{i=1}^n K^i s_0$$

in S .

Let x be the harmonic function in T such that $x=0$ on k_0 , $x=2$ on k_1 ; then $x=1$ on l . Green's formula $\int (\phi d\bar{x} - x d\bar{\phi}) = 0$, where the integral is taken along the boundaries of $T \cap S_0$ and $T \cap S_1$, gives

$$(8) \quad \int_l \phi d\bar{x} = \frac{1}{2} \int_{k_0+k_1} \phi d\bar{x}$$

for any function ϕ , single-valued on T , harmonic in $\bar{T} - l$, permitting a harmonic continuation from both sides over l , and subject to $\int_{k_0} d\bar{\phi} = \int_{k_1} d\bar{\phi}$. Here all curves are positive in the direction which leaves $S_0 - (S_0 \cap T)$ to the left. From (3), (5), and the compactness of T one sees that all functions in the operation $K^i s_0$ satisfy (8), so that, since $\int_l s_0 d\bar{x} = 0$,

$$(9) \quad \int_{k_0+k_1} K^i s_0 d\bar{x} = 0.$$

In view of the construction of K^i , this implies, by Lemma 1,

$$\sup_s |K^{i+1} s_0| = \max_i |K^{i+1} s_0| \leq \max_k |K^i s_0| \leq q \cdot \sup_s |K^i s_0|.$$

Hence the sequence (7) converges uniformly in S towards a harmonic single-

valued limit s^* on S . Write

$$p = s_0 + s^*.$$

By construction of s_n the harmonic function in T which coincides with $s_0 + s_n$ on k is equal to $s_0 + s_{n+1}$ on l . Hence it tends, as $n \rightarrow \infty$, to the harmonic continuation of p , which hereby is defined on the whole surface.

Since s^* is harmonic on l (by $s^* = p$), the condition (4) yields

$$\min_i (s^* - s_n) \leq L(s^* - s_n) \leq \max_i (s^* - s_n).$$

The difference

$$Ls^* - s_n = Ls^* - Ls_n = L(s^* - s_n)$$

(where $s_n = Ls_n$ by $LL = L$) tends to zero uniformly in S , hence

$$s^* = Ls^*.$$

The condition (6) follows:

$$p - s = s^* - Ls = L(s^* - s + s_0) = L(p - s).$$

If $s \not\equiv Ls$, then p is not constant. In fact, the antithesis $p = \text{const.} = c$ would give

$$c - s = L(c - s) = Lc - Ls = c - Ls,$$

whence $s \equiv Ls$. Conversely, if $s \equiv Ls$, then $s_0 \equiv 0$ and $p \equiv 0$.

The function $p = p(z; s, L)$ is unique up to an additive constant. In order to see this, let p_1 and p_2 be two functions p . The difference $p_1 - p_2$ is single-valued and harmonic on R . On S , we have

$$p_1 - p_2 = (p_1 - s) - (p_2 - s) = L(p_1 - s) - L(p_2 - s) = L(p_1 - p_2).$$

Hence $p_1 - p_2$ takes its maximum on l , and this by the maximum principle implies $p_1 - p_2 = \text{const.}$ This proves the theorem.

REMARK. If in the following applications of Theorem 1, the singular function s is given only in some of the regions $S_i \subset S = R - l$, it is understood that in the remaining domains s is selected $\equiv 0$.

5. The operator L_0 . Different normal linear operators L are defined by requiring suitable extremal properties of the function Lv . In this paper an operator L_0 is investigated; other operators will be studied later in another connection [17].

LEMMA 2. On an arbitrary Riemann surface let S be a partial surface with an analytic relative boundary l . On l is given a single-valued harmonic function v .

There exists on S a unique harmonic function u minimizing the Dirichlet integral $D_S[u]$ among all harmonic functions in S taking boundary values v on l , the minimum being

$$(10) \quad D_S[u] = \int_I u d\bar{u}.$$

The function u is a normal linear operator and will be noted by

$$(11) \quad u = L_0 v.$$

The corresponding principal function will be denoted by $p_0(z; s) = p(z; s, L_0)$.

Proof. Let $S_0 \subset S_1 \subset S_2 \subset \dots$ be an exhaustion of \bar{S} such that S_n has an analytic relative boundary $l + k_n$. Let u_n be the harmonic function in S_n defined by $u_n = v$ on l , $\partial u_n / \partial n = 0$ on k_n . (u_n can be constructed by well known methods, e.g. by symmetrizing S_n and solving the first boundary value problem with values v on the two copies of l .) Since $\min_l v \leq u_n \leq \max_l v$, a subsequence, say again $\{u_n\}$, converges towards a function u , harmonic and single-valued in S . Clearly $u = v$ on l ; hence u is, by the Schwarz symmetry principle, also harmonic on l . The convergence of u_n being uniform in each compact set of \bar{S} , it follows, by denoting $D_n = D_{S_n}$ and $D = D_S$,

$$(12) \quad D[u] = \lim_{n \rightarrow \infty} D_n[u] = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} D_n[u_m] \leq \lim_{m \rightarrow \infty} \int_I u_m d\bar{u}_m = \int_I u d\bar{u} < \infty.$$

Green's formula shows immediately that $D_n[u_n]$ is the minimum d_n of D_n with respect to all harmonic functions in \bar{S}_n with boundary values v on l . Hence $d_n \leq d_{n+1} \leq d$, where d is the corresponding greatest lower bound for S , $d \leq D[u] < \infty$. By (12), the statement (10) follows.

The minimizing function u is unique. In fact, if u' , u'' were two functions giving the minimum d for D , then, setting $h = u' - u''$, clearly $D[h] < \infty$, whence $D[h, u'] = 0$. This implies that $D[h] = 0$, since

$$D[u''] = D[u'] + D[h] + 2D[u', h] = D[u'].$$

Hence $u' - u'' = \text{const.} = 0$. It follows in particular that the original sequence $\{u_n\}$, not merely a subsequence, converges.

It is clear, by the corresponding properties of u_n and in view of the uniform convergence, that conditions (3) and (4) are satisfied. Hence u is a normal linear operator (11).

6. Connection with slit mappings. The function p_0 is illustrated by its connection to classical slit mappings.

The principal function p_0 yields the mapping of a planar (genus = 0) Riemann surface onto a plane slit domain as follows:

(1) onto a horizontal slit domain by $p_0 + i\bar{p}_0$ for $s = \text{Re}(1/(z - z_0))$ in a disc around z_0 ,

(2) onto a vertical slit domain by $ip_0 - \bar{p}_0$ for $s = \text{Im}(1/(z - z_0))$ in a disc around z_0 ,

(3) onto a radial slit domain by $e^{p_0 + i\bar{p}_0}$ for $s = \log |z - z_0|$ around z_0 , $s = \log |z - z_1|^{-1}$ around z_1 ,

(4) *onto a circular slit domain by $e^{i\varphi_0 - \bar{\varphi}_0}$ for $s = \arg((z - z_0)/(z - z_1))$ in a disc around z_0 and z_1 , cut along a curve connecting these points.*

The total area of the slits vanishes.

Corresponding mappings are obtained onto circular discs or annuli with radial or circular slits. Analogous results are valid for mappings of surfaces of positive finite genus onto multiple sheeted slit-domains.

§2. EXISTENCE OF REGULAR FUNCTIONS

7. Function classes. In the present paper, only integrals are investigated, i.e. harmonic or analytic functions with a single-valued differential. The following notations will be used, for the sake of brevity, for different classes of integrals.

Greek letters $\alpha, \beta, \gamma, \delta$ will denote classes of (single- or) *multiple-valued regular nonconstant harmonic functions* with the following restrictive properties:

- (13) α all functions,
 β bounded functions,
 γ half bounded functions,
 δ Dirichlet bounded functions.

The phrase "half bounded functions" is here used to mean functions bounded only upwards or only downwards, e.g. the positive functions. (The letter γ can be interpreted in certain analogy to Green's functions.) For a multiple-valued function, the properties β and γ have to be understood in the sense that the existence domain considered can be cut along such curves that the function is single-valued in the cut domain and has there the property β or γ , respectively. "Dirichlet bounded" signifies that the functions have a finite Dirichlet integral.

By a *dividing cycle* is meant a finite set of simple curves dividing the given Riemann surface in two separate parts. The subclasses

$$(14) \quad \alpha_0 \subset \alpha, \quad \beta_0 \subset \beta, \quad \gamma_0 \subset \gamma, \quad \delta_0 \subset \delta$$

denote classes of functions $\alpha, \beta, \gamma, \delta$ with *vanishing periods along each dividing cycle*. This property could be expressed by saying that the function has no periods along any cycle around a boundary part.

The classes of *single-valued* regular nonconstant harmonic functions will be denoted by the corresponding *Latin* letters a, b, g, d ,

$$(15) \quad a \subset \alpha_0, \quad b \subset \beta_0, \quad g \subset \gamma_0, \quad d \subset \delta_0.$$

For *planar* surfaces, clearly,

$$(16) \quad a = \alpha_0, \quad b = \beta_0, \quad g = \gamma_0, \quad d = \delta_0.$$

The classes of regular *analytic* functions will be denoted by pairs of letters, the first expressing the class of the real part, the second the class of the imaginary part. E.g.:

$\beta_0 d$ regular analytic functions with a bounded real part without periods along dividing cycles and a single-valued imaginary part having a finite Dirichlet integral.

Given an open Riemann surface R , the complement $R - \overline{D}$ of any compact domain \overline{D} with an analytic boundary l will be called a *boundary neighborhood* N of R . A property of R will be termed a *boundary property* if it can be expressed as a property of an arbitrarily small boundary neighborhood N , i.e. of the complement of a domain \overline{D} tending to R .

In the following sections the existence of functions on R is investigated in terms of its dependence on the boundary. Relations between different classes will be considered in subsequent papers [16].

We note first:

Given on an arbitrary Riemann surface a domain D with analytic relative boundary l , and on l any single-valued harmonic function v , then on D

$$(17) \quad L_0 v \in b, g, d, \quad \overline{L}_0 v \in \alpha_0.$$

On domains of finite genus

$$(17)' \quad \overline{L}_0 v \in \beta_0, \gamma_0, \delta_0.$$

On planar domains

$$(18) \quad L_0 + i\overline{L}_0 \in ff'$$

where f and f' may be any two of the symbols b, g, d .

8. Existence of harmonic functions. Given a class f of functions, we denote by 0_f the class of Riemann surfaces on which there are no functions belonging to the class f . Now let R be an arbitrary Riemann surface. Consider first the existence of single-valued harmonic functions on R .

THEOREM 2. *A necessary and sufficient condition for*

$$(19) \quad R \in 0_f, \quad f = b, g, d,$$

is that in some boundary neighborhood N each function $\phi \in f$, satisfying $\int_1 d\phi = 0$, is invariant under the operation L_0 :

$$(20) \quad \phi \equiv L_0 \phi.$$

This condition is independent of N , i.e. (19) is a boundary property.

The condition (20) is, for $f = b, d$, equivalent to the following: for any two functions $\phi_1, \phi_2 \in f$ in N , resp. $\phi_1 - L_0 \phi_1$ and $\phi_2 - L_0 \phi_2$ are linearly dependent.

Proof. If there exists on N a function $\phi \in f$, $\int_1 d\phi = 0$, $\phi \not\equiv L_0 \phi$, then, by Theorem 1 and (17), there is on R a function $\phi \in f$. Conversely, for a func-

tion $\phi \in f$ on R clearly $\int_1 d\phi = 0$ and, by the maximum principle,

$$\sup_R \phi = \sup_N \phi > \max_l \phi,$$

whence $\phi \neq L_0 \phi$.

For proving the second statement assume there exist on N two functions $\phi_1, \phi_2 \in f$ such that $\phi_1 - L_0 \phi_1$ and $\phi_2 - L_0 \phi_2$ are linearly independent. Clearly

$$\phi_1 \neq L_0 \phi_1, \quad \phi_2 \neq L_0 \phi_2.$$

If $\int_1 d\phi_1 = 0$ or $\int_1 d\phi_2 = 0$, then the existence of $\phi \in f$ on R is guaranteed by Theorem 1. If both integrals are $\neq 0$ then, for a suitable real constant λ , the function $\psi = (\phi_1 - L_0 \phi_1) + \lambda(\phi_2 - L_0 \phi_2)$ satisfies $\int_1 d\psi = 0$, $\psi \neq L_0 \psi$ in N , and there exists a $\phi \in f$ on R .

Conversely, assume there is a $\phi \in f$ on R . In N the function $\phi' = \phi - L_0 \phi \in f$ is not constant and $\int_1 d\phi' = 0$. Hence ϕ' takes in N both positive and negative values. Take an exhaustion N_n of N , the boundary $l + k_n$ of N_n being analytic. If $\phi \in b$, $\phi' \leq M < \infty$, let ψ_n be the harmonic function in N_n with $\psi_n = 0$ on l , $\psi_n = M$ on k_n . Since $\{\psi_n\}$ is a normal family, some limit function ϕ'' in N satisfies $\phi'' \in b$, $\phi'' \geq \phi'$, $\phi'' \geq 0$, hence $\int_1 d\phi'' \neq 0$. The functions $\phi' = \phi' - L_0 \phi'$ and $\phi'' = \phi'' - L_0 \phi''$ are linearly independent. An analogous reasoning proves the case $\phi \in d$ (cf. no. 11 in [6]).

For multiple-valued harmonic functions the existence theorem is:

THEOREM 3. *On every Riemann surface of positive (finite or infinite) genus there are functions $\beta_0, \gamma_0, \delta_0$. On a planar surface R , these functions exist if and only if R is hyperbolic.*

Proof. If there is a nondividing cycle k on R , divide it into two parts k_1 and k_2 . Let D be a compact domain containing k_1 . Take in \bar{D} for s a harmonic function with the singularities $\arg(z - z_1)$ and $-\arg(z - z_2)$ at the end points z_1, z_2 of k_1 respectively. Let $p_1 = p_0(z; s)$ be the corresponding principal function on R . Then take around k_2 for s a harmonic function with the singularities $\arg(z - z_2)$ and $-\arg(z - z_1)$ at z_2 and z_1 respectively. Denote by p_2 the resulting principal function $p_0(z; s)$. Then $p = p_1 + p_2 \in \beta_0, \gamma_0, \delta_0$ on R .

Assume now R is planar. Then $\beta_0 = b, \gamma_0 = g, \delta_0 = d$ and the nonexistence is implied by condition (20). It is easy to see that this is, for vanishing genus, equivalent with R being of parabolic type.

9. Existence of analytic functions. Consider first multiple-valued analytic functions. The proofs of Theorems 2 and 3 yield immediately:

THEOREM 4. *There are functions ff' for*

$$f = b, g, d, \quad f' = \alpha, \alpha_0$$

on an arbitrarily given Riemann surface if and only if in some boundary neighborhood N there is a function $\phi + i\bar{\phi} \in ff'$, subject to $\int_1 d\phi = 0$, such that

$$\phi \not\equiv L_0\phi.$$

On every Riemann surface of positive (finite or infinite) genus there are functions ff' for

$$f = \beta_0, \gamma_0, \delta_0, \quad f' = \alpha, \alpha_0.$$

On a planar surface these functions ff' , for $f' = \alpha$, exist if and only if the surface is of hyperbolic type.

For $f' = \alpha_0$, the latter functions are single-valued and subject to Theorem 5 below.

Consider now the existence of single-valued analytic functions. The boundary r of a Riemann surface R is said to be *removable* ff' if there are no analytic functions belonging to the class ff' on R [6].

THEOREM 5. *The boundary r of a Riemann surface R of finite genus is removable ff' ,*

$$(21) \quad R \in 0_{ff'}, \quad f = b, d, \quad f' = a, b, d,$$

if and only if in some boundary neighborhood N for each function $\phi + i\bar{\phi} \in ff'$

$$(22) \quad \phi \equiv L_0\phi.$$

Proof. Suppose there is a function $\phi + i\bar{\phi} \in ff'$ on N , $\phi \not\equiv L_0\phi$. Take a planar boundary neighborhood $N_0 \subset N$, which can be mapped conformally and biuniquely onto parts N' of circular discs K in such a way that the relative boundary l of N_0 goes onto the boundary circles l' of K . The map of the boundary r of R is some point set $r' \subset K$. If r' contains a proper continuum, then the existence of a function ff' on R is clear. Suppose r' is totally disconnected. The function $\phi_0 = \phi - L_0\phi$ in N is, by (17)', subject to the condition $\phi_0 + i\bar{\phi}_0 \in ff'_0$, where $f'_0 = \alpha_0, \beta_0, \delta_0$ for $f' = a, b, d$ respectively. Hence by (18) each branch of $\phi_0 + i\bar{\phi}_0 \in ff'$ on N_0 . This implies that ϕ_0 has an infinite set ρ of singular points on r' , since otherwise $\phi_0 \equiv 0$.

Let l_i , $i = 1, \dots, 2q+1$ ($q = \text{genus of } R$), be analytic curves in N_0 bounding disjoint simply-connected domains $S_i \subset K$, each containing some subset $\rho_i \subset \rho$. To the function $s_i = \phi_0 \not\equiv L_0\phi_0$ on S_i corresponds a nonconstant principal function $p_i = p_0(z; s_i)$ on the closed surface $R^* = R - N_0 + K$ such that $p_i + i\bar{p}_i \in ff'_0$. For any real constants c_i ,

$$p = \sum_{i=1}^{2q+1} c_i p_i \neq \text{const.}$$

on R^* since p has singular points. Select the constants c_i so as to satisfy

$$\sum c_i \int_{\alpha_k} d\bar{p}_i = 0$$

along each of the $2q$ nondividing canonical cycles α_k of R . Then $p+i\bar{p}\in ff'$ on R .

Conversely, assume that there is a function $\phi+i\bar{\phi}\in ff'$ on R . Then in N_0 , $\phi_0=\phi-L_0\phi\neq 0$. This completes the proof.

REMARKS. (1) In case of an infinite genus, the removability with respect to single-valued analytic functions is a relative property. It was shown by Ahlfors [2] that there are boundary neighborhoods N which can be completed, by adding along the relative boundary l two different compact parts D_1 or D_2 , to two different Riemann surfaces R_1 or R_2 such that $R_1\in 0_{ff'}$, $R_2\notin 0_{ff'}$. A statement corresponding to Theorem 5 can therefore *not* hold for surfaces of infinite genus.

(2) For surfaces R of finite genus, the notion of a removable boundary ff' coincides with the classical notion of a removable singularity. In fact, the boundary of R being realized as a point set r in a circle K , r is a removable singularity for each function ff' in $K-r$ if and only if there are no functions ff' on R . This is implied in the above reasoning for any class ff' with $f=b, d$; $f'=\alpha, a, b, d$.

The removability of the boundary is closely related to certain uniqueness questions:

A Riemann surface R of finite genus $\in 0_{ff'}$ ($f, f'=b$ or d) possesses the following properties:

- 1°. *All closed extensions of R are conformally equivalent.*
- 2°. *The conformal mapping of R onto a part of a closed surface R^* is unique up to a transformation of R^* onto itself. In particular:*
- 3°. *A plane mapping of a planar surface R is unique up to a linear transformation. Hence:*
- 4°. *All uniformizations of a Riemann surface by a (fixed) planar covering surface R depend linearly on each other.*

These statements are an immediate consequence of Theorem 5 and remark (2).

10. Functions without existence restrictions. As a supplement to Theorem 1, we prove first:

LEMMA 3. *On an open Riemann surface R , consider a noncompact domain D with analytic relative boundary l . Given an arbitrary finite real number c there always exists a single-valued harmonic function u on D such that*

$$(23) \quad \int_l d\bar{u} = c.$$

Proof. For $c=0$ the statement is proved by $u\equiv 0$. For $c\neq 0$ take an exhaustion $\{D_n\}$ of D , D_n having an analytic relative boundary $l+k_n$. Let u_n be the harmonic function in D_n such that $u_n=0$ on l , $u_n=\text{const.}$ on k_n , $u_n=1$ at a fixed point $z_0\in D_0$. For the functions u_{n+p} ($p=1, 2, \dots$), positive on D_{n+1} ,

the Harnack principle yields, on the compact point set $z_0 \cup k_n$,

$$u_{n+p}(z) < M \cdot u_{n+p}(z_0) = M,$$

where M is a constant $< \infty$ independent of p . Hence u_{n+p} is a normal family and there exists on D a single-valued harmonic limit function u . Clearly $u = 0$ on l , $u = 1$ at z_0 , $u > 0$ on D , hence $\int_l d\bar{u} > 0$. By multiplying u by a suitable constant the lemma follows.

THEOREM 6. *Let R be an arbitrary open Riemann surface and D a non-compact domain ($D \subset R$) with analytic relative boundary l . In $S = R - \bar{D}$ let s be a single-valued real function, harmonic on l . Given on S a normal linear operator L there is on R a single-valued function p , harmonic on \bar{D} and such that, on S ,*

$$p - s = L(p - s).$$

Proof. By the previous lemma there is a single-valued harmonic function u on D such that

$$\int_l d\bar{u} = \int_l d\bar{s}.$$

The assumptions of Theorem 1 are thereby satisfied in $R - l$ and the truth of the statement follows.

§3. OPERATOR METHOD FOR MULTIPLE-VALUED FUNCTIONS

11. The first and second boundary value problem. Consider on an arbitrary Riemann surface a compact domain D with an analytic boundary l . Let v be a harmonic single-valued function on l . Denote by D^* the symmetrized domain D , by z_0, z_0^* two symmetric points, by K, K^* circular discs around them.

The principal function $p_0(z; s)$ on D^* for $s = \log |z - z_0|^{-1}$ in K , $s = \log |z - z_0^*|$ in K^* , is the Green's function of D with the logarithmic pole at z_0 and gives the solution u to the Dirichlet problem with boundary values v :

$$(24) \quad u(z_0) = -\frac{1}{2\pi} \int_l v d\bar{p}_0(z; s).$$

The principal function $p_0(z; s)$ on D for $s = \log |z - z_0|^{-1}$ around z_0 , $s = \log |z - z_1|$ around z_1 is a generalized Neumann's function with positive and negative poles at z_0 and z_1 respectively and with a vanishing normal derivative on l . It gives for the Neumann problem the solution u which vanishes at z_1 ,

$$(25) \quad u(z_0) = -\frac{1}{2\pi} \int_l p_0(z; s) \frac{\partial v}{\partial n} ds.$$

Both statements are immediately derived from Green's formula applied to D less small circular discs centered at the points z_0 and z_1 .

The relation (25) will now furnish the solution of the first boundary value problem for a multiple-valued v :

LEMMA 4. *Let v be a harmonic (in general multiple-valued) function on l . The condition*

$$(26) \quad \int_l dv = 0$$

is necessary and sufficient for the existence of a harmonic function on D coinciding with v on l .

Proof. Let u be a harmonic function with normal derivative $\partial u / \partial n = -\partial v / \partial s$ on l , where ∂n is taken in the inner direction of D , ∂s in the positive direction along l with respect to D . Then \bar{u} is a harmonic function in D with $\partial \bar{u} / \partial s = \partial v / \partial s$ on l , hence $\bar{u} = v + c_i$ on the closed curves $l_i \subset l$ ($c_i = \text{const.}$). Let h be the harmonic function in D such that $h = c_i$ on l_i . Then $\bar{u} - h$ is the desired function.

Clearly the condition (26) is necessary.

12. **The generalized operator method.** Theorem 1 has the following counterpart for (in general multiple-valued) analytic functions:

THEOREM 7. *Consider an arbitrary Riemann surface R and a compact domain D on R with an analytic boundary l . Let L be a normal linear operator in $S = R - \bar{D}$ and let $\sigma = s + it$ be a complex-valued function on \bar{S} , analytic, $\sigma = s + i\bar{s}$, on l . The condition*

$$(27) \quad \int_l d\sigma = 0$$

is necessary and sufficient for the existence of a complex-valued function $w = p + iq$ on R , analytic on \bar{D} , $w = p + i\bar{p}$, and such that in S

$$(28) \quad p - s = L(p - s).$$

If $s \neq Ls$ then $w \neq \text{const.}$

Proof. Let $T(\supset \bar{D})$ be a compact domain with an analytic boundary k and such that σ is analytic in $\bar{G} = \bar{T} \cap \bar{S}$. By Lemma 4, there is a harmonic function s' in T coinciding with s on k . The difference $s' - s$ is single-valued in \bar{G} . Write, in S ,

$$(29) \quad s_0 = L(s' - s).$$

Let s'' be the harmonic function in T with the boundary values s_0 on k . Then, in S , Ls'' is a linear operator Ks_0 of s_0 on k . Consider the sequence

$$(30) \quad s_n = \sum_{i=0}^n K^i s_0$$

$(K^0 s_0 = s_0)$ in S .

For any single-valued harmonic function ψ in \bar{G} , subject to the condition $\int_1 d\bar{\psi} = 0$, clearly

$$\int_l \psi d\bar{x} = \int_k \psi d\bar{x},$$

where x is the harmonic measure of k in \bar{G} . It follows, by $\int_k (s' - s) d\bar{x} = 0$ and the compactness of T , that

$$\int_k K^i s_0 d\bar{x} = 0.$$

Hence Lemma 1 is applicable, and the same reasoning as in no. 4 yields the uniform convergence of the sequence (30) towards a harmonic limit s^* in \bar{S} . The harmonic function in T with boundary values $s + s_n$ on k converges uniformly towards the harmonic continuation of $p \doteq s + s^*$. Hence

$$w = p + i(t + \bar{s}^*)$$

is the desired function on R .

13. Unrestricted existence of analytic functions. We close by giving a supplement to Theorem 7.

LEMMA 5. *Consider on an open Riemann surface a noncompact domain S with an analytic relative boundary l . Given an arbitrary finite complex number $c_1 + ic_2$, there always exist analytic functions τ (with single-valued differentials) on S such that*

$$\int_l d\tau = c_1 + ic_2.$$

Proof. Let u_1 and u_2 be two single-valued harmonic functions in \bar{S} such that $\int_1 d\bar{u}_1 = c_1$ and $\int_1 d\bar{u}_2 = c_2$. Then $\tau = (\bar{u}_1 + u_2) + i(\bar{u}_2 - u_1)$ is the desired function.

THEOREM 8. *On an arbitrary open Riemann surface R let D be a compact domain with an analytic relative boundary. Let S_0 and S_1 be disjoint collections of domains which constitute $R - \bar{D}$ ($R = \bar{D} + S_0 + S_1$). In S_0 , let $\sigma = s + it$ be a complex-valued function, analytic, $\sigma = s + i\bar{s}$, on the relative boundary l_0 of S_0 . Given on S_0 a normal linear operator L , there exists a complex-valued function $w = p + iq$ on R , analytic on $S_1 + \bar{D}$, $w = p + i\bar{p}$, and such that on S_0*

$$p - s = L(p - s).$$

Proof. By Lemma 5, there is an analytic function τ on S_1 such that

$$\int_{l_0} d\sigma + \int_{l_1} d\tau = 0$$

where l_1 is the relative boundary of S_1 . The statement then follows by Theorem 7.

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