REMARKS ON SOME MODULAR IDENTITIES

BY

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Introduction. We shall consider a certain class of functions invariant with respect to the substitutions of the congruence subgroup $\Gamma_0(p)$ of the modular group Γ . By specializing these functions, we shall obtain classical identities in the analytical theory of numbers: E.g., the Ramanujan identities for partitions modulo 5, 7 and Mordell's identity for $\tau(n)$. We shall also derive some new identities.

These functions bear some resemblance to those considered by Rademacher in his paper [1]⁽¹⁾ to prove the Ramanujan identities, certain modular equations, etc. The type of function considered, however, seems first to have been studied by Watson in his paper [2].

1. Definitions, notations.

(1.1) Γ is the full modular group; i.e., the group of 2×2 matrices of determinant 1 with rational integral elements.

(1.2) $\Gamma_0(m)$ is the subgroup of Γ characterized as follows: The element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of Γ belongs to $\Gamma_0(m)$ if and only if $c \equiv 0 \pmod{m}$.

$$(1.3) \quad \begin{aligned} \eta(\tau) &= \exp \pi i \tau / 12 \cdot \prod (1 - x^n) \\ &= \exp \pi i \tau / 12 \cdot \left\{ \sum p(n) x^n \right\}^{-1}, \quad x = \exp 2\pi i \tau, \operatorname{Im} \tau > 0. \end{aligned}$$

Here, as in the sequel, all products will be extended from 1 to ∞ and all sums from 0 to ∞ , unless otherwise indicated.

A few words about $\eta(\tau)$ (the Dedekind η -function) are in order. In the interior of the upper half-plane, $\eta(\tau)$ is free from poles and zeros. At $\tau = i\infty$ and at $\tau = 0$, $\eta(\tau)$ is zero. $\eta(\tau)$ is a modular form of dimension $-1/2$, and satisfies the following transformation formula, which will be used extensively:

(1.4) If

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \text{and} \quad c > 0,$$

then

$$\eta(V\tau) = \{ -i(c\tau + d) \}^{1/2} \exp -\pi i N \cdot \eta(\tau),$$

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⁽¹⁾ Numbers in brackets refer to bibliography at end of paper.

where $N = s(a, c) - (a+d)/12c$. Also, $\eta(\tau+1) = \exp \pi i/12 \cdot \eta(\tau)$.

$s(a, c)$ is a "Dedekind sum" and is defined as follows: If we set

$$((x)) = \begin{cases} x - [x] - 1/2 & x \text{ nonintegral} \\ 0 & x \text{ integral,} \end{cases}$$

then $s(a, c) = \sum ((r/c))((ar/c))$, where r runs over a complete set of residues modulo c in the summation.

We shall put

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is well known that S and T are generators of Γ . We observe that

$$S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

$$W^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad T^2 = -I.$$

Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)/(c\tau + d) = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \tau,$$

$$T^2 \tau = \tau.$$

We shall also put

$$z_p = \exp -2\pi i/p\tau, \quad \prod (1 - x^n)^r = \sum p_r(n) x^n.$$

We shall be concerned with series whose coefficients are $p_r(n)$'s.

Notice that $p_{-1}(n) = p(n)$ in this notation.

Some remarkable congruence properties of the $p_r(n)$'s modulo 5, 7 have been found by Ramanathan in his paper [6].

2. Functions on $\Gamma_0(p)$. In what follows, p will always be a prime greater than 3. The fundamental region Q_p of $\Gamma_0(p)$ has only the parabolic points $i\infty, 0$. A function on $\Gamma_0(p)$ (invariant with respect to the substitutions of $\Gamma_0(p)$) which is regular in the interior of Q_p and bounded at $i\infty, 0$ accordingly is constant. For this reason functions on $\Gamma_0(p)$ are amenable to numerical calculation. We need the following elementary theorem, the proof of which we omit:

(2.1) $\Gamma_0(p^2)$ is of index p in $\Gamma_0(p)$, and a set of right representatives for $\Gamma_0(p^2)$ in $\Gamma_0(p)$ is given by:

$$R_k = W^{-pk}, \quad k = 0, 1, \dots, p-1.$$

We also need the following:

(2.2) THEOREM. Let Γ_0, Γ_1 be subgroups of Γ , $\Gamma_0 \supseteq \Gamma_1$, $(\Gamma_0 : \Gamma_1) = \mu < \infty$. Let $R_0, R_1, \dots, R_{\mu-1}$ be a set of right representatives for Γ_1 in Γ_0 . Let $g(\tau)$ be a function on Γ_1 , and let $F(x_0, x_1, \dots, x_{\mu-1})$ be a symmetric function of its variables. Put $f(\tau) = F(g(R_0\tau), g(R_1\tau), \dots, g(R_{\mu-1}\tau))$. Then $f(\tau)$ is a function on Γ_0 .

Proof. Let $M \in \Gamma_0$.

$R_i M$ may be written as $M_i R'_i$, where $M_i \in \Gamma_1$, and R'_i is a representative. Also, if $R_i M = M_i R'_i$, $R_j M = M_j R'_j$, then if $R'_i = R'_j$, we have $R_i R_j^{-1} = M_i M_j^{-1} \in \Gamma_1$, so that $R_i = R_j$. That is, as R_i runs through a complete set of representatives, so does R'_i .

Hence

$$\begin{aligned} f(M\tau) &= F(g(R_0 M\tau), g(R_1 M\tau), \dots, g(R_{\mu-1} M\tau)) \\ &= F(g(M_0 R'_0 \tau), g(M_1 R'_1 \tau), \dots, g(M_{\mu-1} R'_{\mu-1} \tau)) \\ &= F(g(R'_0 \tau), g(R'_1 \tau), \dots, g(R'_{\mu-1} \tau)) \\ &= f(\tau). \end{aligned}$$

The function $\eta(p^2\tau)/\eta(\tau) = x^\nu \prod \{ (1 - x^{p^{2n}})/(1 - x^n) \}$, where $\nu = (p^2 - 1)/24$ (which is an integer since $(p, 6) = 1$), may easily be shown to be a function on $\Gamma_0(p^2)$ by use of the transformation formula (1.4) and some theorems about $s(a, c)$ which may be found in [1] or in [3]. We shall omit the proof. We put

$$(2.3.1) \quad h = h(\tau) = \eta(p^2\tau)/\eta(\tau),$$

$$(2.3.2) \quad S_r = S_r(\tau) = \sum_{n=0}^{p-1} h^r(R_n\tau), \quad r \text{ integral.}$$

By (2.1) and (2.2), S_r is a function on $\Gamma_0(p)$. The S_r 's are the functions we shall consider. Clearly, S_r is regular and bounded in the interior of the upper half-plane (see the remark to (1.3)). Hence to determine the behaviour of S_r completely, we need only know its behaviour at the parabolic points $i\infty, 0$ of Q_p . It will be our purpose in general to compare S_r with

$$(2.3.3) \quad g = g(\tau) = \{ \eta(p\tau)/\eta(\tau) \}^s,$$

where $s = s(p)$ is the least positive even integer such that $s(p-1) \equiv 0 \pmod{24}$. g is also a function on $\Gamma_0(p)$ (see [1]). g has a zero of order $s(p-1)/24$ at $\tau = i\infty$ in x , and a pole of order $s(p-1)/24$ at $\tau = 0$ in z_p . For

$$\begin{array}{ccc} p = 5 & p = 7 & p = 13 \\ s = 6, & s = 4, & s = 2, \end{array}$$

g is a "Hauptmodul" for $\Gamma_0(p)$: I.e., any function on $\Gamma_0(p)$ with polar singularities at most in appropriate uniformizing variables is a rational function of g .

We shall rewrite S_r . We have

$$h(R_n\tau) = \eta(p^2 R_n\tau)/\eta(R_n\tau) = \eta(p^2 W^{-pn}\tau)/\eta(W^{-pn}\tau) = \eta(pW^{-n}p\tau)/\eta(W^{-pn}\tau).$$

For $(n, p)=1$ define n' as the least positive solution of the congruence $nx \equiv 1 \pmod{p}$, and for $p|n$ define $n'=0$.

If $(n, p)=1$, we may rewrite $pW^{-n}p\tau$ as follows:

$$pW^{-n}p\tau = M_1(\tau - n'/p), \quad \text{where } M_1 = \begin{pmatrix} -p & -n' \\ n & (nn' - 1)/p \end{pmatrix}.$$

Hence $h(R_n\tau) = \eta(M_1(\tau - n'/p))/\eta(W^{-pn}\tau)$. Making use of (1.4), we obtain $h(R_n\tau) = p^{-1/2} \exp -\pi i N_1 \cdot \eta(\tau - n'/p)/\eta(\tau)$, where

$$N_1 = \{s(-p, n) - (-n + (nn' - 1)/p)/12n\} - \{s(-1, pn) + 1/6pn\}.$$

It may further be shown by theorems from [1] and [3] that $\exp -\pi i N_1 = \exp -\pi i(p-1)/4 \cdot \exp \pi i n'p/12 \cdot (n'/p)$, where (n'/p) denotes the Legendre-Jacobi symbol of quadratic reciprocity. Hence

$$h(R_n\tau) = p^{-1/2} \exp -\pi i(p-1)/4 \cdot \exp \pi i n'p/12 \cdot (n'/p) \cdot \eta(\tau - n'/p)/\eta(\tau),$$

and so

$$\begin{aligned} (2.4) \quad S_r &= \sum_{n=0}^{p-1} h^r(R_n\tau) = h^r(\tau) + \sum_{n=1}^{p-1} h^r(R_n\tau) \\ &= h^r(\tau) + c^r \eta^{-r}(\tau) \sum_{n=1}^{p-1} (n'/p)^r \cdot \exp \pi i n'p/12 \cdot \eta^r(\tau - n'/p), \end{aligned}$$

where we have put $c = p^{-1/2} \exp -\pi i(p-1)/4$.

Distinguishing cases r even and r odd, and using the well known Gaussian sum formula

$$\sum_{n=1}^{p-1} (n/p) \exp 2\pi i na/p = \exp \pi i(p-1)^2/8 \cdot p^{1/2}(a/p),$$

we find

(2.5.1) For r even,

$$S_r = -c^r + h^r(\tau) + pc^r \prod (1 - x^n)^{-r} \sum_{\lambda \geq 0, \lambda \equiv rn \pmod{p}} p_r(\lambda) x^\lambda.$$

(2.5.2) For r odd,

$$S_r = h^r(\tau) + \exp \pi i(p-1)^2/8 \cdot p^{1/2} c^r \prod (1 - x^n)^{-r} \sum_{\lambda \geq 0} ((rn - \lambda)/p) p_r(\lambda) x^\lambda.$$

The formulae (2.5.1), (2.5.2) furnish the desired information as to the behaviour of the S_r 's at $i\infty$. To study the behaviour of the S_r 's at 0, we subject them to the transformation T and study the TS_r 's at $i\infty$ ⁽²⁾. Proceeding

⁽²⁾ A device employed by Rademacher in [1].

as before, we have for $h(R_n T\tau)$, $(n, p) = 1$:

$$h(R_n T\tau) = c(n'/p) \exp \pi i n' p / 12 \cdot \eta(T\tau - n'/p) / \eta(T\tau).$$

We may rewrite $T\tau - n'/p$ as follows:

$$T\tau - n'/p = M_2(\tau + n p) / p^2, \quad \text{where } M_2 = \begin{pmatrix} -n' & (nn' - 1)/p \\ p & -n \end{pmatrix}.$$

Making use of (1.4) again,

$$\begin{aligned} h(R_n T\tau) &= c(n'/p) \exp \pi i n' p / 12 \cdot \eta(M_2(\tau + n p) / p^2) / \eta(T\tau) \\ &= p^{-1/2} c(n'/p) \exp \pi i n' p / 12 \cdot \exp -\pi i N_2 \cdot \eta((\tau + n p) / p^2) / \eta(\tau), \end{aligned}$$

where $N_2 = s(-n', p) + (n + n') / 12 p$.

After the necessary simplifications, we obtain

$$h(R_n T\tau) = p^{-i} \exp -\pi i n p / 12 \cdot \eta\left(\frac{\tau + n p}{p^2}\right) / \eta(\tau).$$

If we note here that $h(R_n T\tau)$ agrees formally with $h(T\tau)$ for $n=0$, we have

$$\begin{aligned} (2.6) \quad TS_r &= \sum_{n=0}^{p-1} h^r(R_n T\tau) \\ &= p^{-r} \eta^{-r}(\tau) \sum_{n=0}^{p-1} \exp -\pi i n p r / 12 \cdot \eta^r\left(\frac{\tau + n p}{p^2}\right) / \eta(\tau). \end{aligned}$$

Expanding into a power series, we have

$$(2.7) \quad TS_r = p^{-r+1} \prod (1 - x^n)^{-r} \sum_{\lambda \geq 0, \lambda \equiv r\nu(p)} p_r(\lambda) x^{(\lambda - r\nu)/p^2}.$$

A study of (2.5.1), (2.5.2), and (2.7) leads us to the construction of the following table:

(2.8) (a) $r > 0$.

	$i \infty$	0
S_r :	Zero-free and pole-free in x unless r is odd and divisible by p , in which case zero of order 1 in x .	Pole of order $[r\nu/p]$ in $z_p^{(3)}$.
$S_r + c^r$,		Pole of order $[r\nu/p]$ in $z_p^{(3)}$.
r even:	Zero of order $r\nu - p[r\nu/p]$ in $x^{(3)}$.	

(b) $r < 0$. Put $r = -r_1$, $r_1 > 0$. We then have:

	$i \infty$	0
S_r :	Pole of order $r_1\nu$ in x .	Zero of order $1 + [r_1\nu/p]$ in z_p .

3. Applications.

(³) Provided that $p_r(r\nu - p[r\nu/p]) \neq 0$.

(3.1) $r = -1$, $p = 5$.

Then S_{-1} has a pole of order 1 in x at $\tau = i\infty$, and a zero of order 1 in z_5 at $\tau = 0$. This implies that the product gS_{-1} is constant, or that

$$S_{-1} = K_0 g^{-1}.$$

If we replace τ by $-1/5\tau$ and evaluate K_0 , we obtain the first of Ramanujan's identities:

$$\sum p(5n+4)x^n = 5 \prod (1 - x^{5n})^5 (1 - x^n)^{-6}.$$

(3.2) $r = -1$, $p = 7$.

Then S_{-1} has a pole of order 2 in x at $\tau = i\infty$, and a zero of order 1 in z_7 at $\tau = 0$. This implies that $g^2 S_{-1}$ is linear in g , or that

$$S_{-1} = K_0 g^{-2} + K_1 g^{-1}.$$

If we replace τ by $-1/7\tau$ and evaluate K_0, K_1 , we obtain the second of Ramanujan's identities:

$$\sum p(7n+5)x^n = 7 \prod (1 - x^{7n})^3 (1 - x^n)^{-4} + 49x \prod (1 - x^{7n})^7 (1 - x^n)^{-8}.$$

(3.3) $r = -1$, $p = 13$.

Then S_{-1} has a pole of order 7 in x at $\tau = i\infty$, and a zero of order 1 in z_{13} at $\tau = 0$. This implies that $g^7 S_{-1}$ is a polynomial in g of degree 6, or that

$$S_{-1} = K_0 g^{-7} + K_1 g^{-6} + \cdots + K_6 g^{-1}.$$

If we replace τ by $-1/13\tau$ and evaluate $K_0, K_1, \dots, K_6$, we obtain an identity of Zuckerman (see [3]).

(3.4) $r = 24$.

Then $S_{24} + c^{24}$ has a zero of order $p-1$ at least in x at $\tau = i\infty$, and a pole of order $p-1$ at most in z_p at $\tau = 0$. This implies that the quotient $(S_{24} + c^{24}) / \{\eta(p\tau)/\eta(\tau)\}^{24}$ is bounded, which in turn implies that

$$S_{24} = -c^{24} + K_0 \{\eta(p\tau)/\eta(\tau)\}^{24}.$$

If we replace τ by $-1/p\tau$, evaluate K_0 , and set $\tau(n) = p_{24}(n-1)$, we obtain Mordell's identity for $\tau(n)^{(4)}$:

$$\sum \tau(np + p)x^n = \tau(p) \prod (1 - x^n)^{24} - p^{11} x^{p-1} \prod (1 - x^{np})^{24}.$$

(3.5) We can easily generalize Mordell's identity as follows: Choose r even, $0 < r \leq 24$, $r(p-1) \equiv 0 \pmod{24}$. Put $\delta = r(p-1)/24$. Then we have for $S_r + c^r$:

$$S_r + c^r: \quad \begin{array}{cc} i\infty & 0 \\ \text{Zero of order } \delta \text{ at least in } x. & \text{Pole of order } \delta \text{ at most in } z_p. \end{array}$$

This implies that the quotient $(S_r + c^r) / \{\eta(p\tau)/\eta(\tau)\}^r$ is bounded which

(4) See [5].

in turn implies that

$$S_r = -c^r + K_0 \{ \eta(p\tau)/\eta(\tau) \}^r.$$

If we replace τ by $-1/p\tau$ and evaluate K_0 , we obtain

$$\sum p_r(np + \delta)x^n = p_r(\delta) \prod (1 - x^n)^r - p^{r/2-1}x^\delta \prod (1 - x^{np})^r.$$

(3.6) $p=5$.

We have the following table:

	$i\infty$	0
S_1 :	Zero-free and pole-free in x .	Zero-free and pole-free in z_5 .
S_2 :	Zero-free and pole-free in x .	Zero-free and pole-free in z_5 .
S_3 :	Zero-free and pole-free in x .	Zero-free and pole-free in z_5 .
S_4 :	Zero-free and pole-free in x .	Zero-free and pole-free in z_5 .
S_5 :	Zero of order 1 in x .	Pole of order 1 in z_5 .

These imply that S_1, S_2, S_3, S_4 are constant, while S_5 is proportional to $B = \{ \eta(5\tau)/\eta(\tau) \}^6$. Hence in the polynomial $\prod_{i=0}^4 (u - h(R_i\tau)) = u^5 - c_1u^4 + c_2u^3 - c_3u^2 + c_4u - c_5$, the coefficients c_1, c_2, c_3, c_4 must also be constant, while c_5 is linear in $B^{(5)}$.

Putting $u = h(\tau) = \eta(25\tau)/\eta(\tau) = A$, we find that B is a polynomial of degree 5 in A . The actual polynomial turns out to be

$$(3.6.1) \quad B = 25A^5 + 25A^4 + 15A^3 + 5A^2 + A.$$

If we set $A_0 = 5^{1/2}A$, $B_0 = B/A$, then (3.6.1) reads

$$(3.6.2) \quad B_0 = A_0^4 + 5^{1/2}A_0^3 + 3A_0^2 + 5^{1/2}A_0 + 1.$$

(3.6.2) shows the reciprocal nature of the modular equation.

The reason there is an identity (3.6.1) is that $\Gamma_0(25)$ is of genus zero (as are $\Gamma_0(5), \Gamma_0(7), \Gamma_0(13)$) and so a Hauptmodul exists for $\Gamma_0(25)$. (3.6.1) indicates that we may choose $\eta(25\tau)/\eta(\tau)$ for this Hauptmodul⁽⁶⁾. The genus of $\Gamma_0(49)$, however, turns out to be 1, so no analogous identity exists for $\{ \eta(7\tau)/\eta(\tau) \}^4$ in terms of $\eta(49\tau)/\eta(\tau)$. The polynomial relationship between these functions is *quadratic* in $\{ \eta(7\tau)/\eta(\tau) \}^4$.

Some new identities (of the many which are possible) obtained by specializing the S_r 's follow, without comment. Of particular interest are those for which p differs from 5, 7, 13 (since then the subgroups in question are not of genus zero). These identities are isolated instances, not at all characteristic

⁽⁶⁾ The c 's are the elementary symmetric functions on the roots, and the result follows from Newton's formulae relating the c 's and the S 's.

⁽⁷⁾ $\eta(25\tau)/\eta(\tau)$ has a zero of order 1 in x at $\tau = i\infty$, a pole of order 1 in z_{25} at $\tau = 0$, and is finite and different from zero at the other parabolic points $\pm 1/5, \pm 2/5$ of Q_{25} . That is, it has precisely one pole (in the proper uniformizing variable) in Q_{25} . This guarantees that it is a Hauptmodul for $\Gamma_0(25)$.

of the true situation.

1. $r=5, p=5$.

$$\sum p_5(5n)x^n = \prod (1-x^n)^6(1-x^{5n})^{-1(7)}.$$

2. $r=5, p=7$.

$$\sum p_5(7n+3)x^n = 10 \prod (1-x^{7n})(1-x^n)^4 + 49x \prod (1-x^{7n})^5.$$

3. $r=7, p=7$.

$$\sum p_7(7n)x^n = \prod (1-x^n)^8(1-x^{7n})^{-1} + 49x \prod (1-x^n)^4(1-x^{7n})^3(7).$$

4. $r=2, p=11$.

$$\sum p_2(11n+10)x^n = \prod (1-x^{11n})^2.$$

5. $r=4, p=11$.

$$\sum p_4(11n+20)x^n = -11 \prod (1-x^{11n})^4.$$

6. $r=2, p=17$.

$$\sum p_2(17n+24)x^n = - \prod (1-x^{17n})^2.$$

7. $r=6, p=31$.

$$\sum p_6(31n+240)x^n = 961 \prod (1-x^{31n})^6.$$

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(7) Dr. Lehmer informs me that these identities are known to him, though he has never published them.