COMPLEX TAUBERIAN THEOREMS FOR POWER SERIES(1)

BY

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1. **Introduction.** Given a power series (2)

$$f(z) = \sum a_n z^n = (1 - z) \sum s_n z^n \tag{1.1}$$

with radius of convergence 1, we wish to study the relations between the sequence (s) and its associated function $f(z)$ which may be considered as a transformation $T(s)$. Direct (or Abelian) theorems conclude from the sequence (s) to the transformation $T(s)$, while Tauberian theorems infer conclusions from the behavior of $T(s)$ to the behavior of (s) under specified additional conditions (Tauberian conditions).

For the special transformation T which transforms (s) into $f(z)$ according to (1.1) we distinguish two types of Tauberian theorems. Tauberian theorems of real character use assumptions about $f(z)$ where z is on the real axis, and have real Tauberian conditions; for example

$$\lim_{z \to 1-0} f(z) = s \quad \textit{and} \quad a_n \geqq 0 \quad \textit{implies} \quad \lim_{n \to \infty} s_n = s.$$

In this paper we are concerned with Tauberian theorems of complex character in which the assumptions on $f(z)$ are essentially complex.

One of these complex Tauberian theorems was given first by Fatou [3, p. 389]:

THEOREM A. *If the function $f(z)$ defined by* (1.1) *is regular at $z=1$ and $a_n \to 0$ $(n \to \infty)$, then $\sum a_n$ converges.*

Another theorem of this type is due to M. Riesz (see, e.g. [9, p. 64]):

THEOREM B. *If the function $f(z)$ defined by* (1.1) *is regular in the region*

$$S_1: \quad \begin{cases} |z| < R, & R > 1, \\ |\operatorname{arc}(z - 1)| > \theta_0, & 0 < \theta_0 < \pi/2, \end{cases}$$

and continuous in $\overline{S}_1$ (the closure of S_1), then $\sum a_n$ converges.

After recalling some well known facts in summability theory we shall relax the assumptions on $f(z)$ in the Theorems A and B and prove a theorem which contains both as special cases. The condition about the behavior of

Received by the editors August 7, 1952.

(1) The results of this paper are contained in the author's Ph.D. Thesis [4]. Numbers in brackets refer to the bibliography at the end of the paper.

(2) We shall always let $\sum_{n=0}^{\infty} a_n = \sum a_n$.

$f(z)$ in S_1 is thereby localized to a smaller region S_2 in the neighborhood of $z=1$ which is the smallest possible in a certain sense. In §4 other regions are taken instead of S_2.

In §§5 and 6 we assume that $f(z)=\sum a_n z^n$ has a positive radius of convergence and that $z=1$ lies on the boundary of the region of V-summability associated with $f(z)$. In analogy to the above theorems we infer conclusions from the behavior of $f(z)$ in the neighborhood of $z=1$ to the behavior of $V(s)$. In an extended sense these theorems are also complex Tauberian theorems. For V the methods of Euler-Knopp, Borel, and Meyer-König are taken and thus extensions of results of Karamata, Meyer-König, and Obrechkoff are obtained.

2. **The methods** E_p, B, S_α. The methods of Euler-Knopp, Borel, and Meyer-König are useful for the analytic continuation of power series. Assume that a series $\sum a_n$ with partial sums $s_n=\sum_{\nu=0}^{n} a_\nu$ is given and that the associated power series $f(z)=\sum a_n z^n$ has a positive radius of convergence.

a. The method E_p (Euler-Knopp) with fixed parameter p $(0<p<\infty)$ is defined by the triangular matrix (E_p) with elements

$$c_{n\nu} = \frac{1}{2^{pn}}\binom{n}{\nu}(2^p-1)^{n-\nu} \qquad (n,\nu=0,1,2,\cdots;\nu\leqq n). \tag{2.1}$$

The E_p-transformation of the sequence s_n is therefore the new sequence

$$E_p(n;s_\nu) = \frac{1}{2^{pn}}\sum_{\nu=0}^{n}\binom{n}{\nu}(2^p-1)^{n-\nu}s_\nu \qquad (n=0,1,\cdots)$$

and we say that E_p-lim $s_n=s$ (or E_p-$\sum a_n=s$) if $\lim_{n\to\infty}E_p(n;s_\nu)=s$.

For our purposes another definition of the E_p-transformation of the sequence s_n is more suitable. Letting

$$z=\phi_p(w)=\frac{w}{2^p-(2^p-1)w} \tag{2.2}$$

and developing $f(z)$ into powers of w we obtain a power series

$$F(w)=\sum a_n' w^n$$

which is convergent in the neighborhood of $w=0$. It is easy to prove that

$$E_p(n;s_\nu)=\sum_{\kappa=0}^{n}a_\kappa' \quad\text{where}\quad \begin{cases} a_n' = 2^{-p}E_p(n-1;a_{\nu+1}) & (n=1,2,\cdots),\\ a_0' = a_0. \end{cases} \tag{2.3}$$

Hence E_p-$\sum a_n$ exists if and only if $\sum a_n'$ converges, and E_p-lim $a_n=0$ is necessary for the existence of E_p-$\sum a_n$. (Here we use the fact that $E_p(n;a_{\nu+1})\to 0$ $(n\to\infty)$ is equivalent to $E_p(n;a_\nu)\to 0$ $(n\to\infty)$.)

If the method E_p is applied to $\sum a_n z^n$ for different values of z, one obtains a region $\mathfrak{G}_{E_p}$ in the z-plane in the interior of which the power series is E_p-summable to the value $f(z)$, whereas it is certainly not summable for any z which lies outside of $\overline{\mathfrak{G}_{E_p}}$. On the boundary of $\mathfrak{G}_{E_p}$ no general statement can be made and the situation is somewhat similar to the situation on the boundary of the circle of convergence of a power series. $\mathfrak{G}_{E_p}$ can be constructed from the singularities of $f(z)$; we shall use that $f(z)$ is necessarily regular in $\mathfrak{K}_{E_p} = \mathfrak{K}((2^p-1)/(2^{p+1}-1))$ if $z=1$ is on the boundary of $\mathfrak{G}_{E_p}$. (We denote by $\mathfrak{K}(a)$ the region $|z-a|<1-a$ for $0<a<1$.) $\mathfrak{K}_{E_p}$ is the map of $|w|<1$ under the transformation $z=\phi_p(w)$.

b. Two methods of Borel are known. In the case of the "exponential method" we let

$$B(x; s_\nu) = e^{-x} \sum \frac{s_n x^n}{n!} \qquad (x \geqq 0)$$

where the sum exists for $x \geqq 0$ since $\sum a_n z^n$ has a positive radius of convergence; we say that B-lim $s_n = s$ (or B-$\sum a_n = s$) if $\lim_{x\to\infty} B(x; s_\nu) = s$.

The second method is called Borel's "integral method" and is often more suitable to power series. With the given series $\sum a_n$ we associate the function $\phi(t) = \sum a_n t^n/n!$ for $t \geqq 0$ and let

$$B'(x; s_\nu) = \int_0^x e^{-t}\phi(t)dt \qquad (x \geqq 0);$$

we say that B'-lim $s_n = s$ (or B'-$\sum a_n = s$) if $\lim_{x\to\infty} B'(x; s_\nu) = s$.

Concerning the relations between these two methods it is known that

$$\text{(2.4)} \qquad B\text{-lim } s_n = s \quad \text{implies} \quad B'\text{-lim } s_n = s,$$

but not conversely. However, since

$$\text{(2.5)} \qquad B(x; s_\nu) = B(x; a_\nu) + B'(x; s_\nu),$$

the converse of (2.4) is true if and only if $B(x; a_\nu) \to 0 (x \to \infty)$. The relations (2.4) and (2.5) also imply that B-lim $a_n = 0$ is necessary for the existence of B-$\sum a_n$.

The region $\mathfrak{G}_B$ of Borel-summability of the power series $\sum a_n z^n$ is defined analogously to $\mathfrak{G}_{E_p}$ in **a**, and is the same for both methods B and B'. If $z=1$ lies on the boundary of $\mathfrak{G}_B$, then $f(z)$ is necessarily regular in $\mathfrak{K}_B = \mathfrak{K}(1/2)$.

c. The succession of the methods E_p and B is continued in a certain sense by the method of Meyer-König [10, p. 272]). This method depends on a parameter $\alpha (0<\alpha<1)$ and is defined by the matrix (S_α) with elements

$$c_{n\nu} = (1-\alpha)^{n+1} \binom{n+\nu}{n} \alpha^\nu \qquad (n, \nu = 0, 1, \cdots).$$

The S_α-transformation of the sequence s_n

$$S_\alpha(n; s_\nu) = (1-\alpha)^{n+1}\sum \binom{n+\nu}{n}\alpha^\nu s_\nu \qquad (n = 0, 1, \cdots)$$

exists if and only if $s_\nu = O(\nu^{-n}\alpha^{-\nu})$ for $\nu \to \infty$ and all fixed $n = 0, 1, \cdots$; one then says that (S_α) is applicable to the sequence s_n. This obviously implies the regularity of $f(z)$ in $|z| < \alpha$, and the regularity of $f(z)$ in $|z| \leqq \alpha$ is sufficient for the applicability of (S_α) to the sequence s_n.

Later we use an alternative definition of the S_α-transformation. Assume that (S_α) is applicable to the sequence s_n so that $f(z)$ is regular in $|z| < \alpha$; assume in addition that $f(z)$ is regular at $z = \alpha$. Letting

$$z = \frac{\alpha}{1-(1-\alpha)w} \tag{2.6}$$

and developing $f(z)$ into powers of w we obtain a power series

$$F(w) = \sum a_n' w^n$$

which is convergent in the neighborhood of $w = 0$. One finds that

$$S_\alpha(n; s_\nu) = \sum_{\kappa=0}^{n} a_\kappa' \quad \text{where} \quad \begin{cases} a_n' = \dfrac{\alpha}{1-\alpha} S_\alpha(n; a_{\nu+1}) & (n = 1, 2, \cdots), \\ a_0' = a_0 + \dfrac{\alpha}{1-\alpha} S_\alpha(0; a_{\nu+1}). \end{cases} \tag{2.7}$$

Hence S_α-$\sum a_n$ exists if and only if $\sum a_n'$ converges, and S_α-lim $a_n = 0$ is necessary for the existence of S_α-$\sum a_n$.

In the application to power series the situation here is slightly more complicated than in **a** and **b**. The region of S_α-summability has not been investigated in full but it is known that the regularity of $f(z)$ in $\mathfrak{R}_{S_\alpha} = \mathfrak{R}(1/(2-\alpha))$ is necessary for the existence of S_α-$\sum a_n$ (again assuming the regularity of $f(z)$ at $z = \alpha$).

3. **On two theorems of Fatou and M. Riesz.** The starting point for our investigations on the circle of convergence of the power series

$$f(z) = \sum a_n z^n \qquad \left(\limsup_{n\to\infty} |a_n|^{1/n} = 1\right) \tag{3.1}$$

is the Theorems A and B of the introduction. Szász showed [1, pp. 485–486] that in Theorem B the region S_1 can be replaced by another region which has a finite number of corners on $|z| = 1$ instead of one as in the case of S_1(3). We show in Theorem 1 that the assumptions on $f(z)$ can be localized completely to a neighborhood of $z = 1$ if the condition $a_n \to 0$ $(n \to \infty)$ is added.

Let S_2 be the region

(3) Such a region can be expressed by $\bigcap_1^n \{z_\kappa \cdot S_1\}$ with $|z_\kappa| = 1$ $(\kappa = 1, 2, \cdots, n)$.

$$S_2 = S_2(\delta_0, \theta_0): \quad \begin{cases} |z-1| < \delta_0, & \delta_0 > 0, \\ |\operatorname{arc}(z-1)| > \theta_0, & 0 < \theta_0 < \pi/2. \end{cases}$$

THEOREM 1. *Let* (3.1) *be regular and bounded in* S_2, *and* $a_n \to 0$ $(n \to \infty)$. *Then* $\sum a_n = s$ *if* A-$\sum a_n = s$, *i.e. if* $\lim_{z \to 1-0} f(z) = s$.

REMARKS. a. Theorem 1 contains the Theorems A and B and also Szász's extension of Theorem B. The conditions $a_n \to 0$ $(n \to \infty)$ and A-$\sum a_n = s$ are both necessary for $\sum a_n = s$, together they are sufficient for $\sum a_n = s$ if $f(z)$ fulfills the above conditions.

b. Let $K = (e^{i\phi_0}, e^{-i\phi_0})$ be a closed subarc of $|z| = 1$ within $|z-1| < \delta_0$. Under the assumptions of Theorem 1 the series $\sum a_n z^n$ converges for $z = 1$ and Theorem A assures its convergence for every other z on K. We extend Theorem 1 by proving the uniform convergence of (3.1) on K.

THEOREM 2. *Under the assumptions of Theorem* 1 *the series* (3.1) *converges uniformly on* K.

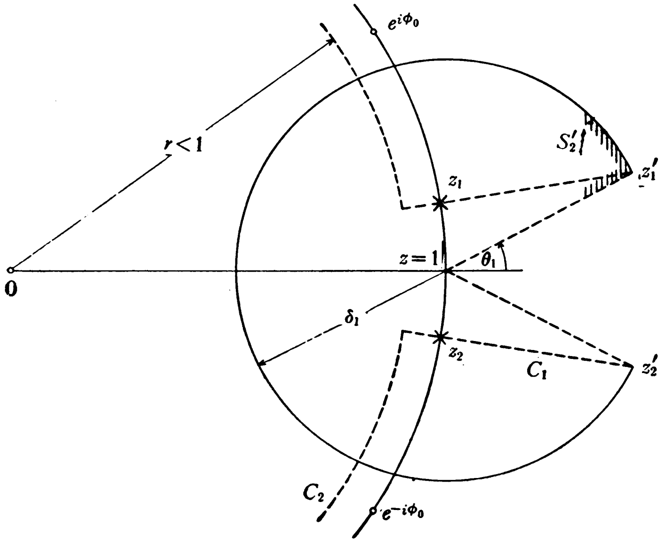

FIG. 1

Proof. By a theorem on bounded functions (see for example [13, p. 65]) one obtains first that $f(z)$ is continuous in $\overline{S_2'}$ where $S_2' = S_2(\delta_1, \theta_1)$ for every δ_1, θ_1 with $0 < \delta_1 < \delta_0$ and $\theta_0 < \theta_1 < \pi/2$. We may assume that $\lim f(z) = 0$ for $z \to 1$ in S_2' and that δ_1 is chosen such that

$$|f(z)| < \epsilon \cdot (\pi \cos \theta_1)/8 \tag{3.2}$$

for $z \in S_2'$ and a given $\epsilon > 0$. Draw the path $C = C_1 + C_2$ as indicated in Fig. 1. (C_2 is the part of C in $|z| < 1$.)

To prove the uniform convergence of $\sum a_n e^{in\phi}$ for $|\phi| \leqq \phi_0$, take first all ϕ with $|\phi| \leqq \phi_1 = (1/2)$ arc z_1. If $\sigma_n(\phi)$ are the arithmetic means of $s_n(\phi) = \sum_{\nu=0}^{n} a_\nu e^{i\nu\phi}$, then

$$| s_n(\phi) - \sigma_n(\phi) | = \frac{1}{n+1}\left| \sum_{\nu=1}^{n} a_\nu \nu e^{i\nu\phi} \right| = \frac{1}{n+1}\left| \frac{1}{2\pi i}\left\{ \sum_{\nu=1}^{n} \nu e^{i\nu\phi} \int_{C_1} \frac{f(z)dz}{z^{\nu+1}} + \int_{C_2} f(z) \sum_{\nu=1}^{n} \frac{\nu e^{i\nu\phi}}{z^{\nu+1}} dz \right\} \right|$$
$$= \frac{1}{2\pi(n+1)} | A_n^{(1)}(\phi) + A_n^{(2)}(\phi) |.$$

Estimation of $A_n^{(1)}$: For $\nu = 1, 2, \cdots$

$$\left| \int_1^{z_1'} \frac{f(z)dz}{z^{\nu+1}} \right| < \epsilon \cdot \frac{\pi \cos \theta_1}{8} \int_0 \frac{d\tau}{| 1 + \tau e^{i\theta_1} |^{\nu+1}}$$
$$< \epsilon \cdot \frac{\pi \cos \theta_1}{8} \int_0^\infty \frac{d\tau}{(1 + \tau \cos \theta_1)^{\nu+1}} = \frac{\pi}{8\nu} \cdot \epsilon,$$

and similarly for the other parts of C_1. Hence

$$| A_n^{(1)}(\phi) | < \frac{n\pi}{2} \cdot \epsilon \qquad (n = 1, 2, \cdots), \tag{3.3}$$

uniformly for $|\phi| \leqq \phi_1$.

Estimation of $A_n^{(2)}$: Evaluating first the finite sum occurring in the expression for $A_n^{(2)}$ we obtain

$$A_n^{(2)}(\phi) = e^{i\phi} \int_{C_2} f(z) \frac{z^{n+1} - (n+1)ze^{in\phi} + ne^{i\phi(n+1)}}{z^{n+1}(z - e^{i\phi})^2} dz.$$

In this integral we have first

$$\int_{C_2} \frac{f(z)dz}{(z - e^{i\phi})^2} = o(n),$$

and we shall prove that also

$$n \int_{C_2} \frac{f(z)dz}{z^n (z - e^{i\phi})^2} = o(n), \tag{3.4}$$

always uniformly for $|\phi| \leqq \phi_1$. Assume for the moment that (3.4) is true. Then

$$| A_n^{(2)}(\phi) | < \frac{n\pi}{2} \cdot \epsilon \qquad (n > n_1), \tag{3.5}$$

uniformly for $|\phi| \leqq \phi_1$, and therefore $|s_n(\phi) - \sigma_n(\phi)| < \epsilon/2$ for $n > n_1$ and

$|\phi| \leqq \phi_1$. Using now the fact [15, pp. 94–95] that $\sigma_n(\phi)$ converges uniformly to $f(e^{i\phi})$ for $|\phi| \leqq \phi_1$, we get

$$| s_n(\phi) - f(e^{i\phi}) | \leqq | s_n(\phi) - \sigma_n(\phi) | + | \sigma_n(\phi) - f(e^{i\phi}) | < \epsilon$$

for $n > n_2$ and all ϕ in $|\phi| \leqq \phi_1$. Since for ϕ with $\phi_1 \leqq |\phi| \leqq \phi_0$ the uniform convergence of $\sum a_n e^{in\phi}$ follows from an extension of Theorem A due to M. Riesz [15, p. 90], the proof of Theorem 2 is completed provided (3.4) is true.

For the proof of (3.4) we apply a method of M. Riesz [15], choosing the numbers $b_1 = b_1(\phi)$ and $b_2 = b_2(\phi)$ such that $H(z_1) = H(z_2) = 0$ for

$$H(z) = H(z, \phi) = \frac{1}{(z - e^{i\phi})^2} + b_1 + b_2 z.$$

One finds that $b_1(\phi)$ and $b_2(\phi)$ are bounded for $|\phi| \leqq \phi_1$. Therefore using (3.3) and $a_n \to 0$ $(n \to \infty)$ we obtain

$$\begin{aligned}\int_{C_2} \frac{f(z)H(z)dz}{z^n} &= \int_{C_2} \frac{f(z)dz}{z^n(z - e^{i\phi})^2} + b_1 \int_C \frac{f(z)dz}{z^n} + b_2 \int_C \frac{f(z)dz}{z^{n-1}} \\ &\qquad - b_1 \int_{C_1} \frac{f(z)dz}{z^n} - b_2 \int_{C_1} \frac{f(z)dz}{z^{n-1}} \\ &= \int_{C_2} \frac{f(z)dz}{z^n(z - e^{i\phi})^2} + o(1),\end{aligned}$$

so that it remains now to show that

$$\lim_{n\to\infty} \int_{C_2} \frac{f(z)H(z)dz}{z^n} = 0, \tag{3.6}$$

uniformly for $|\phi| \leqq \phi_1$. But by Theorem A the series $\sum a_n z^n$ converges for z_1 and z_2 and therefore uniformly on C_2 so that

$$\int_{C_2} \frac{f(z)H(z)dz}{z^n} = \sum_\nu a_\nu \int_{C_2} z^{\nu-n} H(z)dz$$

which is a matrix transformation of the zero-sequence a_ν. We have to show

$$\lim_{n\to\infty} c_{n\nu} = 0 \qquad (\nu = 0, 1, \cdots, \text{fixed}), \tag{3.7}$$

$$\limsup_{n\to\infty} \sum_\nu | c_{n\nu} | < \infty, \tag{3.8}$$

uniformly for $|\phi| \leqq \phi_1$, where

$$c_{n\nu} = c_{n\nu}(\phi) = \int_{C_2} z^{\nu-n} H(z, \phi)dz.$$

Integrating twice by parts we get for $\nu \neq n-1$, $\nu \neq n-2$

$$c_{n\nu} = -\frac{1}{\nu - n + 1}\left\{\frac{z^{\nu-n+2}}{\nu - n + 2}\,\frac{\partial H(z)}{\partial z}\Bigg|_{z_1}^{z_2} - \frac{1}{\nu - n + 2}\int_{C_2} z^{\nu-n+2}\,\frac{\partial^2 H(z)}{\partial z^2}\,dz\right\}.$$

Since $H(z) = H(z, \phi)$ is regular in the finite z-plane except for $z = e^{i\phi}$ where $|\phi| \leqq \phi_1$, we can replace the path C_2 by the arc: $|\phi| \geqq \operatorname{arc} z_1$ of the unit circle. For z on this arc the function $H(z)$ and its first and second derivatives with respect to z are bounded, uniformly for $|\phi| \leqq \phi_1$, by a constant L_1, say. Therefore

$$|c_{n\nu}| \leqq L_1 \frac{2 + 2\pi}{(\nu - n + 1)(\nu - n + 2)} \qquad (\nu \neq n - 1;\ \nu \neq n - 2),$$

uniformly for $|\phi| \leqq \phi_1$, which proves (3.7). But obviously $c_{n,n-1}$ and $c_{n,n-2}$ are uniformly bounded for $|\phi| \leqq \phi_1$, by a constant L_2, say, so that

$$\begin{aligned}\sum_{\nu} |c_{n\nu}| &\leqq 2L_2 + L_1 \sum_{\nu\neq n-1,\nu\neq n-2} \frac{2 + 2\pi}{(\nu - n + 1)(\nu - n + 2)}\\ &\leqq 2L_2 + 2L_1(2 + 2\pi)\end{aligned}$$

which proves (3.8) and therefore completes the proof of Theorems 1 and 2.

4. **Weaker assumptions on** $f(z)$. Until now our assumptions on $f(z)$ concerned its behavior in the regions S_1 and S_2, whose boundaries have an osculation of first order with $|z| = 1$ at $z = 1$. The question arises what can be said in the case that S_1 and S_2 are substituted by the regions S_3 and S_4, whose boundaries have an osculation of second order with $|z| = 1$ at $z = 1$:

$$S_3\colon \begin{cases} |z| < R, & R > 1 \\ |z| < 1 + c_0\phi^2, & c_0 > 0 \end{cases}; \quad S_4\colon \begin{cases} |z - 1| < \delta_0, & \delta_0 > 0 \\ |z| < 1 + c_0\phi^2, & c_0 > 0 \end{cases} \quad (z = |z| \cdot e^{i\phi}).$$

In the neighborhood of $z = 1$ the boundaries of S_3 and S_4 behave like a circle which touches $|z| = 1$ exteriorly at $z = 1$.

First we remark that *it is impossible to substitute* S_1 *by* S_3 *in Theorem* B *or* S_2 *by* S_4 *in Theorem* 1. For there exist power series which define a function $f(z)$ regular in S_3 and continuous in $\overline{S_3}$ and which are not even V-summable for $z = 1$ [5, p. 331]; by V we denote in this paragraph any of the methods E_p $(0 < p < \infty)$, B, S_α $(0 < \alpha < 1)$. We shall show, however, that the following modifications of Theorems B and 1 are true.

THEOREM 3. *If* (3.1) *is regular in* S_3 *and continuous in* $\overline{S_3}$, *then* $\sum a_n$ *converges provided that* V-$\sum a_n$ *exists*(4).

This is a $V \rightarrow K$-Theorem ("K" standing for "convergence") under complex Tauberian conditions.

(4) Theorem 3 and its proof also hold if "continuous in $\overline{S_3}$" is replaced by "bounded in S_3."

Proof. The assumptions on $f(z)$ imply $a_n = O(1/n^{1/2})$ [5, p. 327], which is a Tauberian condition for V-summability.

We now prove the localization of Theorem 3.

THEOREM 4. *Let* (3.1) *be regular and bounded in* S_4, *and* $a_n \to 0$ $(n \to \infty)$. *Then* $\sum a_n = s$ *if* $V\text{-}\sum a_n = s$(5).

Proof. It is sufficient to show

$$\lim_{n\to\infty} \sum_{\nu=n}^{n+[\epsilon_n n^{1/2}]} a_\nu = 0 \qquad \text{for any positive zero-sequence } \epsilon_n, \tag{4.1}$$

because this is a Tauberian condition for V-summability. (For E_p and B see for example [6, p. 312]; for S_α see [4, p. 36].) To prove (4.1) we construct a path similar to the one in Fig. 1, except that the part $(z_2', 1, z_1')$ of C_1 is substituted by a segment of $|z| = 1 + c_1\phi^2$ $(0 < c_1 < c_0)$; the parts (z_2, z_2') and (z_1, z_1') remain rectilinear as in Fig. 1. Then with $k = [\epsilon_n n^{1/2}]$ we have

$$\sum_{\nu=n}^{n+k} a_\nu = \frac{1}{2\pi i}\left[\sum_{\nu=n}^{n+k} \int_{C_1} \frac{f(z)dz}{z^{\nu+1}} + \int_{C_2} f(z) \sum_{\nu=n}^{n+k} z^{-\nu-1} dz\right].$$

As proved in [5], $\int_{C_1} f(z)dz/z^{\nu+1} = O(1/\nu^{1/2})$ $(\nu \to \infty)$, and the second integral equals

$$\int_{C_2} \frac{f(z)dz}{z^n(z-1)} - \int_{C_2} \frac{f(z)dz}{z^{n+k+1}(z-1)}.$$

Both of these integrals tend to zero for $n \to \infty$, which is shown in the same way as (3.4) for $\phi = 0$. Hence

$$\sum_{\nu=n}^{n+k} a_\nu = O\left(\sum_{\nu=n}^{n+k} \frac{1}{\nu^{1/2}}\right) + o(1) = O\left([\epsilon_n n^{1/2}]\cdot\frac{1}{n^{1/2}}\right) + o(1) = o(1) \quad (n \to \infty),$$

which proves (4.1) and therefore Theorem 4.

5. **Tauberian theorems asserting E_p- and S_α-summability.** Consider now a power series

$$f(z) = \sum a_n z^n \tag{5.1}$$

with positive radius of convergence on the boundary of its region of E_p- and S_α-summability. We investigate the question from what assumptions about $f(z)$ in the neighborhood of $z = 1$ we can derive the summability of $\sum a_n$ by means of the methods E_p and S_α. The following assumptions will be frequently used:

(5) Comparing Theorem 4 with Theorem 1 it is noted that the assumptions on $f(z)$ are relaxed in Theorem 4; but since $V\text{-}\sum a_n = s$ implies $A\text{-}\sum a_n = s$ if $\limsup |a_n|^{1/n} = 1$, the assumptions on $\sum a_n$ are strengthened.

(*) $\qquad f(z)$ is regular in $\mathfrak{K}_{E_p} = \mathfrak{K}\left(\dfrac{2^p - 1}{2^{p+1} - 1}\right)$ for $p > 0$;

(**) $\qquad f(z)$ is regular in $|z| < \alpha$ and in $\mathfrak{K}_{S_\alpha} = \mathfrak{K}\left(\dfrac{1}{2 - \alpha}\right)$ for $0 < \alpha < 1$.

As to the necessity of these assumptions for the E_p- and S_α-summability of $\sum a_n$ see §2, **a** and **c**.

Corresponding to Theorem 1 we have

THEOREM 5. *Assume that* (5.1) *fulfills the condition* (*) [*or* (**)] *and is regular and bounded in* S_2, *furthermore* E_p-lim $a_n = 0$ [*or* S_α-lim $a_n = 0$]. *Then* E_p-$\sum a_n = s$ [*or* S_α-$\sum a_n = s$] *if* $\lim_{z\to 1-0} f(z) = s$.

Proof. We restrict ourselves to the proof of the case of E_p-summability. According to §2, **a** the question is whether the series $\sum a_n' w^n$ representing $F(w) = f(\phi_p(w))$ with $\phi_p(w) = w/(2^p - (2^p - 1)w)$ converges for $w = 1$. We shall show that for $F(w) = \sum a_n' w^n$ all the assumptions of Theorem 1 are fulfilled.

(a) The image of the region S_2 in the z-plane under $w = \phi_p^{-1}(z)$ is a region in the w-plane whose boundary has an exterior osculation of order one with $|w| = 1$ at $w = 1$. Hence $F(w)$ (which by (*) is regular in $|w| < 1$) is regular and bounded in some region S_2 in the w-plane.

(b) We have $a_n' \to 0$ $(n \to \infty)$ since E_p-lim $a_n = 0$ (§2, **a**).

(c) Finally $\lim_{w\to 1-0} F(w) = \lim_{z\to 1-0} f(z) = s$.

Hence, by Theorem 1, $\sum a_n' = s$, i.e. E_p-$\sum a_n = s$.

If $f(z)$ is regular at the point $z = 1$ on the boundary of the region of summability, Theorem 5 yields analogues to Theorem A of the introduction.

The proof of Theorem 5 is based upon the following idea. Suppose that certain assumptions on $F(w) = \sum a_n' w^n$ allow one to draw conclusions about $\sum a_n'$, and we "transform" these assumptions on $f(z)$ by $z = \phi_p(w)$. Then from the "transformed" assumptions on $f(z)$ we can conclude to $\sum a_n'$ and therefore to E_p-$\sum a_n$, since E_p-$\sum a_n$ behaves like $\sum a_n'$. Following are two more examples of this general principle.

THEOREM 6. *Assume that* (5.1) *fulfills the condition* (*) [*or* (**)] *and is continuous in* $\overline{\mathfrak{K}_{E_p}}$ [*or* $\overline{\mathfrak{K}_{S_\alpha}}$]. *Then* $C_\epsilon E_p$-$\sum a_n$ [*or* $C_\epsilon S_\alpha$-$\sum a_n$] *exists for every* $\epsilon > 0$.

REMARK. The case where $f(z)$ is regular in $|z| < 1$ and continuous in $\overline{\mathfrak{K}_{E_p}}$ [or $\overline{\mathfrak{K}_{S_\alpha}}$] was treated by Meyer-König [11, p. 352]. He proved that then $E_p C_\epsilon$-$\sum a_n$ [or $S_\alpha C_\epsilon$-$\sum a_n$] exists for $\epsilon = 1, 2, \cdots$. These results are contained in Theorem 6, for since the matrices (E_p) and (C_ϵ) are Hausdorff matrices, $(C_\epsilon E_p) = (E_p C_\epsilon)$; on the other hand it is proved that for $\epsilon = 1, 2, \cdots$ the methods $C_\epsilon S_\alpha$ and $S_\alpha C_\epsilon$ are equivalent [12, p. 450].

Proof of Theorem 6. We restrict ourselves to the case of E_p-summability.

The assumptions on $f(z)$ imply that $F(w)=\sum a_n' w^n$ is regular in $|w|<1$ and continuous in $|w|\leqq 1$ whence by a known result of M. Riesz [15, pp. 94–95] we obtain that C_ϵ-$\sum a_n'$ exists for every $\epsilon>0$, i.e. $C_\epsilon E_p$-$\sum a_n$ exists for every $\epsilon>0$.

If in Theorem 6 the continuity of $f(z)$ for $z\to 1$ is sufficiently strong, we can derive the E_p- [or S_α-] summability of $\sum a_n$.

THEOREM 7. *Assume that $f(z)$ fulfills the condition* (*) [*or* (**)] *and that*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta > 0), \tag{5.2}$$

uniformly for $z\to 1$ in $\mathfrak{K}_{E_p}$ [or $\mathfrak{K}_{S_\alpha}$]. Then E_p-$\sum a_n = s$ [or S_α-$\sum a_n = s$] if E_p-lim $a_n=0$ [*or* S_α-lim $a_n=0$].

The last condition is certainly fulfilled under the stronger assumption that $f(z)$ is continuous in $\overline{\mathfrak{K}_{E_p}}$ [or $\overline{\mathfrak{K}_{S_\alpha}}$], since this means that $F(w)$ is continuous in $|w|\leqq 1$, so that $a_n'\to 0$, i.e. E_p-lim $a_n=0$ [or S_α-lim $a_n=0$].

Proof. For the function $F(w)=\sum a_n' w^n$ we have

$$F(w) = f(\phi_p(w)) = s + o((1-\phi_p(w))^\eta) = s + o((1-w)^\eta) \qquad (\eta > 0),$$

uniformly for $w\to 1$ in $|w|<1$ since $1-\phi_p(w)=(2^p/(2^p-(2^p-1)w))(1-w)$ $\sim(1-w)$ for $w\to 1$; furthermore $a_n'\to 0$ since E_p-lim $a_n=0$. Hence by a known theorem [17, p. 220] the series $\sum a_n'$ converges, i.e. E_p-$\sum a_n$ exists, and its value is s. The proof for the S_α-case is similar.

Assuming that there exists an analogue to Theorem 7 for B-summability (see Theorem 7′, §6) we obtain the following

COROLLARY. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<1$ and*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta > 0),$$

uniformly for $z\to 1$ in $\mathfrak{K}_{E_p}$ [or $\mathfrak{K}_B$, or *$\mathfrak{K}_{S_\alpha}$]. Then E_p-$\sum a_n = s$ [or B-$\sum a_n=s$ or S_α-$\sum a_n = s$].*

For the proof one notes that E_p-lim $a_n=0$ [or S_α-lim $a_n=0$] since $a_n'\to 0$ is implied by the boundedness of $f(z)$ in $\mathfrak{K}_{E_p}$ [or $\mathfrak{K}_{S_\alpha}$]. We shall prove in Theorem 9 that B-lim $a_n=0$ is a consequence of the boundedness of $f(z)$ in $\mathfrak{K}_B$.

The above corollary has some relationship to a theorem which Hardy and Littlewood have stated without proof [7, p. 53]:

Let $f(z)=\sum a_n z^n$ be regular in $|z|<1$ and $f(z)=s+o((1-z)^\eta)$ $(\eta>0)$, uniformly for $z\to 1$ in some circle touching $|z|=1$ interiorly at $z=1$. Then B-$\sum a_n = s$.

Obviously this result and the corollary overlap, i.e. neither one is included in the other(6).

We can combine the above corollary with Theorem 3:

---

(6) For a series $\sum a_n$ with lim sup $|a_n|^{1/n}=1$ we have $E_p\to B\to S_\alpha$.

Let $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *be two circular regions whose bounding circles touch* $|z| = 1$ *at* $z=1$ *exteriorly and interiorly, respectively. Assume that* $f(z) = \sum a_n z^n$ *is regular in* $\mathfrak{R}_1$, *continuous in* $\overline{\mathfrak{R}_1}$ *and strongly continuous for* $z \to 1$ *in* $\mathfrak{R}_2$ (*i.e.*, $f(z) = s + o((1-z)^\eta)$ $(\eta > 0)$, *uniformly for* $z \to 1$ *in* $\mathfrak{R}_2$). *Then* $\sum a_n = s$.

For the proof note that $\mathfrak{R}_2$ is one of the regions of the set $\mathfrak{R}_{E_p}$, $\mathfrak{R}_B$, $\mathfrak{R}_{S_\alpha}$.

6. Tauberian theorems asserting *B*-summability. In this last section we shall deal with a power series

$$f(z) = \sum a_n z^n \tag{6.1}$$

with positive radius of convergence, for which $z=1$ lies on the boundary of its

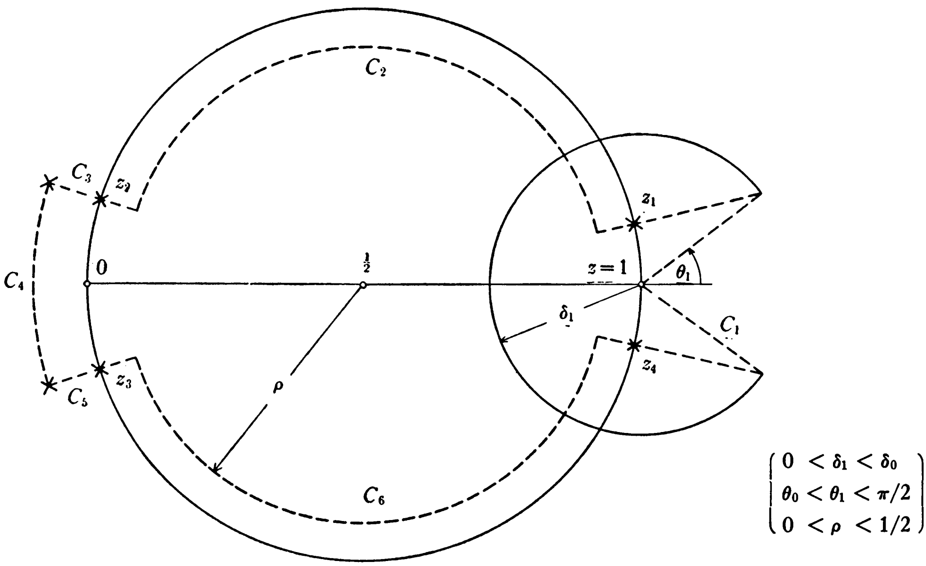

FIG. 2

region of B-summability. This implies the regularity of $f(z)$ in $\mathfrak{R}_B$. The section will be divided into three parts. In part A we prove the analogue to Theorem 1, deriving the B-summability of $\sum a_n$ from assumptions on $f(z)$ in the neighborhood of $z=1$. Also given are sufficient conditions on $f(z)$ which imply B-lim $a_n = 0$. Part B contains a necessary and sufficient condition for the existence of B-$\sum a_n$ and analogues to Theorems 6 and 7 are derived. Finally in part C a theorem of Obrechkoff is discussed.

PART A.

THEOREM 8. *Let* (6.1) *be regular and bounded in* S_2, *and* B-lim $a_n = 0$. *Then* B-$\sum a_n = s$ *if* $\lim_{z \to 1-0} f(z) = s$.

Proof. Since $z=1$ lies on the boundary of the region of B'-summability, B'-$\sum a_n z^n$ exists for $0 < z < 1$, and we have for these values of z

$$f(z) = B'\text{-}\sum a_n z^n = \int_0^\infty e^{-t}\sum \frac{a_n(tz)^n}{n!}\,dt = \frac{1}{z}\int_0^\infty e^{-t/z}\phi(t)dt$$
$$= (\zeta+1)\int_0^\infty e^{-t\zeta}\psi(t)dt = (\zeta+1)I(\zeta) \quad \left(\zeta = \frac{1}{z} - 1\right),$$

where we let $\psi(t) = e^{-t}\phi(t) = e^{-t}\sum a_n\ (t^n/n!) = B(t;\ a_n)$ for $t \geqq 0$. The integral $I(\zeta)$ exists for $\zeta > 0$ and has the limit s for $\zeta \to 0$, so that by a Tauberian theorem for Laplace-integrals (see for example [6, p. 164]) it is sufficient to show that $\int^x \psi(t)dt$ is a slowly oscillating function. We shall show that

$$\lim_{x\to\infty}\int_x^{x+\epsilon(x)x}\psi(t)dt = 0 \tag{6.2}$$

for any positive function $\epsilon(x)$ tending to zero as $x \to \infty$. For (6.2) implies that $I(0) = s$, i.e. that $B'\text{-}\sum a_n = s$, which is equivalent to $B\text{-}\sum a_n = s$ since B-lim $a_n = 0$.

We now prove (6.2). Denote by $C = \sum_{i=1}^6 C_i$ the path as indicated in Fig. 2, and let $|f(z)/z| < M$ for z on C. (Let $f(1) = s$; the points z_2 and z_3 should be chosen such that $f(z)$ is regular on $C - C_1$, thereafter they are fixed.) Then for $t > 0$

$$(6.3)\qquad \begin{aligned} B(t;\ a_n) = \psi(t) &= \frac{1}{2\pi i}e^{-t}\sum\left(\int_C \frac{f(z)dz}{z^{n+1}}\,\frac{t^n}{n!}\right) \\ &= \frac{1}{2\pi i}e^{-t}\int_C \frac{f(z)}{z}\sum\frac{(t/z)^n}{n!}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}e^{-t(1-1/z)}dz, \end{aligned}$$

so that

$$\int_x^{x+\epsilon(x)x}\psi(t)dt = \frac{1}{2\pi i}\int_x^{x+\epsilon(x)x}\int_{C_1+\cdots+C_6}\frac{f(z)}{z}e^{-t(1-1/z)}dzdt$$
$$= \frac{1}{2\pi i}\,[I_1(x) + \cdots + I_6(x)].$$

Divide C_1 into its four rectilinear segments. Then we have for instance

$$\left|\int_1^{1+\delta_1 e^{i\theta_1}}\frac{f(z)}{z}e^{-t(1-1/z)}dz\right| < M\int_0^\infty e^{-\sigma t y}dy = \frac{M}{\sigma t},$$

since for z on $(1,\ 1+\delta_1 e^{i\theta_1})$ we have $\Re(1-1/z) \geqq \sigma|1-z|$ for a constant $\sigma > 0$. With similar estimations for the other parts of C_1 we obtain

$$\int_{C_1}\frac{f(z)}{z}e^{-t(1-1/z)}dz = O\left(\frac{1}{t}\right) \qquad (t\to\infty). \tag{6.4}$$

and hence

$$I_1(x) = \int_x^{x+\epsilon(x)x} O\left(\frac{1}{t}\right) dt = O\left(\frac{1}{x}\right) \cdot \epsilon(x)x = o(1) \qquad (x \to \infty).$$

For the estimation of $I_3(x)$ and $I_5(x)$ note that for instance on C_3 we have $\Re(1-1/z) \geqq \tau|z_2 - z|$ for a constant $\tau > 0$, so that by similar computations as in the case of $I_1(x)$ we obtain $|I_3(x)| + |I_5(x)| = o(1)$ $(x \to \infty)$. For z on C_4 we have $\Re(1-1/z) \geqq \eta > 0$, so that

$$\left| \int_{C_4} \frac{f(z)}{z} e^{-t(1-1/z)} dz \right| < 2\pi M e^{-t\eta} = O\left(\frac{1}{t}\right) \qquad (t \to \infty),$$

whence by the same reasoning as above $I_4(x) = o(1)$ $(x \to \infty)$.

Finally we estimate $I_2(x)$ and $I_6(x)$.

$$\begin{aligned} I_2(x) &= \int_x^{x+\epsilon(x)x} \int_{C_2} \frac{f(z)}{z} e^{-t(1-1/z)} dz dt = \int_{C_2} \int_x^{x+\epsilon(x)x} \frac{f(z)}{z} e^{-t(1-1/z)} dt dz \\ &= \int_{C_2} \frac{f(z)}{z-1} e^{-x(1-1/z)} dz - \int_{C_2} \frac{f(z)}{z-1} e^{-x(1+\epsilon(x))(1-1/z)} dz, \end{aligned}$$

and similarly for $I_6(x)$. Hence it is sufficient to show that

$$\lim_{x \to \infty} \int_{C_2+C_6} \frac{f(z)}{1-z} e^{-x(1-1/z)} dz = 0. \tag{6.5}$$

For this purpose we choose the constants b_1, b_2, b_3, b_4 such that

$$zH(z) = \frac{1}{1-z} + \frac{b_1}{z} + \frac{b_2}{z} \cdot e^{(1-1/z)} + \frac{b_3}{z} \cdot e^{2(1-1/z)} + \frac{b_4}{z} \cdot e^{3(1-1/z)}$$

vanishes at z_1, z_2, z_3, z_4. Then we get

$$\begin{aligned} \int_{C_2+C_6} & zH(z)f(z)e^{-x(1-1/z)} dz \\ &= \int_{C_2+C_6} \frac{f(z)}{1-z} e^{-x(1-1/z)} dz + b_1 \int_C \frac{f(z)}{z} e^{-x(1-1/z)} dz + \cdots \\ &\qquad + b_4 \int_C \frac{f(z)}{z} e^{-(x-3)(1-1/z)} dz \\ &\qquad - b_1 \int_{C_1+C_3+C_4+C_5} \frac{f(z)}{z} e^{-x(1-1/z)} dz - \cdots \\ &\qquad - b_4 \int_{C_1+C_3+C_4+C_5} \frac{f(z)}{z} e^{-(x-3)(1-1/z)} dz. \end{aligned} \tag{6.6}$$

Using (6.4) and the corresponding estimations for the paths C_3, C_4, and C_5, we see that the last four terms in (6.6) tend to zero for $x\to\infty$, while B-lim $a_n=0$ implies by (6.3) that the second to fifth terms tend to zero for $x\to\infty$. Hence it remains to show that

$$\lim_{x\to\infty}\int_{C_2+C_6} zH(z)f(z)e^{-x(1-1/z)}dz = 0. \tag{6.7}$$

Now the function $f(z)$ has the representation

$$f(z) = \frac{1}{z}\int_0^\infty e^{-t/z}\phi(t)dt \qquad (z\in\mathfrak{K}_B),$$

where the Laplace-integral converges also at the regular points z_1, z_2, z_3, z_4 (see [16, pp. 18–19](7)) and therefore uniformly on C_2+C_6 by a familiar Abelian theorem on Laplace-integrals. The integral in (6.7) becomes therefore

$$\int_0^\infty\int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}e^{-t}\phi(t)dzdt = \int_0^\infty \psi(t)\int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}dzdt,$$

which is an integral-transformation of the function $\psi(t)$ which tends to zero for $t\to\infty$. This transform tends to zero for $x\to\infty$ if for any fixed t_1, $t_2>0$

$$\lim_{x\to\infty}\int_t^{t_2} |c(x,t)|dt = 0 \quad\text{and}\quad \limsup_{x\to\infty}\int_0^\infty |c(x,t)|dt < \infty, \tag{6.8}$$

where

$$c(x,t) = \int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}dz \qquad (t\geqq 0,\ x\geqq 0).$$

Finally the relations (6.8) are proved in the same manner as (3.7) and (3.8) in §3, first integrating twice by parts and then estimating $c(x, t)$. This concludes the proof of Theorem 8.

It should be noted that Theorem 8 combines two theorems analogous to the Theorems A and B of the introduction. We mention the following case.

COROLLARY. *Assume that $z=1$ lies on the boundary of the region of B-summability of $f(z)=\sum a_nz^n$ and that $f(z)$ is regular at $z=1$. Then B-$\sum a_n$ exists if* B-lim $a_n=0$.

Sometimes it is useful to have "B-lim $a_n=0$" substituted by an assumption on $f(z)$.

---

(7) This theorem of M. Riesz states: *In $J(w)=\int_0^\infty e^{-tw}\phi(t)dt$ let $\int_0^x\phi(t)dt=o(e^{xc})$ $(c>0)$ $(x\to\infty)$ so that $J(w)$ is regular for $\Re(w)>c$. If $J(w)$ is regular at w_0 with $\Re(w_0)=c$, then $\int_0^\infty e^{-tw_0}\phi(t)dt$ converges.* Here we put $w=1/z$, $c=1$, and $w_0=1$; the assumption $e^{-x}\int_0^x\phi(t)dt\to 0$ $(x\to\infty)$ is clearly fulfilled since B-lim $a_n=0$ and therefore $\phi(t)=o(e^t)$ $(t\to\infty)$.

THEOREM 9. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<r\,(r>0)$ and in $\mathfrak{R}_B$ and assume that $f(z)$ belongs to the class H^1 of the circle $\mathfrak{R}_B$, i.e. that*

$$\int_{-\pi}^{+\pi}\left|f\left(\frac{1}{2}+Re^{i\theta}\right)\right|d\theta \leqq K \qquad \text{for } 0<R<\frac{1}{2}. \tag{6.9}$$

Then B-lim $a_n=0$.

Proof. Again we have

$$2\pi i B(x;\, a_n) = \int_C \frac{f(z)}{z} e^{-x(1-1/z)}dz \qquad (C = C_1+\cdots+C_4).$$

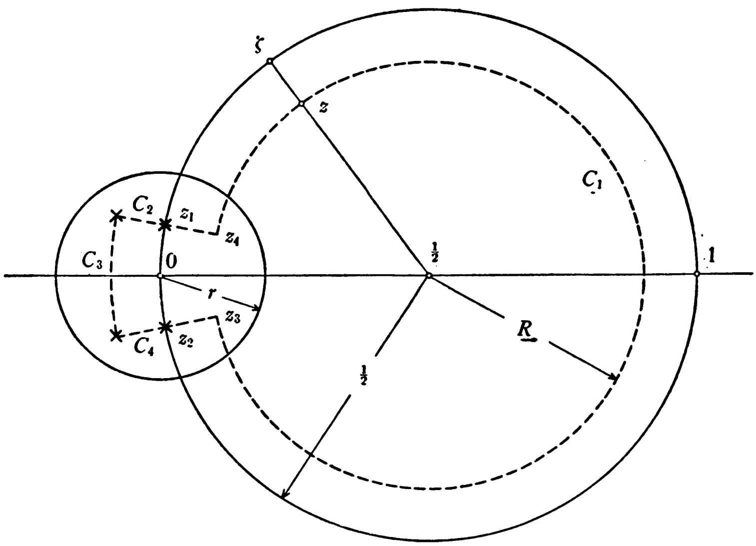

FIG. 3

Given an $\epsilon>0$, we choose C_2 and C_4 so short that

$$\int_{C_2+C_4}\left|\frac{f(z)}{z}\right||dz| < \frac{\epsilon}{3}.$$

Then we have

$$\left|\int_{C_2+C_4}\frac{f(z)}{z}e^{-x(1-1/z)}dz\right| \leqq \int_{C_2+C_4}\left|\frac{f(z)}{z}\right||dz| < \frac{\epsilon}{3} \qquad (x \geqq 0).$$

Furthermore

$$\left|\int_{C_3}\frac{f(z)}{z}e^{-x(1-1/z)}dz\right| = O(e^{-x\eta}) < \frac{\epsilon}{3} \qquad (x \geqq x_1),$$

since for z on C_3 we have $\Re(1-1/z)\geqq\eta>0$. Finally we estimate

$$(6.10)\qquad I(x) = \int_{C_1} \frac{f(z)}{z} e^{-x(1-1/z)}dz.$$

Let x be fixed for the moment and put $z(\theta)=1/2+Re^{i\theta}$, $\zeta(\theta)=1/2+(1/2)e^{i\theta}$, $z_4=z(\alpha)$. Then $\lim_{R\to 1/2} f(z(\theta))$ exists for almost all θ in $(-\pi, +\pi)$ and represents there a Lebesgue-integrable function $f(\zeta(\theta))$ such that

$$(6.11)\qquad \int_{-\pi}^{+\pi} \left| f(z(\theta)) - f(\zeta(\theta)) \right| d\theta \to 0 \qquad \text{for } R \to \frac{1}{2}$$

(see for example [18, p. 162]). Therefore in

$$I(x) = \int_{-\alpha}^{+\alpha} \frac{f(z(\theta)) - f(\zeta(\theta))}{z(\theta)} e^{-x(1-1/z(\theta))}dz(\theta) + \int_{-\alpha}^{+\alpha} \frac{f(\zeta(\theta))}{z(\theta)} e^{-x(1-1/z(\theta))}dz(\theta)$$
$$+ \left(\int_{z_4}^{z_1} + \int_{z_2}^{z_3}\right) \frac{f(z)}{z} e^{-x(1-1/z)}dz$$

the first and third terms tend to zero as $R\to 1/2$, while the second term tends to

$$\int \frac{f(\zeta)}{\zeta} e^{-x(1-1/\zeta)}d\zeta,$$

for $R\to 1/2$, the integral being taken as Lebesgue-integral over the arc: $(z_2, 1, z_1)$ of $|\zeta-1/2|=1/2$. Substituting $w=1/\zeta-1$, we get therefore

$$I(x) = \int_{-w_0}^{+w_0} f^*(w)e^{wx}dw$$

where w is purely imaginary and $f^*(w)$ is Lebesgue-integrable on the finite section $\langle -w_0, +w_0\rangle$ of the imaginary axis. By the Riemann-Lebesgue theorem,

$$|I(x)| < \epsilon/3 \qquad (x \geqq x_2),$$

which completes the proof of Theorem 9.

It is a well known fact that a power series $f(z)=\sum a_n z^n$ with radius of convergence 1 is B-summable at $z=1$ if $f(z)$ is regular at $z=1$. If we combine Theorems 8 and 9 we obtain the following sharper result:

A power series $f(z)=\sum a_n z^n$ *with radius of convergence* 1 *is B-summable at* $z=1$ *if* $f(z)$ *is regular and bounded in* S_2 *and* $\lim_{z\to 1-0} f(z)$ *exists.*

The result of the author [5, p. 331] mentioned previously in §3 shows furthermore that herein *the region* S_2 *cannot be replaced by a region whose boundary has an osculation of second order with* $|z|=1$ *at* $z=1$.

Part B. We give first a necessary and sufficient condition for B-summability.

THEOREM 10. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<r$ $(r>0)$ and in $\mathfrak{K}_B$ and assume that $f(z)$ belongs to the class H^1 of the circle $\mathfrak{K}_B$, i.e. that (6.9) is fulfilled. Then B-$\sum a_n=s$ if and only if*

$$\lim_{x\to\infty}\frac{1}{\pi}\int_{-\tau}^{+\tau}\frac{\sin xt}{t}\,f\left(\frac{1}{1+it}\right)dt = s \tag{6.12}$$

for every fixed $\tau>0$, the integral being taken as Lebesgue-integral.

REMARK. Karamata has proved this theorem in the case $r=1$ and under the assumption that $f(z)$ is bounded in $\mathfrak{K}_B$ [8, pp. 156–157], but his method is not applicable under the more general assumptions of Theorem 10.

Proof. We have

$$2\pi iB(x;s_n) = \int_{C_1+\cdots+C_4}\frac{f(z)}{z(1-z)}\,e^{-x(1-1/z)}dz = I_1(x)+\cdots+I_4(x),$$

where $C=\sum_{i=1}^4 C_i$ is again the path of Fig. 3. As in the proof to Theorem 9 one finds that for a given $\epsilon>0$

$$|I_2(x)|+|I_3(x)|+|I_4(x)|<\epsilon/2 \qquad (x\geqq x_1). \tag{6.13}$$

Consider the auxiliary function

$$H(x) = \int_{C_1}\frac{f(z)}{z(1-z)}\,e^{x(1-1/z)}dz.$$

On the segments (z_4, z_1) and (z_2, z_3) of C_1 we have $\Re(1-1/z)\leqq 0$ and therefore

$$\left|\left(\int_{z_4}^{z_1}+\int_{z_2}^{z_3}\right)\frac{f(z)}{z(1-z)}\,e^{x(1-1/z)}dz\right|<\frac{\epsilon}{8} \qquad (x\geqq 0)$$

if only $|z_4-z_1|<\delta=\delta(\epsilon)$, and since on the remaining part of C_1 the estimation $\Re(1-1/z)\leqq-\eta<0$ holds,

$$|H(x)|<\epsilon/4 \qquad (x\geqq x_2). \tag{6.14}$$

We now estimate

$$I_1(x) = \int_{C_1}\frac{f(z)}{z}\left[\frac{e^{-x(1-1/z)}-e^{x(1-1/z)}}{1-z}\right]dz + H(x) = I(x)+H(x)$$

by a method similar to the one used in the proof of Theorem 9. The bracketed term is regular for all $z\neq 0$ and therefore $I(x)$ can be treated as $I(x)$ in (6.10). Hence

$$I(x) = \int\frac{f(\zeta)}{\zeta}\left[\frac{e^{-x(1-1/\zeta)}-e^{x(1-1/\zeta)}}{1-\zeta}\right]d\zeta,$$

the integral being taken as Lebesgue-integral along the arc: $(z_2, 1, z_1)$ of

$|\zeta-1/2|=1/2$. Letting $\zeta=1/(1+it)$ (t real), we get

$$I(x) = 2i\int_{-T}^{+T} \frac{\sin xt}{t} f\left(\frac{1}{1+it}\right) dt,$$

where $z_2=1/(1+iT)$. By the Riemann-Lebesgue theorem,

$$\left|2\left(\int_{-T}^{-\tau} + \int_{+\tau}^{+T}\right)\right| < \frac{\epsilon}{4} \qquad (x \geqq x_3), \tag{6.15}$$

and hence from (6.13)–(6.15) it follows that for $x \geqq \max(x_1, x_2, x_3)$

$$\left|2\pi i B(x; s_n) - 2i\int_{-\tau}^{+\tau} \frac{\sin xt}{t} f\left(\frac{1}{1+it}\right) dt\right| < \epsilon,$$

which proves Theorem 10.

We prove now the analogues to Theorems 6 and 7 in the case of B-summability.

THEOREM 6′. *Assume that* (6.1) *is continuous in* $\overline{\mathfrak{R}_B}$. *Then* $C_\epsilon B$-$\sum a_n$ *exists for every* $\epsilon>0$.

REMARK. Since $C_\epsilon B$ and BC_ϵ are equivalent methods [2, p. 45], Theorem 6′ contains the result of Karamata [8, p. 157] who proved the BC_ϵ-summability of $\sum a_n$ under the assumption that $f(z)$ is regular in $|z|<1$ and continuous in $\overline{\mathfrak{R}_B}$.

Proof. For z in $\mathfrak{R}_B$ we have $f(z) = (1/z)\int_0^\infty e^{-t/z}\phi(t)dt$, and by a theorem of Riesz [16, p. 20] the integral is C_ϵ-summable for $z=1$, i.e. $C_\epsilon B'$- $\sum a_n=s$, provided that

$$\lim_{x\to\infty} \frac{e^{-x}}{x^\epsilon}\int_0^x (x-t)^\epsilon \phi(t)dt = 0. \tag{6.16}$$

But by Theorem 9 we have B-lim $a_n=0$, i.e. $\phi(t)=\epsilon(t)\cdot e^t$, where $\epsilon(t)\to 0$ $(t\to\infty)$. It is easily shown that the integral transformation with the generating function $c(x, t)=e^{t-x}x^{-\epsilon}(x-t)^\epsilon$ $(0<t\leqq x,\ \epsilon>0)$ transforms every function tending to zero for $t\to\infty$ into one tending to zero for $x\to\infty$, so that (6.16) holds. Now the relation $B(x; s_n)=B(x; a_n)+B'(x; s_n)$ implies $C_\epsilon B(x; s_n)$ $=C_\epsilon B(x; a_n)+C_\epsilon B'(x; s_n)$, and the last two terms tend to zero and to s respectively. This proves Theorem 6′.

If the continuity of $f(z)$ for $z\to 1$ in $\mathfrak{R}_B$ is sufficiently strong, we obtain

THEOREM 7′. *Assume that* (6.1) *belongs to the class* H^1 *of the circle* $\mathfrak{R}_B$, *i.e. that* (6.9) *is fulfilled,*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta > 0),$$

uniformly for $z\to 1$ *in* $\mathfrak{R}_B$. *Then* B-$\sum a_n=s$.

Proof. Without loss of generality we may assume $s=0$, so that for the limit function $f(\zeta)$ existing almost everywhere on $|\zeta-1/2|=1/2$ we have $|f(\zeta)|=|f(1/(1+it))|\leqq A|t|^{\eta}$ for $|t|<t_0$ where $A>0$ is a constant. Given an $\epsilon>0$, we may choose $\tau<t_0$ in (6.12) so small that $\int_0^{\tau}t^{\eta-1}dt<\epsilon\pi/2A$. (If (6.12) holds for some $\tau>0$ it holds for all $\tau>0$ by the Riemann-Lebesgue theorem.) For this τ we obtain

$$\left|\frac{1}{\pi}\int_{-\tau}^{+\tau}\frac{\sin xt}{t}f\left(\frac{1}{1+it}\right)dt\right|\leqq\frac{2A}{\pi}\int_0^{\tau}\frac{dt}{t^{1-\eta}}<\epsilon$$

for all $x\geqq 0$, so that (6.12) is fulfilled.

PART C. This section is concluded with the discussion of a theorem of Obrechkoff [14, p. 1813].

Suppose we have given a power series $f(z)=\sum a_nz^n$ with positive radius of convergence. Let $z=1$ be a singular point of $f(z)$ lying in the interior of L where L is a rectilinear part of the Borel-polygon associated with $f(z)$ (8). Assume that the singularity of $f(z)$ at $z=1$ is such that in the region $S_2=S_2(\delta_0,\theta_0)$ the function $f(z)$ is regular and $|1-z|^{\delta}|f(z)|$ is bounded for some $\delta(0<\delta<1)$.

The theorem of Obrechkoff states that under these assumptions the series $\sum a_nz_0^n$ is B-summable for every regular point $z_0=R_0e^{i\phi_0}$ in the interior of L.

Obrechkoff's proof of this theorem is valid only under restricted conditions. If the circle K: $|z-z_0/2|=R_1(R_1>R_0/2)$ is drawn and $A_1=1+a_1e^{i\theta_0}$ and $A_2=1+a_2e^{-i\theta_0}$ are the two points lying on K with $|\text{arc }(z-1)|=\theta_0$, his proof depends on the fact that a_1 and a_2 tend to zero if R_1 approaches $R_0/2$. This, however, is true if and only if the circle $|z-z_0/2|<R_0/2$ is contained in the region $|\text{arc }(z-1)|>\theta_0$, i.e. if and only if

$$(6.17)\qquad |\text{arc } z_0|+\theta_0\leqq\pi/2.$$

Therefore the theorem of Obrechkoff remains valid if either z_0 is close enough to $z=1$, or if θ_0 may be chosen small enough such that (6.17) holds. (Note that arc $(z_0-1)=\pm\pi/2$ since $z=1$ is in the interior of L; therefore $|\text{arc } z_0|<\pi/2$.)

But if (6.17) holds, the region $|z-z_0/2|<R_0/2$ is contained in the region $|\text{arc }(z-1)|>\theta_0$. Therefore

$$\overline{\lim}\sup|z-1|^{\delta}|f(z)|\leqq M\quad\text{for}\quad z\to 1\ \text{in}\left|z-\frac{z_0}{2}\right|<\frac{R_0}{2}\qquad(0<\delta<1),$$

whilst $f(z)$ remains bounded for the other part of $|z-z_0/2|<R_0/2$; this is because z_0 is an interior point of L and therefore $z=1$ is the only singularity of $f(z)$ on $|z-z_0/2|=R_0/2$. Theorem 9 assures now that B-lim $a_nz_0^n=0$ and hence by Theorem 8 or its corollary the existence of B-$\sum a_nz^n$ follows. This

(8) By "interior of L" we mean that $z=1$ is not a corner of the Borel-polygon.

is a new proof of Obrechkoff's theorem in its modified form.

Added January 28, 1953. It has been investigated by Garten and Karamata [Math. Zeit. vol. 40 (1936) pp. 756–759 and vol. 45 (1939) pp. 635–641] under what conditions on a_n

$$B-\sum a_n = s \text{ implies } B-\sum \alpha_n = s \quad (\alpha_0 = a_0 + a_1,\ \alpha_n = a_{n+1},\ n=1, 2, \cdots). \tag{6.18}$$

Their restrictions on the series [$a_n = o(n^k)$ ($n\to\infty$, k fixed) and $a_n = o(e^{n^\rho})$ ($n\to\infty$, $\rho<1/3$), respectively] are such that the associated function $f(z) = \sum a_n z^n$ is necessarily regular in $|z|<1$. It should be noted that if $f(z)$ does not fulfill this condition, Theorem 9 may be applicable, since B-lim $a_{n+1}=0$ is necessary and sufficient for the validity of (6.18) (see [6, p. 183]).

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