ADJUNCTION OF SUBFIELD CLOSURES TO ORDERED DIVISION RINGS

BY

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1. **Preface.** Let Σ be an ordered division ring, P its prime field (i.e. the field of rationals), P^* the closure of P in its order topology; P^* is then order-isomorphic to the field of real numbers in their natural order. It has been shown by B. H. Neumann [1]⁽¹⁾ that Σ can be extended to an ordered division ring $\Sigma(P^*)$ continuing the order of Σ and containing P^* in its centre.

With a few changes in Neumann's proof the following generalisation will be proved: Let F be an arbitrary subfield of the centre of Σ . Then Σ can be extended to an ordered division ring $\Sigma(F^*)$ continuing the order of Σ and containing the closure F^* of F with respect to its order-topology in its centre.

The new tool which is used throughout the paper is a mapping of Σ into a system consisting of the symbols $-\overline{}$ and $+\overline{}$ and an ordered residue class division ring $\overline{D}$ obtained by a peculiar type of fundamental sequence modulo the corresponding null-sequences.

2. **The mapping $\sigma \rightarrow \overline{\sigma}$.** Let Σ be an ordered division ring with centre Z , F a fixed subfield of Z , and Σ^+ , F^+ the multiplicative groups of the positive elements of Σ , F respectively.

We can assume the order of the additive semigroup of F^+ to be nonarchimedean⁽²⁾. Then the archimedean classes of the additive semigroups of Σ^+ , F^+ form multiplicative groups $a(\Sigma)$, $a(F)$ when the product is defined by the product of representatives of the corresponding classes⁽³⁾. The archimedean classes of the additive semigroups of Σ^+ , F^+ will be called *additive archimedean classes* of Σ^+ , F^+ respectively. The additive archimedean class of an element $\sigma \in \Sigma^+$ in $a(\Sigma)$ will be denoted by $[\sigma]$. Thus $[\sigma]$ consists of those $\tau \in \Sigma^+$ for which there exist positive integers s, t such that $|\sigma| < t\tau$, $\tau < s|\sigma|$.

Let F^* be the closure of F with respect to its order-topology; it can be constructed by means of fundamental sequences and null-sequences in the usual way. Thus F^* is an ordered field containing (an ordered subfield isomorphic to) F such that its additive group is the order-topological completion of the additive group of F . With this t -completion in the sense of Cohen-Goffman⁽⁴⁾ a unique ordinal $\xi^* = \xi(F)$ is associated, and for the construction

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⁽¹⁾ Numbers in brackets refer to the bibliography at the end of the paper.

⁽²⁾ For otherwise the closure F^* of F is order-isomorphic to the field of reals, and F^* can be embedded in Σ according to [1].

⁽³⁾ Obviously this definition is independent of the choice of the representatives.

⁽⁴⁾ Cf. [2].

of F^* it is sufficient to use fundamental well-ordered ξ^* -sequences.

Let $\mathfrak{R}$ be the subring of Σ consisting of all elements $\sigma \in \Sigma$ for which there exists a centre element $\zeta \in F^+$ with $|\sigma| < \zeta$, that is, for which the additive archimedean class $[\sigma]$ of $|\sigma|$ is not greater than all archimedean classes of F^+ . We shall call the elements of $\mathfrak{R}$ *finite*⁽⁵⁾ and the elements of $\Sigma - \mathfrak{R}$ *infinite*⁽⁵⁾.

We shall now deal with certain sequences over $\mathfrak{R}$, similar to those leading to F^* , and start with the following definitions:

A well-ordered ξ^* -sequence $\{\sigma_\alpha\}$ of elements $\sigma_\alpha \in \mathfrak{R}$ is called an *F-fundamental sequence* (abbreviated *FF-sequence*) if for every $\epsilon \in F^+$ there exists an ordinal $\gamma(\epsilon) < \xi^*$ such that $|\sigma_\alpha - \sigma_\beta| < \epsilon$ for $\alpha, \beta > \gamma(\epsilon)$.

An *FF*-sequence $\{\sigma_\alpha\}$ such that for every $\epsilon \in F^+$ there exists an ordinal $\gamma(\epsilon)$ such that $|\sigma_\alpha| < \epsilon$ for $\alpha > \gamma(\epsilon)$ is called an *F-null-sequence* (abbreviated *FN-sequence*).

The *FF*-sequences form a ring $\overline{\mathfrak{R}}$ when addition, subtraction, and multiplication are defined in the usual way. The *FN*-sequences form a two-sided ideal $\overline{I}$ in $\overline{\mathfrak{R}}$. The residue class ring $\overline{\mathfrak{R}}/\overline{I}$ is a division ring $\overline{D}$ ⁽⁶⁾.

Special *FF*-sequences are those where a fixed element $\rho \in \mathfrak{R}$ is repeated ξ^* -times: $\{\rho\}$, clearly their residue class mod $\overline{I}$ defines an element of $\overline{D}$, and we shall denote it by $\bar{\rho}$. Similarly we shall mark subsets of $\overline{D}$ which correspond to subsets of $\mathfrak{R}$ by means of the mapping $\rho \rightarrow \bar{\rho}$ by a bar “—”.

The map $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ in $\overline{D}$ is a division ring. The mapping $F \rightarrow \overline{F}$ is an isomorphism. $\overline{D}$ can be ordered so that the order of $\overline{F}$, induced by $F \cong \overline{F}$, is preserved. $\overline{D}$ contains the closure $\overline{F}^*$ (with respect to the order-topology) of $\overline{F}$ as a subfield⁽⁶⁾.

We shall assume from now on that $\overline{D}$ is ordered with preservation of the order in $\overline{F}$. We shall call the elements of $\mathfrak{R}$ mapped onto $\overline{0}$ *infinitely small*⁽⁶⁾; they form a two-sided maximal prime ideal in $\mathfrak{R}$ which we shall denote by $\mathfrak{P}$.

The aim of our paper is to embed Σ in an ordered division ring Σ^* containing (a field order-isomorphic to) F^* . If the mapping $\mathfrak{R} \rightarrow \overline{\mathfrak{R}}$ is an isomorphism, then Σ^* can be taken to be $\overline{D}$. For if no elements $\sigma \in \Sigma^+$ exist with the property $\sigma < \zeta$ for all $\zeta \in F^+$, then also no elements $\tau \in \Sigma^+$ exist with $\tau > \zeta$ for all $\zeta \in F^+$. Therefore all elements of Σ are finite, $\overline{\mathfrak{R}}$ is order-isomorphic to Σ , and $\overline{D}$ contains $\overline{F}^*$ and $\overline{\mathfrak{R}}$. We shall exclude this trivial case in the following by the assumption $\mathfrak{R} \neq \Sigma$.

Then there exist infinitely small nonzero elements. The construction of $\overline{D}$ sets up a natural mapping of Σ into the system obtained from $\overline{D}$ by adding the two symbols $-\infty, +\infty$ with the usual conventions. The elements $\tau \in \Sigma - \mathfrak{R}$ are mapped onto $\pm\infty$ according as $\tau \geq 0$.

Our mapping $\sigma \rightarrow \bar{\sigma}$ is a generalisation of the mapping $\sigma \rightarrow \wedge \sigma$ in [1] since

⁽⁵⁾ This definition, of course, is dependent on the fixed field F , but for the sake of brevity we shall omit the qualification “with respect to F .”

⁽⁶⁾ The proof runs along the usual lines and is omitted. Cf., for instance, [3, chap. IX].

there $\wedge P^*$ can be constructed by means of fundamental sequences instead of by means of Dedekind sections.

If all elements of $\mathfrak{R}$ are mapped onto elements of $\overline{F}^*$, we are confronted with a situation similar to that in [1], and, in fact, the whole construction of Σ^* including the proofs can be taken over easily from [1].

But, in general, this need not be the case. For there may exist elements $\bar{\rho} \in \overline{\mathfrak{R}}$ such that $\bar{\sigma} \ll \bar{\rho} \ll \bar{1}$ holds for all $\bar{\sigma} \ll \bar{1}$, $\bar{\sigma} \in \overline{F}^*$. Then we shall denote the subring of all elements of $\mathfrak{R}$ mapped into $\overline{F}^*$ by $\mathfrak{S}$ and assume $\mathfrak{S} \neq \mathfrak{R}$. The map $\overline{\mathfrak{S}} = \overline{\mathfrak{R}} \cap \overline{F}^*$ is a field and consists of exactly the same additive archimedean classes as F . Finally we shall denote the subring of all elements of $\mathfrak{R}$ mapped onto the centre of $\overline{\mathfrak{R}}$ by $\mathfrak{Z}$. Then $\overline{\mathfrak{S}} \subseteq \mathfrak{Z}$ holds.

Now (8.4) and (8.5) in [1] are no longer valid in our case. However, we can still take over the construction of Σ^* in [1] and most of the proofs there. We shall have to supply amendments only when (8.4) or (8.5) are applied in [1].

As in [1], we shall denote by P an arbitrary subfield of $Z \cap \mathfrak{R}$ and by $P_{\max}$ one of the maximal subfields of $Z \cap \mathfrak{R}$. We may assume $\overline{P}_{\max} \neq \overline{F}^*$, otherwise no extension would be required. It is then possible to adjoin to $P_{\max}$ an element $\bar{\theta} \in \overline{F}^*$. We shall have to show that the corresponding adjunction of an element θ to $P_{\max}$, and thus to Σ , leads again to a division ring $\Sigma(\theta)$ and that $\Sigma(\theta)$ can be ordered with preservation of the order in Σ . By the successive adjunctions to Σ of all these θ , according to the classical Steinitz procedure⁽⁷⁾, the required extension Σ^* is obtained.

As in [1], in all occurring polynomial domains $\Sigma[x]$, $\mathfrak{R}[x]$, etc., the variable x will be assumed commutative with the coefficients. Finally we shall use the abbreviations:

$$\begin{aligned} \sigma^\tau &= \tau^{-1}\sigma\tau, & \sigma, \tau &\in \Sigma, & \tau &\neq 0, \\ \bar{\sigma}^\tau &= \{\tau^{-1}\sigma\tau\} \bmod \bar{I}, & \sigma &\in \mathfrak{R}, & \tau &\in \Sigma, \tau \neq 0, \\ f^\tau(x) &= \tau^{-1}f(x)\tau, & f(x) &\in \Sigma[x], & \tau &\in \Sigma, \tau \neq 0, \\ \bar{f}^\tau(x) &= \{\tau^{-1}f(x)\tau\} \bmod \bar{I}, & f(x) &\in \mathfrak{R}[x], & \tau &\in \Sigma, \tau \neq 0. \end{aligned}$$

$\mathfrak{R}$ as well as $\mathfrak{R}[x]$ are invariant under all $\tau \in \Sigma$, hence $\bar{\sigma}^\tau$ and $\bar{f}^\tau(x)$ are defined.

Then the following lemmas will help to replace [1, (8.4)] and [1, (8.5)] here:

2.1. LEMMA. *If $\bar{f}(\bar{x}) \in \overline{\mathfrak{R}}[\bar{x}]$, $\bar{\theta} \in \overline{F}^*$, $\bar{f}(\bar{\theta}) > \bar{0}$, then $\bar{f}^\tau(\bar{\theta}) > \bar{0}$ for every non-vanishing $\tau \in \Sigma$.*

Proof. Let $f(x) \in \mathfrak{R}[x]$ be a fixed image of $\bar{f}(\bar{x})$ and $\{\theta_\alpha\}$, $\theta_\alpha \in F$, be a FF -sequence such that $\{\theta_\alpha\} \bmod \bar{I} = \bar{\theta}$. From $\bar{f}(\bar{\theta}) > \bar{0}$ it follows that there exist an $\epsilon \in F^+$ and an ordinal $\gamma (< \phi)$ such that $f(\theta_\alpha) > \epsilon > 0$ for $\phi > \alpha > \gamma$ ⁽⁸⁾. But then $f^\tau(\theta_\alpha) > \epsilon > 0$ holds, too, for $\phi > \alpha > \gamma$, i.e., $\bar{f}^\tau(\bar{\theta}) > \bar{0}$.

⁽⁷⁾ Cf. [4].

⁽⁸⁾ This can be shown in the usual way, cf., for instance, [3, chap. IX].

2.2. COROLLARY. If $\bar{f}(\bar{x}) \in \bar{\mathfrak{R}}[\bar{x}]$, $\bar{\theta} \in \bar{F}^*$, $\bar{f}(\bar{\theta}) = \bar{0}$, then $\bar{f}^\tau(\bar{\theta}) = \bar{0}$ for every nonvanishing $\tau \in \Sigma$.

2.3. LEMMA. If $\bar{g}(\bar{x}) \in \bar{\mathfrak{R}}[\bar{x}]$ is irreducible and monic, $\bar{\theta} \in \bar{F}^*$, $\bar{g}(\bar{\theta}) = \bar{0}$, then $\bar{g}^\tau(\bar{x}) = \bar{g}^\tau(\bar{x})$ for every nonvanishing $\tau \in \Sigma$.

Proof. Suppose $\bar{h}(\bar{x}) = \bar{g}(\bar{x}) - \bar{g}^\tau(\bar{x}) \neq \bar{0}$ for a certain $\tau \in \Sigma$. Then $\bar{h}(\bar{\theta}) = \bar{0}$ holds by (2.2). But this contradicts the irreducibility of $\bar{g}(\bar{x})$ in $\bar{\mathfrak{R}}[\bar{x}]$ since the degree of $\bar{h}(\bar{x})$ is less than the degree of $\bar{g}(\bar{x})$.

2.4. COROLLARY. If $\bar{g}(\bar{x}) \in \bar{\mathfrak{R}}[\bar{x}]$ is irreducible and monic, $\bar{\theta} \in \bar{F}^*$, $\bar{g}(\bar{\theta}) = \bar{0}$, then $\bar{g}(\bar{x}) \in \bar{\mathfrak{Z}}[\bar{x}]$.

3. **Alterations in Neumann's proof.** Now the sections 9–17 in [1] can be taken over mutatis mutandis. Thus all accents have to be replaced by bars, and $\sigma \rightarrow \bar{\sigma}$ is a mapping into $\bar{D}$, whilst in [1], $\sigma \rightarrow \wedge \sigma$ was a mapping into $\wedge P^*$. Only the following alterations are necessary:

In section 9 whenever $g(x)$ occurs the additional assumption has to be made that it is monic and irreducible over $\mathfrak{R}[x]$ and that $\bar{g}(\bar{\theta}) = \bar{0}$ holds, which involves no loss of generality. In the proof of (9.5)⁽⁹⁾, the lemma (8.5) was applied in order to show that $\wedge h'' = \wedge h$ holds. Here the corresponding fact $\bar{h}'' = \bar{h}^{\bar{\sigma}^{-1}} = \bar{h}$ follows from $\bar{f}(\bar{x}) = \bar{f}^{\bar{\sigma}^{-1}}(\bar{x})$ and (2.3).

In section 10 the absence of commutativity in our $\bar{\mathfrak{R}}$ provides no special difficulty since $\bar{g}(\bar{x})$, defined as in (10.41), lies in the centre $\bar{\mathfrak{Z}}[\bar{x}]$ of $\bar{\mathfrak{R}}[\bar{x}]$ because of (2.4). The original proof of (10.73) used (8.5), therefore it has to be altered as follows: $p > 0$ implies $\bar{q}(\bar{\theta}) > \bar{0}$ (according to (10.51)), hence $\bar{q}^\sigma(\bar{\theta}) > \bar{0}$ by (2.1). But $\bar{p}^\sigma(\bar{x}) = \bar{g}(\bar{x})^\lambda \bar{q}^\sigma(\bar{x})$ by (2.3), hence $p^\sigma > 0$.

In section 11, instead of (11.4) only a restricted lemma can be proved.

3.1. LEMMA. If $\sigma \in \mathfrak{S}$, $\rho \in \mathfrak{R}$, then $A_\sigma \rho = o(\rho)$.

Proof. We have $\bar{\sigma} = \bar{\sigma}^\rho$, hence $\sigma^\rho - \sigma = o(1)$, or upon left-hand multiplication by ρ : $\sigma\rho - \rho\sigma = o(\rho)$.

Consequently in (11.51), (11.52), and (11.53), σ and τ must now be restricted to belong to $\mathfrak{S}$. Similarly in section 12 the restrictions $\xi_1, \xi_2, \xi \in \mathfrak{S}$ are necessary in all lemmas.

In section 13, instead of (13.2) we have only the following lemma:

3.2. LEMMA. If $\xi \in \mathfrak{S}$, then $(pq)\xi = p(\xi)q(\xi) + o(q(\xi))$.

This follows from (13.21) and (3.1).

(13.3), (13.4), (13.5), and (13.6) have no value for us since (13.7) cannot be generalised in a way suitable for our purposes. But the following weaker lemma will help us:

⁽⁹⁾ Here and in the following numbers of formulae refer either to our paper or to [1, part II]. No misunderstanding is possible since our paper consists only of the sections 1–3, but [1, part II] starts with section 7.

3.3. COROLLARY. If $\xi \in \mathfrak{S}$, $p(\xi) \neq o(1)$, then $(pq)\xi = p(\xi)q(\xi)(1 + o(1))$.

Proof. $p(\xi) \neq o(1)$ implies $(p(\xi))^{-1} = O(1)$. Hence

$$o(q(\xi)) = o((p(\xi))^{-1}p(\xi)q(\xi)) = o(O(p(\xi)q(\xi)) = o(p(\xi)q(\xi))$$

and (3.2) does the rest.

In the sections 14 and 15, θ and ζ belong already to $\mathfrak{S}$ quite naturally. Thus our proofs are not hampered by the previous restrictions that the arguments of all polynomials must belong to $\mathfrak{S}$. In (15.2) and (15.21), $\wedge \mathfrak{R}$ has to be replaced by $\overline{\mathfrak{S}}$. Of course the normal extension Φ , as introduced in the proof of (15.2), need not lie in $\overline{F^*}(-1)^{1/2}$. But according to Steinitz [4], for every field K an algebraically closed extension field L can be constructed. Thus if $K = \overline{\mathfrak{S}}$, Φ lies in L . In the proof of (15.5) our lemma (3.2) must be applied twice.

The ordering of a simple algebraic extension $\Sigma(\theta)$ of Σ can be defined as in the sections 16 and 17. All proofs but one remain valid, only the proof of (16.73) has to be altered (because it applied (13.7) in [1]) as follows:

$\bar{f}(\bar{x})$, as defined in (9.21), is irreducible over $\mathfrak{R}$. For otherwise $f(x)$ would factorise in $\mathfrak{R}[x]$, in contradiction of (9.8). Put $p_3(x) = \pi_2^{-1}p_1\pi_2(x)$, then $p_3(x) \in \mathfrak{R}[x] - \mathfrak{P}[x]$, according to its definition. Since $p_3(x)$ is of smaller degree than $f(x)$, we have $\bar{p}_3(\bar{\zeta}) \neq \bar{0}$, i.e. $p_3(\zeta) \neq o(1)$. Now (16.73) follows from (3.3).

Finally in the place of (17.5) we have the following result:

3.4. THEOREM. Every ordered division ring Σ can be extended to an ordered division ring $\Sigma(F^*)$, containing the closure F^* of a chosen subfield of F of Σ (with respect to the order-topology in F) with preservation of the order-relations in Σ .

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