ON THE PRIMARY SUBGROUPS OF A GROUP(1)

BY

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1. **Introduction.** In this paper we consider a group theoretic configuration consisting of an operator group G and a lattice of admissible subgroups. If M is the operator system and ϕ the lattice of subgroups, we call G an M-ϕ group and the subgroups in ϕ we call ϕ subgroups. We define M-ϕ isomorphism, normal ϕ chains, ϕ cyclic ϕ subgroups, etc., in the obvious manner.

We call an M-ϕ group P primary if all its ϕ composition factors are M-ϕ isomorphic to the same M-ϕ group F; F is the characteristic of P. Thus if ϕ consists of all subgroups of a finite group G (and if M is void), G is primary with abelian characteristic if, and only if, G is a p-group. The main part of the paper is devoted to the following problem: When is an M-ϕ group the direct sum of primary ϕ subgroups? In §4 we find a necessary and sufficient condition that this be true under very general hypotheses. In studying this question we make use of what we have called ϕ links and unitoral M-ϕ groups. ϕ links are ϕ subgroups which belong to some normal ϕ chain. In §3 we prove various properties for ϕ links. The two main theorems are generalizations of theorems proved by Wielandt in [2]. Wielandt has also defined a unitoral (einköpfig) group as one which has a unique maximal normal subgroup, and we call an M-ϕ group unitoral if it has a unique maximal normal ϕ subgroup. The result about unitoral M-ϕ groups which we need is a generalization of a result proved by Wielandt.

We next consider the analogue of the following theorem for finite groups: A group is the direct product of its Sylow subgroups if, and only if, it is nilpotent. It can be seen readily that the generalization of this theorem to M-ϕ groups is not true without further hypotheses on the nature of ϕ. However, with suitable restrictions on the M-ϕ group G considered we are able to show that G is the direct sum of primary ϕ subgroups with abelian characteristic if, and only if, G is ϕ nilpotent (cf. Theorem 7.5). It is clear that the methods used to prove the theorem for finite groups are not applicable in our case. The following result is obtained and is used in the proof: Every nonzero normal ϕ subgroup of a ϕ nilpotent M-ϕ group G has nonzero intersection with the ϕ center of G. By the use of a rather complicated induction we are able to reduce the problem to the study of a curious type of group —namely, a group which is the sum of two abelian subgroups one of which is normal and whose intersection is zero (cf. §6).

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(1) Part of this paper is a revised version of a portion of a doctoral dissertation presented to Yale University and directed by Professor Reinhold Baer, who suggested the topic.

2. **Definitions and basic theorems.** Let G be an additive group (not necessarily abelian) which admits the system of operators M; and let ϕ be a system of M admissible subgroups of G which forms a complete lattice with respect to intersections and composita (by the compositum of a set of subgroups of a group we mean the subgroup generated by the subgroups of the set). We call the operator group G an M-ϕ group if a complete lattice ϕ has been specified. A ϕ subgroup S of G can be considered as an M-ϕ group if we define the lattice ϕ for S in the following way: the ϕ subgroups of S are the ϕ subgroups of G which are contained in S. If N is a normal ϕ subgroup of G, G/N admits the operator system M and a lattice ϕ of M admissible subgroups of G/N may be defined in this way: the ϕ subgroups of G/N are the subgroups U/N, where U is a ϕ subgroup of G which contains N. Thus G/N is an M-ϕ group. If a is an element of G, the intersection of all ϕ subgroups which contain a is a ϕ subgroup; we call this ϕ subgroup the ϕ cyclic ϕ subgroup generated by a. An M-ϕ group which has no normal ϕ subgroups we call ϕ simple.

Let G and G' be M-ϕ groups, i.e., G and G' both admit the operator system M and a lattice of M admissible subgroups has been defined for both G and G'—this lattice being denoted by ϕ in either case. Then G and G' are M-ϕ isomorphic if there exists an operator isomorphism of G onto G' which induces a lattice isomorphism of the ϕ subgroups of G onto the ϕ subgroups of G'. We write $G \cong_{(M\ \phi)} G'$. M-ϕ homomorphism and M-ϕ automorphism are defined in a similar fashion. It can be easily verified that the analogues of the Homomorphism Theorem and the Isomorphism Theorems hold. We define normal ϕ chain and ϕ composition series in the obvious way and it is clear that the analogue of the Jordan-Hölder Theorem holds. (For a complete statement of most of these definitions and theorems cf. [1].)

The ϕ subgroup S of G is M-ϕ characteristic if every M-ϕ automorphism of G leaves S invariant. It is important to notice that the inner automorphisms of a group are not necessarily M-ϕ automorphisms and hence a ϕ subgroup may be M-ϕ characteristic without being normal. In some of our arguments it is necessary to assume that inner automorphisms are M-ϕ automorphisms (cf., e.g., Theorem 3.5 and Example 3.1). In other places it is sufficient to assume that ϕ contains conjugates, i.e., if S is in ϕ and g is an element of G, then $-g+S+g$ is in ϕ.

Finally we list a few notations which will be used: If, for $i=1, \cdots, n$, A_i are subgroups of a group G, $\mathrm{C}_{i=1}^{n} A_i$ denotes the compositum of the A_i; $\sum_{i=1}^{*n} A_i$ denotes the direct sum. For the direct sum of two subgroups A and B we use $A \oplus B$.

3. **ϕ links.**

Definition 3.1. If S is a ϕ subgroup of the M-ϕ group G, S is said to be a ϕ link for G if there exists a normal ϕ chain connecting S and G.

The following result is easily established:

THEOREM 3.1. *If S and T are ϕ subgroups of the M-ϕ group G and if S is a ϕ link for G, then $S\cap T$ is a ϕ link for T. If T is also a ϕ link for G, $S\cap T$ is a ϕ link for G.*

From the first part of the theorem it follows that if S is a ϕ link for G, S is a ϕ link for any ϕ subgroup which contains it, and thus we may say that S is a ϕ link without fear of ambiguity.

THEOREM 3.2. *Assume that the ϕ links for the M-ϕ group G satisfy the maximum condition and that ϕ contains conjugates. If A and B are ϕ links for G, $\{A, B\}$ is a ϕ link for G.*

Proof. We shall say that the theorem is false for a ϕ link A if there exists a ϕ link B such that $\{A, B\}$ is not a ϕ link. We assume that the theorem is false, and let A be a maximal ϕ link for which the theorem is false.

We define the class of A relative to the ϕ link B as follows: $C_B(A)=k$ if there exists a ϕ link C such that $\{A, B\}\subseteq C$, and a normal ϕ chain of length k connecting A and C; and if there is no normal ϕ chain of length less than k from A to any ϕ link C' containing $\{A, B\}$ ($C_B(A)$ is finite since A can be connected to G by a normal ϕ chain). We use induction on r to prove the following statement: (r) If $C_B(A)=r$, then $\{A, B\}$ is a ϕ link. If $r=0$, (r) is true since then $A=C\supseteq\{A, B\}\supseteq A$ and hence $A=\{A, B\}$. Assume that (r) is true if $r<k$, and $C_B(A)=k$. We distinguish two cases:

(i) A is normal in $\{A, B\}$.

Let $A=A_0\subset \cdots \subset A_i\subset A_{i+1}\subset \cdots \subset A_k=C$ be a normal ϕ chain connecting A and C. Since A_1 is a ϕ link for G and $A_1\supset A$, $\{A_1, B\}$ is a ϕ link. Since B is a ϕ link contained in $\{A_1, B\}$, there exists a normal ϕ chain: $B=U_0\subset \cdots \subset U_i\subset U_{i+1}\subset \cdots \subset U_s=\{A_1, B\}$. Then $\{A, B\}\subseteq \cdots \subseteq\{A, U_i\}\subseteq\{A, U_{i+1}\}\subseteq \cdots \subseteq\{A, U_s\}=\{A_1, B\}$ is a normal ϕ chain; for U_{i+1} is contained in $\{A_1, B\}$, and hence transforms A into itself so that $\{A, U_i\}$ is normal in $\{A, U_{i+1}\}$. Thus $\{A, B\}$ is a ϕ link for $\{A_1, B\}$ and $\{A_1, B\}$ is a ϕ link for G. But this implies that $\{A, B\}$ is a ϕ link for G.

(ii) A is not normal in $\{A, B\}$.

There exists an element b of B such that $A(b)=-b+A+b$ is not contained in A; and by hypothesis $A(b)$ is a ϕ subgroup. Let $A=A_0\subset \cdots \subset A_i\subset A_{i+1}\subset \cdots \subset A_{k-1}\subset A_k=C$ be a ϕ chain; the chain $A(b)\subset \cdots \subset A_i(b)\subset A_{i+1}(b)\subset \cdots \subset A_{k-1}(b)=A_{k-1}$, where $A_i(b)=-b+A+b$, is a normal ϕ chain connecting $A(b)$ and A_{k-1}. Thus $C_{A(b)}(A)\leqq k-1$, and hence by the induction assumption, $A'=\{A(b), A\}$ is a ϕ link. Furthermore, since $A\subset A'$, $\{A', B\}$ is a ϕ link. But $\{A', B\}=\{A, A(b), B\}=\{A, B\}$. Hence $\{A, B\}$ is a ϕ link and (k) is proved.

Thus (r) is true for all r and this contradicts the assumption that the theorem is false for A.

THEOREM 3.3. *Let B and C be ϕ subgroups of the M-ϕ group G, and assume* *that B is a ϕ link for $\{B, C\}$. B and C permute if and only if, for every ϕ link A for $\{B, C\}$ such that $B \subseteq A$, $\{B, A \cap C\} = A$.*

Proof. It is always true that $\{B, A \cap C\} \subseteq A$ and hence the equality holds if and only if $\{B, A \cap C\} \supseteq A$.

Necessity. Assume that $\{B, C\} = B + C$, and let A be any ϕ subgroup of G such that $B \subseteq A \subseteq \{B, C\}$. If a is an element of A, then $a = b + c$, where b and c are elements of B and C, respectively. Hence $-b + a = c$, and c is an element of $A \cap C$. Therefore $A \subseteq B + (A \cap C) \subseteq \{B, A \cap C\}$ and thus $A = B + (A \cap C) = \{B, A \cap C\}$.

Sufficiency. Let $B = B_0 \subset \cdots \subset B_i \subset B_{i+1} \subset \cdots \subset B_n = \{B, C\}$ be a normal ϕ chain connecting B and $\{B, C\}$. By hypothesis, $B_{i+1} = \{B, B_{i+1} \cap C\}$, for $i = 0, \cdots, n-1$. Hence:

$$B_{i+1} = \{B_i, B_{i+1} \cap C\} = B_i + (B_{i+1} \cap C), \quad \text{since } B_i \text{ in normal in } B_{i+1}. \tag{1}$$

We use induction on j to show that $\{B, C\} = B_{n-j} + C$. For $j = 1$, $\{B, C\} = \{B_{n-1}, C\} = B_{n-1} + C$, since B_{n-1} is normal in $\{B, C\}$. Assume that $\{B, C\} = B_{n-j} + C$. From (1) it follows that $B_{n-j} = B_{n-j-1} + (B_{n-j} \cap C)$. Hence $\{B, C\} = B_{n-j} + C = B_{n-j-1} + (B_{n-j} \cap C) + C = B_{n-j-1} + C$, and the induction is complete. In particular, $\{B, C\} = B_0 + C = B + C$.

COROLLARY 3.1. *Let G be an M-ϕ group. A necessary and sufficient condition that any two ϕ links for G permute is that the ϕ links for G form a modular lattice.*

We take up next a generalization of the First Isomorphism Theorem for ϕ links.

THEOREM 3.4. *Let A and B be ϕ subgroups of the M-ϕ group G which are ϕ links for $\{A, B\}$; assume that the double chain condition holds for ϕ links for $\{A, B\}$ and that ϕ contains conjugates. Let $A = A_0 \subset \cdots \subset A_i \subset A_{i+1} \subset \cdots \subset A_n = \{A, B\}$ be a ϕ composition chain connecting A and $\{A, B\}$. Then if $A_i \cap B \neq A_{i+1} \cap B$,*

$$A_{i+1} \cap B / A_i \cap B \underset{(M\text{-}\phi)}{\cong} A_{i+1}/A_i. \tag{2}$$

REMARK. It can readily be shown that G has a ϕ composition series if and only if the ϕ links for G satisfy the double chain condition (cf. [2, p. 219]).

Proof. If $A_i \cap B \neq A_{i+1} \cap B$, then $A_{i+1} \cap B \not\subseteq A_i$. Hence $A_i \subset \{A_i, A_{i+1} \cap B\} \subseteq A_{i+1}$. Thus $A_{i+1} = \{A_i, A_{i+1} \cap B\}$, since A_i is a ϕ link for $\{A_i, A_{i+1} \cap B\}$ and $\{A_i, A_{i+1} \cap B\}$ is a ϕ link for A_{i+1}, and A_{i+1}/A_i is ϕ simple. Therefore, by the First Isomorphism Theorem, (2) holds.

COROLLARY 3.2. *Under the hypotheses of Theorem 3.4, the ϕ composition factors from $A \cap B$ to B are a part of the ϕ composition factors from A to $\{A, B\}$. In particular, if A and B permute, the ϕ composition factors of any ϕ composition* *chain connecting* $A \cap B$ *and* B *are the same as those of any* ϕ *composition chain connecting* A *and* $\{A, B\}$, *and the multiplicity of the factors is the same in both chains.*

If there exists a ϕ composition chain connecting P and Q, we denote its length by $j(P, Q)$. From Corollary 3.2 it follows that if A and B permute, $j(A, \{A, B\}) = j(A \cap B, B)$. The converse is also true, namely:

LEMMA 3.1. *Under the hypotheses of Theorem* 3.4, *if* $j(A, \{A, B\}) = j(A \cap B, B)$, *then* A *and* B *permute.*

Proof. Let S be a ϕ link for G such that $A \subseteq S \subseteq \{A, B\}$. There exists a ϕ composition chain: $A = S_0 \subset \cdots \subset S_j \subset S = S_{j+1} \subset \cdots \subset S_n = \{A, B\}$. For each i, $S_i \cap B \neq S_{i+1} \cap B$, and hence $S_i \subset \{S_i, S_{i+1} \cap B\} \subseteq S_{i+1}$. Therefore, $S_{i+1} = \{S_i, S_{i+1} \cap B\} = S_i + (S_{i+1} \cap B)$, for $i = 0, \cdots, n-1$. $S = S_{j+1} = S_j + (S_{j+1} \cap B) = S_{j-1} + (S_j \cap B) + (S_{j+1} \cap B) = S_{j-1} + (S_{j+1} \cap B) = \cdots = S_0 + (S_{j+1} \cap B) = A + (S \cap B)$. Hence by Theorem 3.3, A and B permute.

THEOREM 3.5. *Assume the hypotheses of Theorem* 3.4 *and assume that inner automorphisms are* M-ϕ *automorphisms; the* ϕ *composition factors from* $A \cap B$ *to* B *and those from* A *to* $\{A, B\}$ *are the same* (*except for multiplicity*).

Proof. We prove by induction the following statement:

(n) If C and D are ϕ links for $\{A, B\}$ and if $j(D, \{C, D\}) < n$, then the factors for any ϕ composition chain connecting $C \cap D$ and D are the same as those for any ϕ composition chain connecting C and $\{C, D\}$.

(1) is obviously true.

If $j(D, \{C, D\}) = 1$, D is normal in $\{C, D\}$ and $\{C, D\}/D \cong_{(M\text{-}\phi)} C/C \cap D$. Thus by the Jordan-Hölder Theorem the factors of any ϕ composition chain from $C \cap D$ to D and those of any ϕ composition chain from C to $\{C, D\}$ are the same (actually, they are M-ϕ isomorphic); therefore, (2) is true.

We now assume (n) (where $n \geqq 2$) and prove $(n+1)$.

Let C and D be ϕ links for $\{A, B\}$ and let $j(D, \{C, D\}) = n$. (A) We assume that $C \cap D$ is normal in D, and that $D/C \cap D$ is ϕ simple. Since $j(D, \{C, D\}) = n \geqq 2$, $C \cap D \neq C$, and hence $C = C \cap \{C, D\} \neq C \cap D$. Let D' be a minimal ϕ link for $\{C, D\}$ such that $D \subset D'$, and $C \cap D' \neq C \cap D$. It is easily seen that the length of a composition chain from $C \cap D$ to $C \cap D'$ is one so that in particular, $C \cap D$ is normal in $C \cap D'$. We have: $D \subset \{C \cap D', D\} \subseteq D'$, and $C \cap D' = C \cap \{C \cap D', D\}$; hence since D' is minimal, $\{C \cap D', D\} = D'$. We distinguish two cases:

(i) D is normal in D'.

Then D and $C \cap D'$ permute; hence $C \cap D'$ is normal in D', and $D'/(C \cap D') \cong_{(M\text{-}\phi)} D/(C \cap D)$. $\{C, D'\} = \{C, D\}$, and $j(D', \{C, D\}) < n$. Therefore, since (n) holds, the factors of a ϕ composition chain from C to $\{C, D\}$ are the same as those of a ϕ composition chain from $C \cap D'$ to D', and hence are all M-ϕ isomorphic to $D/(C \cap D)$.

(ii) D is not normal in D'.

Let S be a maximal ϕ link for D' which contains D as a normal subgroup, and suppose that S is normal in the ϕ subgroup T of D'. Then there exists a t in T such that $-t+D+T=D(t)\neq D$. Since S is normal in T, $-t+S+t=S(t)=S$. Therefore, $D\subset\{D, D(t)\}\subseteq S$. Thus D is normal in $D_1=\{D, D(t)\}$. $C\cap D$ is normal in $D'=\{C\cap D', D\}$ since $C\cap D$ is normal in $C\cap D'$ and in D; hence $C\cap D$ is normal in T. Therefore, $C\cap D\subseteq D(t)\cap D\subset D$, and $C\cap D=D(t)\cap D$. Consequently, $D_1/D\cong_{(M\text{-}\phi)}D(t)/(C\cap D)\cong_{(M\text{-}\phi)}D/(C\cap D)$, since inner automorphisms are M-ϕ automorphisms. $\{C, D_1\}=\{C, D\}$, and $C\cap D_1=C\cap D$, since $D_1\subset D'$, and D' is minimal by definition. $j(D_1, \{C, D_1\})=n-1$; hence by (n) the factors of a ϕ composition chain from C to $\{C, D\}$ are the same as those of a ϕ composition chain from $C\cap D$ to D_1 and are therefore all M-ϕ isomorphic to $D/(C\cap D)$. This completes the proof of the theorem under assumption (A).

(B) We no longer assume that $C\cap D$ is normal in D. Without loss in generality we may take $A=C$ and $B=D$. Let $A_0=A$ and for $i=0, 1, \cdots$ let A_{i+1} be a minimal ϕ link for $\{A, B\}$ such that $A_i\subset A_{i+1}$, and $A_i\cap B\neq A_{i+1}\cap B$.

$(A_{i+1}\cap B)\subset\{A_i, A_{i+1}\cap B\}\subseteq A_{i+1}$ and hence $A_{i+1}\cap B=\{A_i, A_{i+1}\cap B\}\cap B$. But $A_i\subset\{A_i, A_{i+1}\cap B\}\subseteq A_{i+1}$ and A_{i+1} by definition is minimal. Therefore, $A_{i+1}=\{A_i, A_{i+1}\cap B\}$. $j(A_{i+1}\cap B, B)\leqq j(A_{i+1}, \{A, B\})$, and consequently, by the Jordan-Hölder Theorem, $j(A_{i+1}\cap B, A_{i+1})\leqq j(B, \{A, B\})=n$. $(A_{i+1}\cap B)/(A_i\cap B)$ is ϕ simple so that the factors of a ϕ composition chain from A_i to A_{i+1} are all M-ϕ isomorphic to $(A_{i+1}\cap B)/(A_i\cap B)$ by (A). Hence (n+1) is verified and the proof of the theorem is complete.

Corollary 3.3. *Under the hypotheses of Theorem* 3.5, *there exists a* ϕ *composition chain from* A *to* $\{A, B\}$,

$$A=A_0\subset\cdots\subset A_i\subset A_{i+1}\subset\cdots\subset A_n=\{A, B\},$$

with the following two properties:

(i) *if* $A_i\cap B\neq A_{i+1}\cap B$, *then* $(A_{i+1}\cap B)/(A_i\cap B)\cong_{(M\text{-}\phi)}A_{i+1}/A_i$.

(ii) *if* $A_i\cap B=A_{i+1}\cap B$, *then* $A_{i+1}/A_i\cong_{(M\text{-}\phi)}A_{i+2}/A_{i+1}$.

If A and B are permutable, then for each i we have $A_i\cap B\neq A_{i+1}\cap B$ by Corollary 3.5; but if A and B are not permutable, then for some i, $A_i\cap B=A_{i+1}\cap B$ by Lemma 3.1.

Corollary 3.4. *Under the hypotheses of Theorem* 3.5 *the* ϕ *composition factors for* $\{A, B\}$ *are the same as those for* A *and for* B *together.*

An example shows that Theorem 3.5 is not in general true if we omit the hypothesis that inner automorphisms are M-ϕ automorphisms.

Example 3.1. Let G be the subgroup of the symmetric group on six letters which is generated by a, b, c, and d, where

$$a = (1\ 2), \quad b = (3\ 4), \quad c = (1\ 4)(2\ 3), \quad \text{and} \quad d = (5\ 6),$$

and let e be the identity element. Let M be void and let ϕ consist of the following subgroups: $\{e\}$, $A=\{e, a\}$, $B=\{e, c, d, cd\}$, $\{e, ab\}$, $\{e, a, b, ab\}$, $\{e, c, d, cd, ab, abc, abd, abcd\}$, and $G=\{A, B\}$. It is easily verified that ϕ is closed under intersections and composita.

$A\cap B\subset B$ is a ϕ composition chain from $A\cap B$ to B; and $A\subset\{e, a, b, ab\}\subset\{A, B\}$ is a ϕ composition chain from A to $\{A, B\}$. However, $\{e, a, b, ab\}/A$ is not M-ϕ isomorphic to $\{e, c, d, cd\}$.

Since $c^{-1}Ac=\{e, (3\ 4)\}$ is not a ϕ subgroup of G, the inner automorphism induced by c is not an M-ϕ automorphism.

DEFINITION 3.2. If the M-ϕ group H possesses a normal ϕ subgroup N such that $N\neq H$ and such that every ϕ link different from H is contained in N, H is unitoral. H/N is then the tor of H.

LEMMA 3.2. *Let G be an M-ϕ group. Assume that ϕ contains conjugates and that the ϕ links for G satisfy the double chain condition.*

(i) *Let S and T be ϕ links for G with $S\subset T$; if S is normal in T, and T/S is ϕ simple, then there exists a unitoral ϕ link H for T such that $H\not\subset S$. H has tor M-ϕ isomorphic to T/S, and $T=\{S, H\}$.*

(ii) *G is the compositum of a finite number of unitoral ϕ links.*

Proof. (1) Let H be a minimal ϕ link for T which is not contained in S. H is unitoral and has the maximal normal ϕ subgroup $S\cap H$; for if R is a ϕ link for H, R is contained in S, since H is minimal, and thus $R\subseteq S\cap H$. $S\subset\{H, S\}\subseteq T$ and hence, since $\{H, S\}$ is a ϕ link for T, $\{H, S\}=T$. H has tor $H/(H\cap S)\cong_{(M\text{-}\phi)}\{S, H\}/S=T/S$.

(ii) Let $0=G_0\subset\cdots\subset G_i\subset G_{i+1}\subset\cdots\subset G_n=G$ be a ϕ composition series for G. Then by (i) there exists a unitoral ϕ link H_i such that $G_{i+1}=\{G_i, H_i\}$ for $i=0, \cdots, n-1$, and therefore $G=\{H_0, \cdots, H_{n-1}\}$.

4. Primary M-ϕ groups.

DEFINITION 4.1. An M-ϕ group P is primary if it possesses a ϕ composition series all of whose factors are M-ϕ isomorphic. If all the factors are M-ϕ isomorphic to F, then F is the characteristic of P, and P is an F group. We write: char $(P)=F$.

DEFINITION 4.2. The F subgroup P of the M-ϕ group G is a primary component of G if P is not properly contained in any F subgroup of G.

THEOREM 4.1. *Let G be an M-ϕ group and assume that inner automorphisms of G are M-ϕ automorphisms and that the ϕ links for G satisfy the double chain condition. Then*

(i) *If the ϕ links A and B are F subgroups of G, $\{A, B\}$ is an F subgroup of G.*

(ii) *If the ϕ links A and B are primary ϕ subgroups with different characteristics, $\{A, B\}=A\oplus B$.*

Proof. (i) follows from Corollary 3.4.

(ii) Let $F_1=\text{char }(A)$; $F_2=\text{char }(B)$, where $F_1\neq_{(M\text{-}\phi)}F_2$. If b is an element of B, then $A(b)=-b+A+b$ is an F_1 group; for if $0=A_0\subset\cdots\subset A_i\subset A_{i+1}\subset\cdots\subset A_k=A$ is a ϕ composition series for A, and if $A_i(b)=-b+A_i+b$, for $i=1,\cdots,k$, then $0=A_0(b)\subset\cdots\subset A_i(b)\subset A_{i+1}(b)\subset\cdots\subset A_k(b)=A(b)$ is a ϕ composition series for $A(b)$, and $A_{i+1}(b)/A_i(b)\cong_{(M\text{-}\phi)}A_{i+1}/A_i\cong_{(M\text{-}\phi)}F_1$, since inner automorphisms are by hypothesis M-ϕ automorphisms. Also, $A(b)$ is a ϕ link for G, since A is a ϕ link for G. Let $A^*=\mathrm{C}_{b\in B}A(b)=$ the compositum of all $A(b)$, for b in B. Since the ϕ links for G satisfy the ascending chain condition, $A^*=\mathrm{C}_{i=1}^{k}A(b_i)$ for some integer k and for certain elements b_i in B. By (i), A^* is an F_1 group; and A^* is normal in $\{A, B\}$.

Similarly, we may construct $B^*=\mathrm{C}_{a\in A}B(a)$, where $B(a)=-a+B+a$, for a in A. B^* is an F_2 group and is normal in $\{A, B\}$. $A^*\cap B^*=0$, since $A^*\cap B^*$ is a normal ϕ subgroup of A^* and of B^* and $F_1\neq_{(M\text{-}\phi)}F_2$. Hence $\{A, B\}=\{A^*, B^*\}=A^*\oplus B^*$. But $\{A, B\}=A^*+B$. Since $B\subseteq B^*$, $B^*=(A^*\cap B^*)+B=B$. Similarly, $A^*=A$ and, therefore, $\{A, B\}=A\oplus B$.

LEMMA 4.1. *If the group $G=\sum_{i=1}^{*n}U_i$, where U_i is a normal subgroup of G, for $i=1,\cdots,n$, and if $V_j=\sum_{i=1,i\neq j}^{n}U_i$, for $j=1,\cdots,n$, then $\bigcap_{j=1}^{k}V_j=\sum_{i=k+1}^{n}U_i$, if $1\leqq k<n$, and $=0$, if $k=n$.*

DEFINITION 4.3. If P and Q are primary M-ϕ groups, and if char (P) $\neq_{(M\text{-}\phi)}$ char (Q), we say that P and Q are relatively prime.

THEOREM 4.2. *Let G be an M-ϕ group. Assume that ϕ contains conjugates and that $G=\sum_{i=1}^{*n}P_i$, where the P_i are primary ϕ subgroups of G which are relatively prime in pairs. Then if S is a unitoral ϕ link for G, $S\subseteq P_k$ for some k.*

Proof. Let N be the maximal normal ϕ subgroup of S; and let $Q_i=\sum_{j=1,j\neq i}^{n}P_j$. By Lemma 4.1, $\bigcap_{i=1}^{n}Q_i=0$, and therefore there exists a Q_k such that $S\not\subseteq Q_k$. By Corollary 3.4, the ϕ composition factors from $S\cap Q_k$ to S are a part of those from Q_k to $\{S, Q_k\}$ and are hence all M-ϕ isomorphic to $F_k=\text{char }(P_k)$. $S\cap Q_k$ is a ϕ link for S so that $S\cap Q_k\subseteq N$ and $S/N\cong_{(M\text{-}\phi)}F_k$. If $S\not\subseteq Q_l$ for $l\neq k$, $S/N\cong_{(M\text{-}\phi)}F_l$, where F_l is the characteristic of P_l; but this is impossible, since $F_k\neq_{(M\text{-}\phi)}F_l$. Thus $S\subseteq\bigcap_{i=1,i\neq k}^{n}Q_i$, which is equal to P_k by Lemma 4.1.

COROLLARY 4.1. *Under the hypotheses of Theorem 4.2, if P is any primary ϕ link for G, then P is contained in some P_k. Hence there exists only one decomposition of G as the direct sum of primary ϕ subgroups which are relatively prime in pairs.*

Proof. Since P is a ϕ link, its ϕ composition factors are a part of those for G and hence char $(P)=F_k$ for some k. $P=\{H_1,\cdots,H_s\}$, where, for $i=1,\cdots,s$, H_i is a unitoral ϕ link for G, and char $(H_i)=F_k$. Therefore, by Theorem 4.2, $H_i\subseteq P_k$ and hence $P\subseteq P_k$.

COROLLARY 4.2. *Under the hypotheses of Theorem* 4.2, P_k *is* M-ϕ *characteristic, for* $k=1, \cdots, n$.

Proof. Let σ be an M-ϕ automorphism of G, and let $0=S_0\subset \cdots \subset S_i \subset S_{i+1}\subset \cdots \subset S_t=P_k$ be a ϕ composition series for P_k. Then S_{i+1}/S_i is M-ϕ isomorphic to F_k; σ induces an M-ϕ isomorphism of S_{i+1}/S_i onto $S_{i+1}\sigma/S_i\sigma$ and therefore $S_{i+1}\sigma/S_i\sigma$ is M-isomorphic to F_k. Hence $P_k\sigma$ is an F_k group; furthermore, $P_k\sigma$ is normal in $G\sigma=G$. Therefore, by Corollary 4.1, $P_k\sigma=P_k$.

It is not possible to prove in general that the primary ϕ subgroups in the decomposition are primary components, as is shown by a simple example. However, if the primary ϕ subgroups all have abelian characteristic, then they are primary components.

EXAMPLE 4.1. Let F be a finite, simple, nonabelian group; and let p be a prime which divides the order of F. Let P be a cyclic group of order p. Define $G=F\oplus P$. Let M be void, and let ϕ consist of all subgroups of G.

Then $G=F\oplus P$ is a decomposition of G into primary ϕ subgroups and F and P are relatively prime. However F contains a cyclic subgroup P' of order p and $P'\oplus P$ is a primary ϕ subgroup of G which contains P.

THEOREM 4.3. *If the* M-ϕ *group* G *has a* ϕ *composition series of length* n *with abelian factors, then any proper* ϕ *subgroup* H *of* G *has a* ϕ *composition series of length less than* n. *Furthermore, if* F *is a* ϕ *composition factor for* H *of multiplicity* k, F *is a* ϕ *composition factor for* G *of multiplicity* m *and* $m\geqq k$.

Proof. Let

(1) $0=G_0\subset \cdots \subset G_i\subset G_{i+1}\subset \cdots \subset G_n=G$ be a ϕ composition series for G and define $H_i=H\cap G_i$, for $i=0, \cdots, n$. Then the chain

(2) $0=H_0\subseteq H_1\subseteq \cdots \subseteq H_i\subseteq \cdots \subseteq H_n=H$ is a normal ϕ chain for H. Furthermore, if $H_i\neq H_{i+1}$, H_{i+1}/H_i is ϕ simple; for $H_{i+1}/H_i=H\cap G_{i+1}/H\cap G_i\cong_{(M\text{-}\phi)}\{G_i, G_{i+1}\cap H\}/G_i\subseteq G_{i+1}/G_i$, a ϕ simple, abelian group, so that $H_{i+1}/H_i\cong_{(M\text{-}\phi)}G_{i+1}/G_i$. Thus the ϕ chain (2) is a ϕ composition series for H (with some terms repeated possibly), and its length is less than n if there exists an integer j such that $H_j=H_{j+1}$ $(0\leqq j\leqq n-1)$.

Choose j so that $G_j\subseteq H$ and $G_{j+1}\not\subseteq H$; this is always possible since $G_0\subseteq H$, $G_n\not\subseteq H$. Then $G_j\subseteq H\cap G_{j+1}\subset G_{j+1}$ so that $H_{j+1}=G_j=H_j$.

THEOREM 4.4. *Assume that the* M-ϕ *group* $G=\sum_{i=1}^{*n} P_i$, *where* P_i *is a primary* ϕ *subgroup of* G *with abelian characteristic* F_i, *for* $i=1, \cdots, n$, *and where* P_i *and* P_j *are relatively prime for* $i\neq j$. *Then if* P *is a primary* ϕ *subgroup of* G, P *is contained in some* P_i. *Hence each* P_i *is a primary component of* G.

Proof. Let P be an F subgroup of G; by Theorem 4.3, $F\cong_{(M\text{-}\phi)}F_i$ for some i. If $P_i=K_0\subset \cdots \subset K_j\subset K_{j+1}\subset \cdots \subset G$ is a ϕ composition chain connecting P_i and G, $P\cap P_i=P\cap K_0\subseteq \cdots \subseteq P\cap K_j\subseteq P\cap K_{j+1}\subseteq \cdots \subseteq P\cap G=P$ is a ϕ composition chain connecting $P\cap P_i$ and P (with some terms repeated possibly). If $P\cap K_j\neq P\cap K_{j+1}$, $P\cap K_{j+1}/P\cap K_j\cong_{(M\text{-}\phi)}K_{j+1}/K_j$ (this is shown in the proof of Theorem 4.3), and $K_{j+1}/K_j\not\cong_{(M\text{-}\phi)}F_i$. But $P\cap P_i$ is a normal ϕ subgroup of P and P is an F_i group. Hence, for each j, $P\cap K_j =P\cap K_{j+1}$ and, therefore, $P\cap P_i=P$ or $P\subseteq P_i$.

THEOREM 4.5. *Let G be an M-ϕ group and assume that inner automorphisms are M-ϕ automorphisms and that the ϕ links for G satisfy the double chain condition. G is the direct sum of primary ϕ subgroups if, and only if, unitoral ϕ links are primary.*

Necessity. Suppose that $G=\sum_{i=1}^{*n} P_i$, where P_i is primary and char $(P_i) =F_i$, for $i=1, \cdots, n$. Without loss in generality we may assume that $F_i\not\cong_{(M\text{-}\phi)}F_j$ for $i\neq j$. For if $F_i\cong_{(M\text{-}\phi)}F_j$ for some i and j $(i\neq j)$, $P_i\oplus P_j$ is primary with characteristic F_i. If S is a unitoral ϕ link, $S\subseteq P_k$ for some k, by Theorem 4.2, and hence is primary.

Sufficiency. Let $F_1, \cdots, F_s$ be the different ϕ composition factors for G. Let P_i be the compositum of all unitoral ϕ links with characteristic F_i, for $i=1, \cdots, s$; it follows from the ascending chain condition that P_i is the compositum of a finite number of unitoral ϕ links. Since by hypothesis unitoral ϕ links are primary, by Theorem 4.1 (i), P_i is primary with characteristic F_i. By Theorem 4.1 (ii), $S=\mathrm{C}_{i=1}^{s} P_i=\sum_{i=1}^{*s} P_i$ and S is a ϕ link for G. If $S\neq G$, there exists a ϕ composition chain connecting S and G: $S=S_0\subset S_1\subset \cdots \subset S_i\subset \cdots \subset G$. However, by Lemma 3.2, there exists then a unitoral ϕ link H such that $H\not\subseteq S$, $H\subseteq S_1$. But this is impossible as all unitoral ϕ links are contained in S. Hence $G=S=\sum_{i=1}^{*s} P_i$.

COROLLARY 4.3. *Under the hypotheses of Theorem 4.5, if S is a ϕ link for G, and if $G=\sum_{i=1}^{*n} P_i$, where the P_i are ϕ subgroups of G which are relatively prime in pairs, then $S=\sum_{i=1}^{*n} (S\cap P_i)$.*

Proof. Any unitoral ϕ link for S is also a ϕ link for G and is therefore primary. Hence S is the direct sum of primary ϕ subgroups. Let

$$S = \sum_{i=1}^{l}{}^*K_i,$$

where K_i is primary, for $i=1, \cdots, l$. Without loss in generality we may assume that

$$\text{char } (K_i) = \text{char } (P_i) = F_i,$$

for $i=1, \cdots, l$. K_i is a primary ϕ link for G with characteristic F_i. By Corollary 4.1, $K_i\subseteq P_i$. But $K_i\subseteq S$, and we have $K_i\subseteq P_i\cap S$. Hence

$$S = \sum_{i=1}^{l}{}^*K_i \subseteq \sum_{i=1}^{l} (S \cap P_i) \subseteq S,$$

$$S = \sum_{i=1}^{l}{}^{*}(S \cap P_i).$$

COROLLARY 4.4. *Under the hypothesis of Theorem* 4.5, *if G is the direct sum of primary ϕ subgroups, and if N is a normal ϕ subgroup of G, then G/N is the direct sum of primary ϕ subgroups.*

Proof. By Theorem 4.5 it suffices to show that any unitoral ϕ link for G/N is primary. Let T/N be a unitoral ϕ link for G/N with maximal normal ϕ subgroup K/N. Let R be a minimal ϕ link for T which is not contained in K. It follows from the proof of Lemma 3.2 (i) that R is a unitoral; hence R is primary. Furthermore, $\{R, N\}/N$ is a ϕ link for T/N and $R \not\subseteq K$; therefore, $\{R, N\}/N = T/N$. Now by the Second Isomorphism Theorem:

$$\{R, N\}/N \underset{(M\text{-}\phi)}{\cong} R/R \cap N.$$

$R/R \cap N$ is primary, since R is primary. Therefore, T/N is primary.

5. **ϕ nilpotency.** In this section we define ϕ nilpotency for an M-ϕ group and establish a number of results which we shall need later.

DEFINITION 5.1. The ϕ center of the M-ϕ group G is the compositum of all ϕ subgroups which are contained in the center of G, and is denoted by $Z_\phi(G)$.

Clearly $Z_\phi(G)$ is the greatest ϕ subgroup of G which is contained in the center of G. It is a normal, M-ϕ characteristic ϕ subgroup of G.

DEFINITION 5.2. If G is an M-ϕ group, let $Z_0(G) = 0$, and let $Z_{i+1}(G)/Z_i(G) = Z_\phi(G/Z_i(G))$, for $i = 0, 1, \cdots$.

The chain

$$(1) \qquad 0 = Z_0(G) \subseteq Z_1(G) = Z_\phi(G) \subseteq \cdots \subseteq Z_i(G) \subseteq \cdots$$

is an ascending chain of normal, M-ϕ characteristic ϕ subgroups of G. It is called the upper central ϕ chain for G.

DEFINITION 5.3. The M-ϕ group G is ϕ nilpotent if the upper central ϕ chain joins 0 and G—i.e., if $Z_n(G) = G$ for some integer n.

THEOREM 5.1. *If the M-ϕ group G possesses a normal ϕ chain*

$$(2) \qquad 0 = N_0 \subseteq \cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots \subseteq N_k = G,$$

where, for $i = 0, \cdots, k-1$, N_i is normal in G and where $N_{i+1}/N_i \subseteq Z_\phi(G/N_i)$, then G is ϕ nilpotent.

Proof. This theorem may be proved by induction (see [1, §4]). The chain (2) is called a central ϕ chain for G.

THEOREM 5.2. *If the M-ϕ group G is ϕ nilpotent, then any ϕ subgroup of G is a ϕ link.*

Proof. Let S be a ϕ subgroup of G and let (2) be a central ϕ chain for G. Consider the chain

$$(3)\quad S = \{N_0, S\} \subseteq \cdots \subseteq \{N_i, S\} \subseteq \{N_{i+1}, S\} \subseteq \cdots \subseteq \{N_k, S\} = G;$$

this is a normal ϕ chain joining S and G if $\{N_i, S\}$ is a normal subgroup of $\{N_{i+1}, S\}$. In order to show this, it is sufficient to show that for x in N_{i+1} and s in S, $-x+s+x$ is in $\{N_i, S\}$. But since $N_{i+1}/N_i \subseteq Z_\phi(G/N_i)$, $-x+s+x-s=y$ is an element of N_i and, therefore, $-x+s+x=y+s$ is an element of $\{N_i, S\}$. Thus (3) is a normal ϕ chain joining S and G and S is a ϕ link.

THEOREM 5.3. *If G is a ϕ nilpotent M-ϕ group, and N ($\neq 0$) a normal ϕ subgroup of G, then $N \cap Z_\phi(G) \neq 0$.*

Proof. Let $Z_i = Z_i(G)$. Since G is ϕ nilpotent, $G = Z_n$ for some integer n. $N \cap Z_0 = 0$ and $N \cap Z_n = N \neq 0$, so that there exists an integer i such that $N \cap Z_i = 0$ and $N \cap Z_{i+1} \neq 0$. Let $J = N \cap Z_{i+1}$; and let x and g be elements of J and G, respectively. The element $-g-x+g+x = (-g-x+g)+x$ is in N, since $J \subseteq N$ and N is normal in G; it is also in Z_i, since $J \subseteq Z_{i+1}$ and $Z_{i+1}/Z_i = Z_\phi(G/Z_i)$. Thus $-g-x+g+x$ is an element of $N \cap Z_i = 0$, and, therefore, $g+x = x+g$. Hence $J \subseteq Z_\phi(G)$ and $Z_\phi(G) \cap N = J \neq 0$.

6. A preliminary result.

LEMMA 6.1. *Let Q be an abelian group in which every nonzero element has prime order p. Any automorphism of Q of order p leaves some nonzero element fixed.*

Proof. Let $a_1 \neq 0$ be an element of Q, and let σ be an automorphism of Q of order p. Since $\sigma^p = 1$, $a_1(\sigma^p - 1) = 0$. But a_1 has order p so that $a_1(\sigma-1)^p = 0$. Choose the integer r ($0 \leqq r < p$) so that $a_1(\sigma-1)^r \neq 0$, but $a_1(\sigma-1)^{r+1} = 0$. Thus if $a = a_1(\sigma-1)^r$, a is a nonzero element and $a(\sigma-1) = 0$ or $a\sigma = a$.

THEOREM 6.1. *Let G be an M-ϕ group in which every element has finite order, and assume that ϕ contains conjugates. If $G = A+B$, where:*

(i) *A is a normal abelian ϕ subgroup of G.*

(ii) *B is abelian and ϕ simple, and is isomorphic to a ϕ subgroup of A.*

(iii) *$A \cap B = 0$.*

Then $Z_\phi(G) \neq 0$.

Proof. Let A' consist of all a' in A such that $a'+b = b+a'$ for every b in B. A' is a subgroup of A (but not necessarily a ϕ subgroup). We prove the following result:

If $Z_\phi(G) = 0$, and if p is a prime such that A/A' contains an element of order p, then B contains no element of order p.

We break the proof up into a number of steps.

I. *If a_1 and a_2 are in A, $-a_1+B+a_1 = -a_2+B+a_2$ if and only if $a_1 \equiv a_2$ (mod A').*

$-a_1+B+a_1=-a_2+B+a_2$ if, and only if, $a_2-a_1+B+a_1-a_2=B$. Therefore, it is sufficient to show that $-a+B+a=B$ if, and only if, a is in A'. If a is in A', $-a+b+a=b$ for b in B, and hence $-a+B+a=B$. Conversely, if $-a+B+a=B$, then for each b in B there is a b' in B with $-a+b+a=b'$. Thus $-b+b'=-b-a+b+a=(-b-a+b)+a$ and this is in A, since A is normal in G. Hence $-b+b'$ is an element of $A\cap B=0$, and thus $b=b'$ and $-a+b+a=b$ for all b in B. Therefore, a is an element of A'.

II. *If $B_1=-a_1+B+a_1$, where a_1 is in A; and if $B_1=-b+B_1+b$, where b is an element of B, then $b+\bar{b}_1=\bar{b}_1+b$ for all $\bar{b}_1$ in B_1.*

We show first that if x, y are elements of A such that $x\equiv y \pmod{A'}$, $-x+\bar{b}+x=-y+\bar{b}+y$ for all $\bar{b}$ in B. Since $x-y$ is in A', $(x-y)+\bar{b}=\bar{b}+(x-y)$ for all $\bar{b}$ in B. Hence $-y+\bar{b}+y=-x+\bar{b}+x$. Since $B_1=-a_1+B+a_1$, and $B_1=-b+B_1+b$, we have $-a_1+B+a_1=-b-a_1+b+B-b+a_1+b$. Therefore, by I, $a_1\equiv -b+a_1+b \pmod{A'}$ or $b+a_1\equiv a_1+b \pmod{A'}$. Hence we have: $-(b+a_1)+\bar{b}+(b+a_1)=-(a_1+b)+\bar{b}+(a_1+b)$, for $\bar{b}$ in B. Thus $-a_1+\bar{b}+a_1 =-b+(-a_1+\bar{b}+a_1)+b$, or $\bar{b}_1=-b+\bar{b}_1+b$, for $\bar{b}_1$ in B_1; but this implies that $b+\bar{b}_1=\bar{b}_1+b$, for $\bar{b}_1$ in B_1.

III. *If $a+b=b+a$, for some $b\neq 0$ in B and a in A, then a is in A'.* $b=-a+b+a\neq 0$. Thus if $B(a)=-a+B+a$, $B\cap B(a)\neq 0$. But B and $B(a)$ are both ϕ simple abelian groups which have $B\cap B(a)\neq 0$ as a ϕ subgroup; hence $B=B\cap B(a)=B(a)$. From I we deduce that a is in A'.

IV. *If $b\neq 0$ is an element of B, and a is an element of A such that $-b+a+b \equiv a \pmod{A'}$, then $a\equiv 0 \pmod{A'}$.*

Let $-b+a+b-a=a'$, an element of A'; then $a+b-a=a'+b$. If $t=a+b-a$, $t\neq 0$ and $t=-u+t+u$ for each u in B. Hence if $T=a+B-a$, t is an element of $T\cap(-u+T+u)$ for each u in B; but this implies that $T=-u+T+u$ for each u in B. By II, $u+t'=t'+u$ for each t' in T. Thus $C=B+T$ is an abelian ϕ subgroup of G. Since $C\cap A$ is contained in C, every element of $C\cap A$ commutes with every element of B (and commutes with every element of A, since A is abelian); hence $C\cap A$ is contained in $Z_\phi(G)$. But by hypothesis $Z_\phi(G)=0$; hence $C\cap A=0$. But $C=(C\cap A)+B$ so that $B+T=C=B$; thus $T=B$. In particular, $t=a+b-a=a'+b$ is in B. Thus $a'=0$ and $a+b-a=b$ or $a+b=b+a$. By III, a is in A'.

Now let p be a prime such that A/A' contains an element of order p. Let Q/A' be the subgroup of A/A' which consists of all elements of order p (and 0). If B contains an element b of order p, $-b+A'+b=A'$, and $-b+Q+b$, since Q/A' is a characteristic subgroup of A/A'. Let β be the automorphism of Q/A' defined by $x\beta\equiv -b+x+b \pmod{A'}$ for x in Q. If $x\beta\equiv x \pmod{A'}$ for some x in Q, then $-b+x+b\equiv x \pmod{A'}$ and, by IV, x is in A'. Thus β has order p, and $x\beta\equiv x \pmod{A'}$ if, and only if, $x\equiv 0 \pmod{A'}$. But by Lemma 6.1, there exists an x in Q with $x\not\equiv 0 \pmod{A'}$ and $x\beta\equiv x \pmod{A'}$. Hence we have a contradiction and B contains no element of order p. Thus we have proved the statement enunciated at the beginning of the proof.

We now prove the theorem by assuming that $Z_\phi(G)=0$ and showing that this assumption leads to a contradiction. Let K be a ϕ subgroup of A isomorphic to B. On the one hand, if $K\subseteq A'$, $K\subseteq Z_\phi(G)$ which is impossible, since we are assuming that $Z_\phi(G)=0$. On the other hand, if $K\nsubseteq A'$, K contains an element x such that $x+A'$ has prime order p (in A/A'). Since every element of A has finite order, x has finite order k. Furthermore, p divides k; for otherwise there exist integers r and t such that $1=rk+tp$ and $x=rkx+tpx=t(px)$, which implies that x is in A', an impossibility. Hence $k=pk'$ and $k'x$ is an element of K of order p. Since B is isomorphic to K, B contains an element of order p. Thus if $K\nsubseteq A'$, B and A/A' both contain an element of order p, which is impossible if $Z_\phi(G)=0$ (as was shown above). Therefore, $Z_\phi(G)\neq 0$ and the theorem is proved.

7. **ϕ nilpotency and decomposition into primary components.** In this section we shall prove that an M-ϕ group G—which satisfies certain conditions—is the direct sum of primary ϕ subgroups with abelian characteristic if, and only if, G is ϕ nilpotent.

THEOREM 7.1. *Let G be an M-ϕ group which possesses a ϕ composition series with abelian factors, and assume:*

(i) *Every proper ϕ subgroup of G is ϕ nilpotent.*

(ii) *Inner automorphisms of G are M-ϕ automorphisms.*

(iii) *ϕ cyclic ϕ subgroups of G are abelian.*

(iv) $Z_\phi(G)=0$.

Then $G=A+B$, $A\cap B=0$, A is a minimal normal ϕ subgroup of G and is the direct sum of M-ϕ isomorphic, ϕ simple, abelian ϕ subgroups, and B is abelian and ϕ simple.

Proof. Let N be a maximal normal ϕ subgroup of G, and let T be a minimal normal ϕ subgroup. Then G/N is abelian, since G has abelian ϕ composition factors. If $T\cap N=0$, $G=T\oplus N$ and hence $T\subseteq Z_\phi(G)$. But $Z_\phi(G)=0$ so that $T\cap N\neq 0$. $T\cap N$ is a normal ϕ subgroup of G contained in the minimal normal ϕ subgroup T and therefore $T\subseteq N$. By hypothesis N is ϕ nilpotent and hence, by Theorem 5.3, $T\cap Z_\phi(N)\neq 0$. But $Z_\phi(N)$ is M-ϕ characteristic in N and inner automorphisms of G are M-ϕ automorphisms; hence $Z_\phi(N)$ is normal in G and $T\subseteq Z_\phi(N)$.

Now let s be an element of G which is not in N, and let S be the cyclic ϕ subgroup of G generated by s. By hypothesis, S is abelian; and since N is maximal and G/N is abelian, $G=N+S$. If $G\neq T+S$, $T+S$ is ϕ nilpotent. In this case $C=Z_\phi(T+S)\neq 0$ and hence by Theorem 5.3, $T\cap C\neq 0$. Thus $T_1=T\cap C$ is a ϕ subgroup of G which commutes element-wise with $S\subset T+S$. But $T_1\subseteq T\subseteq Z_\phi(N)$ so that T_1 commutes element-wise with $N+S=G$, i.e., $Z_\phi(G)\supseteq T_1\neq 0$. But this is impossible, since by assumption $Z_\phi(G)=0$. Therefore, $G=T+S$.

If S is not ϕ simple, it contains a ϕ simple ϕ subgroup B. If $B\subseteq N$, then

$T+B$ is abelian since $T\subseteq Z_\phi(N)$ and B is abelian. Therefore, $B\subseteq Z_\phi(G)$ since B commutes element-wise with T and with S and $G=T+S$. This proves that $B\not\subseteq N$. Hence $G=N+B$ and the argument used above (replacing S by B) shows that $G=T+B$.

Let A be the sum of all minimal normal ϕ subgroups of T which are M-ϕ isomorphic to a particular minimal normal ϕ subgroup F of T. Then it is easily verified that $A=F_1\oplus\cdots\oplus F_k$, where, for $i=1, 2, \cdots, k$, $F_i\cong_{(M\text{-}\phi)}F$; and that A is M-ϕ characteristic in T (see [1, §3]). Hence A is normal in G and $T=A$. Thus $G=A+B$ and the theorem is proved.

We give an example which shows that a group of the type described in the theorem exists. In the example B is M-ϕ isomorphic to F.

Example 7.1. Let the elements of G be all pairs of real numbers (r, s) and define: $(r, s)=(r', s')$ if, and only if, $r=r'$ and $s=s'$; $(r, s)+(r', s')=(r+r'e^{-s}, s+s')$. Then if A is the subgroup which consists of all pairs $(r, 0)$, and if B is the subgroup which consists of all pairs $(0, s)$, we have $G=A+B$. A and B are isomorphic and, furthermore, A is normal in G.

Let M be void, and let ϕ consist of the subgroups 0, G, A, B and all conjugates of B. We verify that ϕ is a lattice.

$A\cap B'=0$ for any conjugate of B, $B'=-a'+B+a'$ with a' in A. For let a be an element of $A\cap B'$; then $b=a'+a-a'$ is in $A\cap B=0$ so that $a=0$. Now consider $B\cap B'$, where again $B'=-a'+B+a'$ and a' is in A $(a'\neq 0)$. If $b=-a'+\bar{b}+a'$, where b and $\bar{b}$ are in B, then $a'+b=\bar{b}+a'-\bar{b}+\bar{b}$ so that $a'=\bar{b}+a'-\bar{b}$ and $b=\bar{b}$. Let $a'=(r', 0)$ and $b=(0, s)$; then $(r', 0)=(0, s)+(r', 0)-(0, s)=(r'e^{-s}, 0)$ and thus $r'=r'e^{-s}$. Hence $s=0$ and this implies that $b=0$; thus $B\cap B'=0$. Similarly, if $B''=-a''+B+a''$, where $a''\neq a'$ is in A, $B'\cap B''=0$. Therefore, ϕ is closed under intersections.

We now consider the composita of the subgroups in ϕ. We consider first $A+B'$. Since $-a'+G+a'=G$ and $G=A+B$, $G=-a'+A+a'+B'=A+B'$. We consider next $\{B, B'\}$. Since $B\subseteq\{B, B'\}$ and $G=A+B$, $\{B, B'\}=(A\cap\{B, B'\})+B$. Since $A\cap\{B, B'\}\neq 0$, there exists $\bar{a}\neq 0$ in $A\cap\{B, B'\}$; let $\bar{a}=(\bar{r}, 0)$. Let $a=(r, 0)$ be any element of A such that $r\bar{r}$ is positive; then $r=\bar{r}e^{-s}$ for some real number s. If $b=(0, s)$, $a=(r, 0)=(0, s)+(\bar{r}, 0)-(0, s)=b+\bar{a}-b$, so that a is in $\{B, B'\}$. Hence $\{B, B'\}=G$. Similarly, $G=\{B', B''\}$. Hence ϕ is a lattice.

We verify next that G satisfies conditions (i)–(iv) of Theorem 7.1. Since every proper ϕ subgroup of G is abelian, (i) is clearly satisfied; it is also clear that (ii) is satisfied. To show that ϕ cyclic ϕ subgroups are abelian, let $a+b$ be an element of G with a in A and $b(\neq 0)$ in B; we show that the ϕ cyclic ϕ subgroup generated by $a+b$ is abelian. If $a=(r, 0)$ and $b=(0, s)$, then $a+b=(r, s)=-(x, 0)+(0, s)+(x, 0)$, where $x=r/(e^{-s}-1)$. Letting $a'=(x, 0)$, we see that $a+b$ is in $B'=-a'+B+a'$, an abelian ϕ subgroup of G. $Z_\phi(G)$ is a normal ϕ subgroup of G, and $Z_\phi(G)\neq A$; hence $Z_\phi(G)=0$.

We note that the group is a primary M-ϕ group with abelian characteristic. Hence conditions (i)–(iii) are not sufficient to ensure that a primary M-ϕ group with abelian characteristic be ϕ nilpotent.

THEOREM 7.2. *Let P be a primary M-ϕ group with abelian characteristic F and assume*:

(i) *Inner automorphisms of P are M-ϕ automorphisms.*

(ii) *Every element in P has finite order.*

(iii) *ϕ cyclic ϕ subgroups of P are abelian.*

Then P is ϕ nilpotent.

Proof. We use induction on the length of a ϕ composition series for P, $j(P)$. If $j(P)=1$, P is isomorphic to the abelian group F and hence is ϕ nilpotent. Let $j(P)=k$ and assume that the theorem is true for all primary M-ϕ groups Q such that $j(Q)<k$. Since any ϕ subgroup H of P has $j(H)<k$ (by Theorem 4.3) and is primary, any ϕ subgroup of P is ϕ nilpotent. By Theorem 7.1, if $Z_\phi(P)=0$, $P=A+B$, where A is normal in P and is the direct sum of ϕ subgroups which are M-ϕ isomorphic to F and B is M-ϕ isomorphic to F; and $A\cap B=0$. But by Theorem 6.1 this is impossible. Hence $Z_\phi(P)\neq 0$.

Let $Q=P/Z_\phi(P)$; then $j(Q)<k$ and by the induction hypothesis, Q is ϕ nilpotent. Hence P is ϕ nilpotent.

COROLLARY 7.1. *Let P be the direct sum of primary M-ϕ groups with abelian characteristic and assume conditions* (i)–(iii) *of Theorem* 7.2. *Then P is ϕ nilpotent.*

REMARK. In Theorem 7.2, condition (ii) may be replaced by:

(ii′) Any ϕ subgroup has a finite number of conjugates.

In order to prove this it is sufficient to show that Theorem 6.1 still holds if we assume that B has a finite number of conjugates instead of assuming that every element of G has finite order. In Theorem 6.1 for each element a of A we obtain a conjugate of B, $B(a)=-a+B+a$, and two elements which are incongruent modulo A' give different conjugates. Thus if B has a finite number of conjugates, A/A' is a finite group. If $Z_\phi(G)=0$, it follows from IV of the proof of Theorem 6.1 that the elements of B induce distinct automorphisms of A/A'; hence B is a finite group. Thus, in particular, every element of A/A' and of B is of finite order and the proof of Theorem 6.1 may be applied.

COROLLARY 7.2. *If P is a primary M-ϕ group with abelian characteristic, and if P has a ϕ composition series of length* 2, *then P is abelian, provided conditions* (i)–(iii) *of Theorem* 7.2 *hold.*

Proof. Let F be the characteristic of P, and let A be a normal ϕ subgroup of P which is M-ϕ isomorphic to F. Since P is ϕ nilpotent and A is a minimal normal ϕ subgroup, it follows from Theorem 5.3 that A is contained in $Z_\phi(P)$.

Let s be an element of P not in A; and let S be the ϕ cyclic ϕ subgroup of P generated by s. Then S is abelian and $P=A+S$; since $A\subseteq Z_\phi(P)$, P is abelian.

Our results enable us to prove the following converse of Theorem 5.2:

THEOREM 7.3. *Let G be an M-ϕ group which possesses a ϕ composition series. Assume further that:*

(i) *Inner automorphisms of G are M-ϕ automorphisms.*

(ii) *ϕ cyclic ϕ subgroups of G are abelian.*

Then if every ϕ subgroup of G is a ϕ link, G is ϕ nilpotent.

Proof. It is easy to see that (ii) implies that G has abelian ϕ composition factors.

We now use induction on $j(G)$. If $j(G)=1$, G is abelian. Assume the theorem proved for all H with $j(H)<j(G)$. By Theorem 7.1, if $Z_\phi(G)=0$, $G=A+B$, where A and B are abelian ϕ subgroups of G, A is normal in G and $A\cap B=0$. Since B is a ϕ link for G, there exists a ϕ subgroup C of G such that $B\neq C$ and B is normal in C. Then $C=(A\cap C)+B$, and the sum is a direct sum since $(A\cap C)\cap B=0$ and $A\cap C$, B are normal in C. Therefore, $A\cap C\subseteq Z_\phi(G)$ so that $Z_\phi(G)\neq 0$. Hence by the induction assumption, G is ϕ nilpotent.

THEOREM 7.4. *Let G be an M-ϕ group which possesses a ϕ composition series. Assume that inner automorphisms are M-ϕ automorphisms and that unitorial ϕ cyclic ϕ subgroups of G are primary. Then if G is ϕ nilpotent, G is the direct sum of its primary components.*

Proof. By Theorem 4.5, it is sufficient to show that any unitoral ϕ subgroup of G is primary. Let U be a unitoral ϕ subgroup; then U has a normal ϕ subgroup N such that every proper ϕ subgroup of U is contained in N. Choose u in U but not in N; then the ϕ cyclic ϕ subgroup generated by u is not contained in N and hence is equal to U. Thus U is primary.

THEOREM 7.5. *Let P be an M-ϕ group which possesses a ϕ composition series and assume hypotheses* (i), (ii), (iii) *of Theorem 7.2 and*

(iv) *Unitoral ϕ cyclic ϕ subgroups of P are primary.*

Then P is ϕ nilpotent if, and only if, P is the direct sum of primary ϕ subgroups with abelian characteristic.

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