LINEAR FUNCTIONALS ON ALMOST PERIODIC FUNCTIONS

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0. Introduction

0.1 We consider in the present communication the space $\mathfrak{A}$ consisting of all almost periodic continuous functions on the real line. For our purposes, it is convenient to describe $\mathfrak{A}$ as the set of all uniform limits of trigonometric polynomials

0.1.1 $$\sum_{\nu=1}^{n} \alpha_\nu e^{it_\nu x} \qquad (t_\nu \text{ real, } \alpha_\nu \text{ complex}).$$

Our aim is, first, to present two realizations of the space $\mathfrak{A}^*$ of all bounded complex linear functionals on $\mathfrak{A}$, and, second, to use these realizations for the study of positive definite functions which are not necessarily continuous. In this fashion, we obtain a generalization of Bochner's representation theorem for continuous positive definite functions as well as various facts concerning positive definite functions and their structure.

0.2 Throughout the present paper, the symbol R denotes the real numbers, considered either as an additive group or as a field; K the field of complex numbers; T the multiplicative group of complex numbers of absolute value 1; $T^{\mathfrak{m}}$ the complete Cartesian product of $\mathfrak{m}$ groups each identical with T, $\mathfrak{m}$ being any cardinal number greater than 1. The characteristic function of a subset B of a set X is denoted by χ_B. If G is any locally compact Abelian group, we denote the group of all continuous characters of G by the symbol $G]$. $G]$ is given the usual compact-open topology. If X is any topological space, we denote the set of all complex-valued continuous functions on X which are bounded in absolute value by the symbol $\mathfrak{C}(X)$. The space of all trigonometric polynomials 0.1.1 is denoted by $\mathfrak{P}$; the space of all almost periodic continuous functions on R by $\mathfrak{A}$. For a normed complex linear space $\mathfrak{L}$, we denote the space of all bounded complex linear functionals on $\mathfrak{L}$ by the symbol $\mathfrak{L}^*$.

1. The group bR

1.1 Following Weil ([1] and [2, pp. 124–139]) and Anzai and Kakutani [1](1), we imbed the group R in a compact group, by the following construc-

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

tion. Let $\mathfrak{c}$ be the cardinal number of the continuum. Then $T^{\mathfrak{c}}$ is representable as

$$\underset{t\in R}{\boldsymbol{P}}\, T_{(t)}$$

where each $T_{(t)}$ is identical with T. The mapping

1.1.1 $$x \to \{e^{itx}\}_{t\in R} = \Phi(x)$$

is plainly a continuous (but not bicontinuous) isomorphism carrying R into $T^{\mathfrak{c}}$. The group $\Phi(R)$ has a topology induced by that of $T^{\mathfrak{c}}$; we denote the group $\Phi(R)$ with this topology by the symbol R_w. The closure bR of R_w in $T^{\mathfrak{c}}$ is a compact Abelian group, which we call the Bohr compactification of R(2).

1.2 THEOREM. *The group $bR]$ is the group R_d, defined as the additive group R under its discrete topology.*

Let $t\in R$ be a fixed index, and let $\mathbf{x}=\{\mathbf{x}(t)\}_{t\in R}$ be a given element of bR. Since bR is a subgroup of $T^{\mathfrak{c}}$, it is clear that the function π_t, where $\pi_t(\mathbf{x})=\mathbf{x}(t)$, is a continuous character of bR. It is also clear that $\pi_{t_1+t_2}(\mathbf{x})=\pi_{t_1}(\mathbf{x})\cdot\pi_{t_2}(\mathbf{x})$ and $\pi_{-t}(\mathbf{x})=(\pi_t(\mathbf{x}))^{-1}$, for all $\mathbf{x}\in bR$, since these relations are valid in the dense subgroup R_w of bR. It follows that the characters π_t form a subgroup of $bR]$, algebraically isomorphic to R and of course having the discrete topology. To show that there are no other continuous characters on bR, we use the fact that every continuous complex-valued function on bR is arbitrarily uniformly approximable by complex linear combinations of characters $\pi_t(\mathbf{x})$(3). Assume that f is a continuous character on bR distinct from all π_t. Then for every $\epsilon>0$, we have a complex linear combination $\sum_{\nu=1}^{n}\alpha_\nu\pi_{t_\nu}$ such that $\sup_{\mathbf{x}\in bR}\left|f(\mathbf{x})-\sum_{\nu=1}^{n}\alpha_\nu\pi_{t_\nu}(\mathbf{x})\right|<\epsilon$. Denoting Haar measure on bR by ι and taking $\iota(bR)=1$, we have, in view of known orthogonality relations

(2) Note that our terminology differs somewhat from that of Anzai and Kakutani [1]. We also observe that the manoeuver used here exemplifies a process very common indeed in the study of function spaces and function rings. We select a certain subset $\mathfrak{G}$ of the set of all bounded continuous complex functions on a given topological space X, and imbed the space X in a direct product of complex fields, one factor-space for each $g\in\mathfrak{G}$, by $x\to\{g(x)\}_{g\in\mathfrak{G}}$. The closure of the image of X is a compact Hausdorff space on which the space of all continuous complex functions may be identified with the smallest uniformly closed ring containing $\mathfrak{G}$, on X. If $\mathfrak{G}$ is all bounded continuous complex functions and X is completely regular, we obtain the well known space βX (discussed, for example, in Hewitt [1, pp. 53–67]). If X is a locally compact Hausdorff space, and $\mathfrak{G}$ is all continuous complex functions vanishing outside of compact sets, we obtain Alexandroff's completion of X by adding a point "at infinity"; and so on. In the present case, all functions in $\mathfrak{G}$ have absolute value 1, so that we may use T in place of K in forming the mappings referred to.

(3) This follows immediately from the complex form of the Stone-Weierstrass theorem (Stone [1]), as the functions π_t are a separating family, and the ring of all $\sum_{\nu=1}^{n}\alpha_\nu\pi_{t_\nu}$ is closed under complex conjugation.

(Weil [2, p. 78]), that

1.2.1
$$\begin{aligned}\int_{bR} |f(\mathrm{x})|^2 d\iota(\mathrm{x}) &= \int_{bR} [\,|f(\mathrm{x})|^2 - \sum \alpha_\nu \pi_{t_\nu}(\mathrm{x})\overline{f(\mathrm{x})}\,] d\iota(\mathrm{x}) \\ &= \int_{bR} |f(\mathrm{x})| \cdot |f(\mathrm{x}) - \sum \alpha_\nu \pi_{t_\nu}(\mathrm{x})|\, d\iota(\mathrm{x}) \\ &= \int_{bR} |f(\mathrm{x}) - \sum \alpha_\nu \pi_{t_\nu}(\mathrm{x})|\, d\iota(\mathrm{x}) < \epsilon.\end{aligned}$$

This obvious contradiction shows that $bR] = R_d$.

1.3 THEOREM. *The space $\mathfrak{C}(bR)$ is identifiable with the space $\mathfrak{A}$, as a linear space, as a ring, and as a normed linear space under the uniform norm. The correspondence Ψ:*

1.3.1
$$\sum_{\nu=1}^{n} \alpha_\nu e^{it_\nu x} \to \sum_{\nu=1}^{n} \alpha_\nu \pi_{t_\nu} \qquad (\alpha_\nu \in K)$$

is a mapping of the ring of functions $\mathfrak{P}$ onto the ring of all linear combinations of characters π_t, which preserves sums, products, and scalar multiples; and which has the property that

1.3.2
$$\sup_{x\in R} \left| \sum_{\nu=1}^{n} \alpha_\nu e^{it_\nu x} \right| = \sup_{y \in bR} \left| \sum_{\nu=1}^{n} \alpha_\nu \pi_{t_\nu}(y) \right|.$$

Ψ thus admits an extension carrying the completion $\mathfrak{A}$ of $\mathfrak{P}$ onto the completion $\mathfrak{C}(bR)$ of the set of functions $\sum_{\nu=1}^{n} \alpha_\nu \pi_{t_\nu}$, under the norms written in 1.3.2. *This extension is a norm-preserving algebra-isomorphism of $\mathfrak{A}$ onto $\mathfrak{C}(bR)$.*

The only assertion here needing verification is the equality 1.3.2. We note that for fixed $x \in R$ and $t \in R$, $e^{itx} = \pi_t(\Phi(x))$. Hence we have

1.3.3
$$\sup_{x\in R} \left| \sum_{\nu=1}^{n} \alpha_\nu e^{it_\nu x} \right| = \sup_{x \in R} \left| \sum_{\nu=1}^{n} \alpha_\nu \pi_{t_\nu}(\Phi(x)) \right|.$$

As R_w is dense in bR, we have $\sup_{x\in R} |g(\Phi(x))| = \sup_{y \in bR} |g((y)|$ for all $g \in \mathfrak{C}(bR)$, and 1.3.2 is verified.

1.4 THEOREM. *The character group $R_d]$ is the group bR; the value of the character $\mathrm{x} \in bR$ at the point $t \in R_d$ being the number $\pi_t(\mathrm{x})$. This character $\pi_t(\mathrm{x})$ is continuous in t (usual topology) if and only if $\mathrm{x} \in R_w$. Thus the points of R_w correspond to continuous characters of R and the points of $bR \cap R_w'$ to discontinuous (and hence nonmeasurable) characters of R.*

Pontryagin's duality theorem (see Weil [2, p. 108]) shows that $bR = bR]] = R_d]$. The characters of R_d, in accordance with this duality, are all of the form $\pi_t(\mathrm{x})$, with x fixed in bR and t variable in R_d. Since the continuous characters of R all have the form $\pi_t(\Phi(x)) = e^{itx}$ (x fixed in R, t variable in R), it is clear that every function of t $\pi_t(\mathbf{x})$ with $\mathbf{x} \notin R_w$ must produce a discontinuous character of R, and conversely.

We note also the following representation for bR.

1.5 THEOREM. *The elements of bR are all complex functions f on the real line R such that for every $\epsilon > 0$ and $t_1, t_2, \cdots, t_n \in R$ there exists a real number x such that $|e^{it_\nu x} - f(t_\nu)| < \epsilon$ ($\nu = 1, 2, \cdots, n$). The functions in bR are all of the characters of R; those not of the form e^{itx_0} for some fixed $x_0 \in R$ are discontinuous and therefore nonmeasurable.*

1.6 REMARK. The foregoing can all be carried out for an arbitrary locally compact Abelian group G, as Anzai and Kakutani [1] have observed. We have gone through the construction again because our subsequent discussion requires it.

1.7 REMARK. The argument used in 1.2 shows that if G is a compact Abelian group and H is a proper subgroup of G], then there exists $x \neq e$ in G such that $h(x) = 1$ for all $h \in H$. This fails in general for non-compact locally compact Abelian groups, as the group of characters $\{e^{irx}\}$, r rational, shows for the case $G = R$.

2. REPRESENTATIONS FOR $\mathfrak{A}^*$

The function-space $\mathfrak{A}$ has two distinct realizations, according to 1.3. Accordingly, we may expect two distinct realizations for the conjugate space $\mathfrak{A}^*$. The first of these is well known.

2.1 THEOREM. *The space $\mathfrak{A}^*$ is realizable as the space of all complex-valued, countably additive, regular Borel measures $\boldsymbol{\mu}$ on the space bR. For $f \in \mathfrak{A}$ and $M \in \mathfrak{A}^*$, we have*

2.1.1
$$M(f) = \int_{bR} f(\mathbf{x}) d\boldsymbol{\mu}(\mathbf{x})$$

and

2.1.2
$$\|M\| = \int_{bR} d|\boldsymbol{\mu}|(\mathbf{x}) = |\boldsymbol{\mu}|(bR)(^4).$$

The correspondence $M \to \boldsymbol{\mu}$ defined by 2.1.1 is a norm-preserving linear space isomorphism carrying $\mathfrak{A}^*$ onto the space of measures $\boldsymbol{\mu}$ as described.

The existence of a unique $\boldsymbol{\mu}$ satisfying the conditions above follows at once from the Riesz-Radon-Kakutani theorem for real-valued continuous

(4) For measure-theoretic terms not explained here, see Halmos [1]. The measure $|\mu|$ for complex μ is defined by $|\mu|(A) = \sup \sum_{\nu=1}^{n} |\mu(A_\nu)|$ for all Borel sets A, the sup being taken over all finite families $\{A_\nu\}_{\nu=1}^{n}$ of pairwise disjoint Borel sets with union A. The theory of this measure is set forth in Hewitt [3].

functions on compact Hausdorff spaces (see Kakutani [1, p. 1012, Theorem 10]) if we write the functional M as $(M_1-M_2)+i(M_3-M_4)$, where the M_ν are non-negative linear functionals ($\nu=1, \cdots, 4$). The equality 2.1.2 is proved in Hewitt [3, Theorem 1.3]. The last assertion is obvious.

We obtain a new representation for $\mathfrak{A}^*$ by confining ourselves to the space $R_w \subset bR$ and considering our almost periodic functions as being defined only on this space. We first consider some important sets.

2.2 DEFINITION. Let A be any bounded nonvoid subset of R_w, and let α be any real number $\geqq$ sup $A-$inf A. (We use here the ordering of R.) The set A_α is defined as the set of all $x \in R_w$ such that $x+n\alpha \in A$ for some n, $n=0, \pm 1, \pm 2, \pm 3, \cdots$.

2.3 DEFINITION. Let $\mathcal{E}$ be the family of all subsets of R_w having the form $\bigcup_{k=1}^{\infty}(\bigcap_{j=1}^{n_k} G_{j,k})$, where each set $G_{j,k}$ has the form I_α for some open interval I, and let $\mathcal{Z}$ be the family of all sets Z such that $Z' \in \mathcal{E}$. Let $\mathcal{E}^*$ be the smallest algebra of sets containing $\mathcal{E}$.

2.4 According to a theorem of E. Følner [1], the sets in $\mathcal{Z}$ are precisely the sets having the form $E[x; x \in R_w, f(x)=0]$ for $f \in \mathfrak{A}$, and thus the sets in $\mathcal{E}$ are precisely the sets which have the form $E[x; x \in R_w, |f(x)|>0]$ for $f \in \mathfrak{A}$. We shall be concerned with finitely additive, complex-valued, bounded measures defined on the algebra $\mathcal{E}^*$. For the definition and elementary properties of finitely additive real-valued measures, see Yosida and Hewitt [1]. The definition of a complex-valued finitely additive, bounded measure is obtained by writing "complex" for "real" in the definition given *loc. cit.*

2.5 A non-negative finitely additive measure ϕ defined on $\mathcal{E}^*$ is said to be regular if for every $A \in \mathcal{E}^*$ and every $\epsilon>0$, there exists a set $Z \in \mathcal{Z}$ and a set $P \in \mathcal{E}$ such that $Z \subset A \subset P$ and $\phi(P \cap Z')<\epsilon$. A complex finitely-additive measure on $\mathcal{E}^*$ is said to be regular if the positive and negative parts of its real and imaginary parts are all regular. For a complex finitely additive measure ϕ on any algebra of subsets $\mathfrak{M}$ of a set X, we define the measure $|\phi|$ by the relation

2.5.1
$$|\phi|(A) = \sup \sum_{\nu=1}^{n} |\phi(A_\nu)|,$$

the supremum being taken over all finite families of pairwise disjoint sets $\{A_\nu\}_{\nu=1}^{n}$ in $\mathfrak{M}$ whose union is A. It is easy to prove that $|\phi|$ is a non-negative finitely additive measure on $\mathfrak{M}$ and that it is the smallest non-negative measure which exceeds the set-function $|\phi(A)|$ for every $A \in \mathfrak{M}$.

We now state and prove a second representation theorem for the conjugate space $\mathfrak{A}^*$.

2.6 THEOREM. *Let M be an arbitrary element of $\mathfrak{A}^*$. Then there exists a bounded, regular, complex-valued, finitely additive measure μ on the algebra of sets $\mathcal{E}^*$ such that*

2.6.1 $$M(f) = \int_{R_w} f(x)d\mu(x)\text{[5]}$$

for all $f \in \mathfrak{A}$. If μ_1 and μ_2 are distinct measures of the type described, then the functionals

2.6.2 $$\int_{R_w} f(x)d\mu_j(x) \qquad (j = 1, 2)$$

are distinct. For M and μ as in 2.6.1, we have

2.6.3 $$\|M\| = \int_{R_w} d\,|\,\mu\,|\,(x) = |\,\mu\,|\,(R_w).$$

The correspondence $M \to \mu$ defined by 2.6.1 *is a norm-preserving linear space isomorphism carrying $\mathfrak{A}^*$ onto the space of measures of the type specified.*

By making an obvious reduction, we confine ourselves to non-negative linear functionals in obtaining the representation 2.6.1. This representation has been established elsewhere for a general class of function spaces which includes $\mathfrak{A}$ (see Hewitt [2]). We recall the definition of μ when M is given. For a set $P \in \mathcal{E}$, we define $\mu(P)$ as

2.6.4 $$\sup M(f),$$

the supremum being taken over all real-valued $f \in \mathfrak{A}$ such that $0 \leqq f \leqq \chi_P$. For an arbitrary subset S of R_w, we define $\mu(S)$ as

2.6.5 $$\inf_{P \in \mathcal{E}, P \supset S} \mu(P).$$

The sets measurable in the sense of Carathéodory form an algebra containing $\mathcal{E}^*$ on which μ is finitely additive[6]. It is easy to see that for a set $Z \in \mathcal{Z}$, we have

2.6.6 $$\mu(Z) = \inf M(f),$$

the infimum being taken over all real-valued $f \in \mathfrak{A}$ such that $\chi_Z \leqq f$. It is obvious that the μ defined by 2.6.4–2.6.5 is regular. Let μ_1 and μ_2 be distinct real-valued, finitely additive, regular measures on $\mathcal{E}^*$. Then there is a set $P \in \mathcal{E}$ such that $\mu_3 = \mu_2 - \mu_1$ is different from 0 on P; we may assume $\mu_3(P) > 0$. It is easy to see that μ_3 is regular, i.e., the positive and negative parts μ_3^+ and μ_3^- are regular, and we can select a $Z \in \mathcal{Z}$ such that $Z \subset P$ and

(5) This integral is defined by Lebesgue sums just as for countably additive measures.

(6) After the publication of Hewitt [2], we were kindly apprised by Dr. E. Følner of the results in Bochner [2], where a construction is given (Theorem 1, p. 773) for a finitely additive measure very like that in Hewitt [2]. Bochner's measure is defined on a field in general smaller than ours, however, as his example at the bottom of p. 776 shows. Where it is defined, of course, Bochner's measure is identical with ours.

$\mu_3^+(P\cap Z')$, $\mu_3^-(P\cap Z')$ are both less than an arbitrary positive number η. As noted in 2.4, there exist f and $g\in\mathfrak{A}$ such that $f=0$ precisely on Z and $g=0$ precisely on P'. The function $h=|g|/(|f|+|g|)$ is clearly in $\mathfrak{A}$ and satisfies the inequalities $\chi_P\geqq h\geqq\chi_Z$. By selecting η appropriately, we can obtain the inequalities $\int_{R_w}h(x)d\mu_3(x)\geqq\mu_3(Z)-\mu_3^+(P\cap Z')-\mu_3^-(P\cap Z')>0$. Hence distinct real μ's give distinct integrals, and similarly for complex μ's. If we start with an $M\in\mathfrak{A}^*$, define from M a measure μ by 2.6.4–2.6.5, and then a functional M' as the integral $\int_{R_w}f(x)d\mu(x)$, it is proved in Hewitt [2] that $M=M'$. If we start with a regular measure μ, define the functional $M(f)$ as $\int_{R_w}f(x)d\mu(x)$, use M to define a measure μ' by 2.6.4–2.6.5, and then define a functional M' as $\int_{R_w}f(x)d\mu'(x)$, then $M=M'$ and, by the uniqueness established above, $\mu=\mu'$. This shows that the mapping $M\rightarrow\mu$ is one-to-one and carries $\mathfrak{A}^*$ onto the space of all regular, complex, bounded, finitely additive measures on $\mathcal{E}^*$.

It remains to establish 2.6.3. For $f\in\mathfrak{A}$ such that $\sup_{x\in R_w}|f(x)|=\|f\|=1$, let $\sum_{\nu=1}^n f(x_\nu)\mu(A_\nu)$ be within ϵ (in absolute value) of the integral $\int_{R_w}f(x)d\mu(x)$. It is clear that

$$\left|\sum_{\nu=1}^n f(x_\nu)\mu(A_\nu)\right| \leqq \sum_{\nu=1}^n |\mu(A_\nu)| \leqq |\mu|(R_w).$$

Thus $\|M\|\leqq|\mu|(R_w)$. To verify the reverse inequality, let δ be a positive real number and let $\{A_\nu\}_{\nu=1}^n$ be a family of pairwise disjoint elements of $\mathcal{E}^*$ such that $\bigcup_{\nu=1}^n A_\nu=R_w$ and $\sum_{\nu=1}^n|\mu(A_\nu)|+\epsilon>|\mu|(R_w)$. (We may assume that all $\mu(A_\nu)\neq 0$.) Let $Z_\nu\subset A_\nu$ be sets in $\mathcal{J}$ such that $|\mu(Q)|<\epsilon\cdot n^{-1}$ for all $Q\subset A_\nu\cap Z_\nu'$. Let f_ν be a function in $\mathfrak{A}$ which vanishes exactly on Z_ν, and let g_ν be a function in $\mathfrak{A}$ which vanishes exactly on $\bigcup_{j\neq\nu}Z_j$. Define h as

$$\sum_{\nu=1}^n \frac{\overline{\mu(A_\nu)}}{|\mu(A_\nu)|}\cdot\frac{|g_\nu|}{|f_\nu|+|g_\nu|}.$$

A simple computation shows that

$$\int_{R_w} h(x)d\mu(x) \geqq \sum_{\nu=1}^n |\mu(A_\nu)| - \epsilon,$$

and 2.6.3 is thus established.

2.7 THEOREM. *Let μ be any measure of the kind described in* 2.6. *Then μ is completely determined by its values on sets of the form I_α, where I is an interval (a, b) and $b-a\leqq\alpha$.*

Theorem 2.6 shows that the measures μ may be identified with functionals in the space $\mathfrak{A}^*$. It is easy to see that if μ is known for all sets I_α, then the integrals $\int_{-\infty}^{\infty}e^{itx}d\mu(x)$ are determined for all t. Since polynomials $\sum_{\nu=1}^n\alpha_\nu e^{it_\nu x}$ are dense in $\mathfrak{A}$, we see that the functional defined by μ, and hence μ itself, are determined by the numbers $\mu(I_\alpha)$.

2.8 THEOREM. *Let ϕ be any countably additive complex-valued measure of finite total variation defined for all Borel sets in R. Then ϕ is completely determined by its values on sets I_α (I an interval (a, b), $b-a\leqq\alpha$).*

By the uniqueness theorem for Fourier-Stieltjes transforms, ϕ is completely determined by $\int_{-\infty}^{\infty}e^{itx}d\phi(x)$ $(-\infty<t<+\infty)$. Then repeat the observation of 2.7.

3. REPRESENTATION OF POSITIVE DEFINITE FUNCTIONS

Throughout the present section, we identify R_w and R.

3.1 DEFINITION. A complex-valued function $p(t)$ defined on the real line is said to be positive definite if

3.1.1 $$\sum_{\nu=1}^{m}\sum_{\mu=1}^{m} p(t_\mu - t_\nu)\alpha_\mu\bar{\alpha}_\nu \geqq 0$$

for all pairwise different $t_1, t_2, \cdots, t_m\in R$ and all $\alpha_1, \cdots, \alpha_m\in K$.

We recall the well known theorem of Bochner [1, pp. 74–76] dealing with positive definite functions which are continuous, and state this theorem in terms of the measures now under consideration.

3.2 (BOCHNER'S THEOREM). *Let μ be a regular, bounded, non-negative measure on $\mathcal{E}^*$ which is countably additive*(7). *Then the Fourier-Stieltjes transform*

3.2.1 $$\int_{-\infty}^{\infty} e^{itx}d\mu(x)$$

is positive definite and continuous; conversely, if $p(t)$ is any continuous positive definite function, there exists a regular, bounded, non-negative real measure μ on $\mathcal{E}^*$ which is countably additive and for which

3.2.2 $$p(t) = \int_{-\infty}^{\infty} e^{itx}d\mu(x).$$

Our aim here is to generalize 3.2 so as to represent all positive definite functions by means of integrals 3.2.1.

3.3 THEOREM. *Let $p(t)$ be an arbitrary positive definite function on R. Then there exists a unique, regular, bounded, finitely additive, non-negative real measure*

(7) The measure μ, defined only on the algebra of sets $\mathcal{E}^*$, is said to be countably additive if $\sum_{\nu=1}^{\infty}\mu(A_\nu)=\mu(\bigcup_{\nu=1}^{\infty}A_\nu)$ for all families $\{A_\nu\}_{\nu=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{E}^*$ for which $\bigcup_{\nu=1}^{\infty}A_\nu\in\mathcal{E}^*$. It is a classical theorem that such a measure admits a unique countably additive extension over the σ-algebra generated by $\mathcal{E}^*$ (which is clearly the σ-algebra of all Borel sets). See Halmos [1, p. 54].

μ on $\mathcal{E}^*$ such that

3.3.1 $$p(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x), \qquad -\infty < t < +\infty.$$

Conversely, if μ is any such measure, the Fourier-Stieltjes transform

3.3.2 $$\int_{-\infty}^{\infty} e^{itx} d\mu(x)$$

is a positive definite function.

This theorem is easily inferred from the foregoing and the generalized form of Bochner's theorem for arbitrary locally compact Abelian groups. (See Weil [2, pp. 121–122], or Cartan and Godement [1, p. 89].) If we consider the group bR and its character group R_d (1.2), we see that the Radon measures treated by Cartan and Godement are, on the group bR, exactly the measures described in 2.1. The generalized Bochner theorem asserts that every continuous positive definite function $p(t)$ on $bR\,]=R_d$ has the form

3.3.3 $$p(t) = \int_{bR} \pi_t(\mathbf{x}) d\boldsymbol{\mu}(\mathbf{x}),$$

π_t being as in 1.2, and $\boldsymbol{\mu}$ being a non-negative regular Borel measure on bR. Since R_d is discrete, it follows that every positive definite function admits a (unique) representation 3.3.3. From 2.1, we see that 3.3.3 may be rewritten

3.3.4 $$p(t) = M_{\mathbf{x}}(\pi_t(\mathbf{x}))\text{(8)},$$

for a positive linear functional on $\mathfrak{C}(bR)$. In view of the equivalence between $\mathfrak{C}(bR)$ and $\mathfrak{A}$ established in 1.3, we find

3.3.5 $$p(t) = M_x(e^{itx}),$$

where M is a non-negative linear functional on $\mathfrak{A}$. It remains only to apply 2.6 to guarantee the existence of a measure μ on $\mathcal{E}^*$ satisfying the conditions set down and for which 3.3.1 obtains. Uniqueness of μ is obvious, as is the second assertion of the present theorem.

3.4 REMARK. We are now in possession of a triple of mathematical objects: the non-negative, regular, finitely additive, bounded measures on $\mathcal{E}^*$; the non-negative, finite, regular, countably additive Borel measures on bR; and the set of all positive definite functions $p(t)$ on the line R. These objects are equivalent in the sense that they all represent uniquely the non-negative linear functionals on $\mathfrak{A}$. It is an interesting and nontrivial problem to describe specified properties of say a positive definite function in terms of properties

(8) The subscript $\mathbf{x}$ indicates that the linear functional M operates on $\pi_t(\mathbf{x})$ as a function of $\mathbf{x}$.

of the two measures which give rise to it. The following sections are devoted to such problems.

3.5 The analogy between 3.3 and the usual form of Bochner's theorem appears incomplete, inasmuch as the measures appearing in the latter are countably additive Borel measures, while the measures mentioned in 3.3 are defined only on the subalgebra $\mathcal{E}^*$ of the Borel sets. However, a measure needs to be known only on a part of $\mathcal{E}^*$ in order to integrate the functions e^{itx} (2.8); so that we lose nothing by considering our measures only on $\mathcal{E}^*$. On the other hand, every finitely additive measure on $\mathcal{E}^*$ can be extended over all Borel sets so as to remain finitely additive. This extension is highly non-unique, however, and it seems best to express our results in terms of $\mathcal{E}^*$ alone.

3.6 It is known that every finitely additive measure ϕ on an algebra of subsets $\mathfrak{M}$ of a set X (real-valued and bounded) can be written uniquely as a sum $\phi_c+\phi_p$, where ϕ_c is countably additive and ϕ_p is purely finitely additive. (See Yosida and Hewitt [1].) With this in mind, we define pure discontinuity for positive definite functions.

3.7 DEFINITION. A positive definite function p is said to be purely discontinuous if there is no nonzero continuous positive definite function q such that for some positive definite q', $p=q+q'$.

3.8 THEOREM. *Every positive definite function p can be uniquely written in the form p_c+p_d, where p_c is continuous and p_d is purely discontinuous.*

This follows at once from 3.6 and 3.3.

4. RELATIONS BETWEEN THE MEASURES μ AND $\mathfrak{u}$

It is of some interest to sketch the relations existing between the measure μ on R_w and the measure $\mathfrak{u}$ on bR which represent a given linear functional M on $\mathfrak{A}$. We note that R_w is a subset of bR which is the union of a countable number of compact sets (the intervals $-n \leqq x \leqq n$ retain their usual topology in R_w) which are necessarily closed in bR. Hence R_w is an F_σ and is a Borel set. Therefore every Borel set in R_w is $\mathfrak{u}$-measurable, and we can examine $\mathfrak{u}$ both on R_w and on $bR \cap R'_w$. Measures μ, on the other hand, must be examined on R_w alone.

4.1 THEOREM. *If $\mathfrak{u}$ is a non-negative measure as in* 2.1, *then $\mathfrak{u}=\mathfrak{u}_1+\mathfrak{u}_2$, where $\mathfrak{u}_1(A)=\mathfrak{u}(A\cap R_w)$ and $\mathfrak{u}_2(A)=\mathfrak{u}(A\cap R'_w)$ for all Borel sets $A\subset bR$. $\mathfrak{u}_1=0$ if and only if the Fourier-Stieltjes transform* 3.3.3 *is purely discontinuous; and $\mathfrak{u}_2=0$ if and only if the Fourier-Stieltjes transform* 3.3.3 *is continuous.*

The proof is simple and is omitted.

4.2 THEOREM. *The invariant integral J on $\mathfrak{C}(bR)$, normalized so that $J(1)=1$, is represented as Bohr's mean value*

4.2.1 $$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)dx = J(f)$$

on the function-space $\mathfrak{A}$.

This assertion is apparent from the fact that the invariant integral is determined from its values on characters, and on these must be 1 or 0 as the character is 1 or not. As J has this property on $\mathfrak{A}$, it must coincide with the invariant integral.

4.3 THEOREM. *The Haar measure* $\boldsymbol{\iota}$ *of* $R_w \subset bR$ *is zero.*

Let $[a, b]$ be a closed bounded interval in R_w, and let α be a number greater than $b-a$. It is obvious that there exists a continuous function f_α with period α such that f is non-negative, $f=1$ on $[a, b]$, and $\int_t^{t+\alpha} f(x)dx < 2(b-a)$. Being continuous and periodic, f is in $\mathfrak{A}$, and a simple computation shows that $J(f)=(1/\alpha)\int_t^{t+\alpha} f(x)dx$. The function $\mathbf{f}$ obtained by extending f continuously over bR is $\geqq 1$ on $[a, b]\subset R_w$ and has the property that $J(\mathbf{f}) = J(f) < 2(b-a)/\alpha$. The Haar measure $\boldsymbol{\iota}$ of $[a, b]$ is $\leqq J(f) < 2(b-a)/\alpha$ for all $\alpha > (b-a)$, and hence is 0. As $\boldsymbol{\iota}$ is countably additive, it follows that $\boldsymbol{\iota}(R_w) = 0$.

4.4 THEOREM. *A measure* $\boldsymbol{\mu}$ *with continuous Fourier-Stieltjes transform has the property that* $|\boldsymbol{\mu}|(R_w') = 0$, *and is therefore singular with respect to Haar measure on* bR.

We may obviously take $\boldsymbol{\mu} \geqq 0$. If $p(t) = \int_{bR} \pi_t(\mathbf{x}) d\boldsymbol{\mu}(\mathbf{x})$ is continuous, then by Bochner's theorem, $p(t) = \int_{-\infty}^{\infty} e^{itx} d\mu_1(x)$ for some countably additive, non-negative Borel measure μ_1 on R_w. By the uniqueness theorem for Fourier-Stieltjes transforms on bR, we have $\boldsymbol{\mu} = \mu_1$, and thus $|\boldsymbol{\mu}|(R_w') = 0$. Since $\boldsymbol{\iota}(R_w) = 0$, it follows that $\boldsymbol{\mu}$ is singular.

4.5 THEOREM. *If* $\mathbf{f} \in \mathfrak{L}_1(bR)$ *and* $\mathbf{f} \geqq 0$ *almost everywhere, then* $\int_{bR} \pi_t(\mathbf{x}) \mathbf{f}(\mathbf{x}) d\boldsymbol{\iota}(\mathbf{x})$ *is a purely discontinuous positive definite function, and the measure* ϕ *on* $\mathcal{E}^*$ *corresponding to this function is purely finitely additive. In particular, if* $\sum_{\nu=1}^{\infty} |\alpha_\nu| < \infty$ *and* $\{t_\nu\}_{\nu=1}^{\infty}$ *is a sequence of real numbers, then the almost periodic function* $\sum_{\nu=1}^{\infty} \alpha_\nu e^{it_\nu x}$ *is* $\geqq 0$ *everywhere if and only if the function* $\sum_{\nu=1}^{\infty} \alpha_\nu \chi_{\{t_\nu\}}(t)$ *is positive definite; and all such functions are purely discontinuous.*

This follows immediately from 4.1.

4.6 DEFINITION. Let $\mathbf{x}$ be a point in bR. The measure $\boldsymbol{\varepsilon}_{\mathbf{x}}$ is defined for all subsets A of bR by the relation

$$\boldsymbol{\varepsilon}_{\mathbf{x}}(A) = \begin{cases} 1 & \text{for} \quad \mathbf{x} \in A, \\ 0 & \text{for} \quad \mathbf{x} \notin A. \end{cases}$$

4.7 Since $\mathfrak{A}$ is a Banach algebra as well as a Banach space, we may study the measures ω of the kind described in 2.6 which produce algebra-homomorphisms of $\mathfrak{A}$ onto K, i.e., for which

4.7.1 $$\int_{-\infty}^{+\infty} f(x)g(x)d\omega(x) = \int_{-\infty}^{\infty} f(x)d\omega(x)\cdot\int_{-\infty}^{\infty} g(x)d\omega(x).$$

Representing $\mathfrak{A}$ as $\mathfrak{C}(bR)$, we use a well known fact to assert that the measures $\mathfrak{y}$ as described in 2.1 which preserve products are exactly those of the form $\mathfrak{e}_{\mathfrak{x}}$ for some $\mathfrak{x}$ in bR. It is clear that the finitely additive counterparts ω of $\mathfrak{e}_{\mathfrak{x}}$ ($\mathfrak{x}\notin R_w$) on the algebra $\mathcal{E}^*$ must be of a peculiar kind. As we shall see in Theorem 5.2, the measures ω are closely connected with non-Lebesgue measurable functions of t, so that a complete description of these measures is beyond our reach. Nevertheless, we have some information about them.

4.8 THEOREM. *A finitely additive regular measure ω on $\mathcal{E}^*$ is multiplicative, in the sense that* 4.7.1 *is satisfied for all f, $g\in\mathfrak{A}$, if and only if ω assumes only the values* 0 *and* 1.

Let L be any algebra-homomorphism of $\mathfrak{A}$ onto K, and let λ and $\mathfrak{l}$ be the measures on R_w and bR respectively which represent L. As $\mathfrak{l}$ is a measure $\mathfrak{e}_{\mathfrak{x}}$ for some $\mathfrak{x}\in bR$, it is clear that L is a non-negative functional and that λ is a non-negative measure. Since $L(1)=1$, it is clear that $0\leqq\lambda(A)\leqq 1$ for all $A\in\mathcal{E}^*$. If $0<\lambda(A)<1$ for some $A\in\mathcal{E}^*$, there must be a $P\in\mathcal{E}$ such that $0<\lambda(P)<1$. Since λ is regular, there is a set $Z\in\mathcal{J}$ such that $Z\subset P$ and $\lambda(Z)>\lambda(P)/2$. As shown in the proof of 2.6, there is an $f\in\mathfrak{A}$ such that $\chi_Z\leqq f\leqq\chi_P$. It is clear that $\lambda(Z)\leqq L(f)\leqq\lambda(P)$. However, $L(f^n)=(L(f))^n$, and as $\chi_Z\leqq f^n\leqq\chi_P$ $(n=1, 2, 3, \cdots)$ we have $\lambda(P)/2\leqq(\lambda(P))^n$ for $n=1, 2, 3, \cdots$, an evident contradiction. Hence λ assumes only the values 0 and 1.

To establish the converse, suppose that ω is a finitely additive measure (not necessarily regular) on $\mathcal{E}^*$ assuming only the values 0 and 1. Using the identity $(u+v)^2=u^2+2uv+v^2$, we see that it suffices to verify 4.7.1 for the case $f=g$. Suppose that f is real. Let $\alpha=\inf_{x\in R_w} f(x)$, $\beta=\sup_{x\in R_w} f(x)$;

$$A_{n,k} = E[x;\, x\in R_w,\, 2^{-n}\{(2^n-k)\alpha+k\beta\} \leqq f(x) < 2^{-n}\{(2^n-k-1)\alpha+(k+1)\beta\}]$$

for $n=1, 2, 3, \cdots$ and $k=0, 1, \cdots, 2^n-2$; and $A_{n,2^n-1}$ by replacing "$<$" by "$\leqq$" in the last inequality defining $A_{n,k}$. Since ω assumes only the values 0 and 1, it is plain that we have a sequence of sets A_{1,k_1}; A_{2,k_2}; A_{3,k_3}; $\cdots$; A_{n,k_n}; $\cdots$ all of ω-measure 1 and such that $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} 2^{-n}k_n=\alpha$ for any sequence $x_n\in A_{n,k_n}$. It is also plain that $\int_{R_w} f(x)d\omega(x)=\alpha$, and that $\int_{R_w} f^2(x)d\omega(x)=\alpha^2$. Complex f's are handled by a brief computation, and the present theorem is thus proved.

It seems likely that all 0-1 measures on $\mathcal{E}^*$ are regular, but we have been unable to decide this point.

4.9 REMARK. Theorem 4.8 gives us another representation for bR. Indeed, bR is representable as:

4.9.1 the closure of R_w in $T^c(1.1)$;

4.9.2 the space of all algebra-homomorphisms of $\mathfrak{A}$ onto K, given the weakest topology in which these homomorphisms are continuous functions;

4.9.3 the space Ω of all regular finitely additive measures on $\mathcal{E}^*$ assuming only the values 0 and 1. For a given $P\in\mathcal{E}$, a neighborhood Δ_P consists of all ω such that $\omega(P)=1$, and is assigned as an (open) neighborhood of every point it contains. (We define Δ_A analogously for all $A\in\mathcal{E}^*$.)

Representation 4.9.2 is very well known and needs no further discussion. Representation 4.9.3, which we shall use in the sequel, is discussed in 4.10 below. (A representation somewhat like 4.9.3 has been described in another connection (see Yosida and Hewitt [1, Theorems 4.1–4.3]).) 4.9.1 is the only one of the representations in which the group operation is evidently definable. It is far from obvious that 4.9.2 and 4.9.3 actually define compact Abelian groups. Representing bR as $R_d]$ (1.4), we see by a Hamel basis argument that the cardinal number of bR is 2^c. Hence there are 2^c 0-1 measures on $\mathcal{E}^*$, and obviously 2^c of them are not of the form ε_x for $x\in R_w$.

4.10 THEOREM. *The space* Ω *of* 4.9.3 *is homeomorphic to the space bR. Points of* R_w *correspond to measures* ε_x *with* $x\in R_w$. *The correspondence*

4.10.1 $$P\to\Delta_P$$

has the properties that

4.10.2 $$\Delta_{P\cup Q}=\Delta_P\cup\Delta_Q,$$

4.10.3 $$\Delta_{P\cap Q}=\Delta_P\cap\Delta_Q,$$

and

4.10.4 $$(\Delta_P)'=\Delta_{P'}.$$

We first verify 4.10.2 and 4.10.3. If $\omega(P)=\omega(Q)=0$, then $\omega(P\cup Q)=\omega(P\cap Q')+\omega(P'\cap Q)+\omega(P\cap Q)=0$, and therefore $\Delta_{P\cup Q}\subset\Delta_P\cup\Delta_Q$. Conversely, if $\omega(P)=1$, then $1=\omega(P)\leqq\omega(P\cup Q)\leqq 1$, and hence 4.10.2 is verified. If $\omega(P\cap Q)=1$, then we have $1=\omega(P\cap Q)\leqq\omega(P)$, $\omega(Q)\leqq 1$. Conversely, if $\omega(P)=\omega(Q)=1$, then we have

$$1=\omega(P\cup Q)=\omega(P'\cap Q)+\omega(P\cap Q')+\omega(P\cap Q)=\omega(P\cap Q).$$

Hence 4.10.3 holds. From 4.10.3, we see that the intersection of two neighborhoods is a neighborhood, and thus Ω is a neighborhood space. To verify that Ω is a Hausdorff space, let ω_1 and ω_2 be distinct points of Ω, regarded as points of bR. Then there are continuous functions f_1 and f_2 on bR such that f_j is non-negative, $f_j(\omega_j)=1$ $(j=1, 2)$, and $E[f_1>0]\cap E[f_2>0]=0$. Writing P_j as $R_w\cap E[f_j>0]$, we see immediately that $\Delta_{P_1}\cap\Delta_{P_2}=0$, and that $\omega_j \in \Delta_{P_j}$ $(j=1, 2)$. It is evident that the natural mapping of the compact Hausdorff space bR onto the Hausdorff space Ω is continuous, and therefore Ω is compact and homeomorphic to bR.

4.11 THEOREM. *Let M be any bounded linear functional on $\mathfrak{A}$, and let μ and $\mathfrak{u}$ be the measures on R_w and bR respectively which yield integral representations for M. Under the correspondence* 4.10.1, *we have*

4.11.1 $$\mu(P) = \mathfrak{u}(\Delta_P).$$

μ is determined on the algebra of μ-measurable sets by its values on $\mathcal{E}$; and $\mathfrak{u}$ on all Borel sets in bR by its values on $\{\Delta_P\}_{P\in\mathcal{E}}$. Hence μ and $\mathfrak{u}$ are identical on the algebras generated by $\mathcal{E}$ and $\{\Delta_P\}_{P\in\mathcal{E}}$, respectively.

We may obviously suppose that M is a positive functional. In accordance with 2.6.4, we note that $\mu(P) = \sup_{0\leqq f\leqq \chi_P} M(f)$. Now consider the extension $\mathbf{f}$ of f for an arbitrary f such that $0\leqq f\leqq \chi_P$. Clearly $\mathbf{f}$ assumes only values in the interval $[0, 1]$. If $\omega\in(\Delta_P)'=\Delta_{P'}$, then $\int_{-\infty}^{\infty} f(x)d\omega(x) = \int_{P'} f(x)d\omega(x) = 0$, and $\mathbf{f}=0$ at ω. Hence $0\leqq \mathbf{f}\leqq \chi_{\Delta_P}$. Since $\mathfrak{u}(\Delta_P) = \sup M(\mathbf{f})$, over all $\mathbf{f}$ such that $0\leqq \mathbf{f}\leqq \chi_{\Delta_P}$, it follows that $\mu(P)\leqq \mathfrak{u}(\Delta_P)$. Conversely, if $0\leqq \mathbf{f}\leqq \chi_{\Delta_P}$, it is obvious that the contraction f of $\mathbf{f}$ to R_w satisfies the inequalities $0\leqq f\leqq \chi_P$, and from this 4.11.1 follows. To see that μ is determined by its values on $\mathcal{E}$, we note that $\mu(Z) = \mu(R_w) - \mu(Z')$ for $Z\in\mathcal{Z}$, and that μ's values on all $A\in\mathcal{E}^*$ are determined therefrom by regularity. The same argument applies to $\mathfrak{u}$. It is now clear that the algebras generated by $\mathcal{E}$ and $\{\Delta_P\}_{P\in\mathcal{E}}$ are isomorphic and that μ and $\mathfrak{u}$ have equal values for corresponding elements.

4.12 REMARK. Most of the measures μ are purely finitely additive, while all of the measures $\mathfrak{u}$ are of course countably additive. It follows that μ and $\mathfrak{u}$ cannot possibly be in general identical on the σ-algebras generated by $\mathcal{E}$ and $\{\Delta_P\}_{P\in\mathcal{E}}$; and it is of some interest to see what happens to the algebra $\mathcal{E}^*$ when we go over to the corresponding algebra generated by $\{\Delta_P\}_{P\in\mathcal{E}}$. For a non-countably additive, non-negative μ, there exists at least one sequence $P_1\supset \cdots P_n\supset \cdots$ in $\mathcal{E}$ with void intersection for which $\lim_{n\to\infty}\mu(P_n)>0$. The space bR, which contains all Δ_{P_n}, provides points in $\bigcap_{n=1}^{\infty}\Delta_{P_n}$ so that $\mathfrak{u}(\lim_{n\to\infty}\Delta_{P_n}) = \lim_{n\to\infty}\mathfrak{u}(\Delta_{P_n})$, in accordance with the demands of countable additivity. An extreme example of this phenomenon is provided by the functional $L(\mathbf{f}) = \int_{bR}\mathbf{f}(x)\boldsymbol{\alpha}(x)d\mathfrak{l}(x)$, where $\boldsymbol{\alpha}(\neq 0)$ is any function in $\mathfrak{L}_1(bR)$, which for convenience we take to be non-negative. Let the measures corresponding to L be λ and $\boldsymbol{\lambda}$. Write Q_n for the set $E[x;\, x\in R_w, |x|>n]$. It is clear from 2.4 that $Q_n\in\mathcal{E}$, and from Theorems 4.3 and 4.11, we infer that $\lambda(Q_n) = \boldsymbol{\lambda}(\Delta_{Q_n}) = \lambda(R_w) = \boldsymbol{\lambda}(bR)$. Thus $\lim_{n\to\infty}\lambda(Q_n) = \lambda(R_w) \neq \lambda(\lim_{n\to\infty} Q_n) = 0$. However, $\lim_{n\to\infty}\Delta_{Q_n} = bR\cap R_w'$, as one sees immediately, so that we have

$$\boldsymbol{\lambda}(\lim_{n\to\infty}\Delta_{Q_n}) = \boldsymbol{\lambda}(bR\cap R_w') = \boldsymbol{\lambda}(bR) = \boldsymbol{\lambda}\lim_{n\to\infty}(\Delta_{Q_n}).$$

4.13 Finally, we consider the operation of convolution, which can be carried out on measures μ, measures $\boldsymbol{\mu}$, and functionals in $\mathfrak{A}^*$. The formula

4.13.1 $$L * M(f) = L_x(M_y(f(x + y))),$$

where the subscripts x and y indicate the variable with regard to which the functional is to be applied, defines a functional $L * M$, the convolution of L and M, for every L and M in $\mathfrak{A}^*$. This is a particular case of a situation discussed, for example, in Hewitt and Zuckerman [1]. If λ and μ are the measures on $\mathcal{E}^*$ corresponding to L and M, respectively, and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ similarly Borel measures on bR, then we obtain new measures $\lambda * \mu$ and $\boldsymbol{\lambda} * \boldsymbol{\mu}$ corresponding to $L * M$. Plainly $L * M(e^{itx}) = L(e^{itx})M(e^{itx})$. For a Borel set $A \subset bR$, it can be shown that $\boldsymbol{\lambda} * \boldsymbol{\mu}(A) = \int_{bR} \boldsymbol{\lambda}(A - \mathbf{x}) d\boldsymbol{\mu}(\mathbf{x})$; a similar formula is valid for $\lambda * \mu$ and $A \in \mathcal{E}^*$.

5. Examples

We turn now to a description of a few discontinuous positive definite functions and the measures μ and $\boldsymbol{\mu}$ associated with them. We have first:

5.1 Theorem. *Let G be any additive subgroup of R. Let p be any positive definite function on R. Then the function $q = p \cdot \chi_G$ is also positive definite.*

Consider the form

5.1.1 $$\sum_{j,k=1}^{N} q(t_j - t_k)\beta_j\bar{\beta}_k,$$

for $t_1, \cdots, t_N \in R$ and $\beta_1, \cdots, \beta_N \in K$. Let $t_1, \cdots, t_n$ be those t's which are in G. Then $t_j - t_k \notin G$ for $1 \leqq j \leqq n < k \leqq N$, and 5.1.1 is therefore equal to

5.1.2 $$\sum_{j,k=1}^{n} p(t_j - t_k)\beta_j\bar{\beta}_k + \sum_{j,k=n+1}^{N} q(t_j - t_k)\beta_j\bar{\beta}_k.$$

If no t_j, $n \leqq j \leqq N$, has the property that $t_j - t_k \in G$ for some k, $n \leqq k \leqq N$ and $j \neq k$, then the second summand in 5.1.2 reduces to

5.1.3 $$\sum_{j=n+1}^{N} p(0)\beta_j\bar{\beta}_j.$$

If some t_j, say t_{n+1}, has the property that $t_{n+1} - t_k \in G$ for certain k's $> n+1$, let $t_{n+2}, \cdots, t_{n_1}$ be these t_k's. Then, since G is a subgroup, it is plain that $t_j - t_k \in G$ for $n+1 \leqq j$, $k \leqq n_1$. We may thus write

5.1.4 $$\sum_{j,k=n+1}^{N} q(t_j - t_k)\beta_j\bar{\beta}_k = \sum_{j,k=n+1}^{n_1} p(t_j - t_k)\beta_j\bar{\beta}_k + \sum_{j,k=n_1+1}^{N} q(t_j - t_k)\beta_j\bar{\beta}_k.$$

An obvious induction completes the proof, showing that 5.1.1 is always non-negative.

Theorem 5.1 gives us a method of obtaining a large class of positive definite functions, all of them discontinuous if G is a proper subgroup of R. However, we fail to obtain all discontinuous positive definite functions in this fashion, as the next theorem shows.

5.2 THEOREM. *Let $\phi(t)$ be an arbitrary character of the group R_d. Then ϕ is the Fourier-Stieltjes transform* 3.3.2 *of a* 0-1 *measure ω on $\mathcal{E}^*$, or, equivalently, the transform* 3.3.3 *of a point-measure $\epsilon_{\mathbf{x}}$ in bR. Conversely, all such transforms are characters. The character ϕ is continuous if and only if $\mathbf{x}\in R_w$, or equivalently, the measure ω is countably additive.*

This assertion follows from 1.4. Given a character $\phi(t)$ of R_d, we know from 1.4 that $\phi(t)=\pi_t(\mathbf{x})$ for some fixed $\mathbf{x}\in bR$. As $\pi_t(\mathbf{x})$ is exactly the value of $\int_{bR}\pi_t(y)d\epsilon_{\mathbf{x}}(y)$, the first two statements of the present theorem are clear. As $\pi_t(\mathbf{x})=e^{ita}$ for some $a\in R_w$ if and only if $\mathbf{x}\in R_w$, the third statement is also verified.

5.3 REMARK. 5.2 shows, curiously enough, that $\int_{-\infty}^{\infty}e^{itx}d\omega(x)$ is a discontinuous and hence non-Lebesgue measurable character on R_d for every 0-1 measure ω on $\mathcal{E}^*$ which is not countably additive. This circumstance points up the immense difference between countably additive and finitely additive integrals.

5.4 THEOREM. *The positive definite function $\chi_{\{0\}}(t)$ is the Fourier-Stieltjes transform of the functional J described in* 4.2. *The measure $\boldsymbol{\iota}$ on bR corresponding to $\chi_{\{0\}}(t)$ is Haar measure. The corresponding measure ι on $\mathcal{E}^*$ has the property that for an arbitrary bounded Lebesgue measurable set $T\subset R_w$ and $\alpha\geqq\sup T-\inf T$, $\iota(T_\alpha)=(1/\alpha)m(T)$. This defines ι, within the class of all regular bounded measures on $\mathcal{E}^*$.*

The present theorem follows readily from 4.2 and 4.3. We omit the details.

The relation between the measure $\iota(B)$ and the mean value $J(\chi_B)$ for sets $B\subset R_w$ is not simple. For example, H. Bohr [1] has shown that there exists a function $g\in\mathfrak{A}$ such that, writing $Z=E[x; x\in R, g(x)=0]$, the mean value $J(\chi_Z)$ does not exist. Since $\iota(Z)$ must exist, by our construction for ι, it follows that there exist sets Z measurable (ι) for which $J(\chi_Z)$ does not exist. We do not know if the existence of $J(\chi_B)$ for $B\subset R_w$ implies that B is ι-measurable and $\iota(B)=J(\chi_B)$.

Theorem 5.1 shows that the characteristic function of an arbitrary additive subgroup of R_d is positive definite and is hence the Fourier-Stieltjes transform of a measure on $\mathcal{E}^*$. Consider the subgroup $M=\{2\pi n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ of R_d. We find, surprisingly enough, that $\chi_M(t)$ bears the same relation to mean value over the integers that $\chi_{\{0\}}(t)$ bears to mean value over the entire line. We state a general result first. Let N denote the set of all integers in R_w.

5.5 THEOREM. *Let $p(t)$ be any positive definite function on R_d having period*

2π. Then there exists a non-negative regular bounded measure ϕ on $\mathcal{E}^*$ such that $\phi(N')=0$ and such that $p(t)=\int_{-\infty}^{+\infty}e^{itx}d\phi(x)$. Conversely, if ϕ is a regular bounded non-negative measure on $\mathcal{E}^*$ such that $\phi(N')=0$, then the Fourier-Stieltjes transform of ϕ has period 2π.

Since $p(t)$ has period 2π, we may regard it as a function on the additive group of real numbers modulo 2π, *i.e.*, on the circle group T. It follows from arguments for the group N and the group $T=N]$, exactly like those given for R and $R=R]$ in §§1 and 2, that there exists a compact group bN, containing an isomorph of N as a dense subgroup, with character group equal to T_d, which is T in its discrete topology. The entire theory worked out above for functionals on $\mathfrak{A}$ and their two integral representations can be carried over without change to the case of N, bN, and T_d. Thus we assert that there is a finitely additive non-negative measure ϕ_1 defined on a certain algebra $\mathcal{E}_N^*$ of subsets of N with the property that $p(t)=\int_N e^{itn}d\phi_1(n)$, for all $t\in[0, 2\pi)$. We take $\mathcal{E}_N$ as the family of all sets $E[n;\ n\in N,\ f(n)\neq 0]$ for f's which are uniform limits of linear combinations of functions e^{itn} on N $(0\leq t<2\pi)$. The measure ϕ_1 is regular in the obvious sense. It is clear that the sets in $\mathcal{E}_N^*$ are exactly the sets which can be written in the form $A\cap N$ for $A\in\mathcal{E}^*$. It is thus easy to see that ϕ_1 can be regarded as a measure on $\mathcal{E}^*$ and is regular. The integral $\int_{R_w}e^{itx}d\phi_1(x)$ has a meaning for every real t, and in fact gives us the periodic function $p(t)$. The uniqueness theorem 2.6 shows that the measure ϕ_1 and the measure ϕ on $\mathcal{E}^*$ whose transform is $p(t)$ must be identical.

To establish the converse, consider any ϕ such that $\phi(N')=0$. Then

$$\int_{R_w} e^{itx}d\phi(x) = \int_N e^{itn}d\phi(n) = \int_N e^{i(t+2\pi)n}d\phi(n).$$

Therefore the transform of ϕ has period 2π.

5.6 Remark. The resolution of a measure γ on $\mathcal{E}_N^*$ into its countably additive and purely finitely additive parts is of course trivial. Let $\alpha_n=\gamma(\{n\})$, and let $\gamma_c=\sum_{n=-\infty}^{+\infty}\alpha_n\epsilon_n$. Then γ_c is the countably additive part and $\gamma-\gamma_c$ the purely finitely additive part of γ. From this and 5.5, we infer that every continuous positive definite function with period 2π has an absolutely convergent Fourier series with all coefficients non-negative. (This of course is well known.)

We now examine the positive definite function $\chi_M(t)$.

5.7 Theorem. *The function $\chi_M(t)$ is purely discontinuous. The measure ψ on $\mathcal{E}^*$ whose transform is $\chi_M(t)$ is purely finitely additive, has the property that $\psi(N')=0$, and is representable as a mean value:*

5.7.1 $$\int_{R_w} f(x)d\psi(x) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{\nu=-n}^{n} f(\nu),$$

for all $f\in\mathfrak{A}$.

We prove first 5.7.1. Let $L(f)=\int_{R_w} f(x)d\psi(x)$. Then it is clear that $L(e^{itx})=0$ or 1 as $t\neq 2k\pi$ or $t=2k\pi$ for some integer k. An elementary computation shows that

5.7.2 $$\frac{1}{2n+1}\sum_{\nu=-n}^{n} e^{it\nu} = \frac{1}{2n+1}\,\frac{e^{-int}-e^{i(n+1)t}}{1-e^{it}}$$

for $t\neq 2k\pi$. It is thus obvious that 5.7.1 represents L for trigonometric polynomials. By continuity, the same is true for all functions in $\mathfrak{A}$. The remainder of the theorem now follows from 5.5.

5.8 THEOREM. *Let $p(t)$ be any positive definite function, and let μ be the measure on $\mathcal{E}^*$ of which $p(t)$ is the Fourier-Stieltjes transform. Then the positive definite function $p(t)\chi_M(t)$ is the Fourier-Stieltjes transform of the measure $\mu*\psi$.*

This follows at once from 5.7 and 4.13.

It appears that measures on $\mathcal{E}^*$ with transforms equal to the characteristic functions of more complicated subgroups of R_d—e.g., the rational numbers—are very complicated indeed. Nevertheless, Professor M. Riesz has generously communicated to us the following analytic representation for the functional T on $\mathfrak{A}$ whose transform is $\chi_D(t)$ (we write D for the rational numbers):

$$T(f) = \lim_{m\to\infty}\left\{\lim_{n\to\infty}\left\{\frac{1}{2n+1}\sum_{k=-n}^{n} f(2\pi m!k)\right\}\right\}.$$

We have not succeeded in identifying the measure associated with T.

As an application of Theorems 4.5 and 5.1, we give a proof of a known theorem on Fourier series (Zygmund [1, p. 109]).

5.9. THEOREM. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a decreasing, convex sequence of non-negative real numbers such that $\sum_{n=0}^{\infty}\alpha_n<\infty$. Let u be a positive real number. Then $\alpha_0/2+\sum_{n=1}^{\infty}\alpha_n\cos nux$ is non-negative.*

The even function $p(t)$ obtained by linear interpolation between the values α_n at nu ($n=0, 1, 2, \cdots$) is easily seen to be convex, in $\mathfrak{L}_1(-\infty,\infty)$, and continuous. Hence it is positive definite, as its Fourier transform $2\int_0^\infty \cos txp(t)dt$ is evidently non-negative(9). By 5.1, the function $q(t)=\sum_{n=-\infty}^{\infty}\alpha_n\chi_{\{nu\}}(t)$ is also positive definite. Hence it is the Fourier-Stieltjes transform of a non-negative measure on $\mathcal{E}^*$. However, $q(t)$ is the transform of the functional defined for $f\in\mathfrak{A}$ by $H(f)=J(f\cdot h)$, where $\alpha_{-n}=\alpha_n$ and $h=\sum_{n=-\infty}^{\infty}\alpha_n e^{iunx}=\alpha_0+2\sum_{n=1}^{\infty}\alpha_n\cos unx$. Hence h is non-negative.

As another application of Theorem 5.1, we have:

5.10 THEOREM. *Let $f(x)=\sum_{n=1}^{\infty}\gamma_n e^{iu_nx}$ be an almost periodic function*

(9) This was pointed out to the writer by Professor A. Beurling.

such that $\sum_{n=1}^{\infty} |\gamma_n| < \infty$. *For G an arbitrary additive subgroup of R, the inequality* $|\sum_{u_n \in G} \gamma_n| \leqq \sup_{x \in R} |f(x)|$ *is valid.*

The positive definite function $q(t) = \chi_G(t)$ corresponds to a non-negative functional Q of norm 1 on $\mathfrak{A}$, which evidently carries f into the value $\sum_{u_n \in G} \gamma_n$. That is,

$$\left| \sum_{u_n \in G} \gamma_n \right| = |Q(f)| \leqq \|Q\| \cdot \sup_{x \in R} |f(x)| = \sup_{x \in R} |f(x)| \text{ (10)}.$$

A large number of other theorems on almost periodic functions can be established by our present method. We intend to set them forth in a later communication.

6. The space $\mathfrak{A}$ as a convolution algebra

6.1 The space $\mathfrak{A}$, considered as a space of functions on R_w, is of course a complex Banach algebra under pointwise multiplication. It is indeed this fact that makes Theorem 1.3 valid. The ideal theory of $\mathfrak{A}$ is far from simple, but it is more or less known: every closed ideal in $\mathfrak{A}$ is the set of functions f in $\mathfrak{A}$ such that f vanishes on some closed subset of bR. The extremely complicated structure of bR introduces essential complications into the study of ideals in $\mathfrak{A}$.

6.2 However, $\mathfrak{A}$ admits a multiplication under which it is a complex Banach algebra of a quite different type. For f and $g \in \mathfrak{A}$, let the convolution $f * g$ be defined by the relation

6.2.1
$$\begin{aligned} f * g(x) &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x-y) g(y) dy \text{ (11)} \\ &= \int_{-\infty}^{+\infty} f(x-y) g(y) d\iota(y). \end{aligned}$$

It is easy to see that $f * g$ is the contraction to R_w of the function $\boldsymbol{f} * \boldsymbol{g}$ on bR, defined by the relation

6.2.2
$$\boldsymbol{f} * \boldsymbol{g}(\mathbf{x}) = \int_{bR} \boldsymbol{f}(\mathbf{x} - \boldsymbol{y}) \boldsymbol{g}(\boldsymbol{y}) d\boldsymbol{\iota}(\boldsymbol{y}).$$

A general closed ideal in this algebra is identified with an arbitrary subset X of R_d, and consists of all $f \in \mathfrak{A}$ such that

6.2.3
$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} f(x) dx = 0$$

(10) Dr. E. Følner has pointed out that this result can also be obtained by convolving f with suitable Bochner-Fejér polynomials.

(11) It is simple to verify that $f*g$ is actually in $\mathfrak{A}$, making use of the fact that both f and g are uniform limits of trigonometric polynomials.

for all $t \in X$. This assertion is proved by representing $\mathfrak{A}$ as $\mathfrak{C}(bR)$ and using known facts about the algebra of continuous functions on a compact group under convolution.

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