

INVESTIGATIONS IN HARMONIC ANALYSIS⁽¹⁾

BY

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This paper is concerned with the theory of ideals in the algebra L^1 of integrable functions on a locally compact abelian group.

After some preliminaries an analytical proof is given of the known theorem that an analytic function of a Fourier transform represents again a Fourier transform (p. 406). Then, in part I, the continuous homomorphisms of closed ideals I of L^1 upon C , the field of complex numbers, are studied. Any such homomorphism is given by a Fourier transform and, if I_0 is its kernel, the quotient-algebra I/I_0 , normed in the usual way, is not only algebraically isomorphic, but also isometric with C (Theorem 1.2). Another result states that homomorphic groups have homomorphic L^1 -algebras and that a corresponding property of isometry holds (Theorem 1.3).

In part II, which may be read independently of part I, a theorem of S. Mandelbrojt and S. Agmon, which generalizes Wiener's theorem on the translates of a function in L^1 , is extended to groups (Theorem 2.2). Several generalizations of Wiener's classical theorem have been published in the past few years; references to the literature are given on p. 422. The rest of part II is devoted to some applications (pp. 422–425).

In conclusion it should be said that the work is carried out in abstract generality, with the methods, and in the spirit, of analysis, which is then applied to algebra.

To Professors S. Mandelbrojt, B. L. van der Waerden, and A. Weil I owe my mathematical education. The inspiration which I have received in their lectures, in letters, and above all in personal contact, is at the base of this work; may I here express my gratitude.

BASIC CONCEPTS AND RESULTS. NOTATION

Groups and Fourier transforms. The standard work of reference is A. Weil's book [14]; cf. also [4].

For the definition of a topological group see [14, p. 9]. The additive group of real numbers (the "real axis") is denoted by \mathbf{R}^1 , p -dimensional space by \mathbf{R}^p . The group operation will always be denoted multiplicatively (it is hoped that this will not cause confusion in the case of \mathbf{R}^1). The identity (unit element) of the group is denoted by e .

We shall deal only with locally compact abelian groups G . By G we denote

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the dual group, by (x, x) the value for $x \in G$ of the character defined by $x \in G$. For R^1 the symbol (x, x) may be regarded simply as an abbreviation of $\exp(2\pi i x x)$ and R^1 is the same as R^1 .

The measure of a set $S \subset G$ ($S \subset G$) is denoted by $m(S)$ ($m(S)$). L^1, L^2, L^∞ denote, respectively, the space of complex-valued functions on G which are integrable, of integrable square, measurable, and essentially bounded. The last notation, a slight departure from Weil's, is used in [4]. Functions in L^1, L^2, L^∞ will be denoted, respectively, by $f(x), \dots, F(x), \dots, \phi(x), \dots$. We use a dot to indicate the conjugate of a complex number and we write $f^*(x), \phi^*(x)$ for $f^*(x^{-1}), \phi^*(x^{-1})$.

The Faltung of two functions in L^1 is

$$f * g = f * g(x) = \int f(y)g(xy^{-1})dy.$$

We shall also use the Faltung $f * \phi(x)$, $\phi \in L^\infty$. Sometimes we shall write $f(x) * \phi$ or even $f(x) * \phi(x)$, according to convenience. If $\phi_1 = f * \phi$, then $\phi_1^* = f^* * \phi^*$.

The Fourier transform of $f \in L^1$ is⁽²⁾

$$f(x) = \int f(x)(x, x)^{\cdot} dx.$$

This is the definition adopted in [4, p. 87] rather than that used by Weil [14, p. 112]—the difference is irrelevant. For $G = R^1$ it represents a slight, but convenient, deviation from the classical definition.

Banach spaces. For the definition and properties of Banach spaces B over the field of complex numbers, and the notation, we refer to Hille's book [8].

L^1 and L^∞ are Banach spaces, the norms being, respectively,

$$\|f\|_1 = \int |f(x)| dx, \quad \|\phi\|_\infty = \text{ess. sup } |\phi(x)| \quad (x \in G).$$

The dual space of L^1 is L^∞ ; we shall write the general bounded linear functional on L^1 in the form

$$\int f(x)\phi^*(x)dx.$$

The following "distance theorem" is a reformulation of a well known result [8, Theorem 2.9.4]:

Let L be a closed linear subspace of B and define for $x_1 \in B$

⁽²⁾ Thus a function in L^1 is indicated by a letter in boldface type and its Fourier transform by the same letter in italic type. This notation, considerably different from that used by French authors, was dictated by technical necessities of printing. Likewise for the use of the dot instead of the customary bar.

$$\text{dist } \{x_1, L\} = \inf \|x_1 - x_0\| \quad (x_0 \in L).$$

If there is no bounded linear functional $x^*(x)$ satisfying

$$(*) \quad \begin{aligned} x^*(x_1) &= 1, \\ x^*(x_0) &= 0 \end{aligned} \quad \text{for all } x_0 \in L,$$

then $x_1 \in L$.

If there are bounded linear functionals satisfying condition (*), then

$$\text{dist } \{x_1, L\} = 1/\min \|x^*\|,$$

where x^* ranges over all these functionals.

This is an immediate consequence of Theorem 2.9.4 in [8] and the fact that if x^* satisfies (*) then $x^*(x_1 - x_0) = 1$, and hence $\|x^*\| \geq 1/\|x_1 - x_0\|$.

This theorem will be applied, in the space L^1 , to calculate the distance in some cases (cf. [3, chap. III]).

The function $\phi(x)$ is called orthogonal to $f(x)$: $\phi \perp f$, if $\int f(x)\phi^*(x)dx = 0$. It is called orthogonal to the linear subspace $I \subset L^1$: $\phi \perp I$, if $\phi \perp f$ for all $f \in I$. If $\phi \perp L^1$, then ϕ vanishes almost everywhere.

The L^1 -algebra. L^1 is not only a Banach space, but also a commutative Banach algebra [8], multiplication being defined by the Faltung.

A (closed) ideal in the algebra L^1 is defined as a (closed) linear subspace of L^1 which contains, with a function f , all products $f * g$, for any $g \in L^1$. Closed ideals will be denoted by I .

A closed linear subspace is an ideal if and only if it is "invariant under translation," i.e., contains, with a function $f(x)$, all "translates" $f(xy^{-1})$, for any (fixed) $y \in G$. For the sufficiency see [6, p. 127]. The necessity is proved here, for the sake of completeness.

Consider any $f \in I$ and any $g \in L^1$. Then, by hypothesis, $f * g \in I$ and hence if $\int f(x)\phi^*(x)dx = 0$ for all $f \in I$, then

$$\int \phi^*(x)dx \int f(xy^{-1})g(y)dy = 0$$

or

$$\int g(y)dy \int f(xy^{-1})\phi^*(x)dx = 0.$$

Since $g(y)$ is arbitrary, it follows that

$$\int f(xy^{-1})\phi^*(x)dx = 0.$$

Since this holds for any $\phi \perp I$, it follows that $f(xy^{-1}) \in I$, for any (fixed) $y \in G$.

The intersection of the zeros of the Fourier transforms of all functions in

I , which is a closed set in G , is called the co-spectrum of I by L. Schwartz [12]. We shall denote it by Z_I .

By $I(\mathbf{x}, x)^*$ we shall mean the "rotated" ideal obtained by replacing all $f(\mathbf{x}) \in I$ by $f(\mathbf{x})(\mathbf{x}, x)^*$, x being any (fixed) element of G .

PRELIMINARY CONSIDERATIONS

Let S be a compact, symmetrical neighborhood of the identity $e \in G$ and denote by $S(x)$ the characteristic function of S :

$$\begin{aligned} S(x) &= 1, & x \in S, \\ S(x) &= 0, & x \in \mathcal{C}S. \end{aligned}$$

This will be used as a standard notation throughout, and neighborhoods S, S', S_n , etc., will always be supposed to have the above properties (without explicit mention).

Define now

$$\mathbf{S}(\mathbf{x}) = \int S(x)(\mathbf{x}, x) dx.$$

Since $S^{-1} = S$, $\mathbf{S}(\mathbf{x})$ is real and $\mathbf{S}(\mathbf{x}^{-1}) = \mathbf{S}(\mathbf{x})$. If $G = \mathbf{R}^1$ and $S = [-\epsilon/2, \epsilon/2]$, $\mathbf{S}(\mathbf{x}) = (\sin \pi \epsilon \mathbf{x}) / \pi \mathbf{x}$; if G is compact and $S = \{e\}$, then $\mathbf{S}(\mathbf{x}) = 1$.

Consider now the Faltung

$$S * S(x) = m(xS \cap S).$$

By the theorem of Plancherel-Weil

$$S * S(x) = \int \mathbf{S}(\mathbf{x})^2(\mathbf{x}, x)^* d\mathbf{x}.$$

In the applications we shall "normalize" the above Faltung by means of the factor $1/m(S)$. The function thus obtained takes values between 0 and 1; it is 1 for $x=e$ and vanishes outside the neighborhood S^2 .

The function $\mathbf{S}(\mathbf{x})^2$ is non-negative and has the following properties⁽³⁾:

$$(i) \quad m^{-1}(S) \int \mathbf{S}(\mathbf{x})^2 d\mathbf{x} = m^{-1}(S) \|\mathbf{S}(\mathbf{x})^2\|_1 = (m^{-1/2}(S) \|S(x)\|_2)^2 = 1.$$

This follows immediately from the theorem of Plancherel-Weil, applied to $\mathbf{S}(\mathbf{x})$ and $S(x)$.

Given $\epsilon > 0$, for any (fixed) $\mathbf{y} \in G$

$$(ii) \quad m^{-1}(S) \|\mathbf{S}(\mathbf{y}^{-1}\mathbf{x})^2 - \mathbf{S}(\mathbf{x})^2\|_1 < \epsilon,$$

⁽³⁾ We shall denote the norm in L^2 (and in the corresponding space L^2) by $\|\mathbf{F}\|_2 = \{\int |\mathbf{F}(\mathbf{x})|^2 d\mathbf{x}\}^{1/2}$.

provided S is small ($S \subset U$, a neighborhood of e depending on ϵ and y).

By Schwarz's inequality

$$m^{-1}(S) \left\| \{S(y^{-1}x) + S(x)\} \{S(y^{-1}x) - S(x)\} \right\|_1 \\ \leq m^{-1}(S) \|S(y^{-1}x) + S(x)\|_2 \|S(y^{-1}x) - S(x)\|_2.$$

By the theorem of Plancherel-Weil this is equal to

$$m^{-1/2}(S) \|S(x) \{(y, x)^{\bullet} + 1\}\|_2 m^{-1/2}(S) \|S(x) \{(y, x)^{\bullet} - 1\}\|_2 \\ \leq 2 \left(m^{-1}(S) \int_S |(y, x)^{\bullet} - 1|^2 dx \right)^{1/2} < \epsilon,$$

provided $S \subset U$, where the neighborhood U of e is such that $|(y, x) - 1| < \epsilon/2$ for all $x \in U$ and the given y .

Take now two neighborhoods S and S' (cf. the convention concerning the notation on p. 404). Then the product SS' is a neighborhood of the same kind.

Consider the characteristic functions

$$S(x) \quad \text{and} \quad SS'(x)$$

(the use of two letters to denote a function is perhaps somewhat unusual, but convenient here). The normalized Faltung of these functions,

$$m^{-1}(S) S * SS'(x) = m^{-1}(S) m(xS \cap SS'),$$

takes values between 0 and 1; it is 1 for $x \in S'$, vanishes outside $S^2 S'$, and is the Fourier transform of

$$m^{-1}(S) S(x) \cdot SS'(x).$$

We shall need later the following estimate:

$$(iii) \quad \|S(x) SS'(x) - S(x)^2\|_1 \leq \{m(SS') - m(S)\}^{1/2} m^{1/2}(S).$$

Indeed,

$$\|S(x) \{SS'(x) - S(x)\}\|_1 \leq \|S(x)\|_2 \|SS'(x) - S(x)\|_2 = \\ \|S(x)\|_2 \|SS'(x) - S(x)\|_2 = m^{1/2}(S) \{m(SS') - m(S)\}^{1/2}.$$

We shall also need a stronger form of property (ii): Given any (fixed) compact set $C \subset G$, then for any $\epsilon > 0$

$$(ii') \quad m^{-1}(S) \|S(y^{-1}x)^2 - S(x)^2\|_1 < \epsilon$$

for all $y \in C$, provided S is small ($S \subset U$, a neighborhood of e which depends on ϵ and C).

Referring to the proof of (ii), it is required to show that there exists a neighborhood U of e such that $|(y, x) - 1| < \epsilon/2$ for all $y \in C$ and all $x \in U$ (this is obvious in the case $G = \mathbf{R}^1$; in the general case it is precisely the defini-

tion of the topology of the dual group given by Pontrjagin).

The character (y, x) is a continuous function on $G \times G$ [14, p. 101]. Thus, given $\epsilon > 0$, for each point $y_i \in C$ there are neighborhoods V_i of y_i and U_i of e such that $|(y, x) - 1| = |(y, x) - (y_i, e)| < \epsilon/2$ for all $y \in V_i$ and all $x \in U_i$. Since C is compact, a finite number of the neighborhoods V_i already cover C , and we may take for U any neighborhood of e contained in the intersection of the corresponding neighborhoods U_i . Thus (ii') is proved⁽⁴⁾.

Another estimate to be applied later is the following:

Let $C \subset G$ be a compact set and let $\epsilon > 0$ be given. Then

$$(iv) \quad m^{-1}(S) \|S(y^{-1}x)SS'(y^{-1}x) - S(x)SS'(x)\|_1 < \epsilon(m(SS')/m(S))^{1/2}$$

for all $y \in C$, provided both S and S' are chosen small (namely, such that $SS' \subset U$, where U is some neighborhood of e depending on ϵ and C).

The proof is analogous to that of properties (ii) and (ii'), U being such that $|(y, x) - 1| < \epsilon/2$ for all $y \in C$, $x \in U$.

We also have

$$(v) \quad m^{-1}(S) \|S(x)SS'(x)\|_1 \leq (m(SS')/m(S))^{1/2}.$$

ANALYTIC FUNCTIONS OF FOURIER TRANSFORMS

THEOREM. Let $f(x)$ be a function in L^1 and let $C \subset G$ be a compact set. Denote by Γ the set of points z , in the complex plane, of the form $z = f(x)$, $x \in C$. Let now $A(z)$ be a (single-valued) analytic function, regular at all points of Γ . Then there is a function in L^1 whose Fourier transform is equal to $A(f(x))$ for all $x \in C$.

This theorem is proved in two steps: first it is proved locally, i.e., for a sufficiently small neighborhood of any (fixed) point $x_0 \in C$; then the extension to the whole set C is carried out. The proof is modelled after that given by Carleman for the case $G = \mathbf{R}^1$ [3, chap. IV]⁽⁵⁾.

Let x_0 be any (fixed) point of C and expand $A(z)$ in a power series about $z_0 = f(x_0)$. We may write, for x in some (small) neighborhood of x_0 .

$$A(f(x)) = A(f(x_0)) + \sum_{n=1}^{\infty} c_n (f(x) - f(x_0))^n.$$

Consider now the Fourier transform (cf. p. 405)

$$s(x) = m^{-1}(S)S(x) * SS'(x).$$

If both S and S' are small, the function

$$s(x_0^{-1}x)A(f(x_0)) + \sum_{n=1}^{\infty} c_n \{s(x_0^{-1}x)(f(x) - f(x_0))^n\}^n$$

⁽⁴⁾ The reader may now turn immediately to part I (p. 409) or part II (p. 417).

⁽⁵⁾ Professor A. Weil has directed my attention to Segal's paper [13]. Theorem 3.8 in [13] practically contains, as a special case, the theorem given here.

exists for all $x \in G$ and coincides with $A(f(x))$ for $x \in x_0 S'$.

The product

$$s(x_0^{-1}x)(f(x) - f(x_0))$$

is the Fourier transform of

$$q(x) = [s(x)(x, x_0)] * f(x) - s(x)(x, x_0)f(x_0)$$

where

$$s(x) = m^{-1}(S)S(x)SS'(x).$$

Write now

$$\begin{aligned} q(x) &= \int s(xy^{-1})(xy^{-1}, x_0)f(y)dy - s(x)(x, x_0) \int f(y)(y, x_0)dy \\ &= \int (xy^{-1}, x_0) \{s(xy^{-1}) - s(x)\} f(y)dy. \end{aligned}$$

Hence

$$\|q(x)\|_1 \leq \int |f(y)| dy \int |s(xy^{-1}) - s(x)| dx.$$

Given $\epsilon > 0$, let $K \subset G$ be a compact set such that

$$\int_{C_K} |f(y)| dy < \epsilon.$$

By property (iv), pp. 406-408, the neighborhoods S, S' may also be so chosen that

$$\|s(y^{-1}x) - s(x)\|_1 < \epsilon(m(SS')/m(S))^{1/2}$$

for all $y \in K$. Keeping S fixed, we may take S' so small that $m(SS') < 2m(S)^{(6)}$.

Under these conditions

$$\begin{aligned} \|q(x)\|_1 &\leq 2^{1/2}\epsilon \int_K |f(y)| dy + 2\|s(x)\|_1\epsilon \\ &\leq \epsilon \{2^{1/2}\|f\|_1 + 2(2)^{1/2}\} \end{aligned} \quad (\text{cf. p. 406}).$$

Define now

$$q_n(x) = q * q_{n-1}(x), \quad n > 1, \quad q_1(x) = q(x).$$

Then

⁽⁶⁾ This follows from the regularity of Haar measure (cf. [7]) and the fact that for any open set O containing S there is an S' such that $SS' \subset O$.

$$\|q_n\|_1 \leq \|q\|_1^n, \quad n \geq 1.$$

By a suitable choice of the neighborhoods involved, we can thus obtain a function $q(x)$ such that $\|q\|_1 < R$, where R is the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n z^n$. Then the series

$$(x, x_0)s(x)A(f(x_0)) + \sum_{n=1}^{\infty} c_n q_n(x)$$

converges in L^1 -norm and its Fourier transform coincides with $A(f(x))$ for all $x \in x_0 S'$.

The theorem has now been proved locally; to extend it to the set C as a whole, we proceed as follows:

To every point $x_i \in C$ there corresponds a (small) neighborhood $x_i V_i$, V_i being a neighborhood of e , such that there exists a function $f_i \in L^1$ whose Fourier transform $f_i(x)$ coincides with $A(f(x))$ for all $x \in x_i V_i^3$. The reason for considering first $x_i V_i$ and then $x_i V_i^3$ will appear shortly. Since C is compact, a finite number of those neighborhoods, say $x_1 V_1, x_2 V_2, \dots, x_N V_N$, already cover C :

$$C \subset \bigcup_{r=1}^N x_r V_r.$$

Take a neighborhood S so small that $S \subset V_r$ ($1 \leq r \leq N$); then

$$CS \subset \bigcup_{r=1}^N x_r V_r^2.$$

Define now a partition of the set CS as follows:

$$CS_1 = CS \cap x_1 V_1^2, \quad CS_n = CS \cap x_n V_n^2 \cap \bigcap_{r=1}^{n-1} CS_r \quad (2 \leq n \leq N).$$

Thus

$$CS_r \subset x_r V_r^2, \quad CS_r S \subset x_r V_r^3 \quad (1 \leq r \leq N).$$

Consider the functions $f_r(x) \in L^1$ ($1 \leq r \leq N$) whose Fourier transforms $f_r(x)$ coincide "locally" with $A(f(x))$, i.e., for $x \in x_r V_r^3$, respectively.

Let $CS_r(x)$ be the characteristic function of the set CS_r ($1 \leq r \leq N$). Then the Faltung

$$m^{-1}(S)S(x) * CS_r(x)$$

is the Fourier transform of

$$m^{-1}(S)S(x) \cdot CS_r(x)$$

(the notation being the usual), and vanishes outside $CS_r S$. Moreover the sum

$$\sum_{r=1}^N m^{-1}(S)S(x) * CS_r(x)$$

is just

$$m^{-1}(S)S(x) * CS(x) = m^{-1}(S)m(xS \cap CS)$$

and hence is 1 for $x \in C$.

Now the Fourier transform

$$\Sigma(x) = \sum_{r=1}^N f_r(x) [m^{-1}(S)S(x) * CS_r(x)]$$

coincides with $A(f(x))$ for all $x \in C$.

Indeed, if $x \in C$, then in the product

$$f_r(x) [m^{-1}(S)S(x) * CS_r(x)]$$

either the Faltung in the bracket vanishes or else, if it does not, $f_r(x)$ coincides with $A(f(x))$, since $CS_r S \subset x_r V_r^3$.

Thus, for all $x \in C$, $\Sigma(x) = A(f(x)) \cdot 1$ and the theorem is proved.

The following particular case should be especially noted:

If $f(x) \in L^1$ is such that $f(x) \neq 0$ on the compact set $C \subset G$, then there is a function $g(x) \in L^1$ such that

$$g(x) = 1/f(x) \quad \text{for all } x \in C.$$

This theorem is at the base of the results that follow. For that reason it seems worthwhile to have a purely analytical proof; other proofs have been given by Godement [6, Théorème A] and Segal [13, Theorem 3.8].

COROLLARY. Let $f(x)$, $g(x)$, be in L^1 . If $f(x)$ vanishes outside some compact set and the zeros of $g(x)$ are all interior to the set of zeros of $f(x)$, then there is a function $h(x) \in L^1$ such that

$$f(x) = g(x) * h(x).$$

I. CLOSED IDEALS IN L^1 AND THEIR HOMOMORPHISMS

LEMMA 1.1.1. If $f \in L^1$ is such that $f(e) \neq 0$, then every solution $\phi_0 \in L^\infty$ of the integral equation

$$f * \phi_0^*(x) = 0$$

has the following property:

$$\text{ess. sup } |\gamma + \phi_0(x)| > |\gamma|,$$

for any constant γ , unless ϕ_0 is zero almost everywhere.

Consider the function $m^{-1}(S)S(x)^2$. If S is small, we have, by the corollary above since $f(x) \neq 0$ in some neighborhood of e ,

$$m^{-1}(S)S(x)^2 = f * h(x),$$

so that

$$m^{-1}(S)S(x)^2 * \phi_0^*(x) = 0.$$

Now

$$m^{-1}(S)S(x)^2 * |\gamma + \phi_0(x)|^2 \leq \text{ess. sup } |\gamma + \phi_0(x)|^2.$$

The left-hand side may be written⁽⁷⁾

$$\begin{aligned} m^{-1}(S)S(x)^2 * \{ |\gamma|^2 + 2R[\gamma \cdot \phi_0(x)] + |\phi_0(x)|^2 \} \\ = |\gamma|^2 + 2R[\gamma \cdot m^{-1}(S)S(x)^2 * \phi_0] + m^{-1}(S)S(x)^2 * |\phi_0|^2. \end{aligned}$$

But the middle term vanishes and hence, unless ϕ_0 is zero almost everywhere,

$$|\gamma|^2 < \text{ess. sup } |\gamma + \phi_0(x)|^2,$$

which was to be proved.

LEMMA 1.1.2. *If $f \in L^1$ is such that $f(e) = 0$, then the integral equation*

$$f * \phi^*(x) = 1$$

has no solution $\phi \in L^\infty$.

Suppose there is a $\phi \in L^\infty$ such that

$$\int f(y)\phi^*(x^{-1}y)dy = 1 \quad \text{for all } x \in G.$$

Multiply both sides by $m^{-1}(S)S(x)^2$ and integrate with respect to x :

$$\int m^{-1}(S)S(x)^2 dx \int f(y)\phi^*(x^{-1}y)dy = 1$$

or

$$\int f(y)dy \int m^{-1}(S)S(x)^2 \phi^*(x^{-1}y)dx = 1.$$

Since $\int f(y)dy = 0$, the left-hand side is equal to

$$\begin{aligned} \int f(y)dy \int m^{-1}(S)S(x)^2 \{ \phi^*(x^{-1}y) - \phi^*(x) \} dx \\ = \int f(y)dy \int m^{-1}(S) \{ S(y^{-1}x)^2 - S(x)^2 \} \phi^*(x) dx. \end{aligned}$$

Given $\epsilon > 0$, there is a compact set $C \subset G$ such that

(7) $R[z]$ = real part of z .

$$\int_{\mathcal{C}_C} |f(y)| dy < \epsilon.$$

Furthermore, for any $y \in C$,

$$m^{-1}(S) \|S(y^{-1}x)^2 - S(x)^2\|_1 < \epsilon,$$

provided S is small, by property (ii'), p. 405. Thus the last repeated integral is in absolute value smaller than $\epsilon 2\|\phi\|_\infty + \|\phi\|_1 \epsilon \|\phi\|_\infty$. Since this may be made arbitrarily small, no solution $\phi \in L^\infty$ can exist.

Given a function $f \in L^1$, denote by (f) the closed ideal generated by f , i.e., the smallest closed ideal in L^1 containing f . It may be defined explicitly as the closure of the set of all linear combinations of "translates" of $f(x)$:

$$\sum_{n=1}^N a_n f(y_n x)$$

where a_1, a_2, \dots, a_N are arbitrary complex numbers, y_1, y_2, \dots, y_N are any elements of G , and N is any positive integer (cf. p. 403).

Define now the ideal $(f)_e$ as the closure of the set of all linear combinations

$$\sum_{n=1}^N b_n f(y_n x)$$

where the coefficients b_n satisfy the condition

$$\sum_{n=1}^N b_n = 0.$$

The reason for the notation will appear later.

REMARK. It is important to observe that $(f)_e$ may also be defined as the closure of the set of all linear combinations

$$\sum_{n=1}^N a_n \{f(y_n x) - f(x)\}$$

with arbitrary coefficients a_n .

THEOREM 1.1. For any $f(x) \in L^1$

$$\inf \left| f(x) - \sum_{n=1}^N b_n f(y_n x) \right| dx = \left| \int f(x) dx \right|,$$

where N ranges over all positive integers, b_1, b_2, \dots, b_N range over all complex numbers satisfying $\sum_{n=1}^N b_n = 0$, and y_1, y_2, \dots, y_N over the elements of G .

(If the equality sign is replaced by " \geq ", the statement becomes trivial.)

Making use of the definition of the ideal $(f)_e$, the theorem may be stated

concisely thus:

$$\text{dist } \{f, (f)_e\} = |f(e)|.$$

The proof is an application of the distance theorem (p. 402): we have

$$\text{dist } \{f, (f)_e\} = 1/\min \|\phi\|_\infty,$$

where ϕ ranges over all functions in L^∞ satisfying

$$\int f(y)\phi^*(y)dy = 1,$$

$$\int \{f(xy) - f(y)\}\phi^*(y)dy = 0 \quad (x \in G)$$

(here the remark on p. 411 is used). If no such ϕ exists, then $f \in (f)_e$.

Thus we have to investigate the solutions of the integral equation

$$f * \phi^*(x) = 1.$$

Suppose first that $\gamma = \int f(x)dx \neq 0$. Then every solution may be written in the form

$$\phi(x) = 1/\gamma^* + \phi_0(x),$$

where ϕ_0 satisfies $f * \phi_0^*(x) = 0$. By Lemma 1.1.1 the function $\phi(x) = 1/\gamma^*$ is the solution with the smallest norm and hence

$$\text{dist } \{f, (f)_e\} = |f(e)|.$$

If $\int f(x)dx = 0$, then, by Lemma 1.1.2, there is no $\phi \in L^\infty$ satisfying $f * \phi^*(x) = 1$ and hence $f \in (f)_e$. Thus the proof of the theorem is complete.

Theorem 1.1 may be generalized. Let I be the closed ideal generated by the functions f, g, \dots , i.e., the smallest closed ideal containing $(f), (g), \dots$; we express this by writing

$$I = (f, g, \dots).$$

We define now the ideal I_e as the smallest closed ideal containing $(f)_e, (g)_e, \dots$ and write

$$I_e = (f, g, \dots)_e.$$

Thus I_e is the closure of the set of all finite sums of certain linear combinations (see p. 411) of the translates of f, g, \dots , respectively.

Then for any $k \in I$

$$\text{dist } \{k, I_e\} = |k(e)|.$$

Indeed, since $k \in I$, there are, for any $\epsilon > 0$, among the generating functions f, g, \dots , a finite number, say f_1, f_2, \dots, f_R such that the function

$$k_\epsilon(x) = \sum_{r=1}^R \sum_{s=1}^{Sr} a_{rs} f_r(y_{rs}x)$$

(where the choice of the coefficients a_{rs} and the elements $y_{rs} \in G$ depends on ϵ) satisfies the condition

$$\|k(x) - k_\epsilon(x)\|_1 < \epsilon.$$

Consider now the function k_ϵ and the ideal $(k_\epsilon)_\epsilon$: by Theorem 1.1

$$\text{dist } \{k_\epsilon, (k_\epsilon)_\epsilon\} = \left| \int k_\epsilon(x) dx \right|.$$

From the definition of I_ϵ it is clear that

$$(k_\epsilon)_\epsilon \subset I_\epsilon$$

and since for all $h \in I_\epsilon$

$$\int |k_\epsilon(x) - h(x)| dx \geq \left| \int k_\epsilon(x) dx \right|,$$

it follows that

$$\text{dist } \{k_\epsilon, I_\epsilon\} = \left| \int k_\epsilon(x) dx \right|.$$

This implies that

$$\begin{aligned} \text{dist } \{k, I_\epsilon\} &\leq \|k - k_\epsilon\|_1 + \left| \int k_\epsilon(x) dx \right| \\ &\leq \epsilon + \left| \int k(x) dx \right| + \epsilon. \end{aligned}$$

Now the inequality $\text{dist } \{k, I_\epsilon\} \geq \left| \int k(x) dx \right|$ is again trivial and hence, since $\epsilon > 0$ is arbitrary, the result is proved.

We proceed now to the final generalization of Theorem 1.1.

If $I = (f)$, we define the ideal $I_x = (f)_x$ as the closure of the set of all linear combinations

$$\sum_{n=1}^N b_n(y_n, x) \cdot f(y_n x)$$

where x is a fixed element of G , N is any positive integer, $y_n \in G$ ($1 \leq n \leq N$), and the coefficients satisfy the condition

$$\sum_{n=1}^N b_n = 0.$$

If $I = (f, g, \dots)$, we define $I_x = (f, g, \dots)_x$ as the smallest closed ideal containing $(f)_x, (g)_x, \dots$. We have now (in the notation of p. 404):

$$\text{dist } \{k(x), I_x\} = \text{dist } \{k(x)(x, x)^*, I_x(x, x)^*\}.$$

But the ideal $I_x(x, x)^*$ bears to the ideal $I(x, x)^*$ the relation

$$I_x(x, x)^* = (f(x)(x, x)^*, g(x)(x, x)^*, \dots)_e = I(x, x)_e^*,$$

and hence we may apply the first generalization of Theorem 1.1 to the function $k(x)(x, x)^* \in I(x, x)^*$ and the ideal $I(x, x)_e^*$, so that we finally obtain

$$\text{dist } \{k, I_x\} = |k(x)|.$$

The results obtained may now be summarized as follows:

THEOREM 1.2. *Let I be a closed ideal in L^1 . Define, for a fixed $x \in G$, the ideal $I_x \subset I$ as above. Then for any function $f \in I$*

$$\text{dist } \{f, I_x\} = |f(x)|.$$

Hence I_x is the kernel of the homomorphism

$$f \rightarrow f(x)$$

of I into C , the field of complex numbers. Thus I_x is a maximal ideal of I or coincides with I according as $x \in \mathcal{C}Z_I$ or $x \in Z_I$. Moreover, if the quotient-algebra I/I_x is considered as a Banach algebra⁽⁸⁾, then the isomorphism

$$I/I_x \cong C \quad (x \in \mathcal{C}Z_I)$$

preserves the norm.

Conversely: given a continuous homomorphism

$$f \rightarrow \mu(f)$$

of I upon C , then

$$\mu(f) = \int f(x)(x, x_\mu)^* dx,$$

where $x_\mu \in G$ is a uniquely determined element of $\mathcal{C}Z_I$.

To establish the converse, let s be any (fixed) element of G . It is proved in [6, p. 122] that

$$\mu f(s^{-1}x) = (s, x_\mu)^* \mu(f),$$

where x_μ is a uniquely determined element of G .

Consider now the ideal I_{x_μ} . It is the kernel of the (continuous) homomorphism

(8) See, e.g. [8, Theorem 22.11.4].

$$f \rightarrow \int f(x)(x, x_\mu)^* dx.$$

Let I_μ be the kernel of the given homomorphism $f \rightarrow \mu(f)$. If we can prove that $I_{x_\mu} \subset I_\mu$, it will follow that $I_{x_\mu} = I_\mu$ and hence that $\mu(f) = \int f(x)(x, x_\mu)^* dx$.

Suppose there is a function $g \in I_{x_\mu}$ which is not in I_μ . Then $\mu(g) \neq 0$ and, for all $s \in G$,

$$\mu(g(s^{-1}x)) = (s, x_\mu)^* \mu(g),$$

while

$$\int g(x)(x, x_\mu)^* dx = 0.$$

Now $\mu(f)$, considered as a bounded linear functional on I , may be extended to the whole space L^1 , by the complex analogue of the Hahn-Banach theorem [8, Theorems 2.9.2 and 2.9.5]. Hence

$$\mu(f) = \int f(x) \dot{\phi}_\mu(x) dx \quad (\phi_\mu \in L^\infty).$$

We have therefore for the function $g(x)$:

$$\int g(s^{-1}x) \dot{\phi}_\mu(x) dx = (s, x_\mu)^* \mu(g) \quad \text{for all } s \in G,$$

$$\int g(x)(x, x_\mu)^* dx = 0.$$

If we write this in a slightly different form and apply Lemma 1.1.2, we see that it is impossible. This completes the proof.

THEOREM 1.3. *If the group G' is a homomorphic image of G , then the L^1 -algebra on G' is a homomorphic image of the L^1 -algebra on G . The kernel I_0 of this homomorphism consists of all functions in $L^1(G)$ whose Fourier transforms vanish on the dual group $G' \subset G$, and the isomorphism*

$$L^1(G)/I_0 \cong L^1(G')$$

preserves the norm (the measure on G' being properly normalized).

For any $f(x) \in L^1(G)$, define

$$T(f(x)) = f'(x') = \int_g f(xy) dy,$$

where the closed subgroup $g \subset G$ is such that $G' = G/g$ [14, p. 11], and x' denotes the coset xg . Then $f'(x') \in L^1(G')$ and (with proper normalization of

the measure)

$$\int_G f(x) dx = \int_{G'} f'(x') dx'.$$

This result (immediate for $G = \mathbb{R}^1$) is proved in [1]. We have also

$$\|f\|_1 \geq \|f'\|_1.$$

Furthermore, since $G' = G/\mathfrak{g}$, the dual group G' is contained in G and $(y, x') = 1$ whenever $y \in \mathfrak{g}$, $x' \in G'$ (cf. [14, pp. 108–109]). Thus for $x = x' \in G'$,

$$\begin{aligned} \int_G f(x) (x, x')^* dx &= \int_{G'} dx' \int_{\mathfrak{g}} f(xy) (xy, x')^* dy \\ &= \int_{G'} dx' (x', x')^* \int_{\mathfrak{g}} f(xy) dy \\ &= \int_{G'} f'(x') (x', x')^* dx'. \end{aligned}$$

Since a function in $L^1(G')$ is entirely determined by its Fourier transform, it follows that $T(f(x))$ is determined by the values of $f(x)$ for $x = x' \in G'$. Hence we have

$$T(f_1 * f_2) = T(f_1) * T(f_2),$$

i.e., T is a homomorphism of $L^1(G)$ into $L^1(G')$.

The kernel of this homomorphism is the closed ideal I_0 consisting of all functions in $L^1(G)$ whose Fourier transforms vanish on $G' \subset G$.

We have

$$\inf \|f(x) - f_0(x)\|_1 \geq \|f'(x')\|_1 \quad (f_0 \in I_0).$$

To establish the opposite inequality, we use the distance theorem (cf. p. 402):

$$\text{dist } \{f, I_0\} = 1/\min \|\phi\|_\infty$$

where

$$\phi \perp I_0, \quad \int f(x) \phi^*(x) dx = 1.$$

LEMMA. Any $\phi \perp I_0$ has the property that

$$\phi(sx) = \phi(x) \quad (s \in \mathfrak{g})$$

almost everywhere (and conversely).

For any $k(x) \in L^1(G)$, $k(sx) - k(x) \in I_0$, i.e.,

$$[k(sx) - k(x)] * \phi^*(x) = 0$$

or

$$k(x) * [\phi^*(s^{-1}x) - \phi^*(x)] = 0.$$

Since $k \in L^1(G)$ and $s \in \mathfrak{g}$ are arbitrary, the assertion follows (the converse follows from the fact that $f(x) * \phi^* = f'(x') * \phi'^*$ —cf. below).

Thus the function $\phi \perp I_0$ defines a function $\phi'(x')$ on G' and we have

$$1 = \int f(x) \phi^*(x) dx = \int f'(x') \phi'^*(x') dx'.$$

Hence

$$\|\phi'\|_\infty \|f'\|_1 \geq 1$$

and thus

$$\|\phi(x)\|_\infty \geq 1/\|f'\|_1.$$

This gives the opposite inequality.

Thus the Banach quotient-algebra $L^1(G)/I_0$ is not only algebraically isomorphic, but also isometric with the image of $L^1(G)$ in $L^1(G')$ under the homomorphism T . It remains to show that this image is $L^1(G')$ itself.

Since $L^1(G)/I_0$ is complete, it suffices to show that the functions $T(f)$ are dense in $L^1(G')$. But this follows from a proof given by A. Weil [14, pp. 42–43]. Thus the theorem is established.

II. A GENERALIZATION OF WIENER'S THEOREM, WITH APPLICATIONS

By Wiener's theorem is meant Theorem IV in [15]. It was extended by Godement [6, p. 125] to locally compact abelian groups and reads:

Let I be a closed ideal in L^1 . In order that $I = L^1$, it is necessary and sufficient that for every $x_0 \in G$ there exist an $f(x) \in I$ such that $f(x_0) \neq 0$.

In the course of the proof of Wiener's theorem a result is established which may be stated as follows:

An ideal I contains all functions $f_C(x)$ whose Fourier transforms vanish outside some compact set $C \subset \mathcal{C}Z_I$.

(Actually this is stated here in a slightly more general form which is precisely what we shall need. The proof is that of Godement.)

For any $x_i \in C$ there is a function $f_i \in I$ such that $|f_i(x)| \geq 1$ in some small neighborhood $x_i S_i$ of x_i . Since C is compact, it may be covered by a finite number of those neighborhoods, $x_r S_r$ ($1 \leq r \leq N$). Then the Fourier transform of the function $\sum_{r=1}^N f_r(x) * f_r^*(x) \in I$ is strictly positive on C . If we now apply the corollary, p. 409, to $f_C(x)$ and this function, the result follows.

We shall suppose from now on that G has a *denumerable* fundamental system of neighborhoods of the identity (but cf. Appendix). We may then

choose in G a fundamental sequence (S_n) , $n \geq 1$, of compact, symmetrical neighborhoods of e such that $S_{n+1} \subset S_n$. Then every subsequence (S_{n_r}) , $r \geq 1$, will still be a fundamental system.

LEMMA 2.1.1. *For any function $\phi \in L^\infty$ there exists (at least) one constant M such that, for a certain subsequence (n_r) ,*

$$m^{-1}(S_{n_r})S_{n_r}(x)^2 * \phi \rightarrow M \quad (r \rightarrow \infty),$$

for all $x \in G$.

Since the numbers $m^{-1}(S_n) \int S_n(y)^2 \phi(y) dy$ are all bounded, in absolute value, by $\|\phi\|_\infty$, there exists a subsequence (n_r) such that $m^{-1}(S_{n_r}) \int S_{n_r}(y)^2 \cdot \phi(y) dy$ tends to a limit, M , say, as $r \rightarrow \infty$.

We want to show that for any (fixed) $x \in G$, $m^{-1}(S_{n_r})S_{n_r}(x)^2 * \phi \rightarrow M$ ($r \rightarrow \infty$). It will be sufficient to prove that

$$m^{-1}(S_{n_r}) \int S_{n_r}(y)^2 \{ \phi(xy^{-1}) - \phi(y) \} dy \rightarrow 0 \quad (r \rightarrow \infty)$$

for any (fixed) $x \in G$. Now the integral may be written

$$m^{-1}(S_{n_r}) \int \{ S_{n_r}(x^{-1}y)^2 - S_{n_r}(y)^2 \} \phi(y) dy$$

and this, in absolute value, will be smaller than $\epsilon \|\phi\|_\infty$, by property (ii), p. 404, if n_r is sufficiently large. This completes the proof.

LEMMA 2.1.2. *Suppose that $\phi \perp I$ and that the identity $e \in G$ is not a limiting point of Z_I . Then, if the neighborhood S is small ($S^2 \cap Z_I \cap \mathcal{C}\{e\} = \emptyset$),*

$$m^{-1}(S)S(x)^2 * \phi = \text{const.},$$

the constant being independent of S (subject to the above condition).

REMARK. This is of course trivial if G is compact, since then we may take $S = \{e\}$. See also the Appendix at the end of this paper.

Take a neighborhood S_N of the fundamental sequence of neighborhoods, so small that $S^2 S_N \cap Z_I \cap \mathcal{C}\{e\} = \emptyset$. Then the Fourier transform

$$S * SS_N(x) - S * SS_n(x), \quad n > N,$$

vanishes outside the set $S^2 S_N \cap \mathcal{C}S_n$ (cf. p. 405; it may be helpful to plot a figure for the case $G = \mathbf{R}^1$). Applying the result on p. 417, we have

$$S(x) \cdot SS_N(x) - S(x) \cdot SS_n(x) \in I.$$

It is our first aim to prove that

$$S(x) \cdot SS_N(x) - S(x)^2 \in I.$$

Indeed, we have

$$\begin{aligned} S(x) \cdot SS_N(x) - S(x)^2 &= \{S(x) \cdot SS_N(x) - S(x) \cdot SS_n(x)\} \\ &\quad + \{S(x) \cdot SS_n(x) - S(x)^2\}. \end{aligned}$$

The first part is in I (see above); the second is, in L^1 -norm, less than $\{m(SS_n) - m(S)\}^{1/2} m^{1/2}(S)$ (cf. property (iii), p. 405). But this can be made arbitrarily small, by taking n sufficiently large⁽⁹⁾, and thus the partial result is proved.

On the other hand, if we take $S_n \subset S$, then

$$m^{-1}(S)S(x) \cdot SS_N(x) - m^{-1}(S_n)S_n(x) \cdot S_n S_N(x) \in I$$

(cf. p. 405) the reader may again find it helpful to plot a graph of the Fourier transforms for the case $G = \mathbb{R}^1$).

But

$$m^{-1}(S_n)S_n(x) \cdot S_n S_N(x) - m^{-1}(S_n)S_n(x)^2 \in I$$

(set $S = S_n$ in the partial result just proved).

Hence finally

$$m^{-1}(S)S(x)^2 - m^{-1}(S_n)S_n(x)^2 \in I$$

for all $S_n \subset S$, i.e.,

$$\{m^{-1}(S)S(x)^2 - m^{-1}(S_n)S_n(x)^2\} * \phi^* = 0,$$

ϕ being the function of the hypothesis. Hence we have

$$m^{-1}(S)S(x)^2 * \phi = m^{-1}(S_n)S_n(x)^2 \Pi \phi.$$

Let now (n_r) be a subsequence such that

$$m^{-1}(S_{n_r}S_{n_r}(x)^2 * \phi \rightarrow M \quad (r \rightarrow \infty)$$

(Lemma 2.1.1). Since the left-hand side of the above equation is independent of n , it follows that

$$m^{-1}(S)S(x)^2 * \phi = M = \text{const.},$$

which proves the lemma. It also entails that M is unique. In this case we write $M\{\phi(x)\}$ instead of M and call it the mean value of $\phi(x)$.

This terminology is justified by the fact that, if $\alpha(x)$ is not only in L^∞ , but also almost periodic in the sense of von Neumann, then

$$\lim_{n \rightarrow \infty} m^{-1}(S_n) \int S_n(x)^2 \alpha(x) dx$$

⁽⁹⁾ This follows from the regularity of the Haar measure (cf. [7]) and the fact that, for any open set O containing S , the product SS_n will be contained in O for large n .

exists and coincides with the usual mean value. Let A be that mean value. Then, given $\epsilon > 0$, there exist positive numbers c_1, c_2, \dots, c_N , whose sum is 1, and elements $y_1, y_2, \dots, y_N \in G$ such that for all $x \in G$

$$|c_1 \alpha(y_1 x) + \dots + c_N \alpha(y_N x) - A| < \epsilon.$$

We may write the difference

$$m^{-1}(S_n) \int S_n(x)^2 \alpha(x) dx - A$$

in the form

$$\begin{aligned} \sum_{r=1}^N c_r m^{-1}(S_n) \int S_n(x)^2 \{\alpha(x) - \alpha(y_r x)\} dx \\ + m^{-1}(S_n) \int S_n(x)^2 \left\{ \sum_{r=1}^N c_r \alpha(y_r x) - A \right\} dx. \end{aligned}$$

But each term of the first sum tends to 0, as $n \rightarrow \infty$, by property (ii), p. 404, and the second term is in absolute value less than ϵ , for every n . Since $\epsilon > 0$ is arbitrary, the result is proved.

From Lemma 2.1.2 there follows immediately a general result:

THEOREM 2.1. *Let I be a closed ideal in L^1 and suppose that $x_1 \in G$ is not a limiting point of Z_I . Suppose that $\phi \perp I$. Then the mean value of $\phi(x)(x, x_1)^*$ exists and, if S is sufficiently small $(x_1 S^2 \cap Z_I \cap \mathcal{C}\{x_1\} = \emptyset)$,*

$$m^{-1}(S) S(x)^2 * [\phi(x)(x, x_1)^*] = M\{\phi(x)(x, x_1)^*\}.$$

Indeed, we may reduce this case to Lemma 2.1.2 by writing $\phi(x)(x, x_1)^* \perp I(x, x_1)^*$, the right-hand side denoting the "rotated" ideal (cf. p. 404).

Now we are ready to prove a generalization of Wiener's theorem (for groups G whose dual groups have a denumerable fundamental system of neighborhoods of the identity; cf. however the Appendix):

THEOREM 2.2. *Let I be a closed ideal in L^1 , with co-spectrum Z_I (p. 404). If the function $k(x) \in L^1$ is such that its Fourier transform $k(x)$ satisfies the following conditions:*

- (i) $Z_k \supset Z_I$ (Z_k denotes the set of zeros of $k(x)$);
- (ii) $\mathcal{F}(Z_k) \cap \mathcal{F}(Z_I)$ is denumerable ($\mathcal{F}(Z)$ denotes the frontier of Z), then $k(x) \in I$.

This theorem was first proved, for the case $G = \mathbb{R}^1$, by S. Mandelbrojt and S. Agmon [11, Théorème II], making use of Carleman's analytic functions [3, p. 75]. The reader will see for himself the connection between their proof and that given here.

To prove the theorem, it suffices to show (cf. the distance theorem p. 402) that whenever $\phi \perp I$ then $\phi \perp k$.

Consider a $\phi \perp I$ and define the function $\phi_1(x)$ by $\phi_1(x) = k^*(x) * \phi(x)$. If we can prove that $\phi_1 \perp L^1$, then it will follow that $k * \phi^*(x) = 0$, and $\phi \perp k$.

Let I_1 be the set of all $f \in L^1$ such that $f * \phi_1^* = 0$; this is a closed ideal. We shall prove:

- (1) $Z_{I_1} \subset \mathcal{F}(Z_k) \cap \mathcal{F}(Z_I)$,
- (2) Z_{I_1} has no isolated points.

From (1) it follows, under the hypothesis of the theorem, that Z_{I_1} is denumerable, and from (2) that it is perfect. This implies that $Z_{I_1} = \emptyset$, and hence (Wiener's theorem) $I_1 = L^1$. Thus the theorem will be proved if we can establish (1) and (2).

(1) Since $g * \phi^* = 0$ for any $g \in I$, we have

$$g * \phi_1^* = g * k * \phi^* = k * g * \phi^* = 0.$$

Thus $\phi_1 \perp I$, i.e., $I \subset I_1$, and hence $Z_{I_1} \subset Z_I$. Now we shall show that no interior point of Z_k can belong to Z_{I_1} .

Indeed, if x_0 is an interior point of Z_k , let S be so small that $x_0 S^2 \subset Z_k$, and consider the function $(x, x_0)S(x)^2$. This function is in I_1 , since $[(x, x_0)S(x)^2] * \phi_1^* = [(x, x_0)S(x)^2] * k * \phi^* = 0$ (observe that the Fourier transform of $[(x, x_0)S(x)^2] * k$ vanishes identically). Thus x_0 is not in Z_{I_1} (cf. [11, Lemme I]).

Therefore Z_{I_1} is in Z_I and contains no point interior to Z_k . Since $Z_I \subset Z_k$, the only points that can possibly belong to Z_{I_1} are those points of $\mathcal{F}(Z_I)$ which are in $\mathcal{F}(Z_k)$. Thus (1) is proved.

(2) Let us suppose that $x_1 \in \mathcal{F}(Z_I) \cap \mathcal{F}(Z_k)$ is not a limiting point of Z_{I_1} . Consider first the case $x_1 = e$. By Theorem 2.1

$$m^{-1}(S)S(x)^2 * \phi_1 = M\{\phi_1\}$$

or

$$m^{-1}(S)S(x)^2 * \phi_1^* = M^*\{\phi_1\},$$

$$\text{i.e., } m^{-1}(S)S(x)^2 * k * \phi^* = M^*\{\phi_1\}.$$

Take now $S = S_n$. By Lemma 2.1.1 there exists a subsequence (n_r) such that $m^{-1}(S_{n_r})S_{n_r}(x)^2 * \phi^*$ tends (boundedly) to a constant c ($r \rightarrow \infty$). But then the left-hand side of the last equation tends to $c \int k(x) dx = c \cdot k(e) = 0$. Thus $M^*\{\phi_1\} = 0$, i.e., $S(x)^2 \in I_1$, and hence e is not in Z_{I_1} (see also Appendix).

The general case may be reduced to the preceding one by considering instead of $\phi_1(x)$ the function

$$\phi_1(x)(x, x_1)^{\bullet} = [k(x)(x, x_1)^{\bullet}]^* * [\phi(x)(x, x_1)^{\bullet}].$$

The ideal corresponding to that function is just the "rotated" ideal (cf.

p. 404) $I_1(\mathbf{x}, x_1)^*$, with co-spectrum $x_1^{-1}Z_{I_1}$, and the co-spectrum of $k(\mathbf{x})(\mathbf{x}, x_1)^*$ is $x_1^{-1}Z_k$. Thus, by the special case just proved, e is not in $x_1^{-1}Z_{I_1}$, i.e., x_1 is not in Z_{I_1} , and the proof of (2) is complete.

REMARKS. Condition (i) of the theorem is obviously necessary as well; moreover, for compact groups it is also sufficient, condition (ii) being then superfluous, since the dual group of a compact group is discrete. Thus *for compact groups G there is a one-to-one correspondence between the closed ideals of L^1 and the subsets of G which invert the inclusion* (cf. [10, p. 392]). On the other hand L. Schwartz has proved [12] that condition (i) alone is not sufficient in general, if G is not compact. This shows the role of condition (ii).

The reader is referred to Mackey's survey [10, pp. 392–395] for a discussion of previous generalizations of Wiener's theorem. Besides the papers by Mandelbrojt-Agmon [11] and Godement [6], we mention here in particular those of Ditkin [5], Segal [13], and Kaplansky [9]⁽¹⁰⁾.

We shall now apply the preceding results to study the functions $\phi \in L^\infty$ orthogonal to a given ideal I , when the co-spectrum Z_I has a particularly simple structure.

If Z_I is finite, then any $\phi \perp I$ is a linear combination, with constant coefficients, of the characters of G defined by the elements of Z_I .

Let Z_I consist of x_1, x_2, \dots, x_N . If $k(\mathbf{x})$ is any function in L^1 , then (for S suitably small)

$$k(\mathbf{x}) - k(\mathbf{x}) * \sum_{n=1}^N (\mathbf{x}, x_n) m^{-1}(S) S(\mathbf{x})^2$$

will be in I (Theorem 2.2), i.e.,

$$\left[k(\mathbf{x}) - k(\mathbf{x}) * \sum_{n=1}^N (\mathbf{x}, x_n) m^{-1}(S) S(\mathbf{x})^2 \right] * \phi^* = 0$$

for any $\phi \perp I$. By Theorem 2.1 we may then write

$$k(\mathbf{x}) * \left\{ \phi^*(\mathbf{x}) - \sum_{n=1}^N a_n(\mathbf{x}, x_n) \right\} = 0$$

where $a_n = M\{\phi(\mathbf{x})(\mathbf{x}, x_n)^*\}$.

Since $k(\mathbf{x})$ is an arbitrary function in L^1 , it follows that

$$\phi(\mathbf{x}) = \sum_{n=1}^N a_n(\mathbf{x}, x_n)$$

⁽¹⁰⁾ It should be pointed out that some of these papers contain theorems proved for arbitrary locally compact abelian groups. Since the present paper was submitted for publication, Dr. Henry Helson has kindly sent me a reprint of his paper *Spectral synthesis of bounded functions*, Arkiv för Matematik vol. 1 (1951) pp. 497–602. This paper contains also a generalization of Wiener's theorem (for arbitrary locally compact abelian groups). The theorem is stated there in a weaker form, but the proof would actually yield the result given here.

almost everywhere. Cf. [6, Hypothèse A, p. 136] and [9].

If Z_I is discrete, then any uniformly continuous $\phi \perp I$ is almost periodic, with spectrum in Z_I (and conversely).

Let $k_C(x)$ be any function in L^1 whose Fourier transform vanishes outside some compact set C . Then, in the same way as before,

$$k_C(x) * \phi^* = \sum_i k_C(x_i) a_i(x, x_i) \quad (x_i \in Z_I \cap C).$$

Now the functions of the type k_C are dense in L^1 [13, Theorem 2.6]. Thus for any $k \in L^1$ the function $k * \phi$ is almost periodic, with spectrum in Z_I . In particular, set $k(x) = m^{-1}(V)V(x)$, where $V(x)$ is the characteristic function of a neighborhood V of e . Given $\epsilon > 0$, there is a V such that $|m^{-1}(V)V(x) * \phi - \phi(x)| < \epsilon$, for all $x \in G$, by the assumed uniform continuity of $\phi(x)$. Hence $\phi(x)$ itself is almost periodic, with spectrum in Z_I (the converse follows from the approximation theorem for almost periodic functions).

For the case $G = \mathbf{R}^1$ this was proved by A. Beurling [2].

If now $G = \mathbf{R}^p$, let V_n be a sequence of (p -dimensional) cubes, with center at the origin and sides of length $1/n$. Then $m^{-1}(V_n)V_n(x) * \phi \rightarrow \phi(x)$ almost everywhere ($n \rightarrow \infty$). Since $m^{-1}(V_n)V_n(x) * \phi$ is almost periodic, there are trigonometric polynomials $P_n(x)$, with frequencies in Z_I , such that $|m^{-1}(V_n)V_n(x) * \phi - P_n(x)| < 1/n$, $n \geq 1$. Thus $P_n(x) \rightarrow \phi(x)$ almost everywhere ($n \rightarrow \infty$) and $|P_n(x)| < \|\phi\|_\infty + 1$ for all n . Hence, for $G = \mathbf{R}^p$, if Z_I is discrete, then the set of all $\phi \perp I$ consists of all those functions in L^∞ which are limits (almost everywhere) of sequences of uniformly bounded trigonometric polynomials with frequencies in Z_I .

In the case $G = \mathbf{R}^1$ this was proved by Carleman [3, p. 115], with an additional restriction on Z_I .

If Z_I is discrete, then to any $\phi \perp I$ there corresponds (by Theorem 2.1) the "Fourier series"

$$\phi(x) \sim \sum a_i(x, x_i)$$

where $a_i = M\{\phi(x)(x, x_i)^*\}$ and the summation extends over all $x_i \in Z_I$. The "sum" is purely formal—no ordering of the terms, or convergence, is implied.

If all the Fourier coefficients are zero, then $\phi(x)$ vanishes almost everywhere ("uniqueness theorem"), since then $\phi \perp L^1$.

"Bessel's inequality" holds in the sense that

$$\sum |a_i|^2 \leq \liminf_{n \rightarrow \infty} m^{-1}(S_n) \int S_n(x)^2 |\phi(x)|^2 dx,$$

the set of all $a_i \neq 0$ being denumerable. The proof is the same as in the classical case.

The question now arises under what conditions it is true that

$$\lim_{n \rightarrow \infty} m^{-1}(S_n) \int S_n(x)^2 |\phi(x)|^2 dx = \sum |a_i|^2$$

("Parseval's equation").

Consider the function

$$\alpha(x) = m^{-1}(S) \int S(y)^2 \phi(yx) \phi^*(y) dy,$$

which is orthogonal to I . We have, for large n ,

$$\begin{aligned} M\{\alpha(x)(x, x_i)^*\} &= \int m^{-1}(S_n) S_n(x)^2 (x, x_i)^* dx \int m^{-1}(S) S(y)^2 \phi(yx) \phi^*(y) dy \\ &= \int m^{-1}(S) S(y)^2 \phi^*(y) dy \int m^{-1}(S_n) S_n(x)^2 (x, x_i)^* \phi(yx) dx \\ &= \int m^{-1}(S) S(y)^2 \phi^*(y) dy \int m^{-1}(S_n) S_n(yx^{-1})^2 (yx^{-1}, x_i) \phi(x) dx. \end{aligned}$$

But, in view of Theorem 2.1, this is just

$$\begin{aligned} &\int m^{-1}(S) S(y)^2 \phi^*(y) a_i(y, x_i) dy \\ &= a_i m^{-1}(S) \int S(y^{-1})^2 (y^{-1}, x_i) \phi^*(y) dy \\ &= a_i \sum_{\kappa} \Delta(\iota, \kappa) \dot{a}_{\kappa} \quad (\Delta(\iota, \kappa) = m^{-1}(S) S * S(x_i x_{\kappa}^{-1})), \end{aligned}$$

by an argument already familiar. The last sum (over all $x_{\kappa} \in Z_I$) has actually only a finite number of terms.

Set now $S = S_n$ and write correspondingly $\alpha_n(x)$, $\Delta_n(\iota, \kappa)$. We are going to indicate some restriction on Z_I which will guarantee the absolute convergence of

$$\sum_{\iota, \kappa} \Delta_n(\iota, \kappa) a_{\iota} \dot{a}_{\kappa}.$$

Suppose Z_I is such that, for some S , $xS \cap Z_I = x$, for all $x \in Z_I$ (Z_I may then be called "*uniformly discrete*"). Then we have, for sufficiently large n ,

$$\alpha_n(e) = \sum_{\iota} |a_{\iota}|^2,$$

by the uniqueness theorem.

Suppose now that Z_I is the union of a finite number N of sets, each uniformly discrete. Then for sufficiently large n , say $n \geq n_0$, the following holds:

if one of the indices ι, κ is kept fixed, then $\Delta_n(\iota, \kappa) > 0$ for at most N values of the other one.

Define now the function $T(\iota, \kappa)$ by

$$\begin{aligned} T(\iota, \kappa) &= 1 && \text{if } \Delta_{n_0}(\iota, \kappa) > 0, \\ T(\iota, \kappa) &= 0 && \text{if } \Delta_{n_0}(\iota, \kappa) = 0. \end{aligned}$$

Then both $\sum_{\kappa} T(\iota, \kappa)$ and $\sum_{\iota} T(\iota, \kappa)$ are at most equal to N . Hence

$$\begin{aligned} \sum_{\iota, \kappa} T(\iota, \kappa) |a_{\iota}| |a_{\kappa}| &\leq \sum_{\iota, \kappa} T(\iota, \kappa) (|a_{\iota}|^2 + |a_{\kappa}|^2) / 2 \\ &\leq N \sum_{\iota} |a_{\iota}|^2. \end{aligned}$$

Hence the series

$$\sum_{\iota, \kappa} \Delta_n(\iota, \kappa) a_{\iota} a_{\kappa}$$

will converge absolutely, and (some order of the terms having been fixed) uniformly with respect to n . By the uniqueness theorem

$$\alpha_n(e) = \sum_{\iota, \kappa} \Delta_n(\iota, \kappa) a_{\iota} a_{\kappa}.$$

Thus

$$M\{| \phi |^2\} = \lim_{n \rightarrow \infty} m^{-1}(S_n) \int S_n(x)^2 | \phi(x) |^2 dx$$

exists and

$$M\{| \phi |^2\} = \sum_{\iota} |a_{\iota}|^2.$$

Thus, if Z_I may be decomposed into a finite number of sets each of which is uniformly discrete (p. 424), then "Parseval's equation" holds for all $\phi \perp I$. It follows that the Fourier series of ϕ converges "in the mean" to ϕ .

For the case $G = \mathbf{R}^1$, this means that all $\phi \perp I$ are almost periodic B^2 (Besicovitch), since in this case $M\{| \phi |^2\}$, as defined here, coincides with the mean used by Besicovitch, in view of a theorem of Wiener [16, Theorem 21].

APPENDIX⁽¹¹⁾

It is possible to modify the proof of Lemma 2.1.2, avoiding Lemma 2.1.1 entirely and dropping the assumption that the dual group possess a denumerable fundamental system of neighborhoods of the identity (first axiom of countability). The modification is as follows:

In Lemma 2.1.2 it is proved that the function

⁽¹¹⁾ Added September 15, 1952.

$$m^{-1}(S)S(x)^2 * \phi,$$

for S sufficiently small, is independent of S , without any recourse to the first axiom of countability (it is convenient, of course, to change the notation, writing, say, S' , S'' instead of S_n , S_N , respectively, where $S' \subset S''$).

To show that the function above is also independent of x , take $x = e$ and set

$$m^{-1}(S)S(e)^2 * \phi = M.$$

Now the difference

$$m^{-1}(S)S(x)^2 * \phi - M$$

is equal to

$$m^{-1}(S')(S'(x)^2 * \phi - m^{-1}(S')S'(e)^2 * \phi),$$

for any $S' \subset S$, and hence we have

$$\begin{aligned} |m^{-1}(S)S(x)^2 * \phi - M| &= m^{-1}(S') \left| \int \{S'(xy^{-1})^2 - S'(y^{-1})^2\} \phi(y) dy \right| \\ &\leq m^{-1}(S') \|S'(x^{-1}y)^2 - S'(y)^2\|_1 \|\phi\|_\infty \\ &\leq \epsilon \|\phi\|_\infty, \end{aligned}$$

by property (ii), p. 404, if S' is small enough. Thus Lemma 2.1.2 is established by a method which is more general as well as simpler.

The only other place where the first axiom of countability is used is on p. 421 where it is proved that in the equation

$$m^{-1}(S)S(x)^2 * k * \phi^* = M^* \{\phi_1\}$$

the constant $M^* \{\phi_1\}$ is zero. Keeping S fixed and setting

$$m^{-1}(S)S(x)^2 * \phi = \psi(x),$$

we have

$$k * \psi^* = M^* \{\phi_1\}$$

and also

$$\int k(x) dx = 0.$$

Now in Lemma 1.1.2 (part I) it is proved, without any countability restriction, that the last two equations imply

$$M^* \{\phi_1\} = 0.$$

It results that the proof of the extension of the theorem of Mandelbrojt-Agmon (Theorem 2.2) is valid for locally compact abelian groups in general.

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