# SIMULTANEOUS PARTITIONINGS OF TWO SETS(1)

BY

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**1. Introduction and definitions.** In [5](2) R. H. Bing introduced the concept of partitioning a set. This concept has been applied by him and others as a means of obtaining many notable results [1; 2; 3; 4; 5; 6; 7; 8; 9]. In this paper an extension of this concept to the case of two sets is considered.

We shall consider only subsets of a metric space, particularly continuous curves and similar sets. A set $M$ is partitionable if for each $\epsilon > 0$, there exists a finite collection $G = \{g_1, \cdots, g_k\}$ of disjoint open connected subsets of $M$, such that $\sum_{i=1}^{k} g_i$ is dense in $M$ and $\delta(g_i) < \epsilon$ for each $i$. The mesh of $G$ equals max $\delta(g_i)$, $1 \leqq i \leqq k$. Each $g_i$ is an element of $G$, written $g_i \in G$. $G$ is called an $\epsilon$-partitioning of $M$. If each element of $G$ has property S, $G$ is called an $S$, $\epsilon$-partitioning of $M$. $\{G_i\}$ is a decreasing sequence of $S$-partitionings of $M$ if (1) for each $i$, $G_i$ is an $S$-partitioning of $M$ and each element of $G_i$ is contained in an element of $G_{i-1}$ (that is, $G_i$ is a refinement of $G_{i-1}$), and (2) the limit, as $i$ increases without limit, of the mesh of $G_i$ is 0. Suppose $H$ and $G$ are two partitionings of a set $M$, $G$ a refinement of $H$. Let $h \in H$ and let $g_1, g_2, \cdots, g_k$ be all the elements of $G$ which are contained in $h$. $g_i$ is called an interior element of $G$ if $\bar{g}_i \subset h$. Otherwise $g_i$ is a border element of $G$. $G$ is a core refinement of $H$ if for each $h \in H$ the sum of the closures of all interior elements of $G$ contained in $h$ is connected and intersects the closure of each border element of $G$ contained in $h$.

Let $M$ and $N$ be two partitionable sets such that $N \subset M$. If $G = \{g_1, g_2, \cdots, g_k\}$ is a partitioning of $M$ such that $G' = \{g_1N, g_2N, \cdots, g_kN\}$ constitutes a partitioning of $N$, then $G$ is called a simultaneous partitioning of $M$ and $N$. If $g_i$ and $g_iN$ both have property S for each $i$, $G$ is a simultaneous $S$-partitioning of $M$ and $N$. If $\delta(g_i) < \epsilon$ for each $i$, $G$ is a simultaneous $S$, $\epsilon$-partitioning of $M$ and $N$. $\{G_i\}$ is a decreasing sequence of simultaneous core partitionings of $M$ and $N$ if the following conditions are satisfied. (1) $\{G_i\}$ is a decreasing sequence of $S$-partitionings of $M$, and if $g \in G_i$ then $gN$ has property S. (2) $G_i$ is a core refinement of $G_{i-1}$ for each $i$. (3) Let $g \in G_{i-1}$ and let $g_1, g_2, \cdots, g_k, g_{k+1}, g_{k+2}, \cdots, g_{k+n}$ be the elements of $G_i$ which are contained in $g$ and which intersect $N$ where $\bar{g}_i \subset g$ if $i \leqq k$ and $\bar{g}_i \not\subset g$ if $i > k$. Then $\sum_{i=1}^{k} \mathrm{Cl}(g_iN)$ is connected and intersects $\mathrm{Cl}(g_iN)$ for each $i$ from $k+1$ to

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(2) Numbers in brackets refer to the Bibliography at the end of the paper.

$k+n$. The question arises, what results regarding partitionings have analogies in the case of simultaneous partitionings?

2. **Preliminary results.** First let us recall some facts which have been previously established. An important one which characterizes partitionable sets is: Theorem A. A set $M$ is partitionable if and only if it has property S. This means that if we wish to partition a set $M$ into elements which themselves can be further partitioned, then these elements must have property S. Thus a fundamental result (Theorem B) is that if $M$ is a partitionable set and $\epsilon$ is an arbitrary positive number, then there exists an $S$, $\epsilon$-partitioning of $M$. A consequence of this is that if $M$ is partitionable, then there exists a decreasing sequence of $S$-partitionings of $M$ (Theorem C). Of course many other results have been obtained concerning partitionings, and many applications of this concept have been found.

LEMMA 1. *Let $M$ be a continuous curve, $U$ an open partitionable connected subset of $M$, and $H_1, H_2, \cdots, H_k$ a finite number of connected disjoint closed subsets of $U$ with property* S. *Then there exists a finite collection $\{W_1, W_2, \cdots, W_k\}$ of disjoint connected open subsets of $U$ with property* S, *whose sum is dense in $U$, with $W_i \supset H_i$ for each $i$.*

**Proof.** Let $G_1$ be an $S$, 1-partitioning of $U$. Let $D=\sum_{i=1}^{k} H_i$. Let $\{p_i\}$' $i=1, 2, \cdots, n$, be a collection of points obtained by selecting a point from each element of $G_1$. Let $A_1, A_2, \cdots, A_n$ be $n$ arcs in $U$ such that $A_i \supset p_i$ for each $i$, each $A_i$ intersects exactly one component of $D$, and $A_iA_j=0$ if $A_i$ and $A_j$ intersect different components of $D$. For each $i$ from 1 to $k$, let $S_1(H_i)$ equal $H_i$ plus all the above arcs which intersect $H_i$.

Let $S_1'(H_i)=\{x \mid x \in S_1(H_i)$ and $\rho(x, \text{bdy } U) \geqq 1\}$. There exists a positive number $d_1$ such that $\rho(S_1(H_i), S_1'(H_j))>d_1$ if $i \neq j$. About each point of $S_1'(H_i)$ there exists an open connected set with property S containing $p$ of diameter less than min $(1, d_1/3)$. The closure of each such set thus is contained in $U$. A finite number of these sets cover $S_1'(H_i)$. Let $T_1'(H_i)$ equal the sum of their closures. Let $T_1(H_i)=T_1'(H_i)+S_1(H_i)$. Do this for each $i$ from 1 to $k$. Then each $T_1(H_i)$ is a closed connected partitionable subset of $U$ and $T_1(H_i) \cdot T_1(H_j)=0$ if $i \neq j$.

Let $G_2$ be an $S$, 1/2-partitioning of $U$ which is a refinement of $G_1$. Let $D_1=\sum_{i=1}^{k} T_1(H_i)$. Proceed in a manner exactly similar to that described in the above two paragraphs with only the added condition that each new arc must be contained in an element of $G_1$. Thus sets $T_2(H_i)$, $i=1, 2, \cdots, k$ are obtained which are each closed partitionable connected subjects of $U$ and such that $T_2(H_i) \cdot T_2(H_j)=0$ if $i \neq j$.

Proceed similarly to obtain sets $T_n(H_i)$, $i=1, 2, \cdots, k$; $n=3, 4, \cdots$, using $S$-partitionings $G_n$ of mesh less than $1/2^{n-1}$. Define $W_i=\sum_{n=1}^{\infty} T_n(H_i)$ for each $i$. All the conclusions are clearly satisfied except possibly that each $W_i$ has property $S$. To prove this, recall $W_i=\sum_{n=1}^{\infty} T_n(H_i)$. $T_{n+1}(H_i)$ is obtained from $T_n(H_i)$ by adding a finite number of arcs, each of diameter less than $1/2^{n-2}$ and a finite number of connected sets with property S each of diameter less than $1/(2^{n-2})$. Hence any point in $T_{n+1}(H_i)$ can be joined to $T_n(H_i)$ by a connected set in $T_{n+1}(H_i)$ of diameter less than $1/(2^{n-3})$. Given $\epsilon>0$ take $\epsilon/3$. Since $\sum_{n=1}^{\infty} 1/2^n=1$, there exists a positive integer $N$ such that each point of $W_i$ can be joined to $T_N(H_i)$ by a connected set in $W_i$ of diameter less than $\epsilon/3$. Since $T_N(H_i)$ has property S it is the sum of a finite number of connected sets each of diameter less than $\epsilon/3$, say $C_1, C_2, \cdots, C_m$. Let $C_i'$ equal $C_i$ plus all points of $W_i$ which can be joined to it by a connected set of diameter less than $\epsilon/3$ which lies in $W_i$. Then the diameter of $C_i'$ is less than $\epsilon$ and $\sum_{i=1}^{m} C_i' = W_i$, thus proving that $W_i$ has property S.

LEMMA 2. *Let $U$ be an open partitionable connected subset of a continuous curve $M$, and $V$ a closed partitionable connected subset of $U$. Let $H$ and $K$ be two sets in $U$ such that $\rho(H, K)>0$. Then $H$ and $K$ can be expanded into larger closed subsets of $U$, $H'$, and $K'$, such that $\rho(H', K') >0$, and $H'$, $K'$, $H'V$, $K'V$ all have property* S.

**Proof.** Let $\rho(H, K)=\epsilon>0$. Let $G$ ($G'$) be an $S$, $\epsilon/6$-partitioning of $U$ ($V$). Let $H'$ equal $H$ plus the closures (relative to $U$) of all elements of $G$ whose closures intersect $H$. Define $K'$ similarly. Let $H''$ equal $H'$ plus the closures (relative to $V$) of all elements of $G'$ whose closures intersect $H'$. Define $K''$ similarly. Then $\rho(H'', K'')>0$ and $H''$, $K''$, $H''V$, and $K''V$ all have property S.

THEOREM 1. *Let $U$ be an open connected partitionable subset of a continuous curve $M$, $V$ a closed partitionable subset of $U$, and $H$ and $K$ two subsets of $U$ such that $\rho(H, K)>0$. Then there exists a collection $\{W_1, W_2, \cdots, W_n\}$ of open disjoint connected subsets of $U$ with property* S *whose sum is dense in $U$, such that no $W_i$ intersects both $H$ and $K$ but each $W_i$ intersects either $H$ or $K$, and such that each $W_iV$ is either void or an open connected subset of $V$ with property* S, *and $\sum_{i=1}^{n} W_iV$ is dense in $V$.*

**Proof.** We may clearly suppose that $H$ and $K$ are closed subsets of $U$. It is also no loss in generality to assume that $(H+K)$ intersects each component of $V$. To prove this, let $C$ be a component of $V$ which contains no point of $(H+K)$. Then there exists an arc $A$ in $U$ intersecting $C$ and one of the sets $H$ and $K$ but lying at a positive distance from the other set. $A$ may be added to the set which it intersects. This procedure may be repeated until each component of $V$ contains a point of $(H+K)$. Also by Lemma 2 we may assume that $H$ and $K$ are closed subsets of $U$ such that that $H$, $K$, $HV$, and $KV$ all have property S. Let $\{H_i\}$, $i=1, 2, \cdots, n$, $\{K_i\}$, $i=1, 2, \cdots, m$, $\{A_i\}$, $i=1, 2, \cdots, r$, $\{B_i\}$, $i=1, 2, \cdots, s$, be the components of $H$, $K$, $HV$, and $KV$ respectively.

Let $G_1$ be an $S$, 1-partitioning of $V$. Let $\{p_i\}$, $i=1, 2, \cdots, n_1$, be a collection of points obtained by selecting a point from each element of $G_1$. Let $\{Z_i\}$, $i=1, 2, \cdots, n_1$, be a collection of arcs in $V$ such that $Z_i$ contains $p_i$ for each $i$, each $Z_i$ intersects exactly one component of $(HV+KV)$, and $Z_iZ_j=0$ if $Z_i$ and $Z_j$ intersect different components of $(HV+KV)$. Let $A_i'$ (or $B_i'$ or $H_i'$ or $K_i'$) equal $A_i$ (or $B_i$ or $H_i$ or $K_i$) plus all the arcs $Z_i$ which intersect $A_i$ (or $B_i$ or $H_i$ or $K_i$). Let $H'=\sum_{i=1}^n H_i'$ and $K'=\sum_{i=1}^m K_i'$.

Let $G_1'$ be an $S$, 1-partitioning of $U$. Let $\{q_i\}$, $i=1, 2, \cdots, m_1$, be the set of points obtained by selecting a point from each of the elements of $G_1'$ which does not intersect $V$. Join $q_i$ to some component of $(H'+K')$ by an arc entirely in $U-V$ except possibly for an end point if this is possible. If not, join $q_i$ to $V$ by an arc intersecting $V$ at only one point, say $y$, and then join $y$ by an arc in $V$ to some component of $(\sum_{i=1}^r A_i'+\sum_{i=1}^s B_i')$. This can be done in such a way that we have a finite family of arcs $N_1, N_2, \cdots, N_{m_1}$, such that $N_i$ contains $q_i$ for each $i$, each $N_i$ intersects exactly one component of $(H'+K')$ and at most one component of $(\sum_{i=1}^r A_i' + \sum_{i=1}^s B_i')$, and such that $N_iN_j=0$ if $N_i$ and $N_j$ intersect different components of $(H'+K')$ or different components of $(\sum_{i=1}^r A_i'+\sum_{i=1}^s B_i')$. For each $i$ from 1 to $r$ let $S_1(A_i)$ equal $A_i'$ plus the intersection of $V$ with all the above arcs which intersect $A_i'$. Similarly define $S_1(B_i)$ for each $i$ from 1 to $s$. For each $i$ from 1 to $n$ $(m)$ let $S_1(H_i)$ $(S_1(K_i))$ equal $H_i'$ $(K_i')$ plus all the above arcs which intersect $H_i'$ $(K_i')$.

Let $E_1=\{x \mid x\in(\sum_{i=1}^r S_1(A_i)+\sum_{i=1}^s S_1(B_i))$ and $\rho(x, \text{bdy } U)\geqq 1\}$. This is a compact set. Let $\{D_i\}$, $i=1, 2, \cdots, r+s$, be the components of $(\sum_{i=1}^r S_1(A_i)+\sum_{i=1}^s S_1(B_i))$. Let $d_1$ be a positive number such that $\rho(E_1D_i, D_j)>d_1$ if $i\neq j$. Around each point $p$ of $E_1$ there exists an open connected subset of $V$ containing $p$ with property S, of diameter less than min $(1/2, d_1/3)$. The closure of each such set thus lies in $V$. A finite number of such sets cover $E_1$. Let $T_1(A_i)$ equal $S_1(A_i)$ plus the closures of all these finite number of sets which intersect $S_1(A_i)$, for each $i$ from 1 to $r$. Define $T_1(B_i)$ similarly for each $i$ from 1 to $s$. Then each $T_1(A_i)$ and each $T_1(B_i)$ is a connected partitionable closed subset of $V$, and no two of these sets intersect.

Let $F_1=\{x \mid x\in(\sum_{i=1}^n S_1(H_i)+\sum_{i=1}^m S_1(K_i))$ and $\rho(x, \text{bdy } U)\geqq 1\}$. Again $F_1$ is compact. Let $\{D_i'\}$, $i=1, 2, \cdots, n+m$, be the components of $(\sum_{i=1}^n S_1(H_i)+\sum_{i=1}^m S_1(K_i))$. Let $d_1'$ be a positive number such that $\rho(F_1D_i', D_j')>d_1'$ if $i\neq j$. About each point $p$ of $F_1$ there exists an open partitionable connected subset of $U$ containing $p$ of diameter less than min $(1/2, d_1'/3)$, whose closure lies in $U$ and does not intersect $V-(\sum_{i=1}^r T_1(A_i)+\sum_{i=1}^s T_1(B_i))$. A finite number of such sets cover $F_1$. For each $i$ from 1 to $n$ let $T_1(H_i)$ equal $S_1(H_i)$ plus the closures of all these finite number of sets which intersect $S_1(H_i)$, plus all components of $(\sum_{i=1}^r T_1(A_i)+\sum_{i=1}^s T_1(B_i))$ which intersect $S_1(H_i)$. Define $T_1(K_i)$ similarly for each $i$ from 1 to $m$. Note that each $T_1(H_i)$ and each $T_1(K_i)$ is a closed connected partitionable subset of $U$ and no two of these sets intersect.

Let $G_2$ be an $S$, 1/2-partitioning of $V$ which is a refinement of $G_1$. Proceed in exactly the same manner as in the second paragraph of this proof to obtain sets $T_1'(A_i), i=1, 2, \cdots, r, T_1'(B_i), i=1, 2, \cdots, s, T_1'(H_i), i=1, 2, \cdots, n$, and $T_1'(K_i)$, $i=1, 2, \cdots, m$, with only the added restriction that each arc concerned must be contained in some element of $G_1$ and hence be of diameter less than 1.

Let $G_2'$ be an $S$, 1/2-partitioning of $U$ which is a refinement of $G_1'$. Proceed in a manner similar to that described in the third paragraph of this proof to obtain sets $S_2(A_i)$, $S_2(B_i)$, $S_2(H_i)$, $S_2(K_i)$, with only the added condition that if $M_i$ is one of the arcs concerned then $M_iV$ is contained in the closure of some element of $G_1$ and $M_i - V$ is contained in some element of $G_1'$.

Let $E_2 = \{x \mid x \in (\sum_{i=1}^{r} S_2(A_i) + \sum_{i=1}^{s} S_2(B_i))$ and $\rho(x, \text{bdy } U) \geqq 1/2\}$. Let $\{X_i\}$, $i=1, 2, \cdots, r+s$, be the components of $(\sum_{i=1}^{r} S_2(A_i) + \sum_{i=1}^{s} S_2(B_i))$. Let $d_2$ be a positive number such that $d_2 < \rho(X_i, X_j E_2)$ if $i \neq j$. Let $\eta_2 = \min(1/4, d_2/3)$. About each point $p$ of $E_2$ there exists an open connected partitionable subset of $V$ of diameter less than $\eta_2$ containing $p$. A finite number of these sets cover $E_2$. Let $T_2(A_i)$ equal $S_2(A_i)$ plus the closures of all these finite number of sets which intersect $S_2(A_i)$, for each $i$ from 1 to $r$. Define $T_2(B_i)$ similarly for each $i$ from 1 to $s$.

Let $\{Y_i\}$, $i=1, 2, \cdots, n+m$, be the components of $(\sum_{i=1}^{n} S_2(H_i) + \sum_{i=1}^{m} S_2(K_i))$ and let $d_2'$ be a positive number such that $d_2' < \rho(Y_i, Y_j F_2)$ if $i \neq j$ where $F_2 = \{x \mid x \in (\sum_{i=1}^{n} S_2(H_i) + \sum_{i=1}^{m} S_2(K_i))$ and $\rho(x, \text{bdy } U) \geqq 1/2\}$. Let $\eta_2' = \min(1/4, d_2'/3)$. About each point $p$ of $F_2$ there exists an open connected partitionable subset of $U$ which contains $p$ and is of diameter less than $\eta_2'$, whose closure lies in $U$ and does not intersect $V - (\sum_{i=1}^{r} T_2(A_i) + \sum_{i=1}^{s} T_2(B_i))$. A finite number of these sets cover $F_2$. For each $i$ from 1 to $n$, let $T_2(H_i)$ equal $S_2(H_i)$ plus the closures of all these finite number of sets which intersect $S_2(H_i)$ plus all components of $(\sum_{i=1}^{r} T_2(A_i) + \sum_{i=1}^{s} T_2(B_i))$ which intersect $S_2(H_i)$. Define $T_2(K_i)$ similarly for each $i$ from 1 to $m$.

Thus we obtain sets $T_2(A_i)$, $i=1, 2, \cdots, r$, $T_2(B_i)$, $i=1, 2, \cdots, s$, $T_2(H_i)$, $i=1, 2, \cdots, n$, $T_2(K_i)$, $i=1, 2, \cdots, m$, all of which are closed partitionable connected subsets of $U$. Moreover each $T_2(A_i)$ is a component of $(\sum_{i=1}^{r} T_2(A_i) + \sum_{i=1}^{s} T_2(B_i))$ as is each $T_2(B_i)$. Also each $T_2(H_i)$ is a component of $(\sum_{i=1}^{n} T_2(H_i) + \sum_{i=1}^{m} T_2(K_i))$, as is each $T_2(K_i)$.

This procedure is repeated using $S$-partitionings $G_3, G_4, \cdots, G_3, G_4', \cdots$, each $G_i$ (or $G_i'$) being a refinement of $G_{i-1}$ (or $G_{i-1}'$), and each $G_i$ (or $G_i'$) being of mesh less than $1/2^{i-1}$. Define $R_i = \sum_{j=1}^{\infty} T_j(H_i)$, $i=1, 2, \cdots, n$, $S_i = \sum_{j=1}^{\infty} T_j(K_i)$, $i=1, 2, \cdots, m$, $Q_i = \sum_{j=1}^{\infty} T_j(A_i)$, $i=1, 2, \cdots, r$, $L_i = \sum_{j=1}^{\infty} T_j(B_i)$, $i=1, 2, \cdots, s$.

Then the sets $R_i$ and $S_i$ are each open subsets of $U$ since each point of $R_i$ for example is an interior point of some $T_n(H_i)$. Also each of these sets is connected and their sum $Y$ is dense in $U$ and has property S for the same reasons that each $W_i$ in Lemma 1 has property $S$. Each $R_i$ and each $S_i$ is a component of $Y$ and hence has property S also. Similarly each $Q_i$ and each $L_i$ is an open connected subset of $V$ with property S. Also $(\sum_{i=1}^r Q_i+\sum_{i=1}^s L_i)$ is dense in $V$ and actually $\sum_{i=1}^r Q_i=(\sum_{i=1}^n R_i)V$ and $\sum_{i=1}^s L_i=(\sum_{i=1}^m S_i)V$ so that each $Q_i$ $(L_i)$ is a closed subset of some $R_j$ $(S_j)$.

By Lemma 1, there exist sets $W_1, W_2, \cdots, W_{r_1}$, $r_1\geqq r$, in $\sum_{i=1}^n R_i$ such that $W_iV=Q_i$ for $1\leqq i\leqq r$ and $W_iV=0$ if $i>r$, $\sum_{i=1}^{r_1} W_i$ is dense in $\sum_{i=1}^n R_i$ and each $W_i$ is an open connected subset of $U$ with property S. Similarly there exist sets $W_{r_1+1}, \cdots, W_{r_1+s_1}$, $s_1\geqq s$, such that $W_{r_1+i}\cdot V=L_i$ for $1\leqq i\leqq s$ and $W_{r_1+i}\cdot V=0$ if $i>s$, $\sum_{i=1}^{s_1} W_{r_1+i}$ is dense in $\sum_{i=1}^m S_i$ and each $W_{r_1+i}$ is an open connected subset of $U$ with property S.

3. **The fundamental theorems.**

Theorem 2. *Let $M$ be a compact partitionable set and $N$ a closed partitionable subset. Then for every $\epsilon>0$ there exists a simultaneous $S$, $\epsilon$-partitioning of $M$ and $N$.*

**Proof.** It may easily be shown that it is sufficient to prove the theorem for the case when $M$ and $N$ are both connected (and hence continuous curves).

Given $\epsilon>0$ take $\epsilon/4$. Pick a finite set of points, $\{p_i\}$, $i=1, 2, \cdots, n$, such that if $p\in M$ then $D(p, p_i)<\epsilon/4$ for some $i$. Define $A_i=\{x\mid x\in M$ and $D(x, p_i)\leqq\epsilon/4\}$, and $B_i=\{x\mid x\in M$ and $D(x, p_i)\geqq\epsilon/2\}$. Then the sets $A_i$ and $B_i$ are at positive distance apart for each $i$. In particular $\rho(A_1, B_1)>0$. By Theorem 1 there exist disjoint open connected subsets of $M$, say $V_1, V_2, \cdots, V_m$, such that no $V_i$ intersects both $A_1$ and $B_1$ but each $V_i$ intersects $(A_1+B_1)$, $\sum_{i=1}^m V_i$ is dense in $M$ and has property S, each $V_iN$ is either void or an open connected subset of $N$ with property S and $\sum_{i=1}^m V_iN$ is dense in $N$. Clearly if $V_iA_1\neq 0$, $\delta(V_i)<\epsilon$. Let $X_1$ $(Y_1)$ be the sum of those $V_i$ which intersect $A_1$ $(B_1)$.

Let $V_n$ be an arbitrary component of $Y_1$. Then $V_nA_1=0$. Consider $V_nA_2$ and $V_nB_2$. If $V_nN$ is not void these two sets satisfy the conditions of Theorem 1 for $H$ and $K$ with $U=V_n$ and $V=V_nN$ and so we can apply the theorem again. If $V_nN=0$, it can be $S$, $\epsilon$-partitioned (Theorem 4, p. 1104 of [5]). The procedure can be repeated until the components of $Y_1$ are exhausted. This gives us a simultaneous $S$-partitioning $G$ of $Y_1$ and $Y_1N$. Let $X_2$ equal the sum of all elements of $G$ which are of diameter less than $\epsilon$. Let $Y_2$ equal the sum of the remaining elements of $G$. Proceeding similarly we obtain sets $X_3, Y_3, \cdots, X_{n-1}, Y_{n-1}$. Then the components of $(Y_{n-1}+\sum_{i=1}^{n-1} X_i)$ are the elements of a simultaneous $S$, $\epsilon$-partitioning of $M$ and $N$.

The following results may be obtained in the same manner and will be used several times.

Theorem 3. *Let $M$ and $N$ be two continuous curves such that $N\subset M$ and let $U$ be an open partitionable connected subset of $M$ such that $UN$ is connected* *and partitionable. Then for every* $\epsilon > 0$ *there exists a simultaneous* $S$, $\epsilon$*-partitioning of* $U$ *and* $UN$.

THEOREM 4. *Let* $M$ *be a compact partitionable set and* $N$ *a closed partitionable subset. Then there exists a sequence* $\{G_i\}$ *of simultaneous* $S$*-partitionings of* $M$ *and* $N$ *such that* $G_i$ *is a refinement of* $G_{i-1}$ *of mesh less than* $1/i$.

4. **Remarks and examples.** Theorem 2 would follow immediately (using Lemma 1) from Theorem B if for every $S$, $\epsilon$-partitioning $G$ of $M$ and every element $g$ of $G$ which intersects $N$, $gN$ had property S. However this is not the case. Very simple examples can be found where $G$ is an $S$-partitioning of $M$, $g \in G$, and $gN$ does not have property S. These and other examples show that it is not possible to prove Theorem 2 by working with $M$ only or with $N$ only at first, but rather the sets $N$ and $M$ must be considered simultaneously as was done. These examples can easily be constructed and so are not included.

A COUNTER EXAMPLE. One might wonder if the restriction that $N$ must be contained in $M$ could be removed. That is, consider the following theorem.

FALSE STATEMENT D. *Let* $M$ *and* $N$ *be two arbitrary continuous curves,* $\epsilon$ *an arbitrary positive number, and* $U$ *an open connected partitionable subset of* $(M+N)$ *such that* $UM$ *and* $UN$ *are both connected and partitionable. Then there exists an* $S$, $\epsilon$*-partitioning* $G = \{g_1, g_2, \cdots, g_k\}$ *of* $U$ *such that* $\{g_iM\}$, $i=1, 2, \cdots, k$, *and* $\{g_iN\}$, $i=1, 2, \cdots, k$, *are* $S$*-partitionings of* $UM$ *and* $UN$ *respectively.*

It can easily be shown that a consequence of this theorem is that if $p$ is an arbitrary point of $(M+N)$ and $\epsilon$ an arbitrary positive number, there exists an open connected set $U$ such that $p \in U$, $\delta(U) < \epsilon$, and $UM$ and $UN$ are both connected and partitionable. However consider a straight line $L$ in euclidean 3-space and points $p$ and $p_1$ on $L$. Let $p_2$ be the mid-point of $[p, p_1]$ and in general $p_i$ be the mid-point of $[p, p_{i-1}]$. Consider a sequence $\{q_i\}$ of points defined as follows. Points $q_1, q_2, \cdots, q_6$ are the points, $p_{6k}, p_{6k-3}, p_{6k+2}, p_{6k-1}, p_{6k+4}, p_{6k+1}$, respectively, with $k=1$. The points $q_7$ to $q_{12}$ are the points $p_{6k}, p_{6k-3}, p_{6k+2}, p_{6k-1}, p_{6k+4}, p_{6k+1}$, respectively, with $k=2$, and so on. Let $\{A_i\}$ be a sequence of arcs such that (1) the end points of $A_i$ are $q_i$ and $q_{i+1}$, (2) each pair of these arcs are disjoint except possibly for a common end point, and (3) $\rho(x, [q_i, q_{i+1}]) < 1/2^i$ if $x \in A_i$. Let $N = p + \sum_{i=1}^{\infty} A_i$ and let $M = [p, p_1]$. Then there do not exist arbitrarily small open subsets $U$ of $(M+N)$ containing $p$ such that $UM$ and $UN$ are both connected. The proof is omitted. This shows that Theorem D is indeed a false theorem and thus that the results of this paper are not valid for two arbitrary continuous curves.

5. **Some additional theorems.**

THEOREM 5. *Let* $p$ *be an arbitrary point of a compact partitionable set* $M$ *and let* $N$ *be a closed partitionable subset of* $M$ *and let* $\epsilon$ *be an arbitrary positive* *number. Then there exists in $M$ an open partitionable connected set $U$ such that $p\in U$, $\delta(U)<\epsilon$, and $UN$ is either void or connected and partitionable.*

**Proof.** (a) *If* $p\in(M-N)$ the proof is left to the reader. (b) Suppose $p\in N$. Given $\epsilon$, take $\epsilon/2$. By Theorem 2 there exists a simultaneous $S$, $\epsilon/2$-partitioning $G$ of $M$ and $N$. If $p$ belongs to an element of $G$, that element serves as the set $U$. Otherwise let $\{g_i\}$, $i=1, 2, \cdots, k$, be the elements of $G$ such that $p\in\mathrm{Cl}\,(g_i)$. Let $V$ be the interior of $\sum_{i=1}^{k}\mathrm{Cl}\,(g_i)$. If $VN$ is connected we are done. If not $VN$ has a finite number of components and we may apply Lemma 1.

THEOREM 6. *Let $M$ be a compact partitionable set, $N$ a closed partitionable subset, $p$ an arbitrary point of $M$, and $\delta$ and $\epsilon$ any two positive numbers such that $\delta<\epsilon$. Then there exists an open set $U$ such that $U$ and $UN$ both have property* S *and $A\subset U$ and $\rho(U, B)>0$, where $A=\{x\mid x\in M$ and $D(x, p)\leqq\delta/2\}$ and $B=\{x\mid x\in M$ and $D(x, p)\geqq\epsilon/2\}$. (If $B$ is void omit the conclusion $\rho(U, B)>0$.)*

**Proof.** If $B=0$, let $U=M$. Otherwise let $\gamma=(\epsilon-\delta)/2$. Let $G$ be a simultaneous $S$-partitioning of $M$ and $N$ of mesh less than $\gamma/3$. Let $U$ equal the interior of the sum of the closures of all elements of $G$ whose closures intersect $A$. This set is of the required type. $U$ clearly has property S. $UN$ also has property $S$ because it is the sum of sets $gN$ where $\bar{g}A\neq 0$, plus a subset of the limit points of this sum. The other conclusions of Theorem 6 are clearly satisfied.

THEOREM 7. *Let $M$ be a compact partitionable set and $N$ a closed partitionable subset of $M$. Then there exists a decreasing sequence $\{G_i\}$ of simultaneous $S$-partitionings of $M$ and $N$ such that if $g$ and $h$ are two elements of $G_i$ for which $gN\neq 0$, $hN\neq 0$, and $\rho(gN, hN)>0$, then $\rho(g, h)>0$.*

**Proof.** We prove the theorem by showing that if partitionings $G_1, G_2, \cdots, G_{n-1}$ have been chosen so that the conclusions of Theorem 7 are satisfied for them, then $G_n$ can be chosen of the required type.

Let $H$ be any simultaneous $S$, $1/n$-partitioning of $M$ and $N$ which is a refinement of $G_{n-1}$. Let $\{h_i\}$, $i=1, 2, \cdots, k$, be the elements of $H$ which intersect $N$ and $\{h_i\}$, $i=k+1, \cdots, k+t_1$, be the remaining elements. Consider $h_1$. If $\rho(h_1, h_i)=0$ only if (a) $i>k$ or (b) $i\leqq k$ and $\rho(h_1N, h_iN)=0$, pass on to $h_2$. However if, for some $i\leqq k$, $\rho(h_1, h_i)=0$ but $\rho(h_1N, h_iN)>0$, let $h_m$ be one such $h_i$ for which $\rho(h_1N, h_iN)$ is a minimum. Consider a simultaneous $S$-partitioning $H'$ of $M$ and $N$ which is a refinement of $H$ of mesh less than $(1/4)\cdot\rho(h_1N, h_mN)$. For each $i$ from 1 to $k$ let $h_i'$ equal the interior of the sum of the closures of all elements of $H'$ whose closures intersect $h_iN$ and let $\{h_i'\}$, $i=k+1, \cdots, k+t_2$, be the remaining elements of $H'$. Then if $i\leqq k$, $\rho(h_1', h_i')>0$ unless $\rho(h_1'N, h_i'N)=0$.

Now omitting $h_1'$ and considering only the remaining elements of $H'' = \{h_i'\}$, $i=1, 2, \cdots, k+t_2$, treat $h_2'$ in a similar manner to the way in which $h_1$ was treated. After at most $k-1$ steps a partitioning is obtained which satisfies the conditions on $G_n$.

In fact this theorem may be strengthened so that if $\delta$ equals the minimum of $\rho(g, h)$ where $g$ and $h$ vary over all elements of $G_i$ which intersect $N$ and are such that $\rho(gN, hN)>0$, then the diameter of $g'$ is less than $\delta/H$ where $H$ is an arbitrary positive integer, and $g'$ is an arbitrary element of $G_i$ which does not intersect $N$.

THEOREM 8 (CORE PARTITIONINGS). *Let $M$ be a compact partitionable set and $N$ a closed partitionable subset. Then there exists a decreasing sequence of simultaneous core partitionings $\{G_i\}$ of $M$ and $N$.*

**Proof.** Let $G_1$ be a simultaneous $S$, 1-partitioning of $M$ and $N$, let $g \in G_1$ and let $G_1'$ be the partitioning of $g$ consisting of the single element $g$. If $gN=0$, it has previously been shown [7 and 8, Theorem 3, p. 1119] that there exists a core refinement of $G_1'$ of mesh less than 1/2. Thus suppose $gN \neq 0$.

Let $H$ be a simultaneous $S$, 1/2-partitioning of $g$ and $gN$. Let $T$ be a connected set in $g$ such that $TN \neq 0$, $T$ intersects every element of $H$, and $\rho(M-g, T)>0$. If $TN$ is not a connected set in $gN$ which intersects each element $h$ of $H$ for which $hN \neq 0$, a set $S$ may be obtained from $T$ by adding a finite number of dendrons in $gN$ chosen in such a way that $SN$ is a connected set in $gN$ which intersects each element $h$ of $H$ for which $hN \neq 0$. Note that $\rho(M-g, S)=\delta>0$.

Let $H'$ be a simultaneous $S$-partitioning of $g$ and $gN$ which is a refinement of $H$ of mesh less than $\delta/3$ and which satisfies the following condition. If $h' \in H'$, $h'N \neq 0$, and $\rho(h', S)=0$, then $\rho(h'N, S)=0$ also. To show that $H'$ exists we may consider an arbitrary simultaneous $S$-partitioning $K$ of $g$ and $gN$ which is a refinement of $H$ of mesh less than $\delta/3$. If there exists an element $k$ of $K$ such that $kN \neq 0$, $\rho(k, S)=0$, and $\rho(kN, S)=\gamma>0$, it is merely necessary to consider a simultaneous $S$-partitioning of $k$ and $kN$ of mesh less than $\gamma/3$. If this procedure is repeated for all elements of $K$ similar to $k$, then the collection of all sets thus obtained plus all the unchanged elements of $K$ constitutes a partitioning satisfying the conditions on $H'$.

Let $C$ equal the sum of the closures of all elements of $H'$ whose closures intersect $S$, say $C=\sum_{i=1}^{k} \mathrm{Cl}\,(h_i')$ and let $D=\sum_{i=1}^{k} \mathrm{Cl}\,(h_i'N)$. Then $C$ and $D$ are both connected sets in $g$. $C$ is connected because for each $i$ from 1 to $k$, $h_i'$ has a limit point in $S$. $D$ is connected because for each $i$ from 1 to $k$ for which $h_i'N \neq 0$, $h_i'N$ has a limit point in $SN$.

Consider the border elements $h_1, h_2, \cdots, h_m$ of $H$ which are contained in $g$. Consider an arbitrary component $K$ of $(\sum_{i=1}^{m} h_i)-C$. Thus $K$ will be contained in some element of $H$. If $KN=0$, leave $K$ alone. If $KN \neq 0$, $KN$ has a finite number of components, $X_1, X_2, \cdots, X_n$, and $K$ may be partitioned into open disjoint connected partitionable subsets $W_1, W_2, \cdots, W_n$ such that $W_i \supset X_i$ for each $i$ (Lemma 1). Treat each component of $(\sum_{i=1}^m h_i) - C$ in this manner. Let $\{R_i\}$, $i=1, 2, \cdots, r$, and $\{S_i\}$, $i=1, 2, \cdots, s$, be the sets thus obtained which have a boundary point in common with bdy $g$, where $R_iN=0$ and $S_iN$ is nonvoid and connected. These sets plus all the sets obtained by the same process whose closures lie in $g$, plus all elements of $H'$ which are contained in $(C+(g-\sum_{i=1}^m h_i))$, constitute a simultaneous $S$, 1/2-partitioning $G$ of $g$ and $gN$. The border elements of $G$ are $\{R_1, \cdots, R_r, S_1, \cdots, S_s\}$ and all the remaining elements are interior elements (border and interior elements with respect to $G_1$).

Suppose $E$ equals the sum of the closures of all the interior elements of $G$, defined above, say $E=\sum_{i=1}^t \mathrm{Cl}\,(g_i)$ and let $F=\sum_{i=1}^t \mathrm{Cl}\,(g_iN)$. Then $E$ and $F$ are both connected, as we shall now prove. Suppose $E$ is not connected. Let $E'$ be a component of $E$ which does not intersect $C$. Suppose $E'$ contains an element $g'$ of $G$ which is contained in a border element $h$ of $H$. If $g'N=0$, $g'$ is a component of $h-C$ and hence $g'$ has a limit point in $C$, since $h$ is connected. If $g'N\neq 0$, $g'N$ is a component of $(h-C)N$ and hence has a limit point in $CN$ because $hN$ is connected. In each case $E'$ intersects $C$, contrary to assumption. $E'$ cannot contain an element of $H'$ which is contained in an interior element $h$ of $H$ for in this case $E'$ would contain $h$ and hence again intersect $C$, a contradiction. Hence $E$ is connected. We may proceed in exactly the same manner to prove that $F$ is connected. The same arguments show that if $g''$ is an arbitrary border element of $G$, then $E$ contains a limit point of $g''$ and $F$ contains a limit point of $g''N$ unless the latter set is void.

Once $G$ has been defined, this shows how the other elements of $G_1$ may be treated. By dealing with each of them in the same manner, we obtain a simultaneous $S$, 1/2-partitioning $G_2$ of $M$ and $N$ of the required type. The method is general and so the proof is complete.

**6. Unanswered questions.** Consider the following two theorems which may possibly be true, and if so, might lead to further results.

CONJECTURE E. *Let $M$ and $N$ be two continuous curves such that $N\subset M$, $U$ a connected, uniformly locally connected* (*ulc*), *open subset of $M$ such that $UN$ is ulc and connected, and $\epsilon$ an arbitrary positive number. Then there exists a simultaneous $\epsilon$-partitioning $G$ of $U$ and $UN$, such that if $g\in G$ then $g$ and $gN$ both are ulc.*

This is a stronger result than Theorem 2 since for subsets of a compact metric space ulc implies property S but not conversely.

CONJECTURE F. *Let $M$ and $N$ be two continuous curves such that $N\subset M$, and $\epsilon$ an arbitrary positive number. Then there exists a simultaneous $S$, $\epsilon$-partitioning $G$ of $M$ and $N$ such that if $g\in G$ then $\bar{g}N=\mathrm{Cl}\,(gN)$.*

## Bibliography


1. R. H. Bing, *A characterization of 3-space by partitionings*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 15–27.

2. ———, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 812–919.

3. ———, *A convex metric with unique segments*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 167–174.

4. ———, *Complementary domains of continuous curves*, Fund. Math. vol. 36 (1949) pp. 303–318.

5. ———, *Partitioning a set*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1101–1110.

6. ———, *Partitioning continuous curves*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 536–556.

7. R. H. Bing and E. E. Floyd, *Coverings with connected intersections*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 387–391.

8. E. E. Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

9. ———, *A note of correction*, Proc. Amer. Math. Soc. vol. 2 (1951) p. 838.

10. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942.

11. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, New York, 1949.


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