ON THE STRUCTURE OF UNITARY GROUPS

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1. Let K be an arbitrary sfield with an *involution* J, that is, a one-to-one mapping $\xi \to \xi^J$ of K onto itself, distinct from the identity, such that $(\xi+\eta)^J = \xi^J + \eta^J$, $(\xi\eta)^J = \eta^J\xi^J$, and $(\xi^J)^J = \xi$. Let E be an n-dimensional right vector space over K $(n \geqq 2)$; an *hermitian* (resp. *skew-hermitian*) *form* over E is a mapping $(x, y) \to f(x, y)$ of $E \times E$ into K which, for any x, is linear in y, and such that $f(y, x) = (f(x, y))^J$ (resp. $f(y, x) = -(f(x, y))^J$). This implies that $f(x, y)$ is additive in x and such that $f(x\lambda, y) = \lambda^J f(x, y)$. The values $f(x, x)$ are always *symmetric* (resp. *skew-symmetric*) elements of K, that is, elements α such that $\alpha^J = \alpha$ (resp. $\alpha^J = -\alpha$). The orthogonality relation $f(x, y) = 0$ relative to f is always symmetric.

We shall always suppose that the form f is *nondegenerate*, or in other words that there is no vector in E other than 0 orthogonal to the whole space. Moreover, when the characteristic of K is 2, the distinction between hermitian and skew-hermitian forms disappears, and $f(x, x)$ is symmetric for every $x \in E$; in that case we shall make the *additional assumption* that $f(x, x)$ has always the form $\xi + \xi^J$ ("trace" of ξ) for a convenient $\xi \in K$; this assumption is automatically verified when the restriction of J to the center Z of K is not the identity, but not necessarily in the other cases.

A *unitary transformation* u of E is a one-to-one linear mapping of E onto itself such that $f(u(x), u(y)) = f(x, y)$ identically; these transformations constitute the *unitary group* $U_n(K, f)$. In a previous paper [5, pp. 63–82](1), I have studied the structure of that group in the two simplest cases, namely those in which K is commutative, or K is a reflexive sfield and the form f is hermitian; the present paper is devoted to the study of $U_n(K, f)$ in the general case.

2. We shall need the following lemma:

LEMMA 1. *If the sfield K is not commutative, it is generated by the set S of the symmetric elements, except when K is a reflexive sfield of characteristic $\neq 2$, and S is identical with the center Z of K.*

Let L be the subsfield of K generated by S; we are going to prove that if L is not contained in Z, then $L = K$. Suppose the contrary, and let α be an element in K not belonging to L; let M be the 2-dimensional right vector space over L having 1 and α as a basis; we are going to prove that M is a *sfield*. We first notice that L is identical with the *subring* of K generated by S;

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(1) Numbers in square brackets refer to the bibliography at the end of the paper.

for if $\zeta \neq 0$ is an element of that subring, it is clear that ζ^J also belongs to it; but $\zeta\zeta^J = \delta$ is in S, hence $\zeta^{-1} = \zeta^J\delta^{-1}$ belongs to the ring generated by S, which proves that L is identical with that ring. We next notice that $\alpha^J + \alpha = \beta \in S \subset L$, and $\alpha\alpha^J = -\alpha^2 + \alpha\beta = \gamma \in L$ and therefore $\alpha^2 = \alpha\beta - \gamma$. On the other hand, if ξ is any element in S, $\alpha\xi + (\alpha\xi)^J = \alpha\xi + \xi\alpha^J$ is in L, and therefore $\alpha\xi - \xi\alpha$ is in L; by induction on k, it follows that if $\zeta = \xi_1\xi_2 \cdots \xi_k$, where $\xi_j \in S$ for $1 \leqq j \leqq k$, the element $\alpha\zeta - \zeta\alpha$ is in L. These remarks prove that M is a *subring* of K, invariant by the involution J, and the same argument as the one made for L proves that M is a sfield. Now for any $\zeta \in L$, $\alpha\zeta + (\alpha\zeta)^J = \alpha\zeta + \zeta^J\alpha^J$ is in $S \subset L$, and replacing α^J by its value shows that $\alpha\zeta - \zeta^J\alpha$ is in L; but as $\zeta^J\alpha - \alpha\zeta^J$ also belongs to L, we see that $\alpha(\zeta - \zeta^J)$ is in L; this is of course possible only when $\zeta = \zeta^J$. In other words, we come to the conclusion that $L = S$; in particular, if ξ and η are any two elements of S, $\xi\eta$ is in S, and therefore $(\xi\eta)^J = \eta^J\xi^J = \eta\xi$ is equal to $\xi\eta$; this means that L is a *commutative* field.

To go on with the argument, let us first suppose that the characteristic of K is $\neq 2$; then, as $\alpha = (\alpha + \alpha^J)/2 + (\alpha - \alpha^J)/2$, $\alpha - \alpha^J$ is not in L, and we can replace α by $\alpha - \alpha^J$ in the preceding sequence of arguments. We then have $\alpha^J = -\alpha$, and $\alpha^2 = -\gamma \in L$. The mapping $\zeta \rightarrow \alpha\zeta - \zeta\alpha$ is a *derivation* of the field L; if we put $D\zeta = \alpha\zeta - \zeta\alpha$, we have $D^2\zeta = \alpha^2\zeta - 2\alpha\zeta\alpha + \zeta\alpha^2 \in L$ for every $\zeta \in L$, which gives $\alpha\zeta\alpha \in L$, since the characteristic of L is $\neq 2$. But we may write $\alpha\zeta\alpha = \alpha^2\zeta - \alpha \cdot D\zeta$ and as $\alpha^2 \in L$, this gives $\alpha \cdot D\zeta \in L$, which is possible only if $D\zeta = 0$ for every $\zeta \in L$. This proves that every element $\alpha \in K$ commutes with every element of L, in other words, that L is in the *center* of K, contrary to assumption.

We next take up the case in which the characteristic of K is 2. From the relation $\alpha^3 = \alpha\beta\alpha - \gamma\alpha = \alpha^2\beta - \alpha\gamma$, one derives immediately $D\beta = D\gamma = 0$, in other words, β and γ commute with α; replacing α by $\beta^{-1}\alpha$, we can therefore suppose that $\alpha^2 = \alpha + \gamma$, with $D\gamma = 0$. Let N be the subfield of L defined by the equation $D\xi = 0$ (commuting subfield of α or center of M). The relation $\alpha^2 = \alpha + \gamma$ implies that $D^2\xi = D\xi$ for every $\xi \in L$, or in other words, that $\xi + D\xi \in N$ for all $\xi \in L$. On the other hand, $D(\xi^2) = 2\xi \cdot D\xi = 0$ because the characteristic is 2, hence $\xi^2 \in N$ for $\xi \in L$. Now, if $\zeta = \alpha\xi + \eta$ is any element of M, with $\xi \in L$ and $\eta \in L$, an easy computation shows that $\zeta\zeta^J = \gamma\xi^2 + \xi\eta + D(\xi\eta) + \eta^2$ and therefore $\zeta\zeta^J \in N$; on the other hand $\zeta + \zeta^J = \xi + D\xi$ is also in N. If $N \neq L$, this means that M is a *reflexive* sfield over its center N [5, p. 72]. But in a reflexive sfield of characteristic 2, the symmetric elements constitute a 3-dimensional subspace over the center, whilst here they are the elements of L, which is only 2-dimensional over N; the assumption $N \neq L$ is therefore untenable. But if $N = L$, α commutes again with every element of L, in other words, L is again the center of K, contrary to assumption.

We have still to examine the exceptional case in which S is contained in Z. For every element $\xi \in K$, $\xi + \xi^J$ and $\xi^J\xi$ are then in the center Z, and therefore, as $\xi^2 - (\xi + \xi^J)\xi + \xi^J\xi = 0$, every element of K has degree 2 over the center

Z. It is well known that this is possible only if K has rank 4 over Z. Moreover if $\gamma \in Z$ and ζ is not in Z, $\gamma\zeta + (\gamma\zeta)^J = \gamma(\zeta + \zeta^J) + (\gamma^J - \gamma)\zeta^J$ is in Z, which implies $\gamma^J = \gamma$; this shows that K is a *reflexive* sfield [5, p. 72], and $S = Z$; but this is possible only when K has a characteristic $\neq 2$ (loc. cit.), and that completes the proof of Lemma 1.

3. From the involution J, we can deduce other involutions T of K by the general process of setting $\xi^T = p^{-1}\xi^J p$, where p is a symmetric or skew-symmetric element of K (with respect to J); if $p^J = \epsilon p$ ($\epsilon = 1$ or $\epsilon = -1$), the relation $\xi^T = \xi$ is then equivalent to $p\xi = \epsilon(p\xi)^J$; in other words, the T-symmetric elements of K are of the form $p^{-1}\eta$, where η is J-symmetric if $\epsilon = 1$ and η is J-skew-symmetric if $\epsilon = -1$. This enables one to reduce to each other the hermitian and skew-hermitian forms, by a change of the involution (when the characteristic of K is not 2). Indeed, if $f(y, x) = -(f(x, y))^J$, consider the form $g(x, y) = p^{-1}f(x, y)$, where p is skew-symmetric; then g is linear in y, and one has $g(y, x) = -p^{-1}(f(x, y))^J = -p^{-1}(pg(x, y))^J = (g(x, y))^T$. For the sake of convenience, we shall always suppose in the following that the form f is *skew-hermitian* for J.

The notions of orthogonal basis, of isotropic vector, of isotropic and totally isotropic subspaces of E are defined as usual (see [5]); the *index* ν of f is the maximum dimension of the totally isotropic subspaces, and one has $2\nu \leqq n$. When a plane $P \subset E$ is not totally isotropic but contains an isotropic vector $a \neq 0$, then there exists in P a second isotropic vector b such that $f(a, b) = 1$; P is then said to be a *hyperbolic plane*, and the restrictions of f to any two hyperbolic planes are equivalent. Moreover, Witt's theorem is still valid (see [6, pp. 8–9]; in the case of characteristic 2, this, as well as the preceding property, is due to the restrictive assumption on f to be "trace-valued"); we shall formulate it in the following form: *if V and W are any two subspaces of E such that the restrictions of f to V and W are equivalent, then there is a unitary transformation u such that $u(V) = W$.*

4. Let us recall that a *transvection* is a linear transformation of the type $x \to x + a\rho(x)$, where ρ is a linear form, not identically 0, and such that $\rho(a) = 0$. If we write that such a transformation is unitary, we get

$$(\rho(x))^J f(a, y) + f(x, a)\rho(y) + (\rho(x))^J f(a, a)\rho(y) = 0$$

identically in x and y; with $x = a$ this gives $f(a, a)\rho(y) = 0$, hence $f(a, a) = 0$, the vector a must be *isotropic*; then we get

$$(\rho(x))^J f(a, y) + f(x, a)\rho(y) = 0$$

which, for fixed x such that $\rho(x) \neq 0$, shows that $f(x, a) \neq 0$, and $\rho(y) = \lambda f(a, y)$; finally, we have

$$(f(a, x))^J \lambda^J f(a, y) + f(x, a)\lambda f(a, y) = 0$$

identically, and as $f(a, x) = -(f(x, a))^J$, this yields $\lambda^J = \lambda$. In other words, unitary transvections exist only if $\nu \geqq 1$, and then are of the form $x \rightarrow x + a\lambda f(a, x)$, where a is an arbitrary isotropic vector, and λ an arbitrary symmetric element in K; the hyperplane of points of E invariant by the transvection is the hyperplane orthogonal to a.

Let H be a nonisotropic hyperplane, a a vector orthogonal to H. Then every unitary transformation u leaving invariant every element of H is such that $u(a) = a\mu$, with $\mu^J \alpha \mu = \alpha$, where $\alpha = f(a, a)$; we shall say that such a transformation is a *quasi-symmetry*. There always exist quasi-symmetries of hyperplane H, not reduced to the identity; this is obvious if K has a characteristic $\neq 2$, for then the ordinary symmetry $(\mu = -1)$ has that property. If K has characteristic 2, one has by assumption $\alpha = \beta + \beta^J$, with $\beta \neq \beta^J$; then $\mu = \beta^{-1}\beta^J$ satisfies $\mu^J \alpha \mu = \alpha$, and $\mu \neq 1$.

These remarks already enable us to determine the *center* Z_n of the group $\boldsymbol{U}_n(K, f)$. Indeed, a transformation v belonging to the center must permute with every quasi-symmetry, hence leave invariant every nonisotropic line; and if there are isotropic lines, v must permute with every unitary transvection, hence leave invariant every isotropic line as well. Therefore v leaves invariant every line, which means that it is a homothetic mapping $x \rightarrow x\gamma$, with γ in the center Z of K and $\neq 0$; moreover, in order that such a mapping be unitary, it is necessary and sufficient that $\gamma^J \gamma = 1$.

5. From now on, we are going to suppose that $\nu \geqq 1$. Let T_n be the subgroup of $\boldsymbol{U}_n(K, f)$ *generated by unitary transvections*; as a transform vuv^{-1} of a transvection u is again a transvection, it is clear that T_n is a *normal* subgroup of $\boldsymbol{U}_n$. Let W_n be the center of T_n (we shall determine its structure in §11). We shall now prove the following theorem.

THEOREM 1. *If the sfield K has more than 25 elements*(2), *the group T_n/W_n is simple for $n \geqq 2$ and $\nu \geqq 1$.*

Our proof will be modeled after that of [5, Theorem 4, p. 55], and will proceed in several steps.

1°. We first prove that *if a normal subgroup G of T_n contains all transvections of $\boldsymbol{U}_n$ having the same vector a, then $G = T_n$.* In order to do this, we shall prove the following lemma.

LEMMA 2, *If a and b are any two noncollinear isotropic vectors, there exists a transformation $u \in T_n$ such that $u(a) = b\mu$ for a convenient scalar $\mu \in K$.*

If we suppose the lemma proved, and consider an arbitrary transvection $x \rightarrow v(x) = x + a\alpha f(a, x)$, it is readily verified that uvu^{-1} is the transvection $x \rightarrow x + b\mu\alpha\mu^J f(b, x)$; but as α can take any value in the set S of symmetric elements, so can $\mu\alpha\mu^J$. Therefore G contains all transvections of b, and in consequence is identical to T_n, since b is an arbitrary isotropic vector.

(2) The theorem is still true when K has at most 25 elements, except when $K = \boldsymbol{F}_4$, $n = 2$ and $n = 3$, and $K = \boldsymbol{F}_9$, $n = 2$ [5, p. 70].

To prove the lemma, let us first suppose that $f(a, b) \neq 0$; then there is a scalar $\mu \neq 0$ such that $a+b\mu = c$ is isotropic. Indeed, the relation $f(a+b\mu, a+b\mu) = 0$ gives the condition $\mu^J f(b, a) + f(a, b)\mu = 0$ which is satisfied by taking $\mu = (f(a, b))^{-1}$, owing to the relation $f(b, a) = -(f(a, b))^J$. The transvection $x \to u(x) = x + cf(c, x)$ sends then a into $-b\mu$, for $f(c, a) = \mu^J f(b, a) = -1$.

Suppose next that $f(a, b) = 0$; this means that the plane containing a and b is totally isotropic, hence $n \geqq 3$. Therefore there exists a vector z such that $f(a, z) \neq 0$ and $f(b, z) \neq 0$; the plane containing a and z is hyperbolic, and contains therefore a vector a_1 not collinear to a and isotropic; moreover a_1 cannot be orthogonal to b, otherwise z would also be orthogonal to b; therefore one has $f(a, a_1) \neq 0$ and $f(a_1, b) \neq 0$; applying the preceding result, there is a transvection u_1 transforming a into a scalar multiple of a_1, and a transvection u_2 transforming a_1 into a scalar multiple of b; the transformation $u = u_2 u_1$ satisfies the conditions of the lemma.

6. Our next step will be to prove that:

2°. *Theorem* 1 *is true for* $n=2$, $\nu \geqq 1$. The assumption implies that there is a basis of E consisting of 2 isotropic vectors e_1, e_2 such that $f(e_1, e_2) = 1$. If u is a unitary transformation,

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

its matrix with respect to the basis (e_1, e_2), the elements of U satisfy the following conditions

$$\text{(1)} \qquad \alpha^J\gamma - \gamma^J\alpha = 0, \qquad \beta^J\delta - \delta^J\beta = 0, \qquad \alpha^J\delta - \gamma^J\beta = 1,$$

and conversely, the matrices satisfying these relations are unitary. We observe that from (1) one deduces the following relations

$$\text{(2)} \qquad \alpha\beta^J - \beta\alpha^J = 0, \qquad \gamma\delta^J - \delta\gamma^J = 0.$$

Indeed, let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let U^* be the transposed matrix of U^J; then (1) is equivalent to the matrix relation $U^*AU = A$, whence $A^{-1} = U^{-1}A^{-1}(U^*)^{-1}$, and therefore $UA^{-1}U^* = A^{-1}$; but as $A^{-1} = -A$, the last relation implies (2) (this short derivation of (2) from (1) was indicated by the referee). The transvections of vector e_2 have matrices of the type

$$B(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

where $\lambda \in S$; the transvections of vector e_1 have matrices of the type

$$C(\mu) = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

with $\mu \in S$. We want to prove that if a normal subgroup G of T_2 contains a transformation u not in the center W_2, then $G = T_2$; it will be enough, by virtue of part 1°, to show that *all* matrices $C(\mu)$ belong to G.

Let us first suppose that the matrix U is such that $\beta \neq 0$. Then the matrix

$$(B(\lambda))^{-1}UB(\lambda) = B(-\lambda)UB(\lambda)$$
$$= \begin{pmatrix} \alpha + \beta\lambda & \beta \\ \gamma' & \delta' \end{pmatrix}$$

belongs to G, for any $\lambda \in S$. It follows from the first relation (2) that $\beta^{-1}\alpha \in S$; taking $\lambda = -\beta^{-1}\alpha$, we see that we can always limit ourselves to the case in which $\alpha = 0$; the third relation (1) then yields $\gamma = -(\beta^{-1})^J$.

Supposing therefore that $\alpha = 0$, we next determine a linear transformation v of E such that $u(v(e_1)) = e_1\xi$, and $v(u(e_1)) = e_1\eta$, ξ and η being at first arbitrary elements $\neq 0$ in K. An easy computation shows that the matrix of v with respect to e_1, e_2 is equal to

$$V = \begin{pmatrix} -\gamma^{-1}\delta\beta^{-1}\xi & \eta\gamma^{-1} \\ \beta^{-1}\xi & 0 \end{pmatrix}.$$

We now want v to be in the group T_2; this, by the third condition (1), is possible only if we have

$$(\eta\gamma^{-1})^J\beta^{-1}\xi = -1. \tag{3}$$

Conversely, if ξ and η satisfy (3) and $\beta^{-1}\xi \in S$, then $v \in T_2$. To prove this, we first remark that there is $\sigma \in S$ such that

$$VB(\sigma) = \begin{pmatrix} 0 & -(\zeta^{-1})^J \\ \zeta & 0 \end{pmatrix},$$

with $\zeta = \beta^{-1}\xi$; indeed, this relation is equivalent to $\sigma = \gamma\eta^{-1}\gamma^{-1}\delta\beta^{-1}\xi$; but it follows from the second relation (2) that $\gamma^{-1}\delta \in S$, and on the other hand, (3) shows that $\gamma\eta^{-1} = -(\beta^{-1}\xi)^J$; therefore, the element σ is in S.

Further, we have, for $\zeta \in S$,

$$C(-\zeta^{-1})B(\zeta)C(-\zeta^{-1}) = \begin{pmatrix} 0 & -\zeta^{-1} \\ \zeta & 0 \end{pmatrix},$$

hence $VB(\sigma)$ is in T_2, which proves that V is in T_2.

The transformation $u_1 = u^{-1}v^{-1}uv$ is then in G, and its matrix has the form

$$U_1 = \begin{pmatrix} \rho & \beta' \\ 0 & (\rho^{-1})^J \end{pmatrix},$$

where $\rho = \beta^J \zeta \beta \zeta$. Finally the matrix $W = U_1 C(\theta) U_1^{-1} C(-\theta)$ is in G for every $\theta \in S$, and is equal to

$$\begin{pmatrix} 1 & \rho\theta\rho^J - \theta \\ 0 & 1 \end{pmatrix};$$

in other words, it is a matrix $C(\mu)$ with $\mu = \rho\theta\rho^J - \theta$.

7. We first want to prove that it is possible to choose ζ and θ in the set S of symmetric elements such that $\mu \neq 0$. This will certainly be the case if $\rho\rho^J \neq 1$, with $\theta = 1$. We have therefore to show that, under the assumptions of Theorem 1, it is impossible that $\rho\rho^J = 1$ for every $\zeta \in S$. This is immediate if the subfield Z_0 of the center Z, which consists of the symmetric elements of Z (and is such that Z is a separable quadratic extension of Z_0, or identical to Z_0), has more than 5 elements; for if $\zeta \in Z_0$, the relation $\rho\rho^J = 1$ reduces to $\zeta^4(\beta^J\beta)^2 = 1$, which can be verified by at most 4 different elements of Z_0. We are therefore reduced to the case in which Z_0 has at most 5 elements, which means that Z has at most 25 elements; moreover, we can suppose that K is noncommutative, and therefore infinite. In the identity $\rho\rho^J = 1$, if we replace ζ by 1, we get $(\beta^J\beta)^2 = 1$, hence $\beta^J = \beta^{-1}$ or $\beta^J = -\beta^{-1}$; in any case, β^J and β commute. If $\beta^J + \beta = 0$, we have $\beta^4 = 1$; if $\beta + \beta^J \neq 0$, we can replace ζ by $\beta + \beta^J$, and we get $(\beta + \beta^J)^4 = 1$. In every case, β is a root of an algebraic equation with coefficients in Z, and as Z is finite, so is the commutative field $Z(\beta)$. Let L be the subsfield of K consisting of the elements of K which commute with β; as $Z(\beta)$ has finite degree over Z, K has finite degree over L, and therefore L is an infinite sfield [2, p. 104]; moreover, as $Z(\beta^J) = Z(\beta)$, L is invariant under the involution J. Now, if we take ζ in $S \cap L$, the relation $\rho\rho^J = 1$ reduces to $\zeta^4 = 1$, in other words $\zeta^2 = 1$ or $\zeta^2 = -1$. If we apply this to $\zeta = \xi + \eta$, where ξ and η are arbitrary in $S \cap L$, we conclude that $\xi\eta + \eta\xi$ is in the center Z of K, from which it immediately follows that the sfield M generated by ξ and η over Z has at most rank 4 over Z; as Z is finite, this sfield must be commutative. In other words, any two elements of $S \cap L$ commute; it then follows from Lemma 1 that either L is commutative, or is a reflexive sfield, and then has necessarily an infinite center which is identical to $S \cap L$. In any case, the relation $\zeta^4 = 1$, valid for $\zeta \in S \cap L$ (and $\zeta \neq 0$) shows that $S \cap L$ must be finite; this is possible only when L is commutative; but then $S \cap L$ is a subfield of L such that L has degree 2 over $S \cap L$, and as L is infinite, $S \cap L$ would also have to be infinite; we thus have reached a contradiction, which ends this part of the argument.

8. We now have proved that there exists in S an element $\mu_0 \neq 0$ such that $C(\mu_0)$ belongs to G. We want to show next that $C(1)$ also belongs to G. In order to do this, we repeat the whole argument of §§6 and 7, starting with

the matrix $C(\mu_0)$ instead of U, and, therefore, this time the element $\beta=\mu_0$ is symmetric. If we can take ζ in the center Z, we thus get an element ρ which is symmetric and such that $\rho^2\neq 1$. If not, which is the case only when Z_0 has at most 5 elements, the commutative field $Z(\beta)$ is either finite or infinite. If it is infinite, we can again take a symmetric ζ in $Z(\beta)$ such that ρ is symmetric and $\rho^2\neq 1$. If on the contrary $Z(\beta)$ is finite, an argument similar to that of §7, where $Z(\beta)$ replaces Z, proves that in the subsfield L of K commuting with β it is possible to find a symmetrical element ζ such that $\zeta^4\beta^4\neq 1$, and then $\rho=\beta^2\zeta^2$ is again symmetric and such that $\rho^2\neq 1$. Now, in the method of §6, we can take $\theta=(\rho^2-1)^{-1}$; then ρ and θ commute, and the matrix we obtain in that way is $C(1)$.

Finally, let μ be *any* symmetric element $\neq 0$, and consider the subsfield N of K commuting with μ; we are going to prove that there exists in N a symmetric element ζ such that $\zeta^4\neq 1$. This is certainly the case if the center of N (which contains the commutative field $Z(\mu)$) is infinite (or has more than 25 elements). On the other hand, if the center of N is finite and is distinct from N, in particular $Z(\mu)$ is finite, and then N is necessarily infinite; but then the argument of §7 shows that it is impossible that $\zeta^4=1$ for every symmetric element in N. The symmetric element ζ being thus chosen, we apply again the procedure of §6, starting this time from the matrix $C(1)$ instead of U; we take then $\rho=\zeta^2$, and ρ is symmetric and such that $\rho^2\neq 1$. Moreover, ρ commutes with μ and with ζ (which commute together); therefore, if we take this time $\theta=\mu(\zeta^4-1)^{-1}$, θ is symmetric, and we have $\rho\theta\rho^J-\theta=\mu$.

9. To end the proof of step 2°, we still have to consider the cases in which $\beta=0$ in the matrix U. Suppose first that $\gamma\neq 0$; then, if

$$Q=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

we notice that $Q=C(-1)B(1)C(-1)$ belongs to T_2 and that

$$QUQ^{-1}=\begin{pmatrix}\delta & -\gamma\\ 0 & \alpha\end{pmatrix},$$

and we are reduced to the preceding case. Finally, if $\beta=\gamma=0$, we have $\delta=(\alpha^{-1})^J$ by the third relation (1); then the matrix $C(\mu)UC(-\mu)$ belongs to G, and it is equal to

$$\begin{pmatrix}\alpha & \mu(\alpha^{-1})^J-\alpha\mu\\ 0 & (\alpha^{-1})^J\end{pmatrix}.$$

We are therefore reduced to the former case if there is a symmetric μ such that $\mu(\alpha^{-1})^J-\alpha\mu\neq 0$. If not, U commutes with every matrix $C(\mu)$, and it is easily verified that it also commutes with every matrix $B(\lambda)$. But this is possible only if U is in the center W_2 of T_2, owing to the following lemma:

LEMMA 3. *The group T_2 is generated by the transvections $B(\lambda)$ and $C(\mu)$.*

To prove that lemma, consider an arbitrary isotropic vector $x = e_1\alpha + e_2\beta$ in E; one has then $\alpha^J\beta - \beta^J\alpha = 0$. Suppose $\beta \neq 0$; then $\alpha\beta^{-1}$ is a symmetric element. But then the transvection $C(\mu)$, with $\mu = -\alpha\beta^{-1}$, transforms x into a vector collinear with e_2, and this shows that every transvection of vector x is transformed by $C(\mu)$ into a transvection of vector e_2, that is, a transvection $B(\lambda)$. This of course proves the lemma, and ends the proof of step 2° of Theorem 1.

10. It is now easy to prove that Theorem 1 is true for any $n \geqq 3$. Let G be a normal subgroup of T_n, and u a transformation in G which does not belong to the center W_n. Then u does not belong to Z_n, in other words it is not a homothetic mapping. From that, we shall deduce that there exists an isotropic vector x such that $u(x)$ and x are not collinear. This will be proved if we show that when u leaves invariant every isotropic line, it leaves invariant every line (and is therefore a homothetic mapping), according to the following lemma:

LEMMA 4. *For $n \geqq 3$ and $\nu \geqq 1$, every nonisotropic line in E is the intersection of two hyperbolic planes.*

To prove the lemma, let x be a nonisotropic vector, and y an isotropic vector. Let z be a vector which is orthogonal neither to x nor to y and is not in the plane determined by x and y. Then the plane P determined by y and z is a hyperbolic plane, and it contains therefore a second isotropic vector y_1 such that $f(y, y_1) = 1$. Moreover, any vector $y_2 = y\alpha + y_1\beta$ is isotropic if $\alpha^J\beta - \beta^J\alpha = 0$, and therefore there exists such a vector y_2 which is collinear with neither of y and y_1 (take for instance $\alpha = \beta = 1$). Among the three isotropic vectors y, y_1, y_2, two at least are not orthogonal to x, since x is not orthogonal to P. Therefore two of the three planes Q, Q_1, Q_2 determined by x and the vectors y, y_1, y_2, respectively, are hyperbolic planes, which proves the lemma.

We can now resume the end of the proof of Theorem 1. Let x be an isotropic vector such that x and $u(x)$ are not collinear. Suppose first that $f(x, u(x)) = 0$. Then there exists a vector z which is orthogonal to $u(x)$ but not to x. The plane P determined by x and z is a hyperbolic plane, hence contains an isotropic vector y which is not collinear to x. From Lemma 2, there exists a transvection $v \in T_n$ transforming x into a scalar multiple $y\lambda$ of y; moreover the vector of that transvection is in P, hence orthogonal to $u(x)$, and therefore $v(u(x)) = u(x)$. The transformation $u_1 = vu^{-1}v^{-1}u$ belongs to G, and one has $u_1(x) = y$. This proves that we can always suppose that $u \in G$ is such that $f(x, u(x)) \neq 0$.

Let then w be a transvection of vector x; uwu^{-1} is a transvection of vector

$u(x)$, and as x and $u(x)$ are not collinear, these two transvections do not commute. Let Q be the hyperbolic plane determined by x and $u(x)$; the transformation $u_2 = w^{-1}uwu^{-1}$ belongs to G, and leaves invariant every vector in the subspace Q^* orthogonal to Q. It therefore belongs to the subgroup Γ of $\boldsymbol{U}_n(K, f)$ which leaves invariant every vector of Q^*, and is obviously isomorphic to the unitary group $\boldsymbol{U}_2(K, f_1)$, where f_1 is the restriction of f to the plane Q; we shall identify Γ with that group. Moreover, u_2 is the product of two transvections, hence belongs to the group $T_2(K, f_1)$; finally, it is not in the center of that group, since it does not commute with w. Now step 2° of the proof shows that G contains every transformation of $T_2(K, f_1)$, in particular every transvection of vector x. Applying step 1° of the proof, we see that $G = T_n$, and Theorem 1 is completely proved.

11. We can supplement Theorem 1 by proving the following theorem.

THEOREM 2. *Under the same assumptions as in Theorem* 1, *the center* W_n *of the group* T_n *is the intersection* $T_n \cap Z_n$.

Indeed, if $n \geqq 3$, every transformation $u \in W_n$ must commute with every transvection, hence leave invariant every isotropic line. It then follows from Lemma 4 that u leaves invariant every line, hence is a homothetic mapping.

For $n = 2$, if e_1 and e_2 are two isotropic vectors constituting a basis of E such that $f(e_1, e_2) = 1$, the matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of u with respect to that basis must commute with every one of the matrices $B(\lambda)$ and $C(\mu)$ (notations of §6); this, as is readily seen, means that

$$U = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^{-1})^J \end{pmatrix},$$

where α is such that $\alpha\lambda = \lambda(\alpha^{-1})^J$ for every symmetric element $\lambda \in K$. Taking $\lambda = 1$ gives $\alpha^J = \alpha^{-1}$, and therefore α must commute with every symmetric element. From Lemma 1, we deduce therefore that α is in the center Z of K (and therefore that $u \in T_2 \cap Z_2$) with the possible exception of the case in which K is a reflexive sfield of characteristic $\neq 2$, and Z is identical to the set S of symmetric elements. But in that case we remark that the matrices $B(\lambda)$ and $C(\mu)$ have their elements in Z, and from Lemma 3 it follows that the same is true for every matrix of the group T_2; hence if the matrix U belongs to T_2, α is again in Z, and this ends the proof of Theorem 2.

12. The remainder of this paper is devoted to the study of the quotient group $\boldsymbol{U}_n/T_n$; the results we obtained in that direction are far from complete, and part of them are valid only under the additional assumption that the sfield K has *finite rank* over its center Z.

We begin by proving a lemma which is valid for any sfield K. A *plane rotation* is a transformation $u \in \boldsymbol{U}_n$ which leaves invariant every element of a nonisotropic $(n-2)$-dimensional subspace Q; the plane Q^* orthogonal to Q is then called *the plane of the rotation u*. A *hyperbolic rotation* is a plane rotation whose plane is hyperbolic. We then prove the following lemma.

LEMMA 5. *For $\nu \geqq 1$, every unitary transformation is a product of hyperbolic rotations.*

The lemma being obvious for $n=2$, we prove it by induction on n, as in [5, p. 66]. Let u be any unitary transformation, and let x be a nonisotropic vector such that the hyperplane H orthogonal to x contains isotropic vectors. If $u(x)=x$, u leaves H invariant, and we can apply induction to its restriction to H, since the index of the restriction of the form f to H is $\geqq 1$ by assumption; the lemma is then proved. If $u(x) \neq x$, there is always a hyperbolic plane P containing the vector $u(x)-x$: indeed, if $a=u(x)-x$ is not isotropic, there is an isotropic vector b not orthogonal to a (Lemma 4), and then the plane P determined by a and b is hyperbolic; if on the contrary a is isotropic, there is a nonisotropic vector c not orthogonal to a, and the plane P determined by a and c is hyperbolic. Now, as $u(x)-x$ is in P, we can write $x=z+y$, $u(x)=z+y'$, where y and y' are in P, and z in the $(n-2)$-dimensional subspace P^* orthogonal to P. Moreover, as $f(u(x), u(x))=f(x, x)$, we have also $f(y, y)=f(y', y')$. From Witt's theorem applied to the restriction of f to the plane P, it follows that there exists a plane rotation v of plane P such that $v(y)=y'$, hence also $v(x)=u(x)$, since $v(z)=z$. But then $v^{-1}u$ leaves x invariant, and we are reduced to the first case: $v^{-1}u$ is thus a product of hyperbolic rotations, and so is therefore u.

13. We shall use Lemma 5 to prove that in certain cases the subgroup T_n is *identical to $\boldsymbol{U}_n$*: Lemma 5 shows that this will be done if we can prove that *every hyperbolic rotation is a product of transvections*. In particular, we shall have proved that $\boldsymbol{U}_n = T_n$ for every dimension n if we can prove that $\boldsymbol{U}_2 = T_2$ (for $\nu \geqq 1$, of course). We therefore begin by investigating the relations between the group $\boldsymbol{U}_2$ and its subgroup T_2.

As in §6, we consider a basis of E consisting of two isotropic vectors e_1, e_2 such that $f(e_1, e_2)=1$; let

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the matrix of a unitary transformation u with respect to that basis; the relations (1) and (2) are then satisfied. As α and β are not both 0, there is a $\sigma \in S$ such that in

$$UB(\sigma) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

$\alpha'=\alpha+\beta\sigma\neq 0$; we can therefore already suppose that $\alpha\neq 0$; then it follows from the first relation (2) that $\mu=\alpha^{-1}\beta$ and from the first relation (1) that $\lambda=\gamma\alpha^{-1}$ are both symmetric. But then the matrix

$$B(-\lambda)UC(-\mu) = \begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}$$

(owing to the third relation (1)). If we observe that T_2 is a normal subgroup of $\boldsymbol{U}_2$, and that T_2 is generated by the matrices $B(\xi)$ and $C(\eta)$ (Lemma 3), we finally see that every matrix U in the group $\boldsymbol{U}_2$ can be written as a product VW, where W belongs to the group T_2, and V has the form

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}.$$

In order that $T_2=\boldsymbol{U}_2$, it is therefore necessary and sufficient that every matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}$$

belong to T_2. Now, for every pair of elements λ, μ in S, we have

$$C(\mu)B(\lambda) = \begin{pmatrix}1+\mu\lambda & \mu\\ \lambda & 1\end{pmatrix};$$

if we apply the preceding method to that matrix, we see that every matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}$$

with $\alpha=1+\mu\lambda=(\lambda^{-1}+\mu)\lambda$ belongs to T_2.

This proves that $T_2=\boldsymbol{U}_2$ *if every element* $\neq 0$ *in* K *is a product of elements of* S.

14. Let us suppose in this section that K has *finite rank* m^2 over its center Z. We recall that K is said to be of the *first kind* if J leaves invariant every element of Z, of the *second kind* if the restriction of J to Z is not the identity (it is then an involution in Z). Moreover, when K is of the first kind and of characteristic $\neq 2$, the dimension of S over Z is equal to $m(m+1)/2$ or $m(m-1)/2$ [7]; the easiest way to see this is to extend Z to a splitting field L of K; the involution J is extended to $K_{(L)}$ in an obvious way (the elements of L being invariant by J), and by taking a basis of K over Z consisting of symmetric or skew-symmetric elements, one sees readily that the dimension over L of the space of symmetric elements of $K_{(L)}$ is equal to the dimension over Z of the space of symmetric elements of K. But $K_{(L)}$ is the algebra of matrices of order m over L, and an involution of that algebra leaving in-

variant the elements of L is known, namely the mapping $X \rightarrow {}^tX$, where ${}^tX$ is the transposed matrix of X; therefore [1, p. 896], one has $X^J = P^{-1} \cdot {}^tX \cdot P$, where P is either a symmetric or a skew-symmetric matrix. Hence, the relation $X^J = X$ means that PX is symmetric (resp. skew-symmetric) if P is symmetric (resp. skew-symmetric); this proves at once our assertion. Similarly, it is shown that when the characteristic of K is 2, the dimension of S over Z is always $m(m+1)/2$ when K is of the first kind.

We can now prove the following theorem.

THEOREM 3. *When K is a sfield of the first kind, of finite rank m^2 over its center Z and of characteristic $\neq 2$, and such that the space S of symmetric elements in K has dimension $m(m+1)/2$ over Z, then $\boldsymbol{U}_n = T_n$ for every $n \geqq 2$.*

All we have to prove (according to the final remark of §13) is that, for every $\zeta \in K$, there exist two elements ξ, η in S such that $\zeta = \xi\eta$. If $\theta = \eta^{-1}$, this amounts to saying that there exists an element $\theta \in S$ such that $\zeta\theta$ is symmetric, which means that $\zeta\theta - \theta\zeta^J = 0$. But the mapping $\theta \rightarrow \zeta\theta - \theta\zeta^J$ of S into K is linear with respect to Z, and maps S into the space A of skew-symmetric elements, which is supplementary to S in K, hence has a dimension equal to $m(m-1)/2$; as $m(m+1)/2 > m(m-1)/2$, the kernel of the linear mapping $\theta \rightarrow \zeta\theta - \theta\zeta^J$ is not reduced to 0, and this ends our proof.

As a corollary, we obtain Theorem 6 of [5] when K is a reflexive sfield of characteristic $\neq 2$: the passage from an hermitian to a skew-hermitian form over K, explained in §3, replaces the involution $\xi \rightarrow \bar{\xi}$ in K by an involution for which the symmetric elements are the skew-symmetric elements of $\xi \rightarrow \bar{\xi}$, hence form a subspace of dimension 3 over the center Z.

15. Turning now to the case in which the sfield K, of finite rank m^2 over Z, is a sfield of the first kind but such that S has dimension $m(m-1)/2$ over Z (this property implying that K has a characteristic $\neq 2$), we have to set aside the case $m = 2$, in which $S = Z$, and therefore S cannot generate the group K^* of elements $\neq 0$ in K. When $m > 2$, it seems likely (due to Lemma 1) that S generates K^*, but I have not been able to prove that conjecture, and in the absence of any further assumptions, the structure of the group $\boldsymbol{U}_n/T_n$ remains unknown in that case. I shall therefore consider only the case $m = 2$; in other words, K is then a sfield of *generalized quaternions* over Z, and the involution J is the (unique) involution of K for which the elements of Z are the only symmetric elements.

Let us first consider the case $n = 2$; then T_2 is simply the *unimodular group* $\boldsymbol{SL}_2(Z)$ [4, p. 30]. Moreover, as every element $\alpha \in K$ is such that $(\alpha^{-1})^J = \alpha \cdot (N(\alpha))^{-1}$, where $N(\alpha) = \alpha\alpha^J \in Z$, it follows from §13 that every matrix U in the group $\boldsymbol{U}_2$ can be written αX, where X is an arbitrary matrix in $\boldsymbol{GL}_2(Z)$ such that $\det(X) = (N(\alpha))^{-1}$, and α is an arbitrary element in K^*. We observe in addition that α and X are permutable, and that α is determined by U up to a factor $\lambda \in Z^*$ (the matrix X being then multiplied by λ^{-1}). We can therefore describe the structure of the group $\boldsymbol{U}_2$ in the following way: consider in the direct product $K^* \times \boldsymbol{GL}_2(Z)$ the subgroup Γ consisting of the pairs (α, X) such that $N(\alpha)\cdot\det(X)=1$, and let Δ be the subgroup of Γ consisting of the pairs (λ, λ^{-1}), where $\lambda \in Z^*$; then *$\boldsymbol{U}_2$ is isomorphic to the factor group Γ/Δ*. We observe that $\boldsymbol{U}_2$ contains as a normal subgroup the multiplicative group $\boldsymbol{U}_1$ of elements of norm 1 in K, and that $\boldsymbol{U}_1$ and T_2 commute and have as their intersection the two elements 1 and -1, which constitute the center W_2 of T_2; the quotient group $\boldsymbol{U}_2/T_2$ contains $\boldsymbol{U}_1/W_2$ as a subgroup, hence T_2 is certainly not the commutator subgroup of $\boldsymbol{U}_2$.

16. There are reasons to believe that the preceding structure of the group $\boldsymbol{U}_2(K, f)$ when K is a sfield of generalized quaternions and f a skew-hermitian form is exceptional among the corresponding groups $\boldsymbol{U}_n(K, f)$ for $n>2$, much as the 4-dimensional orthogonal groups among the orthogonal groups of other dimensions. The evidence I can supply in favor of that view is summed up in the following theorem:

THEOREM 4. *If K is a sfield of characteristic $\neq 2$, and the index ν of the form f is at least 2 (which implies $n \geqq 4$), then T_n is the commutator subgroup of $\boldsymbol{U}_n(K, f)$.*

To prove that theorem, we shall establish two lemmas.

LEMMA 6. *Let P be a hyperbolic plane, Γ the group of hyperbolic rotations of plane P. Then (for $\nu \geqq 2$) the factor group $\Gamma/(\Gamma \cap T_n)$ is abelian.*

Let e_1, e_2 be two isotropic vectors forming a basis of P, with $f(e_1, e_2)=1$; it is then possible to find two other isotropic vectors e_3, e_4 orthogonal to P and such that $f(e_3, e_4)=1$ (because $\nu \geqq 2$). Let Q and R be the totally isotropic planes determined by e_1, e_3 and e_2, e_4 respectively; if $u \in \boldsymbol{U}_n$ leaves invariant both planes Q and R, and V and W are the matrices of the restrictions of u to Q and R, with respect to the bases e_1, e_3 and e_2, e_4 respectively, one has $W=(V')^J$, V' being the contragredient of V. We are going to prove that there are transformations $u \in T_n$ of the preceding type, and such that $V=B(\lambda)$, where λ is *any element of* K. Let $a=e_2\alpha+e_3\beta$ be any vector in the totally isotropic plane determined by e_2 and e_3, and consider the transvection w such that $w(x)=x+af(a, x)$; it leaves invariant e_2 and e_3, and is such that

$$w(e_1) = e_1 - e_2\alpha\alpha^J - e_3\beta\alpha^J, \qquad w(e_4) = e_4 + e_2\alpha\beta^J + e_3\beta\beta^J.$$

Let $a_1=e_2\alpha_1+e_3\beta_1$ be a second isotropic vector, w_1 the transvection such that $w_1(x)=x-a_1f(a_1, x)$; then $u=w_1w$ leaves invariant e_2 and e_3 and is such that

$$u(e_1) = e_1 + e_2(\alpha_1\alpha_1^J - \alpha\alpha^J) + e_3(\beta_1\alpha_1^J - \beta\alpha^J),$$

$$u(e_4) = e_4 + e_2(\alpha\beta^J - \alpha_1\beta_1^J) + e_3(\beta\beta^J - \beta_1\beta_1^J).$$

If we take $\alpha_1=\alpha$ and $\beta_1=-\beta$, u leaves invariant Q and R, and is such that $u(e_1)=e_1-2e_3\beta\alpha^J$; as the characteristic of K is not 2, it is possible to take α and β such that $-2\beta\alpha^J=\lambda$, for any element $\lambda\in K$, and the matrix of the restriction of u to Q is then $B(\lambda)$. Similarly, it can be proved that $u\in T_n$ exists such that $V=C(\mu)$ for any $\mu\in K$. Therefore T_n contains all the transformations $u\in \boldsymbol{U}_n$ leaving invariant Q and R and such that the matrix of the restriction of u to Q is *any matrix V in the unimodular group* $\boldsymbol{SL}_2(K)$ [4, p. 30]; in particular, for *any element γ in the commutator subgroup of K^**, $u\in T_n$ exists such that

$$V=\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix},$$

[4, p. 29], which means that u is a hyperbolic rotation of plane P, such that its matrix in P is

$$\begin{pmatrix}\gamma & 0\\ 0 & (\gamma^{-1})^J\end{pmatrix}.$$

Now we have seen in §13 that every hyperbolic rotation of plane P has a matrix (with respect to e_1, e_2) which can be written as the product of a matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}$$

(with $\alpha\in K^*$) and a matrix of $\Gamma\cap T_n$. If, to every $\alpha\in K^*$, we associate the class of the matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix},$$

modulo the subgroup $\Gamma\cap T_n$, we define a homomorphism of K^* *onto* $\Gamma/(\Gamma\cap T_n)$, and the preceding result shows that the kernel of that homomorphism *contains the commutator subgroup C of K^**; hence $\Gamma/(\Gamma\cap T_n)$ is *isomorphic to a quotient group of the abelian group K^*/C.*

17. LEMMA 7. *Let P_1 and P_2 be any two hyperbolic planes. Then (for $\nu\geqq 2$) there exists a transformation $w\in T_n$ such that $w(P_1)=P_2$.*

It follows from Lemma 2 that there exists a transformation in T_n sending an isotropic vector in P_1 into an isotropic vector in P_2; we can therefore assume in the following proof that there exists a common isotropic vector e_2 in P_1 and P_2. We now consider separately several cases.

(a) *The dimension $n=4$.* Let e_1 be a second isotropic vector in P_1 such that $f(e_1, e_2)=1$, and let e_3, e_4 be determined as in the proof of Lemma 6. There exists in P_2 an isotropic vector e_1' such that $f(e_1', e_2)=1$; we can write $e_1'=e_1$

$+e_2\beta+e_3\gamma+e_4\delta$, and the condition $f(e_1', e_1')=0$ is equivalent to

$$\beta - \beta^J + \gamma^J\delta - \delta^J\gamma = 0$$

which can be written $\beta+\gamma^J\delta=(\beta+\gamma^J\delta)^J$, and means therefore that the expression $\beta+\gamma^J\delta$ is a symmetric element λ. Now, it has been proved in the proof of Lemma 6 that the transformation w_1 leaving invariant e_2 and e_3, and such that

$$w_1(e_1) = e_1 + e_3\gamma, \qquad w_1(e_4) = e_4 - e_2\gamma^J,$$

belongs to T_n. Similarly (exchanging the parts played by e_3 and e_4), the transformation w_2 leaving invariant e_2 and e_4, and such that

$$w_2(e_1) = e_1 + e_4\delta, \qquad w_2(e_3) = e_3 - e_2\delta^J,$$

belongs to T_n. The transformation w_1w_2, which belongs to T_n, is such that $w_1w_2(e_2)=e_2$, and $w_1w_2(e_1)=e_1+e_3\gamma+e_4\delta-e_2\gamma^J\delta$. Let finally v be the transvection $x\rightarrow x-e_2\lambda f(e_2, x)$, which leaves invariant e_2, e_3, e_4 and is such that $v(e_1)=e_1+e_2\lambda$; the transformation $w=vw_1w_2$ belongs to T_n, leaves e_2 invariant, and is such that

$$w(e_1) = e_1 + e_2(\lambda - \gamma^J\delta) + e_3\gamma + e_4\delta = e_1'.$$

Therefore $w(P_1)=P_2$, and the lemma is proved in that case.

(b) $n>4$ *and the* 3-*dimensional subspace* $M=P_1+P_2$ *is isotropic.* This means that there exists in M at least an isotropic vector c orthogonal to M; such a vector cannot be in P_1, since P_1 is not isotropic. Therefore the three vectors c, e_1, e_2 (e_1 being defined as in (a)) constitute a basis for M, such that $f(e_1, e_2)=1$, $f(e_1, c)=f(e_2, c)=0$. There exists then in E a fourth isotropic vector d such that $f(c, d)=1$, $f(e_1, d)=f(e_2, d)=0$ [5, p. 18], and the four vectors e_1, e_2, c, d form the basis of a nonisotropic 4-dimensional subspace N of E containing P_1 and P_2 and such that the restriction of the form f to N has an index equal to 2. The result of case (a) proves then the lemma.

(c) $n>4$ *and the space* M *is not isotropic.* There exists then in M a nonisotropic vector c orthogonal to P_1. As the index $\nu\geqq 2$, the restriction of f to the $(n-2)$-dimensional subspace P_1^* orthogonal to P_1 has an index $\geqq 1$, by Witt's theorem. Therefore (Lemma 4), there exists a hyperbolic plane Q contained in P_1^* and containing c. The subspace $N=P_1+Q$ is then a nonisotropic 4-dimensional subspace of E, such that the restriction of f to N has index 2, and N contains P_1 and P_2. The proof of the lemma then follows as in case (b).

18. To end the proof of Theorem 4, let us consider a fixed hyperbolic plane P. We are going to show that every unitary transformation v can be written su, where s is a hyperbolic rotation of plane P, and u belongs to T_n. The result is true if v is a hyperbolic rotation of plane P', for by Lemma 7 there exists $t\in T_n$ such that $t(P)=P'$, and therefore $v=tst^{-1}$, where s is a

rotation of plane P; but we can also write $v=s(s^{-1}ts)t^{-1}$, and as T_n is a normal subgroup, $s^{-1}ts\in T_n$. Suppose now that v is a product of p hyperbolic rotations (Lemma 5), and use induction on p. Let $v=w_1w_2$, where w_1 is a hyperbolic rotation and w_2 is a product of $p-1$ hyperbolic rotations; we can write by assumption $w_1=s_1u_1$, $w_2=s_2u_2$, hence $v=s_1u_1s_2u_2=s_1s_2(s_2^{-1}u_1s_2)u_2$, and this proves our contention. We have thus shown that the group U_n/T_n is isomorphic to $\Gamma/(\Gamma\cap T_n)$, hence abelian (and isomorphic to a quotient group of K^*/C). Theorem 4 then follows from the fact that T_n/W_n is a simple group (Theorem 1).

19. In special cases it is possible to obtain more precise information. Let us suppose for instance that K is the sfield of *ordinary quaternions* over a *Euclidean ordered field* Z (i.e., an ordered field in which every positive element has a square root in Z). The usual theory of quaternions can then be carried out exactly as when Z is the field $\boldsymbol{R}$ of real numbers; we know therefore that every quaternion $\xi\neq 0$ can be written in one and only one way $\xi=\rho\zeta$, where $\rho\in Z$, $\rho>0$, and $\rho^2=N(\xi)$, hence $N(\zeta)=1$; moreover, every quaternion of norm 1 is a commutator; finally, if ξ and η are two quaternions of norm 1 and scalar 0, there is a third quaternion α of norm 1 such that $\xi=\alpha\eta\alpha^{-1}$. We suppose as usual that J is the only involution in K leaving invariant the elements of Z, and that f is skew-hermitian. We can then show that there exists an orthogonal basis in E with respect to which $f(x, y)=\sum_{k=1}^{n}\xi_k^J i\xi_k$. Indeed, there exists an orthogonal basis (e_k) for f, and with respect to that basis, $f(x, y)=\sum_{k=1}^{n}\xi_k^J\alpha_k\xi_k$, with $\alpha_k^J=-\alpha_k$, which means that the scalar of the quaternion α_k is 0. We can write $\alpha_k=\rho_k\beta_k$, with $\rho_k>0$, $N(\beta_k)=1$, and $\beta_k^J=-\beta_k$, and therefore $\alpha_k=\rho_k\gamma_k i\gamma_k^{-1}$, where $N(\gamma_k)=1$, hence $\gamma_k^J=\gamma_k^{-1}$. If we replace e_k by $e_k(\rho^{1/2})^{-1}\gamma_k$, we obtain for $f(x, y)$ the canonical expression $\sum_{k=1}^{n}\xi_k^J i\xi_k$. This proves that *all nondegenerate skew-hermitian forms over E are equivalent*, hence their index is $[n/2]$. In particular, for $n\geqq 4$, $\nu\geqq 2$, and therefore Theorem 4 applies. But here every matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & (\alpha^{-1})^J\end{pmatrix}$$

can be written

$$\begin{pmatrix}\gamma & 0\\ 0 & (\gamma^{-1})^J\end{pmatrix}\begin{pmatrix}\rho & 0\\ 0 & \rho^{-1}\end{pmatrix},$$

where $N(\gamma)=1$, hence γ is a commutator, and $\rho\in Z$; as the matrix

$$\begin{pmatrix}\rho & 0\\ 0 & \rho^{-1}\end{pmatrix}$$

belongs to $\boldsymbol{SL}_2(Z)$, the proof of Lemma 6 shows that we have here $\Gamma=\Gamma\cap T_n$, hence $\boldsymbol{U}_n=T_n$. When $Z=\boldsymbol{R}$, this is equivalent to one of E. Cartan's theorems on the real forms of the simple Lie groups [3, p. 286].

20. We end by mentioning some relations between our results and the properties of the commutator subgroup C of a sfield K with involution.

THEOREM 5. *Let K be a sfield of characteristic $\neq 2$, of finite rank over its center Z, and let J be an involution in K leaving invariant the elements of Z. Then, for every $\xi \in K^*$, ξ and ξ^J are in the same class modulo the commutator subgroup C of K^*.*

Let m^2 be the rank of K over its center, and let us suppose first that the set S of symmetric elements in K has dimension $m(m+1)/2$ over Z. Then we have seen in §14 that every element $\xi \in K^*$ can be written $\xi = \alpha\beta$, where α and β are in S; accordingly $\xi^J = \beta^J \alpha^J = \beta\alpha$, hence $\xi^J \xi^{-1} = \beta\alpha\beta^{-1}\alpha^{-1}$, which proves our contention in that case. If on the contrary S has dimension $m(m-1)/2$ over Z, and p is a skew-symmetric element of K, then $\xi \rightarrow \xi^T = p^{-1}\xi^J p$ is an involution in K for which the symmetric elements form a space of dimension $m(m+1)/2$ over Z (§3); therefore ξ and ξ^T are in the same class modulo C, and the same is true for ξ and ξ^J, since ξ and $p^{-1}\xi p$ are in the same class modulo C.

The situation is reversed when K is a sfield of the second kind:

THEOREM 6. *Let K be a sfield of finite rank over its center Z, and let J be an involution in K which does not leave invariant every element of Z. Then there exist elements ξ in K^* such that ξ and ξ^J are not in the same class modulo C.*

The theorem being obvious when K is commutative, we can suppose that K is not commutative, hence that Z is an infinite field. The theorem will be proved if we exhibit a homomorphism ϕ of K^* onto an abelian group, such that $\phi(\xi^J) \neq \phi(\xi)$ for some $\xi \in K^*$. Let $N(\xi)$ be the norm of an element ξ in the regular representation of K (considered as an algebra over its center Z); $\xi \rightarrow N(\xi)$ is then a homomorphism of K^* into Z^*. If $r = m^2$ is the rank of K over Z, we have $N(\xi) = \xi^r$ for every $\xi \in Z^*$; we have only therefore to verify that if the element $\omega \in Z$ constitutes with the identity a basis of Z over the subfield Z_0 of J-invariant elements, then the elements $(x + y\omega)^r$ and $(x + y\omega^J)^r$ cannot be identical for all values of x and y in Z_0. But as $\omega^J \neq \omega$, this follows at once from the fact that Z_0 is an infinite field.

Theorem 6 has as a consequence that *when K is a sfield of the second kind, the groups U_n and T_n (for $\nu \geqq 1$) are always distinct.* To prove this, we have only to verify that the determinant [4] of some unitary matrix is not the identity element in K^*/C; but this is obvious for the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^{-1})^J \end{pmatrix}$$

if α and α^J are not in the same class modulo C.

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