COMPLEX BOUNDARY VALUE PROBLEMS

BY

P. R. GARABEDIAN AND D. C. SPENCER

1. **Introduction.** In this paper we study the theory of functions of several complex variables from the point of view of partial differential equations of elliptic type. The basic objective is to obtain a generalization of the tools of potential theory which will apply to the Cauchy-Riemann equations in several variables. Thus we investigate further the kernel function and its relation to the Green's function introduced earlier [8; 9].

Our point of departure here is a discussion of boundary value problems for harmonic differential forms in Euclidean space carried through by Hodge [12]. We find that in terms of two new complex differential operators $\bar{d}$ and $\bar{\delta}$ one can develop a theory of boundary value problems, analogous formally to Hodge's work, which centers about the Cauchy-Riemann equations. We are enabled to bring together systematically and concisely the concepts of the Cauchy formula, the Green's and Neumann's forms, the kernel form, and harmonic integrals, and to build from them an elegant generalization for several complex variables of the basic ideas of potential theory.

A number of new phenomena arise for the complex differential forms of our theory. While we rely strongly on the theory of the Fredholm integral equation, our boundary value problems lead to singular equations with infinitely many eigenfunctions. Thus to obtain a complete existence proof, we resort to a combination of the method of integral equations and a procedure of orthogonal projection developed around the Bergman kernel function [10]. Indeed, it is the recent results relating the kernel function to the Green's function which motivate our entire presentation. Thus we obtain a single theory which connects harmonic forms, integral equations, Cauchy's formula, and the kernel function in terms of suitable boundary value problems.

Many special cases and examples of our general theory prove to be of particular interest. A few of these are given special attention in a final section of the paper.

2. **Definitions, notation, and formalism.** We shall consider the $2k$-dimensional Euclidean space of k complex variables $z_j = x_j + iy_j$ and a cell M in that space with smooth boundary ∂M. We shall set $y_j = x_{k+j}$, and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

Presented to the Society, September 7, 1951; received by the editors July 12, 1951.

while $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$.

In the formalism of E. Cartan, one introduces the differential form

$$dx_1 \cdots dx_{2k} = \left(\frac{i}{2}\right)^k dz_1 \cdots dz_k d\bar{z}_1 \cdots d\bar{z}_k$$

as volume element. The expression

$$\phi_p = a_{i_1 \cdots i_p} dz_{i_1} \cdots dz_{i_p}$$

will be called a pure p-form, where summation over all combinations of indices with $i_1 < i_2 < \cdots < i_p$ is to be understood. In particular, with

$$r^2 = \sum_{j=1}^{k} | z_j - t_j |^2,$$

we introduce the singular p-form

$$\omega_p(z, t) = \frac{1}{r^{2k-2}} dz_{i_1} \cdots dz_{i_p} d\bar{t}_{i_1} \cdots d\bar{t}_{i_p}.$$

For pure forms ϕ_p, the conjugation operator * yields

$$*\phi_p = \frac{2^p}{(2i)^k} a_{i_1 \cdots i_p} d\bar{z}_{i_{p+1}} \cdots d\bar{z}_{i_k} dz_1 \cdots dz_k,$$

where $i_1, \cdots, i_k$ is an even permutation of the indices $1, \cdots, k$. In particular (without summation with respect to $i_1 i_2 \cdots i_p$),

$$dz_{i_1} \cdots dz_{i_p} * (dz_{i_1} \cdots dz_{i_p})^- = 2^p dx_1 \cdots dx_{2k},$$

and

$$* * \phi_p = (-1)^p \phi_p.$$

Also,

$$* d\bar{z}_{i_{p+1}} \cdots d\bar{z}_{i_k} dz_1 \cdots dz_k = (2i)^k (-2)^{-p} dz_{i_1} \cdots dz_{i_p}.$$

In addition to the usual differential operators d and $\delta = *d*$ of Hodge [13], we define here the complex operators $\bar{d}$ and $\bar{\delta}$ by the formulas

$$\bar{d}\phi_p = \sum_{m=p+1}^{k} \frac{\partial a_{i_1 \cdots i_p}}{\partial z_{i_m}} dz_{i_m} dz_{i_1} \cdots dz_{i_p},$$

$$\bar{\delta}\phi_p = \frac{1}{2} * d * \phi_p = \sum_{m=1}^{p} (-1)^{m+1} \frac{\partial a_{i_1 \cdots i_p}}{\partial \bar{z}_{i_m}} dz_{i_1} \cdots [dz_{i_m}] \cdots dz_{i_p},$$

where $[dz_{i_m}]$ indicates that this differential is to be omitted. Note that with $\zeta = d\bar{z}_1 \cdots d\bar{z}_k$ we obtain

$$\zeta \bar{d}\phi_p = \zeta d\phi_p = (-1)^k d(\zeta\phi_p).$$

Also, $\bar{d}\bar{d}\phi_p = \bar{\delta}\bar{\delta}\phi_p = 0$. The importance of the operators $\bar{d}$ and $\bar{\delta}$ stems from the fact that the Cauchy-Riemann equations for a function f analytic in the k complex variables $z_1, \cdots, z_k$ can be written $\bar{\delta}(fdz_1 \cdots dz_k) = 0$ or $\bar{d}\bar{f} = 0$.

Two identities will be of fundamental importance in our work. They are

$$\Delta\phi_p = \frac{1}{4}\sum_{m=1}^{2k} \frac{\partial^2 a_{i_1\cdots i_p}}{\partial x_m^2} dz_{i_1} \cdots dz_{i_p} = (\bar{d}\bar{\delta} + \bar{\delta}\bar{d})\phi_p$$

and

$$\bar{d}\omega_p(z, t) = -(\bar{\delta}\omega_{p+1}(t, z))^-,$$

where in the last relation $\bar{d}$ refers to differentiation with respect to z variables and $\bar{\delta}$ refers to differentiation with respect to t variables. For some positive constant σ_k one obtains by Poisson's equation the further identity

$$\phi_p(z) = -2^{-p}\sigma_k\Delta_z \int_M \phi_p(t) * (t)(\omega_p(t, z))^-.$$

We shall have need for the scalar product

$$(\phi_p, \psi_p) = \int_M \phi_p * (\psi_p)^-$$

of a pair of pure forms ϕ_p, ψ_p. Green's theorem yields for this the relations

$$(\bar{d}\phi_p, \psi_{p+1}) + 2(\phi_p, \bar{\delta}\psi_{p+1}) = \int_{\partial M} \phi_p * (\psi_{p+1})^-,$$

$$2(\bar{\delta}\psi_{p+1}, \phi_p) + (\psi_{p+1}, \bar{d}\phi_p) = \int_{\partial M} (*\psi_{p+1})(\phi_p)^-.$$

Similar identities can be obtained in curved cells M by Stokes' theorem.

Finally, one obtains by a ready calculation

$$[*(\zeta\phi_p)]\bar{\zeta} = (-2)^k * \phi_p.$$

At a point of ∂M, one can introduce tangential coordinates $s_1, \cdots, s_{2k-1}$ and a normal coordinate n. Then ϕ_p can be written

$$\phi_p = \alpha_{i_1\cdots i_p} ds_{i_1} \cdots ds_{i_p} + \beta_{i_1\cdots i_{p-1}} dn ds_{i_1} \cdots ds_{i_{p-1}}.$$

We define

$$T\phi_p = \alpha_{i_1\cdots i_p} ds_{i_1} \cdots ds_{i_p}.$$

3. Combined method of integral equations and orthogonal projection. Suppose that

$$\theta_{k+p} = T[\alpha_{i_1\cdots i_p} dz_{i_1} \cdots dz_{i_p} d\bar{z}_1 \cdots d\bar{z}_k]$$

is a given form on ∂M. We shall prove in this section that there exists in M a unique pure form A_p such that

$$T(\zeta A_p) = \theta_{k+p}, \qquad \text{on } \partial M,$$
$$\bar{\delta}\bar{d}A_p = 0, \qquad \text{in } M,$$
$$A_p = \bar{\delta}Q_{p+1}, \qquad \text{in } M,$$

for some pure form Q_{p+1}. Our method of proof is based on the classical procedure of integral equations together with an application of a method of orthogonal projection developed recently in connection with the theory of kernel functions [10]. The result will serve as a basis for the study of analytic functions of several complex variables by the methods of the Fredholm integral equation, of harmonic forms, and of the general theory of partial differential equations of elliptic type.

Let ϕ_p be a given pure form defined in the neighborhood of ∂M, and let this form be extended into M in some manner. Then by Poisson's equation

$$-2^{-p}\sigma_k\Delta \int_M \phi_p * (\omega_p)^- = \begin{cases} \phi_p, & \text{in } M, \\ 0, & \text{outside } M. \end{cases}$$

Also,

$$-2^{-p}\sigma_k\Delta \int_M \phi_p * (\omega_p)^- = -2^{-p}\sigma_k\bar{\delta}\bar{d} \int_M \phi_p * (\omega_p)^- - 2^{-p}\sigma_k\bar{d}\bar{\delta} \int_M \phi_p * (\omega_p)^-$$
$$= 2^{-p}\sigma_k\bar{\delta} \int_M \phi_p * (\bar{\delta}\omega_{p+1})^- + 2^{-p}\sigma_k\bar{d} \int_M \phi_p * (\bar{d}\omega_{p-1})^-.$$

Thus from Green's theorem

$$2^{-p}\sigma_k\bar{\delta}(\phi_p, \bar{\delta}\omega_{p+1}) + 2^{-p}\sigma_k\bar{d}(\phi_p, \bar{d}\omega_{p-1})$$
$$= 2^{-p-1}\sigma_k\bar{\delta} \int_{\partial M} \phi_p * (\omega_{p+1})^- + 2^{-p}\sigma_k\bar{d} \int_{\partial M} (*\phi_p)(\omega_{p-1})^-$$
$$- 2^{-p-1}\sigma_k\bar{\delta}(\bar{d}\phi_p, \omega_{p+1}) - 2^{-p+1}\sigma_k\bar{d}(\bar{\delta}\phi_p, \omega_{p-1})$$
$$= \begin{cases} \phi_p, & \text{in } M, \\ 0, & \text{outside } M. \end{cases}$$

Now $Td(\zeta\omega_{p-1}) = (-1)^k T\zeta\bar{d}\omega_{p-1}$, and it may be verified that

$$T\zeta 2^{-p}\sigma_k\bar{d} \int_{\partial M} (*\phi_p)(\omega_{p-1})^-$$

is continuous across ∂M. Thus

$$T\zeta 2^{-p-1}\sigma_k\bar{\delta}\int_{\partial M}\phi_p * (\omega_{p+1})^-$$

must jump by $T\zeta\phi_p$ on ∂M.

If we set

$$(1)\qquad A_p = 2^{-p-1}\sigma_k\bar{\delta}\int_{\partial M}\phi_p * (\omega_{p+1})^-, \qquad \text{in } M,$$

we find, by the usual argument, that $T\zeta A_p$ has on ∂M the boundary values

$$T\zeta A_p = P\left[T2^{-p-1}\sigma_k\zeta\bar{\delta}\int_{\partial M}\phi_p * (\omega_{p+1})^-\right] + \frac{1}{2}T\zeta\phi_p,$$

where $P[\ ]$ indicates that the integral inside the brackets is to be taken in the sense of the Cauchy principal value. Clearly

$$\bar{\delta}\bar{d}A_p = 2^{-p-1}\sigma_k\bar{\delta}\bar{d}\bar{\delta}\int_{\partial M}\phi_p * (\omega_{p+1})^- = 2^{-p-1}\sigma_k\bar{\delta}\Delta\int_{\partial M}\phi_p * (\omega_{p+1})^- = 0,$$

since $\Delta\omega_{p+1}=0$. Thus A_p will solve our boundary value problem provided

$$(2)\qquad P\left[T2^{-p-1}\sigma_k\zeta\bar{\delta}\int_{\partial M}\phi_p * (\omega_{p+1})^-\right] + \frac{1}{2}T\zeta\phi_p = \theta_{k+p}$$

on ∂M. We seek therefore to determine ϕ_p as a solution of the singular integral equation (2).

The solution of the integral equation depends upon a discussion of the transposed equation [11]. The transposed, or associated, integral equation arises when we attempt to solve in the exterior of M the boundary value problem

$$d\bar{\delta}B_{p+1} = 0,$$
$$T * B_{p+1} = (\theta_{2k-p-1})^-, \qquad \text{on } \partial M,$$

in the form

$$(3)\qquad B_{p+1} = 2^{-p-1}\sigma_k\bar{d}\int_{\partial M}(*\phi_{p+1})(\omega_p)^-.$$

Indeed, $T * \bar{\delta}\omega_{p+2} = 2^{-1}(-1)^{p+1}Td * \omega_{p+2}$ and

$$T * 2^{-p-2}\sigma_k\bar{\delta}\int_{\partial M}\phi_{p+1} * (\omega_{p+2})^-$$

is continuous across ∂M, while

$$T * 2^{-p-1}\sigma_k\bar{d}\int_{\partial M}(*\phi_{p+1})(\omega_p)^-$$

must jump by $T*\phi_{p+1}$ on ∂M. Therefore, the boundary values of $T*B_{p+1}$ on ∂M from the exterior of M are

$$T*B_{p+1} = P\left[T2^{-p-1}\sigma_k * \bar{d}\int_{\partial M}(*\phi_{p+1})(\omega_p)^-\right] - \frac{1}{2}T*\phi_{p+1}.$$

Noting that $(-2)^k*\phi_{p+1} = [*(\zeta\phi_{p+1})]\bar{\zeta}$, we obtain for the determination of ϕ_{p+1} the singular integral equation

$$P\left[T2^{-p-1}\sigma_k*\int_{\partial M}(*\zeta\phi_{p+1})\bar{\zeta}(\bar{\delta}\omega_{p+1})^-\right] + \frac{1}{2}T[(*\zeta\phi_{p+1})\bar{\zeta}] = -(-2)^k(\theta_{2k-p-1})^-. \tag{4}$$

The integral equations (2) and (4) to which we are thus led are each the transpose of the other, since the kernel is in the first case

$$T(z)T(t)\zeta(z)\bar{\delta}(z)*(t)\omega_{p+1}(z, t)$$

and in the second case

$$T(z)T(t)*(z)(\zeta(t))^-(\bar{\delta}(t)\omega_{p+1}(t, z))^-.$$

It follows [11] that (2) can be solved for ϕ_p if and only if $\theta_{k+p} = T(\zeta A_p)$ is orthogonal to every solution ϕ_{p+1} of (4) in the homogeneous case $\theta_{2k-p-1} = 0$, in the sense

$$\int_{\partial M}(*\zeta\phi_{p+1})(\theta_{k+p})^- = \int_{\partial M}(*\zeta\phi_{p+1})\bar{\zeta}(A_p)^- = 0,$$

or

$$\int_{\partial M}A_p*(\phi_{p+1})^- = 0.$$

This result is a consequence of the relation

$$\begin{aligned}
&\int_{\partial M}(*\zeta\phi_{p+1})(\theta_{k+p})^- \\
&\quad = P\left[2^{-p-1}\sigma_k\int_{\partial M}\int_{\partial M}\{*(z)\zeta(z)\phi_{p+1}(z)\}(\zeta(z))^-(\bar{\delta}(z)\phi_p(t)*(t)\omega_{p+1}(z, t))^-\right] \\
&\qquad + \frac{1}{2}\int_{\partial M}\{*(z)\zeta(z)\phi_{p+1}(z)\}(\zeta(z))^-(\phi_p(z))^- \\
&\quad = P\left[\int_{\partial M}\left\{2^{-p-1}\sigma_k*\int_{\partial M}(*\zeta\phi_{p+1})\bar{\zeta}(\bar{\delta}\omega_{p+1})^- + \frac{1}{2}[*\zeta\phi_{p+1}]\bar{\zeta}\right\}(\phi_p)^-\right] \\
&\quad = 0,
\end{aligned}$$

where ϕ_{p+1} runs through precisely the class of solutions of the homogeneous equation (4) and the inner bracket therefore vanishes identically.

We turn now to the study of the solutions ϕ_{p+1} of the homogeneous equation (4) in order to analyze the possible boundary forms θ_{k+p}. It is immediately evident that (3) is a form with boundary values $T(*B_{p+1})=0$ from the exterior $\tilde{M}$ of M. Therefore

$$2\int_{\tilde{M}} \bar{\delta}B_{p+1} * (\bar{\delta}B_{p+1})^- = -\int_{\partial M} (*B_{p+1})(\bar{\delta}B_{p+1})^- = 0,$$

where the application of Green's theorem in $\tilde{M}$ is justified by the good behavior of B_{p+1} at infinity. Here the minus sign before the integral on the right is a consequence of the relation $\partial\tilde{M} = -\partial M$. We conclude that

$$\bar{\delta}B_{p+1} = 0, \qquad \text{in } \tilde{M}.$$

Therefore the form

$$Q_p = 2^{-p-1}\sigma_k \int_{\partial M} (*\phi_{p+1})(\omega_p)^-$$

satisfies the differential equation $\bar{\delta}\bar{d}Q_p=0$ in $\tilde{M}$. A second application of Green's theorem yields

$$\int_{\tilde{M}} \bar{d}Q_p * (\bar{d}Q_p)^- = -\int_{\partial M} Q_p * (\bar{d}Q_p)^- = -\int_{\partial M} Q_p * (B_{p+1})^- = 0.$$

Thus $B_{p+1}=\bar{d}Q_p=0$ in $\tilde{M}$.

Since $T*B_{p+1}$ jumps by $T*\phi_{p+1}$ on ∂M, we see that $T*B_{p+1}$ has from the interior of M the boundary values

$$T*B_{p+1} = T*\phi_{p+1}.$$

Also

$$\begin{aligned}\bar{\delta}B_{p+1} &= 2^{-p-1}\sigma_k\bar{\delta}\bar{d}\int_{\partial M} (*\phi_{p+1})(\omega_p)^- \\ &= -2^{-p-1}\sigma_k\bar{d}\bar{\delta}\int_{\partial M} (*\phi_{p+1})(\omega_p)^- \\ &= 2^{-p-1}\sigma_k\bar{d}\int_{\partial M} (*\phi_{p+1})(\bar{d}\omega_{p-1})^- \\ &= (-1)^p 2^{-p-1}\sigma_k\bar{d}\int_{\partial M} d(*\phi_{p+1})(\omega_{p-1})^-\end{aligned}$$

by Stokes' theorem, and since the tangential derivative

$$T\zeta\bar{d}\omega_{p-1} = (-1)^k Td(\zeta\omega_{p-1})$$

is of order of magnitude r^{-2k+2}, we find that

$$T\zeta\bar{\delta}B_{p+1} = (-1)^{p+k}2^{-p-1}\sigma_k T d\left\{\zeta\int_{\partial M} d(*\phi_{p+1})(\omega_{p-1})^-\right\}$$

is continuous across ∂M. Hence $T\zeta\bar{\delta}B_{p+1}=0$ there.

A final application of Green's theorem gives

$$2(\bar{\delta}B_{p+1}, \bar{\delta}B_{p+1}) = \int_{\partial M}(*B_{p+1})(\bar{\delta}B_{p+1})^- = 0,$$

and we conclude that

$$\bar{\delta}B_{p+1} = 0, \qquad \text{in } M.$$

Thus $T*\phi_{p+1}=T*B_{p+1}$ is obtained from a form B_{p+1} in the interior of M which satisfies there the differential equations

$$\bar{d}B_{p+1} = \bar{\delta}B_{p+1} = 0.$$

Our conclusion is that the boundary expressions $T\zeta A_p$ of pure forms A_p in M with

$$\bar{\delta}\bar{d}A_p = 0, \qquad A_p = \bar{\delta}Q_{p+1} \qquad \text{for some } Q_{p+1},$$

generate the entire class of forms $\theta_{k+p}=T(\zeta C_p)$ on ∂M such that

$$\int_{\partial M} C_p*(B_{p+1})^- = 0$$

for every pure B_{p+1} in M with $\bar{d}B_{p+1}=\bar{\delta}B_{p+1}=0$ there.

In order to show that even this restriction is not necessary, we proceed to apply a method of orthogonal projection.

Let C_p be any pure form in M with

$$(\bar{d}C_p, \bar{d}C_p) < \infty.$$

We denote by α_{p+1} the pure form in M such that

$$(5) \qquad (\bar{d}C_p - \alpha_{p+1}, \bar{d}C_p - \alpha_{p+1}) = \text{minimum}$$

under the restriction $\bar{\delta}\bar{d}\alpha_{p+1}=\bar{\delta}\alpha_{p+1}=0$. Since

$$\Delta\alpha_{p+1} = \bar{\delta}\bar{d}\alpha_{p+1} + \bar{d}\bar{\delta}\alpha_{p+1} = 0,$$

the coefficients of the competing forms are harmonic functions, and the extremal form α_{p+1} exists.

We make a permissible variation

$$\alpha_{p+1} + \epsilon\bar{\delta}\omega_{p+2}$$

of α_{p+1} with singularity outside M, and we deduce that the pure form

(6) $$E_{p+1} = -2^{-p-1}\sigma_k(\bar{d}C_p - \alpha_{p+1}, \omega_{p+1})$$

satisfies in the exterior of M the relation

$$\bar{d}E_{p+1} = 2^{-p-1}\sigma_k(\bar{d}C_p - \alpha_{p+1}, \bar{\delta}\omega_{p+2}) = 0.$$

The coefficients of E_{p+1} are clearly harmonic functions there.

On the other hand, we obtain in M

$$\begin{aligned}\bar{\delta}\bar{d}\bar{\delta}E_{p+1} &= \bar{\delta}\Delta E_{p+1}\\ &= -2^{-p-1}\sigma_k\bar{\delta}\Delta\int_M(\bar{d}C_p - \alpha_{p+1}) * (\omega_{p+1})^-\\ &= \bar{\delta}(\bar{d}C_p - \alpha_{p+1}) = \bar{\delta}\bar{d}C_p,\end{aligned}$$

or, better,

$$\bar{\delta}\bar{d}(C_p - \bar{\delta}E_{p+1}) = 0.$$

We set, therefore,

(7) $$A_p = C_p - \bar{\delta}E_{p+1}$$

and study $T\zeta A_p$ on ∂M.

First, it is not hard to see by arguments from potential theory that $\bar{\delta}E_{p+1}$ is continuous across ∂M [10]; so also is the tangential derivative $T\zeta\bar{d}\bar{\delta}E_{p+1}$. But outside M

$$\bar{d}\bar{\delta}E_{p+1} = -\bar{\delta}\bar{d}E_{p+1} = 0,$$

and hence $Td(\zeta\bar{\delta}E_{p+1})=0$ on ∂M. Thus $T\zeta A_p$ and $T\zeta C_p$ differ on ∂M by a closed form only.

But more can be proved. If β_{p+1} is any form in M such that $\bar{d}\beta_{p+1}=\bar{\delta}\beta_{p+1}=0$, we obtain from

$$\begin{aligned}\beta_{p+1} &= -2^{-p-1}\sigma_k\Delta\int_M\beta_{p+1} * (\omega_{p+1})^-\\ &= 2^{-p-1}\sigma_k\bar{d}\int_M\beta_{p+1} * (\bar{d}\omega_p)^- + 2^{-p-2}\sigma_k\bar{\delta}\int_M\beta_{p+1} * (\bar{\delta}\omega_{p+2})^-\\ &= 2^{-p-1}\sigma_k\bar{d}\int_{\partial M}(*\beta_{p+1})(\omega_p)^- + 2^{-p-2}\sigma_k\bar{\delta}\int_{\partial M}\beta_{p+1} * (\omega_{p+2})^-\end{aligned}$$

the generalized Cauchy formula

$$\beta_{p+1} = -2^{-p-1}\sigma_k\int_{\partial M}(*\beta_{p+1})(\bar{\delta}\omega_{p+1})^- - 2^{-p-2}\sigma_k\int_{\partial M}\beta_{p+1} * (\bar{d}\omega_{p+1})^-.$$

Substituting this in the variational relation $(\bar{d}C_p-\alpha_{p+1}, \beta_{p+1})=0$, we obtain

$$- 2^{-p-1}\sigma_k \int_M \int_{\partial M} (\bar{d}C_p - \alpha_{p+1}) * (\beta_{p+1})^- * \bar{\delta}\omega_{p+1}$$

$$- 2^{-p-2}\sigma_k \int_M \int_{\partial M} (\bar{d}C_p - \alpha_{p+1})(\beta_{p+1})^- * * \bar{d}\omega_{p+1}$$

$$= - 2^{-p-1}\sigma_k \int_{\partial M} * (\beta_{p+1})^- \bar{\delta} \int_M (\bar{d}C_p - \alpha_{p+1}) * (\omega_{p+1})^-$$

$$- 2^{-p-2}\sigma_k \int_{\partial M} (\beta_{p+1})^- * \bar{d} \int_M (\bar{d}C_p - \alpha_{p+1}) * (\omega_{p+1})^-$$

$$= \int_{\partial M} * (\beta_{p+1})^- \bar{\delta} E_{p+1} + \frac{1}{2} \int_{\partial M} (\beta_{p+1})^- * \bar{d}E_{p+1} = 0.$$

But since $\bar{d}E_{p+1}$ vanishes on ∂M, this yields

$$\int_{\partial M} \bar{\delta}E_{p+1} * (\beta_{p+1})^- = 0$$

for every β_{p+1} in M with $\bar{d}\beta_{p+1} = \bar{\delta}\beta_{p+1} = 0$.

The method of orthogonal projection therefore gives a form A_p such that $T\zeta A_p$ differs from the arbitrary values $T\zeta C_p$ by a form $T\zeta\bar{\delta}E_{p+1}$ which satisfies the orthogonality relation

$$\int_{\partial M} \bar{\delta}E_{p+1} * (\beta_{p+1})^- = 0.$$

This condition is precisely the one which we developed with the procedure of integral equations, and thus our previous theory shows that $T\zeta\bar{\delta}E_{p+1}$ is a permissable boundary form for our Dirichlet problem. A combination of the methods leads us to conclude that it is always possible to find a solution of the given boundary value problem. The only assumption required is that the given boundary values $T\zeta C_p$ be generated by a form expressible as $C_p = \bar{\delta}R_{p+1}$, so that the solution A_p has also the form $A_p = \bar{\delta}Q_{p+1}$.

Uniqueness of the solution is quite elementary. For if $T\zeta A_p = 0$ on ∂M, then

$$(\bar{d}A_p, \bar{d}A_p) = \int_{\partial M} A_p * (\bar{d}A_p)^- = 0,$$

so that $\bar{d}\bar{\delta}Q_{p+1} = \bar{d}A_p = 0$. Hence

$$(\bar{\delta}Q_{p+1}, \bar{\delta}Q_{p+1}) = \frac{1}{2} \int_{\partial M} (* Q_{p+1})(\bar{\delta}Q_{p+1})^- = \frac{1}{2} \int_{\partial M} (* Q_{p+1})(A_p)^- = 0,$$

and $A_p = \bar{\delta}Q_{p+1} = 0$ in M.

A quite analogous treatment can be given for the related Neumann boundary value problem

$$\begin{aligned} d\bar{\delta}B_{p+1} &= 0, && \text{in } M, \\ T * B_{p+1} &= (\theta_{2k-p-1})^-, && \text{given on } \partial M, \\ B_{p+1} &= \bar{d}Q_p && \text{for some } Q_p \text{ in } M, \end{aligned}$$

and existence and uniqueness of the solution B_{p+1} can be obtained. In this case, of course, the Dirichlet integral is

$$(\bar{\delta}B_{p+1}, \bar{\delta}B_{p+1}).$$

4. **Green's form, Neumann's form, and the kernel form.** The existence theorems of the previous section show that we can define a pure Green's p-form $G_p(z, t)$ and a pure Neumann's p-form $N_p(z, t)$ in M as follows. In M

$$G_p(z, t) = 2^{-p-1}\sigma_k\bar{\delta}(\omega_{p+1} + \text{regular terms}),$$
$$\bar{\delta}\bar{d}G_p(z, t) = 0,$$

while on ∂M

$$T\zeta G_p(z, t) = 0.$$

In M

$$N_p(z, t) = 2^{-p+1}\sigma_k\bar{d}(\omega_{p-1} + \text{regular terms}),$$
$$\bar{d}\bar{\delta}N_p(z, t) = 0,$$

and on ∂M

$$T * N_p(z, t) = 0.$$

These conditions determine G_p and N_p uniquely.

The regular kernel form

$$K_p(z, t) = \bar{d}G_{p-1}(z, t) + \bar{\delta}N_{p+1}(z, t) \tag{8}$$

clearly satisfies the analyticity relations $\bar{d}K_p = \bar{\delta}K_p = 0$ in M, and if β_p also satisfies $\bar{d}\beta_p = \bar{\delta}\beta_p = 0$, we obtain

$$\begin{aligned} (\beta_p, K_p) &= (\beta_p, \bar{d}G_{p-1}) + (\beta_p, \bar{\delta}N_{p+1}) \\ &= \beta_p + \int_{\partial M} (*\beta_p)(G_{p-1})^- + \frac{1}{2}\int_{\partial M} \beta_p * (N_{p+1})^- \end{aligned}$$

from Green's theorem and the Cauchy formula. Thus K_p has the characteristic reproducing property

$$\beta_p = (\beta_p, K_p)$$

and is symmetric,

$$K_p(z, t) = (K_p(t, z))^- = \kappa_{i_1 \cdots i_p, j_1 j_2 \cdots j_p} dz_{i_1} \cdots dz_{i_p} \cdot d\bar{t}_{j_1} \cdots d\bar{t}_{j_p}.$$

If $\beta_p^{(\nu)}(z)$ is any complete orthogonal system for the class of pure forms β_p with

$$\bar{d}\beta_p = \bar{\delta}\beta_p = 0, \qquad (\beta_p, \beta_p) < \infty,$$

in the sense

$$(\beta_p^{(\nu)}, \beta_p^{(\mu)}) = \delta_{\nu\mu} = \begin{cases} 0, & \nu \neq \mu, \\ 1, & \nu = \mu, \end{cases}$$

then we obtain in the usual manner the Bergman expansion

$$K_p(z, t) = \sum_{\nu=1}^{\infty} \beta_p^{(\nu)}(z)(\beta_p^{(\nu)}(t))^-.$$

In particular, the coefficient

$$\bar{\kappa}_0 = 2^k \kappa_{12 \cdots k, 12 \cdots k}$$

is the well known analytic kernel function of the k complex variables $z_1, \cdots, z_k$ in M [1].

It is of interest to reverse our point of view and obtain the Green's and Neumann's forms G_{p-1}, N_{p+1} from the kernel form K_p. We note that K_p is the form β_p minimizing

$$(\beta_p, \beta_p)$$

for fixed β_p at $z=t$, under the differential restraints $\bar{d}\beta_p = \bar{\delta}\beta_p = 0$. An application of the theory of the problem of Bolza in the calculus of variations tells us, therefore, that there is a Lagrange multiplier λ and two pure Lagrange multiplier forms λ_{p-1} and μ_{p+1} in M such that

$$(K_p, h_p) + \lambda(h_p)^- + (\lambda_{p-1}, \bar{\delta} h_p) + (\mu_{p+1}, \bar{d} h_p) = 0$$

for every variation h_p of K_p. If we take $h_p = \beta_p$ with $\bar{d}\beta_p = \bar{\delta}\beta_p = 0$, the reproducing property of the kernel yields $\lambda = -1$. Green's theorem gives

$$\left(K_p - \frac{1}{2}\bar{d}\lambda_{p-1} - 2\bar{\delta}\mu_{p+1}, h_p\right) + \int_{\partial M} \left[\frac{1}{2}\lambda_{p-1} * (h_p)^- + (*\mu_{p+1})(h_p)^-\right] = 0,$$

and from the arbitrary nature of h_p, we obtain

$$K_p = \frac{1}{2}\bar{d}\lambda_{p-1} + 2\bar{\delta}\mu_{p+1}, \qquad \text{in } M,$$

$$T\zeta\lambda_{p-1} = T * \mu_{p+1} = 0, \qquad \text{on } \partial M.$$

Thus $\lambda_{p-1} = 2G_{p-1}$ and $\mu_{p+1} = (1/2)N_{p+1}$, whence we have an interpretation of the Green's and Neumann's forms as Lagrange multipliers for the extremal problem of Bolza type which characterizes the kernel K_p. This formal procedure serves to motivate our earlier discussion of boundary value problems involving p-forms.

We turn next to the representation of the solution of the boundary value problems in terms of G_p and N_p. We shall confine ourselves to the Dirichlet problem as solved by the Green's form.

By definition of G_p, there is a form

$$\Gamma_{p+1} = 2^{-p-1}\sigma_k\omega_{p+1} + \text{regular terms}$$

in M such that $G_p = \bar{\delta}\Gamma_{p+1}$. Γ_{p+1} is uniquely determined by the requirement

$$(\Gamma_{p+1}, h_{p+1}) = 0$$

for every h_{p+1} such that $\bar{\delta}h_{p+1} = 0$. If $\bar{\delta}\bar{d}A_p = 0$ in M, and if $A_p = \bar{\delta}Q_{p+1}$ for a uniquely determined form Q_{p+1} with

$$(Q_{p+1}, h_{p+1}) = 0$$

whenever $\bar{\delta}h_{p+1} = 0$, then we obtain

$$\begin{aligned}
&\int_{\partial M} A_p * (\Gamma_{p+1})^- - 2^{-p-2}\sigma_k \int_{\partial M} Q_{p+1} * (\bar{d}\omega_{p+1})^- \\
&= \int_{\partial M} A_p * (\Gamma_{p+1})^- - \int_{\partial M} (*Q_{p+1})(\bar{\delta}\Gamma_{p+1})^- - 2^{-p-2}\sigma_k \int_{\partial M} Q_{p+1} * (\bar{d}\omega_{p+1})^- \\
&= (\bar{d}A_p, \Gamma_{p+1}) + 2(\bar{\delta}Q_{p+1}, \bar{\delta}\Gamma_{p+1}) - 2(\bar{\delta}Q_{p+1}, \bar{\delta}\Gamma_{p+1}) \\
&\quad - (Q_{p+1}, \bar{d}\bar{\delta}\Gamma_{p+1}) - 2^{-p-1}\sigma_k(Q_{p+1}, \bar{\delta}\bar{d}\omega_{p+1}) \\
&\quad - 2^{-p-2}\sigma_k(\bar{d}Q_{p+1}, \bar{d}\omega_{p+1}) + Q_{p+1} \\
&= -(Q_{p+1}, \bar{d}\bar{\delta}[\Gamma_{p+1} - 2^{-p-1}\sigma_k\omega_{p+1}]) \\
&\quad + 2^{-p-2}\sigma_k\bar{\delta}(\bar{d}Q_{p+1}, \omega_{p+2}) + Q_{p+1} \\
&= 2^{-p-2}\sigma_k\bar{\delta}(\bar{d}Q_{p+1}, \omega_{p+2}) + Q_{p+1}.
\end{aligned}$$

By application of the operator $\bar{\delta}$ to this identity, we derive the formula

$$A_p = \bar{\delta}\int_{\partial M} A_p * (\Gamma_{p+1})^- \tag{9}$$

for the solution of the Dirichlet problem.

The corresponding formula for the solution of the Neumann problem is

$$B_{p+1} = \frac{1}{2}\bar{d}\int_{\partial M} (*B_{p+1})(H_p)^-, \tag{10}$$

where

$$H_p = 2^{-p}\sigma_k\omega_p + \text{regular terms}$$

is the unique form such that $N_{p+1}=\bar{d}H_p$ and such that

$$(H_p,\, h_p) = 0$$

whenever $\bar{d}h_p=0$.

The symmetry of the Green's form $G_p(z,\, t)$ is obtained in a similar way. We have

$$\int_{\partial M}\bar{\delta}\Gamma_{p+1}*(\Gamma_{p+1})^- - \int_{\partial M}(*\Gamma_{p+1})(\bar{\delta}\Gamma_{p+1})^-$$

$$\begin{aligned} &- 2^{-p-2}\sigma_k\int_{\partial M}\Gamma_{p+1}*(\bar{d}\omega_{p+1})^- + 2^{-p-2}\sigma_k\int_{\partial M}(*\bar{d}\omega_{p+1})(\Gamma_{p+1})^- \\ &= \Gamma_{p+1} - (\Gamma_{p+1})^- + 2^{-p-2}\sigma_k(\bar{d}\omega_{p+1},\, \bar{d}\Gamma_{p+1}) \\ &\quad - 2^{-p-2}\sigma_k(\bar{d}\Gamma_{p+1},\, \bar{d}\omega_{p+1}), \end{aligned}$$

so that application of the operator $\bar{\delta}(z)(\bar{\delta}(t))^-$ yields the rule

$$(\bar{\delta}(t))^- G_p(z,\, t) = \bar{\delta}(z)(G_p(t,\, z))^-.$$

Likewise, by the analogous argument for Neumann's problem,

$$(\bar{d}(t))^- N_p(z,\, t) = \bar{d}(z)(N_p(t,\, z))^-.$$

The boundary value problems can be reformulated as orthogonal decomposition theorems in the Hilbert space L^2 of pure forms $C_p=\bar{\delta}R_{p+1}$ with

$$(\bar{d}C_p,\, \bar{d}C_p) < \infty$$

and the Hilbert space Λ^2 of pure forms $D_p=\bar{d}S_{p-1}$ with

$$(\bar{\delta}D_p,\, \bar{\delta}D_p) < \infty,$$

and a Dirichlet principle can be stated. We introduce for this purpose the subclass of L^2 generated by elements Φ_p such that

$$T\zeta\Theta_p = 0, \qquad \text{on } \partial M,$$

and the subclass of Λ^2 generated by elements Ψ_p such that

$$T*\Psi_p = 0, \qquad \text{on } \partial M.$$

Then for each $A_p\in L^2$ with $\bar{\delta}\bar{d}A_p=0$ in M, we have the orthogonality relation

$$(\bar{d}\Phi_p,\, \bar{d}A_p) = \int_{\partial M}\Phi_p*(\bar{d}A_p)^- = 0,$$

whereas for each $B_p\in\Lambda^2$ with $\bar{d}\bar{\delta}B_p=0$ in M, we obtain

$$(\bar{\delta}\Psi_p,\, \bar{\delta}B_p) = \frac{1}{2}\int_{\partial M}(*\Psi_p)(\bar{\delta}B_p)^- = 0.$$

Thus it follows from our theory of the Dirichlet problem for $\bar{\delta}\bar{d}A_p=0$ and the Neumann problem for $\bar{d}\bar{\delta}B_p=0$ that each $C_p \in L^2$ has a unique orthogonal decomposition

$$C_p = A_p + \Phi_p$$

and each $D_p \in \Lambda^2$ has a unique orthogonal decomposition

$$D_p = B_p + \Psi_p.$$

The Dirichlet principle is an immediate consequence of these decompositions. It states that for given C_p, the form A_p such that $T\zeta A_p = T\zeta C_p$ on ∂M and

$$(\bar{d}A_p, \bar{d}A_p) = \text{minimum}$$

satisfies the partial differential equations $\bar{\delta}\bar{d}A_p=0$. Similarly, for given D_p, the form B_p such that $T * B_p = T * D_p$ on ∂M and

$$(\bar{\delta}B_p, \bar{\delta}B_p) = \text{minimum}$$

satisfies the system of equations $\bar{d}\bar{\delta}B_p=0$.

We close this section with a remark concerning the generalized Cauchy formula

$$\beta_p = -2^{-p}\sigma_k \int_{\partial M} (*\beta_p)(\bar{\delta}\omega_p)^- - 2^{-p-1}\sigma_k \int_{\partial M} \beta_p * (\bar{d}\omega_p)^-,$$

valid for pure forms β_p such that $\bar{d}\beta_p = \bar{\delta}\beta_p = 0$ in M. If one introduces the class of pure forms U_p which satisfy in M the Laplace equation $\Delta U_p=0$, one obtains the identity

$$\int_{\partial M} (*\beta_p)(\bar{\delta}U_p)^- + \frac{1}{2}\int_{\partial M} \beta_p * (\bar{d}U_p)^-$$
$$= 2(\bar{\delta}\beta_p, \bar{\delta}U_p) + \frac{1}{2}(\bar{d}\beta_p, \bar{d}U_p) + (\beta_p, \Delta U_p) = 0,$$

for each U_p, when $\bar{d}\beta_p = \bar{\delta}\beta_p = 0$. Conversely, suppose β_p is defined on ∂M and satisfies

$$\int_{\partial M} (*\beta_p)(\bar{\delta}U_p)^- + \frac{1}{2}\int_{\partial M} \beta_p * (\bar{d}U_p)^- = 0$$

for each U_p with $\Delta U_p = 0$ in M. If we extend β_p into M as a solution of Laplace's equation $\Delta\beta_p=0$, we obtain

$$2(\bar{\delta}\beta_p, \bar{\delta}U_p) + \frac{1}{2}(\bar{d}\beta_p, \bar{d}U_p) = 0.$$

For $U_p=\beta_p$, this yields $\bar{d}\beta_p=\bar{\delta}\beta_p=0$. Thus a necessary and sufficient condition that a pure form β_p defined on ∂M represent the boundary values of a form in M satisfying $\bar{\delta}\beta_p=\bar{d}\beta_p=0$ is that

$$\int_{\partial M}(*\beta_p)(\bar{\delta}U_p)^- + \frac{1}{2}\int_{\partial M}\beta_p*(\bar{d}U_p)^- = 0$$

for each solution U_p of Laplace's equation in M. If the equations $\bar{d}\beta_p=\bar{\delta}\beta_p=0$ are viewed as a generalization of the Cauchy-Riemann equations, this result can be interpreted as a generalized Cauchy theorem.

A similar reasoning shows that if $\Delta U_p=0$ in M and $T\zeta\bar{\delta}U_p=T*\bar{d}U_p=0$ on ∂M, then $\bar{d}U_p=\bar{\delta}U_p=0$ in M.

5. **Special cases and examples.** It is evident that the Dirichlet problem for forms A_p with $\bar{\delta}\bar{d}A_p=0$ is equivalent to the Neumann problem for forms B_{k-p} with $\bar{d}\bar{\delta}B_{k-p}=0$. We indicate here the significance of these problems for some special values of p and k.

For $p=0$, the Dirichlet problem is that for the classical Laplace equation in $2k$ real variables, and our work adds nothing to the known theory. But for $p=k-1$, we obtain the most important case of our theory and develop through integral equations the recently discovered formalism of boundary value problems for the Cauchy-Riemann equations in k complex variables [8]. Indeed, here with forms $\beta_k=\beta_{p+1}$ the equation $\bar{d}\beta_p=0$ is vacuous, while the relation $\bar{\delta}\beta_k=0$ represents precisely the Cauchy-Riemann equations for the single coefficient $b_{i_1}\dots{}_{i_k}$ of

$$\beta_k = b_{i_1\dots i_k}dz_{i_1}\cdots dz_{i_k},$$

which must therefore be analytic in the k complex variables $z_1, \cdots, z_k$. The Neumann's form N_{k+1} is absent, and the Green's form $G_p=G_{k-1}$ can be taken as the natural generalization of the Green's function in one variable, by virtue of the kernel identity

$$K_k = \bar{d}G_{k-1}.$$

An even more direct generalization appears in terms of the form Γ_k with $\bar{\delta}\Gamma_k=G_{k-1}$, and the kernel identity can also be written

$$K_k = \bar{d}\bar{\delta}\Gamma_k = \Delta\Gamma_k.$$

For further discussion the reader is referred to previous papers [9]. However, the formulation of the theory given here has special interest because it introduces the method of the Fredholm integral equation, thus completing the application of the apparatus of elliptic partial differential equations to the study of functions of several complex variables.

Since we are using the notation of the exterior differential calculus of E. Cartan, it is worthwhile to compare at this point the complex operators $\bar{d}$, $\bar{\delta}$ and Hodge's operators d, δ. The distinct advantage of our operators is that they yield not only Laplace's equation when $p=0$, but also the Cauchy-Riemann equations for k complex variables when $p=k$. Note, however, that Hodge's harmonic forms have applications in magnetostatics for three dimensions, etc. It is, of course, true that all our results have a direct analogue for Hodge's forms in k-dimensional space. In particular, the kernel form can be introduced in that theory and the method of kernel orthogonal projection goes through, but it seems unnecessary to mention details here, since the case of the complex operators $\bar{d}$, $\bar{\delta}$ serves to present the leading ideas.

It is of particular interest to write down the fundamental integral equation (2) when $p=0$, $k=1$, the classical case. The boundary ∂M is a curve in the complex z-plane with unit tangent $z'(s)$, and we obtain

$$\frac{1}{2\pi i}\oint_{\partial M} \frac{\phi(z)dz}{z-t} + \frac{1}{2}\phi(t) = \theta(t) \tag{11}$$

for the interior Dirichlet problem. Such singular integral equations of Hilbert type have been amply treated in the literature [14]. The homogeneous transposed equation is

$$\frac{(z'(s))^-}{2\pi i}\oint_{\partial M} \frac{\psi(t)ds}{\bar{t}-\bar{z}} + \frac{1}{2}\psi(z) = 0,$$

which has eigenfunctions of the form $\psi(z)=(g(z)z'(s))^-$, with $g(z)$ analytic in M. Thus one obtains from a discussion of this integral equation the orthogonal decomposition

$$\theta(z) = f(z) + (g(z)z'(s))^- \tag{12}$$

of any square integrable function $\theta(z)$, defined on the curve ∂M, in terms of functions $f(z)$ and $g(z)$ analytic in M. This decomposition is basic in a discussion of the Szegö kernel function and was treated originally by an alternative method of conformal mapping [10; 16].

An intermediate case of the generalized Cauchy-Riemann equations $\bar{d}\beta_{p+1}=\bar{\delta}\beta_{p+1}=0$ which has special importance occurs when $p=0$, $k=2$. Here we have

$$\beta_1 = a_1dz_1 + a_2dz_2$$

and the equations $\bar{d}\beta_1=\bar{\delta}\beta_1=0$ can be written

$$\frac{\partial a_1}{\partial \bar{z}_1} = -\frac{\partial a_2}{\partial \bar{z}_2}, \qquad \frac{\partial a_1}{\partial z_2} = \frac{\partial a_2}{\partial z_1}.$$

These will be recognized as the equations of Fueter [6; 7] for quaternion functions. Thus we obtain as a particular case his theory, which was the first to center about a Cauchy formula of the type given in the present work.

Returning to the original boundary value problem

$$T\zeta A_p = \theta_{k+p}, \qquad \text{given on } \partial M,$$
$$\bar{\delta}\bar{d}A_p = 0, \qquad \text{in } M,$$
$$A_p = \bar{\delta}Q_{p+1}, \qquad \text{in } M,$$

we note that it is precisely the last, less usual, condition which makes the solution unique, with coefficients which are analytic as functions of real variables. Thus we might say that this condition gives a strongly elliptic system of partial differential equations, whereas without this condition the problem is only weakly elliptic. One has, in fact,

$$\Delta A_p = \bar{\delta}\bar{d}A_p + \bar{d}\bar{\delta}\bar{\delta}Q_{p+1} = 0,$$

so that the coefficients of A_p are harmonic functions, and our boundary value problem can be formulated as a question of the determination of a system of harmonic functions satisfying certain differential relations and correspondingly fewer boundary conditions than usual.

We shall discuss here briefly the weakly elliptic system

$$\bar{\delta}\bar{d}A_p = 0$$

for the case $p=1$, $k=2$ in a product region M, and we shall prove that a solution can be found for prescribed A_p on ∂M, and not just for prescribed $T\zeta A_p$. The solution is not, however, unique or regular, since any form $A_p = \bar{d}Q_{p-1}$ satisfies the equation. Our result is not restricted, of course, to $k=2$, but holds in general for $p=k-1$, as can be shown by a somewhat more involved reasoning.

To solve $\bar{\delta}\bar{d}A_1=0$ for $k=2$, we recall that the method of orthogonal projection yields for given pure C_1 in the product region M a form

$$E_2 = -2^{-2}\sigma_2(\bar{d}C_1 - \alpha_2, \omega_2),$$

with $\bar{d}E_2=0$ outside M, and with $\bar{\delta}\bar{d}\bar{\delta}E_2=\bar{\delta}\bar{d}C_1$ in M. Since $k=2$, the relation $\bar{d}E_2=0$ outside M is trivial, and also $\bar{d}\bar{\delta}E_2=0$ is immediate. If we can construct outside M a form F_0 such that $\bar{d}F_0=\bar{\delta}E_2$, then we can find the desired form A_1. For we can extend F_0 smoothly into M, and we can set

$$A_1 = C_1 - \bar{\delta}E_2 + \bar{d}F_0.$$

Then

$$\bar{\delta}\bar{d}A_1 = \delta\bar{d}C_1 - \bar{\delta}\bar{d}\bar{\delta}E_2 + \bar{\delta}\bar{d}\bar{d}F_0 = 0$$

in M, and $A_1=C_1$ on ∂M by the continuity of $\bar{\delta}E_2$ and $\bar{d}F_0$ across ∂M, since $\bar{\delta}E_2=\bar{d}F_0$ outside M.

We turn, then, to the construction of F_0, a question related to the problem of Cousin for analytic functions of two complex variables. We note that

$$\bar{\delta}E_2 = 2^{-2}\sigma_2(\bar{d}C_1 - \alpha_2, \bar{d}\omega_1),$$

where

$$\bar{d}\omega_1 = -\frac{\partial r^{-2}}{\partial t_2} dt_1 dt_2 d\bar{z}_1 + \frac{\partial r^{-2}}{\partial t_1} dt_1 dt_2 d\bar{z}_2.$$

But

$$\frac{\partial r^{-2}}{\partial t_2} = -\frac{\partial r^{-2}}{\partial z_2} = \frac{\partial}{\partial \bar{z}_1}\left\{\lambda r^{-2}\frac{\bar{z}_1 - \bar{t}_1}{z_2 - t_2} + (\lambda - 1)r^{-2}\frac{\bar{z}_2 - \bar{t}_2}{z_1 - t_1}\right\},$$

$$\frac{\partial r^{-2}}{\partial t_1} = -\frac{\partial r^{-2}}{\partial z_1} = -\frac{\partial}{\partial \bar{z}_2}\left\{\lambda r^{-2}\frac{\bar{z}_1 - \bar{t}_1}{z_2 - t_2} + (\lambda - 1)r^{-2}\frac{\bar{z}_2 - \bar{t}_2}{z_1 - t_1}\right\}$$

for any λ, and hence

$$\bar{d}(t)\omega_1(t, z) = -(\bar{d}(z))^{-}\left\{\lambda r^{-2}\frac{\bar{z}_1 - \bar{t}_1}{z_2 - t_2} + (\lambda - 1)r^{-2}\frac{\bar{z}_2 - \bar{t}_2}{z_1 - t_1}\right\} dt_1 dt_2.$$

Thus,

$$\bar{\delta}E_2 = -\bar{d}2^{-2}\sigma_2\left(\bar{d}C_1 - \alpha_2, \left\{\lambda r^{-2}\frac{\bar{z}_1 - \bar{t}_1}{z_2 - t_2} + (\lambda - 1)r^{-2}\frac{\bar{z}_2 - \bar{t}_2}{z_1 - t_1}\right\} dt_1 dt_2\right).$$

The coefficient of λ depends only on the scalar product

$$\left(\bar{d}C_1 - \alpha_2, \frac{dt_1 dt_2}{(z_1 - t_1)(z_2 - t_2)}\right),$$

which can be shown to vanish by a variational argument when z_1 is not in the projection M_1 of M on the z_1-plane and z_2 is not in the projection M_2 of M on the z_2-plane, by virtue of the extremal property of α_2. Therefore, we can take

$$F_0 = -2^{-2}\sigma_2\left(\bar{d}C_1 - \alpha_2, \left\{\lambda r^{-2}\frac{\bar{z}_1 - \bar{t}_1}{z_2 - t_2} + (\lambda - 1)r^{-2}\frac{\bar{z}_2 - \bar{t}_2}{z_1 - t_1}\right\} dt_1 dt_2\right),$$

with $\lambda = 1$ when z_1 is in M_1 and $\lambda = 0$ when z_2 is in M_2, and with arbitrary λ in all other cases. This gives, indeed, a uniquely defined function F_0 outside M, since F_0 is independent of λ when z_1 is outside M_1 and z_2 is outside M_2. Thus our discussion of the boundary value problem is completed.

An analogous argument could be given when M is no longer a product region if it were true that $\bar{d}\phi_p = 0$ implies $\phi_p = \bar{d}\psi_{p-1}$ in the large. However, the example $\phi_p = \bar{\delta}\omega_k$ shows that this is not the case.

References

1. S. Bergman, *Sur les fonctions orthogonales de plusieurs variables complexes*, Mémorial des Sciences Mathématiques, vol. 106, Paris, 1947.

2. ———, *The kernel function and conformal mapping*, Mathematical Surveys, no. 5, New York, American Mathematical Society, 1950.

3. S. Bergman and M. Schiffer, *A representation of Green's and Neumann's functions in the theory of partial differential equations of second order*, Duke Math. J. vol. 14 (1947) pp. 609–638.

4. P. Bidal and G. de Rham, *Les formes differentielles harmoniques*, Comment. Math. Helv. vol. 19 (1946) pp. 1–49.

5. S. Bochner, *Analytic and meromorphic continuation by means of Green's formula*, Ann. of Math. vol. 44 (1943) pp. 652–673.

6. R. Fueter, *Über einen Hartogs'schen Satz*, Comment. Math. Helv. vol. 12 (1939) pp. 75–80.

7. ———, *Über einen Hartogs'schen Satz in der Theorie der analytischen Funktionen von n komplexen Variablen*, Comment. Math. Helv. vol. 14 (1942) pp. 394–400.

8. P. R. Garabedian, *A new formalism for functions of several complex variables*, Journal d'Analyse Mathématique vol. 1 (1951) pp. 59–80.

9. ———, *A Green's function in the theory of functions of several complex variables*, Ann. of Math. vol. 55 (1952) pp. 19–33.

10. P. R. Garabedian and M. Schiffer, *On existence theorems of potential theory and conformal mapping*, Ann. of Math. vol. 52 (1950) pp. 164–187.

11. G. Giraud, *Équations et systèmes d'équations où figurent des valeurs principales d'intégrales*, C. R. Acad. Sci. Paris vol. 204 (1937) pp. 628–630.

12. W. V. D. Hodge, *A Dirichlet problem for harmonic functionals, with applications to analytic varieties*, Proc. London Math. Soc. vol. 36 (1934) pp. 257–303.

13. ———, *The theory and applications of harmonic integrals*, Cambridge, 1941.

14. F. Noether, *Über eine Klasse singulärer Integralgleichungen*, Math. Ann. vol. 82 (1921) pp. 42–63.

15. M. Schiffer, *The kernel function of an orthonormal system*, Duke Math. J. vol. 13 (1946) pp. 529–540.

16. ———, *Various types of orthogonalization*, Duke Math. J. vol. 17 (1950) pp. 329–366.

17. W. Wirtinger, *Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen*, Math. Ann. vol. 97 (1927) pp. 357–375.

Stanford University,
Stanford, Calif.
Princeton University,
Princeton, N. J.