ISOLATED SINGULARITIES OF SOLUTIONS OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS(1)

BY

ROBERT FINN

INTRODUCTION

A significant role in the theory of linear elliptic second-order partial differential equations in two independent variables has been played by the concept of the fundamental solution. Such a function is a single-valued solution of the equation, regular except for an isolated point at which it possesses a logarithmic singularity. For the special case of the Laplace equation $\phi_{xx}+\phi_{yy}=0$ this solution can be uniquely characterized as the only (non-constant) solution which exhibits radial symmetry. The requirement of radial symmetry leads to an ordinary differential equation whose solution (up to a constant) is $\mu \log (x^2+y^2)^{1/2}$. This function admits an important hydrodynamical interpretation as the velocity potential of an incompressible, non-viscous, two-dimensional source-flow. An amount $2\pi\mu$ of fluid is visualized as flowing in unit time out of a source at the origin (i.e. across every closed curve surrounding the source) and into a corresponding sink at infinity.

The requirement of incompressibility may be relaxed by assuming a relation—called the *equation of state*—between the density ρ of the fluid and the velocity $|\nabla\phi|$. For adiabatic flows this relation takes the form $\rho = [1-((\gamma-1)/2)|\nabla\phi|^2]^{1/(\gamma-1)}$, where γ is the ratio of specific heats of the fluid. In this case the potential ϕ satisfies not the Laplace equation but the nonlinear equation

$$(1) \qquad (\rho\phi_x)_x + (\rho\phi_y)_y = 0.$$

Because of the complicated nature of the nonlinearity in this equation, its analysis has presented many difficulties, and various attempts have been made to suitably approximate the coefficients. It was observed by Chaplygin [1] in 1904 that a local approximation to the adiabatic equation of state can be obtained by setting $\rho=[1+\phi_x^2+\phi_y^2]^{-1/2}$. In this case (1) becomes the minimal surface equation

Presented to the Society, September 7, 1951; received by the editors November 8, 1951 and, in revised form, August 20, 1952.

(1) This paper is based on a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Syracuse University, June, 1951. Part of the work was performed under contract no. NR 045 049 of the Office of Naval Research. The author wishes to express his deepest gratitude to Professor A. Gelbart for guidance throughout years of graduate study; he wishes also to thank Professor L. Bers for many stimulating conversations.

$$(1 + \phi_y^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_x^2)\phi_{yy} = 0 \tag{2}$$

so that solutions of this equation admit interpretation as the velocity potential of a compressible fluid.

It is natural to seek source-type flows for equations (1) and (2), and this is easily done by use of the requirement of radial symmetry. For equation (2) we obtain in this manner the solution $\phi(x, y) = \mu \log (r + (r^2 - \mu^2)^{1/2})$, $r = (x^2 + y^2)^{1/2}$. Unlike the fundamental solution of the Laplace equation, this function is defined only exterior to a circle of radius μ, on the boundary of which the velocity components become infinite. A similar situation is found in the case of equation (1). Thus in compressible flow the source behaves not as a point singularity but as a solid nucleus, interior to which the flow cannot be continued.

In order to see the reason for this behavior, let us define the *mass-flow* at a point as the product of the density ρ of the fluid by the velocity at the point, $m = \rho |\nabla\phi|$. It is the amount of fluid which flows in unit time across a segment of unit length orthogonal to the direction of flow. It is clear that if a finite amount of fluid is to emanate in unit time from a point source, then the mass flow must become infinite at the source. On the other hand, for the minimal surface equation, $m = (\phi_x^2 + \phi_y^2)^{1/2} \cdot (1 + \phi_x^2 + \phi_y^2)^{-1/2}$, which is less than 1 for all solutions of the equation. Similarly, the coefficients of equation (1) are not in general defined for large values of $|\nabla\phi|$, so that an a priori bound is placed on the mass flow for any solution.

It is not hard to conjecture that there must in general be severe restrictions on the kind of isolated singularities admitted by solutions of (1) and (2). A very detailed discussion of this question for the minimal surface equation (2) has been given by Bers [2; 3; 4] who has proved in particular that every isolated singularity of a minimal surface having a simply covered plane projection is removable. Earlier work in this direction has been done by S. Bernstein [5] who proved that a surface $z(x, y)$ of nonpositive curvature which is everywhere bounded in magnitude is a constant. As a corollary, Bernstein obtained the result that if $\phi(x, y)$ represents a minimal surface for all (x, y) then $\phi(x, y)$ is a linear function, $\phi(x, y) = ax + by + c$. Radó [6] pointed out a topological difficulty in Bernstein's geometrical theorem and gave an independent proof of the property of minimal surfaces. Still another proof has recently been given by Bers [2]. The topological difficulty in Bernstein's proof has been overcome by E. Hopf [7] and by Mickle [8].

In the present paper a broad class of nonlinear partial differential equations is characterized by an analogue of the mass-flow criterion discussed above. In §1 the nature of the restriction imposed by this criterion is discussed and it is shown that the solution must satisfy a generalized maximum principle. In §2 conditions for removability of isolated singularities are formulated. In particular a new proof is obtained for the theorem of Bers already cited. The methods employed are entirely different from those used by Bers, and it is believed that the proof presented is conceptually clearer and simpler. In §3 certain types of multi-valued solutions are discussed and finally some applications to the equation of gas-dynamics (1) are made in §4. Examples are given to show that the results obtained are best-possible in certain senses and extensions to the case of n-independent variables are stated.

1. A GENERALIZED MAXIMUM-MINIMUM PRINCIPLE

1.1. **Equations in two independent variables.** Let Θ and Λ be functions of x, y, ϕ, and the derivatives of ϕ with respect to x and y. Let us agree to say that a function $\phi(x, y)$ is an *admissible solution* of the partial differential equation

$$\Theta_x + \Lambda_y = 0 \tag{3}$$

at a point (x_0, y_0) if there exists a neighborhood N of (x_0, y_0) interior to which

(i) $\phi(x, y)$ is twice continuously differentiable,
(ii) Θ and Λ become continuously differentiable functions of x and y,
(iii) equation (3) is satisfied identically,
(iv) $\Theta\phi_x + \Lambda\phi_y \geqq 0$, equality holding if and only if $\phi_x = \phi_y = 0$.

We shall say that $\phi(x, y)$ is an admissible solution of (3) in a domain D provided that it is an admissible solution at every point of D.

We shall consider in this section the behavior of functions $\phi(x, y)$ which are single-valued admissible solutions of (3) in a neighborhood of an isolated singular point. For simplicity of notation, the singular point will occasionally be supposed at the origin. Points in the (x, y)-plane will alternatively be denoted by the symbol (x, y) or by the letters X, Y, $\cdots$.

It is easy to construct solutions of equations of type (3) with isolated singularities, e.g. if $\phi(x, y) = \log(x^2+y^2)$, then $\phi_{xx}+\phi_{yy}=0$ for $0<x^2+y^2<\infty$. One may also find solutions (although not of the Laplace equation) which are bounded at isolated singular points, e.g. if $\phi(x, y) = (x^2+y^2)^{1/4}$, then $(|\nabla\phi|\phi_x)_x + (|\nabla\phi|\phi_y)_y = 0$ for $0<x^2+y^2<\infty$. However it will be seen that a simple inference from Green's Identity permits the characterization of a broad class of equations of type (3) whose admissible solutions are severely limited in their behavior near isolated singular points.

THEOREM I. *Let $\phi(x, y)$ be a single-valued admissible solution of* (3) *in $0<x^2+y^2<R^2$. Let*

$$H_r = \operatorname{lub} \phi(x, y), \quad \text{for } 0 < x^2 + y^2 < r^2 < R^2,$$
$$L_r = \operatorname{glb} \phi(x, y), \quad \text{for } 0 < x^2 + y^2 < r^2.$$

Let

$$H = \lim_{r\to 0} H_r, \qquad L = \lim_{r\to 0} L_r.$$

Suppose[2] *that* $(\Theta^2+\Lambda^2)^{1/2}=o(1/r)$.

Then in every neighborhood of the origin there exist points X_1 *and* X_2 *such that*

$$\phi(X_1) \geqq H, \qquad \phi(X_2) \leqq L.$$

In particular, $\phi(x, y)$ *is bounded in* $0<x^2+y^2<R^2$.

The following corollaries are immediate consequences of this theorem:

I.1. *An admissible solution of* (3) *cannot have a strict maximum or minimum interior to a region of regularity.* It is essential that the solution be admissible. The statement is in general not true for solutions that do not satisfy condition (iv).

I.2. *Let* $\Theta=\alpha\phi_x+\beta\phi_y$, $\Lambda=\gamma\phi_x+\delta\phi_y$, $4\alpha\delta-(\beta+\gamma)^2>0$. *Then the condition* $(\alpha^2+\beta^2+\gamma^2+\delta^2)^{1/2}|\nabla\phi|=o(1/r)$ *is sufficient to ensure the existence in every neighborhood of the origin of points* X_1 *and* X_2 *such that*

$$\phi(X_1) > H, \qquad \phi(X_2) < L.$$

That $\phi(x, y)$ is admissible follows from the condition $4\alpha\delta-(\beta+\gamma)^2>0$. Further, a direct application of the inequality of Schwarz shows that $(\Theta^2+\Lambda^2)^{1/2}\leqq(\alpha^2+\beta^2+\gamma^2+\delta^2)^{1/2}|\nabla\phi|$. The removal of the equality signs from the conclusion of Theorem 1 follows from the well known fact that a solution $\phi(x, y)$ of a second order elliptic equation $a\phi_{xx}+2b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y=0$ satisfies the maximum-minimum principle in the strict sense.

I.3. *A single-valued solution of the minimal surface equation*

$$\left(\frac{\phi_x}{(1+\phi_x^2+\phi_y^2)^{1/2}}\right)_x + \left(\frac{\phi_y}{(1+\phi_x^2+\phi_y^2)^{1/2}}\right)_y = 0 \tag{2}$$

is bounded in a neighborhood of an isolated singular point.

In this case $(\Theta^2+\Lambda^2)^{1/2}<1$ uniformly in ϕ_x, ϕ_y. It will later appear that such a singularity is in fact removable.

We now present several lemmas in preparation for the proof of Theorem 1.

LEMMA 1. *Let* $\phi(x, y)$ *be a solution of* (3) *on and interior to a simple closed differentiable arc* Γ. *Then*

$$\oint_\Gamma [\Theta dy - \Lambda dx] = 0. \tag{4}$$

Proof. The integrand is an exact differential.

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(2) It is sufficient that there exist a sequence of simple closed arcs about X, converging in length to zero, on which $(\Theta^2+\Lambda^2)^{1/2}=o(1/r)$.

LEMMA 2. *The conclusion of Lemma* 1 *remains valid if* $\phi(x, y)$ *is singular at an isolated point* X *of* D, *provided that, at* X, $(\Theta^2+\Lambda^2)^{1/2}=o(1/r)$.

Proof. Let C_r be a circle of radius r about X and interior to Γ. Then

$$\oint_{C_r} [\Theta dy - \Lambda dx] = \oint_{\Gamma} [\Theta dy - \Lambda dx]$$

by Lemma 1. The result follows by letting $r\to 0$, since

$$\begin{aligned} \left| \oint_{C_r} [\Theta dy - \Lambda dx] \right| &\leqq \oint_{C_r} (\Theta^2 + \Lambda^2)^{1/2} ds \\ &\leqq \max_{C_r} \left\{ (\Theta^2 + \Lambda^2)^{1/2} \right\} \cdot 2\pi r = o(1/r)\cdot r \end{aligned}$$

by Schwarz's Inequality.

LEMMA 3. *If* $\phi(x, y)$ *is an admissible solution of* (3) *on and interior to a simple closed differentiable arc* Γ, *then*

$$\oint_{\Gamma} \phi[\Theta dy - \Lambda dx] \geqq 0$$

and equality holds if and only if $\phi(x, y) \equiv$ *constant in* D.

Proof. If $\psi(x, y)$ is continuously differentiable in $D+\Gamma$, then

$$\iint_D [\Theta\psi_x + \Lambda\psi_y] dxdy = \oint_{\Gamma} \psi[\Theta dy - \Lambda dx]. \tag{5}$$

Set $\psi(x, y)\equiv\phi(x, y)$.

LEMMA 4[3]. *Let* R *be a bounded region and let* $F(x, y)$ *be a function twice continuously differentiable in the closure of* R. *Let* $X\in R$ *and suppose that* $F(X) = h_0$, $F(x, y) \leqq h < h_0$ *on the boundary* B *of* R. *Choose* h_1 *and* h_2 *so that* $h<h_1<h_0 <h_2$. *Let* D *be the set of all points* (x, y) *in* R *at which* $h_1<F(x,y)<h_2$. *Clearly* D *is bounded and contains* X. *Let* Γ_D *be the boundary of* D. $\Gamma_D=\Gamma_{1D}+\Gamma_{2D}$, *where* $F(x, y)=h_1$ *on* Γ_{1D}, $F(x, y)=h_2$ *on* Γ_{2D}. *Then if* $\delta>0$ *and if* $g(x, y)$ *is any function continuous in* R, *there exists a connected domain* $D(\delta)\subset D$ *with the following properties*:

(1) $X\in D(\delta)$,

[3] By use of a theorem of Carleman [10] and the theory of transformation of second order equations in two independent variables to canonical form, it is possible to prove—at least in the case considered in I.2—that the gradient of a nonconstant solution of (3) vanishes only at isolated points. This property leads to a somewhat more elegant proof of Theorem I without use of Lemma 4. However, a natural generalization of Lemma 4 permits the extension of Theorem I to equations in n dimensions. Since Carleman's result is not valid in more than two dimensions (e.g. the gradient of the potential of an infinite circular cylinder vanishes on the axis), it appears that Lemma 4 is more appropriate to the problem.

(2) *the boundary* $\Gamma(\delta)$ *of* $D(\delta)$ *consists of a finite number of differentiable sub-arcs of* Γ_D *and a finite number of straight line segments.* $\Gamma(\delta)$ *may be expressed as the union of two disjoint components,* $\Gamma(\delta)=\Gamma_1(\delta)+\Gamma_2(\delta)$, *where* $\Gamma_1(\delta)$ *and* $\Gamma_2(\delta)$ *have distances less than* δ *from* Γ_{1D} *and* Γ_{2D}, *respectively.*

$$\oint_{\Gamma_i(\delta)} |F(x, y) - h_i| \, |g(x, y)| \, ds < \delta/2, \qquad i = 1, 2. \tag{3}$$

Proof. Let d be the distance between Γ_{1D} and Γ_{2D}, d_X the distance from X to Γ_D. Choose δ smaller than the minimum of d_X and $d/2$. Cover $D+\Gamma_D$ with a net of squares of side $\epsilon\delta/2^{1/2}$, $0<\epsilon<1$. Denote those squares that contact Γ_D by $S(\delta)$. Denote by $S_\alpha(\delta)$ those squares of $S(\delta)$ on which $F_x^2+F_y^2\neq 0$ at any point of Γ_D, denote by $S_\beta(\delta)$ the remaining squares of $S(\delta)$.

By the implicit function theorem, each point of $\Gamma_D\cap S_\alpha(\delta)$ is the center of a circle through which passes a unique differentiable arc of Γ_D and which contains no other points of Γ_D. By the Heine-Borel theorem a finite number of such circles suffices to cover $S_\alpha(\delta)\cap\Gamma_D$. Hence there is interior to each square of $S_\alpha(\delta)$ a finite number of nonintersecting differentiable arcs of Γ_D, each of which either meets $S_\beta(\delta)$ at the boundary of the square, forms a closed Jordan arc interior to the square, or continues uniquely into adjacent squares of $S_\alpha(\delta)$. Thus the set $\Gamma_D\cap S_\alpha(\delta)$ consists of a finite number of differentiable nonintersecting arcs.

Let $D(\delta)$ be the union of all open connected sets containing X, which contain no points of $[\Gamma_D\cap S_\alpha(\delta)]\cup S_\beta(\delta)$. Since any arc joining Γ_D to X must intersect this latter set, it follows that $D(\delta)$ is an open connected set containing X, interior to which $h_2>F(x, y)>h_1$. Hence $D(\delta)\subset D$. Further, $D(\delta)$ is bounded by certain arcs $\Gamma_\alpha(\delta)$ of $\Gamma_D\cap S_\alpha(\delta)$ and by a finite number of segments $\Gamma_\beta(\delta)$ of the boundary of $S_\beta(\delta)$. Every point of these segments has distance $\leqq\epsilon\delta$ from Γ_D. Hence the curve $\Gamma(\delta)=\Gamma_\alpha(\delta)+\Gamma_\beta(\delta)$ has the required proximity to Γ_D for any ϵ in the interval $0<\epsilon<1$.

It is clear that $\Gamma_\alpha(\delta)=\Gamma_{1\alpha}(\delta)+\Gamma_{2\alpha}(\delta)$, $\Gamma_\beta(\delta)=\Gamma_{1\beta}(\delta)+\Gamma_{2\beta}(\delta)$, with similar expressions for $S_\alpha(\delta)$ and $S_\beta(\delta)$, in the sense described above. If $\delta<d/2$, the sets with subscript 1 will be disjoint from those with subscript 2. We define $\Gamma_1(\delta)=\Gamma_{1\alpha}(\delta)+\Gamma_{1\beta}(\delta)$, etc.

If N_δ denotes the number of squares of $S_\beta(\delta)$, then there is a constant K such that $N_\delta<K/(\epsilon\delta)^2$. Therefore, if L_δ denotes the total length of the polygonal arcs on the boundary of $S_\beta(\delta)$, $L_\delta<4K/\epsilon\delta$. On the other hand, $S_{1\beta}\subset S_\beta$ and on each square of $S_{1\beta}(\delta)$ there is at least one point (x_1, y_1) of Γ_D at which $F_x^2+F_y^2=0$. By Taylor's theorem,

$$F(x, y) = h_1 + \frac{(x-x_1)^2}{2}F_{xx}[x_1+\theta(x-x_1), y_1+\theta(y-y_1)]$$

$$+ (x-x_1)(y-y_1)F_{xy}[\quad] + \frac{(y-y_1)^2}{2}F_{yy}[\quad], \qquad 0<\theta<1,$$

where the arguments appearing in the terms $F_{xy}[\ ]$ and $F_{yy}[\ ]$ do not differ from that of F_{xx}. Therefore, if M is chosen superior to the maximum of $|F_{xx}| + |F_{xy}| + |F_{yy}|$ in R,

$$(6) \qquad |F(x, y) - h_1| < 4M[(x - x_1)^2 + (y - y_1)^2] < K_1(\epsilon\delta)^2$$

for all points (x, y) in the square of $S_{1\beta}(\delta)$ containing (x_1, y_1). An analogous argument is valid for the set $S_{2\beta}(\delta)$.

On $\Gamma_{1\alpha}(\delta)$, $F(x, y) = h_1$, since $\Gamma_\alpha(\delta) \subset \Gamma_D$. On $\Gamma_{1\beta}(\delta)$, (6) is valid. Therefore,

$$\oint_{\Gamma_1(\delta)} |F - h_1| \, |g| \, ds = \oint_{\Gamma_{1\beta}(\delta)} |F - h_1| \, |g| \, ds < K_1(\epsilon\delta)^2 \cdot \max |g| \cdot 4K/\epsilon\delta$$

and an identical inequality is valid for $\oint_{\Gamma_2(\delta)} |F - h_2| \, |g| \, ds$. The demonstration is completed by choosing $\epsilon < 1/4KK_1 \max |g|$.

The proof of Theorem I now follows by reductio ad absurdum. Suppose in fact that $\phi(x, y)$ satisfies the conditions of Theorem I in a neighborhood of a point X, and that there exists a neighborhood N of X such that if $p \in N - X$, $\phi(p) < H$. Let C be a circle about X which lies interior to N. Let h be the maximum value attained by $\phi(x, y)$ on the boundary of C. By assumption $h < H$. Let ν be an integer, let C_ν be the circle of radius $1/\nu$ about X. Let Y be a point interior to C at which $\phi(Y) = h_0 > h$. If ν is chosen sufficiently large Y will be exterior to C_ν. Choose h_1 and h_2 so that $h < h_1 < h_0 < h_2 < H$. Let G_ν be the set of all points interior to C but exterior to C_ν, at which $h_1 < \phi(x, y) < h_2$. Let $T_{1\nu}$ and $T_{2\nu}$ be those (disjoint) parts of the boundary of G_ν which contain no points of the boundary of C_ν, and on which $\phi(x, y) = h_1$ and $\phi(x, y) = h_2$, respectively. By Lemma 4, if $\delta(\nu)$ is a positive quantity smaller than half the distance between $T_{1\nu}$ and $T_{2\nu}$ and smaller than the distance of Y to either of these sets, there exists a connected subdomain D_ν containing Y, bounded by piecewise smooth arcs $\Gamma_{1\nu}$ and $\Gamma_{2\nu}$ within distance $\delta(\nu)$ from $T_{1\nu}$ and $T_{2\nu}$ respectively, and by certain sub-arcs C_ν^* of the boundary of C_ν. Further,

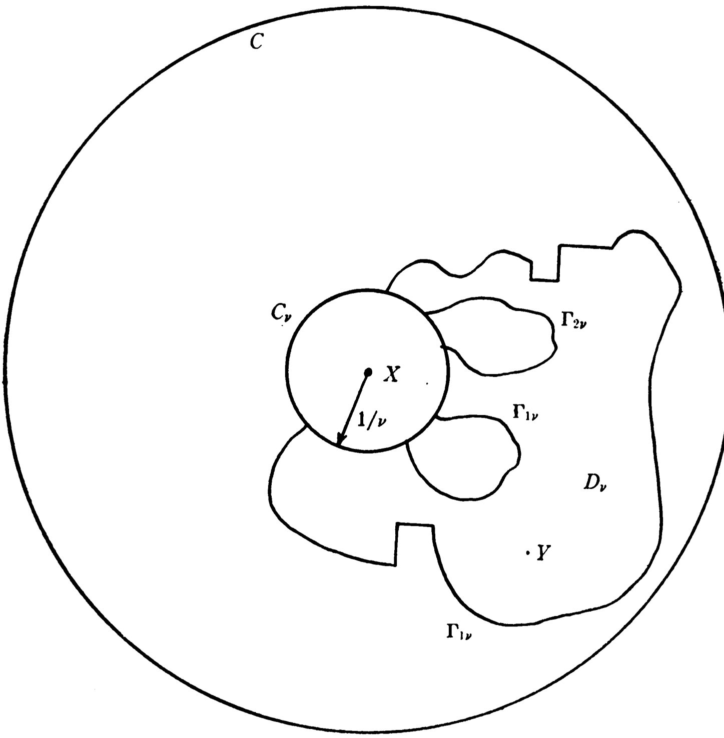
$$\int_{\Gamma_{1\nu}} |\phi(x, y) - h_1| \{ |\Theta dy| + |\Lambda dx| \} < \delta(\nu),$$

$$\int_{\Gamma_{2\nu}} |\phi(x, y) - h_2| \{ |\Theta dy| + |\Lambda dx| \} < \delta(\nu).$$

The set $\Gamma_\nu = \Gamma_{1\nu} + \Gamma_{2\nu}$ consists of a finite number of nonintersecting components, each of which either meets the boundary of C_ν in exactly two points, or else forms a simple closed curve exterior to C_ν. Interior to the domain D_ν bounded by $\Gamma_\nu + C_\nu^*$, $h_1 < \phi(x, y) < h_2$.

Choose a fixed $\nu = \nu_0$ and choose $\delta(\nu_0)$ sufficiently small so that $\phi(x, y) \not\equiv$ constant in D_{ν_0}. Set $D = D_{\nu_0}$. It may clearly be assumed that for all $\nu \geqq \nu_0$

the construction of D_ν is so made that $D \subset D_\nu$. In fact, it is sufficient that the net of squares used in the construction of D_ν, $\nu \geqq \nu_0$, is a subdivision of the net corresponding to ν_0.



The desired contradiction is now obtained by an application of Lemmas 1, 2, and 3 to D_ν and its boundary. In fact, let Γ^* represent an arbitrary component of $\Gamma_{1\nu}$. It has been noted that Γ^* may be completed to a simple closed curve by adjoining to it an appropriate subarc C^* of C_ν. Thus, if $\psi(x, y)$ is the function conjugate to $\phi(x, y)$,

$$\psi(x, y) = \int [\Theta dy - \Lambda dx],$$

$$\begin{aligned} \sum \int_{\Gamma^*} \phi d\psi &= \sum \int_{\Gamma^*} h_1 d\psi + \eta \qquad (|\eta| \leqq \delta(\nu)) \\ &= \sum \int_{\Gamma^* + C^*} h_1 d\psi - \sum \int_{C^*} h_1 d\psi + \eta \\ &= -h_1 \sum \int_{C^*} d\psi + \eta \end{aligned}$$

by Lemmas 1 and 2, where the summation is taken over all the (finite number of) components Γ^* of $\Gamma_{1\nu}$. Setting, by hypothesis,

$$(\Theta^2 + \Lambda^2)^{1/2} \leqq \epsilon(1/\nu)\cdot\nu$$

where $\epsilon \to 0$ as $\nu \to \infty$,

$$\sum \left| \int_{\Gamma^*} \phi d\psi \right| \leqq | h_1 | \cdot\epsilon(1/\nu)\cdot\nu\cdot 2\pi/\nu + \delta(\nu)$$

which is arbitrarily small if ν is chosen sufficiently large and $\delta(\nu)$ sufficiently small. An identical argument may of course be applied to the components of $\Gamma_{2\nu}$ and hence to the collection $\Gamma_{1\nu}+\Gamma_{2\nu}$. Further, on the arcs C_ν^*, $h_1 \leqq \phi(x, y) \leqq h_2$ since C_ν^* lies on the boundary of D_ν. Therefore

$$\sum \left| \int_{C_\nu^*} \phi d\psi \right| \leqq \{ | h_1 | + | h_2 | \}\cdot\epsilon(1/\nu)\cdot\nu\cdot 2\pi/\nu$$

which approaches zero with increasing ν. On the other hand, Lemma 3 implies that

$$0 < \iint_D [\Theta\phi_x + \Lambda\phi_y] dxdy \leqq \iint_{D_\nu} [\Theta\phi_x + \Lambda\phi_y] dxdy$$

$$= \oint_{\Gamma_\nu + C_\nu^*} \phi[\Theta dy - \Lambda dx] = \oint_{\Gamma_\nu + C_\nu^*} \phi d\psi.$$

The last term has been seen to be arbitrarily small for sufficiently large ν and sufficiently small $\delta(\nu)$. But the first term is independent of ν. This can of course be only if the assumption $\phi(x, y) < H$ in N is erroneous. An identical argument disposes of the assumption $\phi > L$ in N, and the proof of Theorem I is complete.

The following strengthened conclusion follows immediately from the method of proof:

I.4. *If, in the conditions for admissibility,* (iii) *is replaced by* (iii′) $\Theta_x + \Lambda_y \geqq 0$ ($\Theta_x + \Lambda_y \leqq 0$), *then in the generalized sense of Theorem* I, $\phi(x, y)$ *cannot have a positive maximum* (*negative minimum*) *at* X.

A particular interpretation of Theorem I is that if $\Theta^2 + \Lambda^2 < K^2 < \infty$ for every solution $\phi(x, y)$, then a single-valued solution cannot become unbounded at an isolated singular point. In particular, such an equation admits no fundamental solution. The examples of the minimal surface and gas-dynamic equations have already been cited. Further examples may readily be constructed by the reader.

1.2. Equations in more than 2 independent variables. No essential change of argument is required to extend Theorem I to equations in n independent variables,

$$\sum_{i=1}^{n} (\Theta_i)_{x_i} = 0. \tag{7}$$

As before, a solution $\phi(x_1, \cdots, x_n)$ of (7) will be called *admissible* if ϕ and $\{\Theta_i\}$ satisfy suitable differentiability requirements and if $\sum_{i=1}^n (\Theta_i\phi_{x_i}) \geqq 0$, equality holding if and only if $\sum_{i=1}^n \phi_{x_i}^2 = 0$. It is a purely formal matter to extend Lemma 4 to the case of a function $F(x_1, \cdots, x_n)$ defined over an n-dimensional region R. In this case the boundary $\Gamma(\delta)$ consists of a finite number of smooth bounded hypersurfaces and a finite number of hyperplanes, such that

$$\int \cdots \int_{\Gamma_i(\delta)} |F(x_i, \cdots, x_n) - h_i| \, |g| \, dS < \delta/2, \qquad i = 1, 2.$$

The equation (7) no longer serves to insure the existence of a conjugate function ψ, but this function was introduced only to simplify notation and did not enter in an essential way into the proof of Theorem I. The only modification required is in the growth criterion at the singular point.

I.5. *If $\phi(x_1, \cdots, x_n)$ is an admissible solution of* (7) *in a neighborhood N of an isolated singular point X, the conclusion of Theorem* I *remains valid provided that near X, $(\sum_{i=1}^n \Theta_i^2)^{1/2} = o(1/r^{n-1})$.*

2. Removable singularities

2.1. Single-valued solutions in two variables. The hypotheses introduced in Theorem 1 are of a very general nature, and it can hardly be expected that they should lead to results much more specific than that obtained. Indeed, consider the equation

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0$$

where $\rho = e^{-(x^2+y^2)^{-\alpha/2}/\alpha^2}(x^2+y^2)^{(1-\alpha^2)/2\alpha}$, $\alpha > 0$. A direct computation shows that the function $\phi(x, y) = [1 + (x^2+y^2)^{\alpha/2}]y/(x^2+y^2)^{1/2}$ is an admissible solution for $0 < x^2+y^2 < \infty$, and that $\rho|\nabla\phi| \to 0$ as $x^2+y^2 \to 0$. $\phi(x, y)$ satisfies the conclusions of Theorem 1, but it can hardly be regarded as a solution at the origin, for it is discontinuous at this point.

It is clear then that these results cannot be sharpened without suitably restricting the nature of the nonlinearity in the functions Θ and Λ. It is of interest that relatively weak assumptions of this nature suffice to yield very precise information on the behavior of a solution at a singular point.

THEOREM II. *Let $\Theta = f(\phi_x, \phi_y)$, $\Lambda = g(\phi_x, \phi_y)$, let the equation*

$$\Theta_x + \Lambda_y = 0 \tag{3}$$

be of elliptic type. Let $\phi(x, y)$ be a solution of (3) *singular at an isolated point X and single-valued in a neighborhood N of X. Then if at X, $(\Theta^2+\Lambda^2)^{1/2} = o(1/r)$, the singularity at X is removable.*

II.1. *A solution of the minimal surface equation*

$$\left[\frac{\phi_x}{(1+\phi_x^2+\phi_y^2)^{1/2}}\right]_x+\left[\frac{\phi_y}{(1+\phi_x^2+\phi_y^2)^{1/2}}\right]_y=0, \tag{2}$$

single-valued in a domain D, admits only removable isolated singularities in D.

Proof of Theorem II. Let $\phi(x, y)$ be a solution of (3) singular at an isolated point X. Let D be a circular domain with center at X and radius r sufficiently small so that $\bar{D}\subset N$. Let the boundary of D be Γ. Since $\phi(x, y)$ is assumed twice differentiable, it follows [11] that its values on Γ satisfy a three-point condition([4]). Hence [14] there exists a function $\phi_0(x, y)$ which assumes on Γ values identical to those of $\phi(x, y)$ and is a solution of (3) throughout D. We shall show that $\phi(x, y)\equiv\phi_0(x, y)$ in $D-X$.

Let $\Theta=f(\phi_x, \phi_y)$, $\Lambda=g(\phi_x, \phi_y)$, $\Theta_0=f(\phi_{0_x}, \phi_{0_y})$, $\Lambda_0=g(\phi_{0_x}, \phi_{0_y})$, let $D_{r-\epsilon}$ and D_ϵ be circular domains about X of radii $r-\epsilon$ and ϵ, respectively, let $\Gamma_{r-\epsilon}$ and Γ_ϵ be their respective boundaries. The following identity results immediately from an integration by parts:

$$\begin{aligned}\iint_{D_{r-\epsilon}-D_\epsilon} &\{(\phi-\phi_0)_x[\Theta-\Theta_0]+(\phi-\phi_0)_y[\Lambda-\Lambda_0]\}dxdy \\ &=\oint_{\Gamma_{r-\epsilon}}(\phi-\phi_0)\{[\Theta-\Theta_0]dy-[\Lambda-\Lambda_0]dx\} \\ &\quad-\oint_{\Gamma_\epsilon}(\phi-\phi_0)\{[\Theta-\Theta_0]dy-[\Lambda-\Lambda_0]dx\}.\end{aligned} \tag{8}$$

Since, on Γ, $\phi_0(x, y)$ satisfies a three-point condition, $\phi_{0_x}^2+\phi_{0_y}^2$ is uniformly bounded in magnitude [11; 12] in D. But $\phi(x, y)$ is regular in $\bar{D}-X$, and it follows that as $\epsilon\to 0$ the first integral on the right approaches zero, since $\phi(x, y)\to\phi_0(x, y)$. Since $\phi_0(x, y)$ is regular at X, the second integral on the right will approach zero provided that $\phi(x, y)$ is bounded near X. By Theorem I, $\phi(x, y)$ will be bounded near X if it is an admissible solution, that is, if $\Theta\phi_x+\Lambda\phi_y\geqq 0$, equality holding only if $\phi_x=\phi_y=0$. Let us defer this question for a moment, and assume that $\phi(x, y)$ is bounded. Then

([4]) A set of boundary values is said to satisfy a three-point condition with constant Δ if the space curve defined by these values has the property that any plane which meets this curve in three or more points has maximum inclination less than Δ. It has been proved by Radó [12] and von Neumann [13] that Δ serves as a uniform bound on the gradient among all solutions of elliptic equations $a\phi_{xx}+2b\phi_{xy}+c\phi_{yy}=0$ which assume the given boundary values.

The existence of the solution $\phi_0(x, y)$ is proved in the paper of Leray and Schauder [13] under the assumption that the boundary values admit third derivatives which satisfy a Hölder condition. However, it has been proved by L. Nirenberg in a paper to be published shortly that the solution exists for arbitrary continuous prescribed values which satisfy a three-point condition.

$$\iint_D \{(\phi - \phi_0)_x[\Theta - \Theta_0] + (\phi - \phi_0)_y[\Lambda - \Lambda_0]\}dxdy$$

exists and is equal to zero. We shall show that the integrand I of this expression is of definite sign and vanishes only if $\phi(x, y) \equiv \phi_0(x, y)$ in D.

Consider a fixed point (x, y) of D, set

$$\eta(x, y) = \phi(x, y) - \phi_0(x, y), \qquad \eta_x(x, y) = \omega_0, \qquad \eta_y(x, y) = \lambda\omega_0.$$

Then

$$I = \omega_0\{\Theta(\phi_{0x} + \omega_0, \phi_{0y} + \lambda\omega_0) - \Theta_0 + \lambda[\Lambda(\phi_{0x} + \omega_0, \phi_{0y} + \lambda\omega_0) - \Lambda_0]\}.$$

The quantity ω_0 will now be considered as an independent variable, all other quantities remaining fixed, and the subscript removed. Thus,

$$I = \omega F(\omega),$$
$$F'(\omega) = \Theta_p + \lambda\Theta_q + \lambda\Lambda_p + \lambda^2\Lambda_q$$

where $p = \phi_x$, $q = \phi_y$. By hypothesis, (3) has elliptic character. Therefore

$$\Theta_p\Lambda_q > [\Theta_q + \Lambda_p]^2/4.$$

In particular $\Theta_p\Lambda_q > 0$ and it is no loss of generality to assume $\Theta_p > 0$, $\Lambda_q > 0$. On the other hand

$$\lambda(\Theta_p\Lambda_q)^{1/2} \leqq \frac{\Theta_p + \lambda^2\Lambda_q}{2}$$

so that

$$(\Theta_p + \lambda^2\Lambda_q)^2 \geqq 4\lambda^2\Theta_p\Lambda_q > \lambda^2[\Theta_q + \Lambda_p]^2$$

or

$$\Theta_p + \lambda^2\Lambda_q > |\lambda(\Theta_q + \Lambda_p)|.$$

Thus $F'(\omega) > 0$, $F(\omega) \gtrless 0$ according as $\omega \gtrless 0$, and it follows that for every point (x, y) in D, $I = \omega_0 F(\omega_0) \geqq 0$, equality holding if and only if $\omega_0 = 0$. But $\iint_D I dxdy = 0$ and hence $\omega_0(x, y) \equiv 0$ in D. That is, $\nabla\phi(x, y) \equiv \nabla\phi_0(x, y)$ in D. But $\phi(x, y) \equiv \phi_0(x, y)$ on Γ, hence $\phi(x, y) \equiv \phi_0(x, y)$ in $D - X$. The proof of Theorem 2 has thus been reduced to proving that $\phi(x, y)$ is bounded near X.

We have already observed that this will follow if $\phi(x, y)$ is an admissible solution. It is easily seen that in general this will not be the case. This difficulty is avoided by observing that a solution of (3) remains a solution if Θ and Λ are changed by additive constants. It may thus be assumed that $\Theta(0, 0) = \Lambda(0, 0) = 0$. The proof that $\Theta\phi_x + \Lambda\phi_y$ is definite then proceeds exactly as the above proof that I is definite, the comparison function being in this case the solution $\phi_0(x, y) \equiv 0$. Theorem II is thus completely proved.

The following corollary is an immediate consequence of the method of proof above.

II.2. *Let* $\Theta=f(x, y, \phi_x, \phi_y)$, $\Lambda=g(x, y, \phi_x, \phi_y)$, $\Theta(x, y, 0, 0)\equiv\Lambda(x, y, 0, 0)\equiv 0$. *Then removability of an isolated singularity is reduced to solvability of the first boundary problem for a sufficiently small domain, the other hypotheses of Theorem* II *remaining unchanged.*

Conditions for solvability have been given by various authors, notably by Leray [15].

2.2. **A direct approach to the problem.** The proof of Theorem II given in (2.1) has the virtue of elegance and conceptual simplicity; a more direct approach suggested to the author by Professor M. Shiffman yields a sharper result in the case that (3) arises from a variational problem $\delta\iint F(\phi_x, \phi_y)dxdy=0$. Specifically, we shall prove the following theorem:

THEOREM III. *Let* $\Theta=f(\phi_x, \phi_y)$, $\Lambda=g(\phi_x, \phi_y)$ *be such that the domain of values* (ϕ_x, ϕ_y) *for which* (3) *is elliptic is convex. Let* $\phi(x, y)$ *be a solution of* (3) *singular at an isolated point* X *and single-valued in a neighborhood* N *of* X. *Suppose that for this solution* (3) *is of elliptic type in* N. *Denote by* ϕ_α *the directional derivative of* $\phi(x, y)$ *in the direction* α. *Then if, at* X, $(\Theta^2+\Lambda^2)^{1/2}=o(1/r)$, *the conclusions of Theorem* I *are valid for* ϕ_α.

The point of this theorem is that the equation is no longer assumed elliptic for all (ϕ_x, ϕ_y), but only for the particular solution considered. There is no requirement that the solution should remain properly within the elliptic domain.

III.1. $\phi(x, y)$ *satisfies a uniform Lipschitz condition in* N *and is therefore continuous at* X. *The magnitude* $(\phi_x^2+\phi_y^2)^{1/2}$ *of the gradient of* $\phi(x, y)$ *satisfies the conclusion of Theorem* I.

III.2. *Suppose that* (3) *is the Euler-Lagrange equation for the variational problem* $\delta\iint F(p, q)dxdy=0$. *Then the singularity at* X *is removable.*

For Theorem III implies that the closure of the set of values (ϕ_x, ϕ_y) achieved in N is a compact subset of the (open) domain of ellipticity. It follows that the discriminant of (3) is bounded away from zero in N. Under these conditions it has been proved by Shiffman [16] that $\phi(x, y)$ has continuous derivatives at X. It then follows from the work of Morrey [17] and of E. Hopf [18] that $\phi(x, y)$ has second derivatives which satisfy a Hölder condition, i.e. that the singularity at X is removable.

Proof of Theorem III. A rotation of axes transforms (3) into an equation of the same form and leaves invariant the elliptic character of the equation. Therefore it is sufficient to demonstrate the theorem for the derivative ϕ_x. We shall suppose the singularity to be at the origin. Denote by Φ_h the quantity $(\phi(x+h)-\phi(x))/h$. Φ_h is singular at the two points $(0, 0)$ and $(-h, 0)$. If h_0 is sufficiently small, a circle Γ of radius $3h_0$ about the origin will contain in its interior both of these singular points but no other singularities of Φ_h, for all $h<h_0$. Construct circles Γ_1 and Γ_2, both of radius $h/2$, about $(0, 0)$ and $(-h, 0)$ respectively.

Since Θ and Λ do not depend explicitly on the coordinates (x, y), $\phi(x+h)$ is a solution of (3) whenever $\phi(x)$ is. Therefore if D is any domain of regularity bounded by a smooth curve B,

$$\oint_B [(\Theta - \Theta_h)dy - (\Lambda - \Lambda_h)dx] = 0, \tag{9}$$

$$\begin{aligned} \iint_D \{(\Phi_h)_x(\Theta - \Theta_h) + (\Phi_h)_y(\Lambda - \Lambda_h)\}dxdy \\ = \oint_B \Phi_h[(\Theta - \Theta_h)dy - (\Lambda - \Lambda_h)dx] \end{aligned} \tag{10}$$

where Θ_h denotes $\Theta[\phi_x(x+h, y), \phi_y(x+h, y)]$, etc.

We have already seen that the integrand on the left-hand side of (10) is of definite sign. Further, in the circle Γ_1, Θ_h and Λ_h are regular, hence $(\Theta-\Theta_h)^2+(\Lambda-\Lambda_h)^2$ has at the origin the same order of magnitude as $\Theta^2+\Lambda^2$. It follows that the proof of Theorem 1 may be carried out step by step in Γ_1, replacing the identities (4) and (5) by (9) and (10), respectively. Thus, if $H=\sup \Phi_h(x, y)$, $L=\inf \Phi_h(x, y)$ as $x^2+y^2\to 0$, there will exist in every neighborhood of the origin points X_1 and X_2 for which

$$\Phi_h(X_1) \geqq H, \qquad \Phi_h(X_2) \leqq L.$$

A similar analysis is valid for the singularity in Γ_2.

This result can be sharpened. An application of the mean-value theorem shows [19] that the difference of two solutions of (3) is a solution of a certain linear elliptic equation of the form $a\phi_{xx}+2b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y=0$. Therefore Φ_h satisfies the maximum principle in the strict sense at every point of regularity. The equality signs can thus be removed from the above relations, and it follows readily from this that the maximum and minimum of Θ_h are achieved on Γ. We are now at liberty to let $h\to 0$. Since, on Γ, $\Phi_h(x, y)$ converges uniformly to $\phi_x(x, y)$, the stated result follows immediately.

The proofs presented above of Theorems II and III have required that (3) be of elliptic type when considered as a *nonlinear* equation which does not involve x and y explicitly. On the other hand, suppose—to take a simple case—that (3) has the form

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0, \qquad \rho = f(\phi_x, \phi_y) > 0. \tag{1}$$

Given a solution of (1), then ρ becomes a function of (x, y), so that every solution of (1) is also a solution of the elliptic equation

$$\rho\Delta\phi + \rho_x\phi_x + \rho_y\phi_y = 0$$

and hence shares all behavior properties of solutions of elliptic equations, even though ρ may be such that (1) is of hyperbolic type. The question naturally arises, was it necessary to assume that (3) is elliptic, or could weaker assumptions (e.g. that $\phi(x, y)$ should be a solution of *some* elliptic equation) have sufficed? The answer is provided by the following example:

Let

$$\rho(\phi_x, \phi_y) = \left\{\frac{\phi_y(1+\alpha) + \left[\phi_y^2(1-\alpha)^2 - 4\alpha\phi_x^2\right]^{1/2}}{2\alpha}\right\}^{-((1+\alpha)/\alpha)}, \qquad 0 < \alpha < 1.$$

A direct computation then verifies that

$$\phi(x, y) = y(x^2 + y^2)^{(\alpha-1)/2}$$

is a solution of (1) regular in $0 < x^2+y^2 < \infty$. For this solution,

$$\rho = (x^2+y^2)^{(1-\alpha^2)/2\alpha}, \quad \text{and} \quad (\Theta^2+\Lambda^2)^{1/2} = \rho\,|\nabla\phi| = (x^2+y^2)^{(1-2\alpha)/2\alpha}(\alpha^2y^2+x^2)^{1/2},$$

which $\to 0$ as $x^2+y^2 \to 0$. Thus all hypotheses of Theorems II and III are satisfied except for ellipticity; yet $\phi(x, y)$ cannot be considered a solution at the origin, for its derivatives of first order become unbounded there.

It should be noted here that the coefficients of the linear elliptic equation satisfied by $\phi(x, y)$ become singular at the origin, i.e. the equation approaches the boundary of the domain of ellipticity. If this were not the case then $\phi(x, y)$ would be identical in a circle about the origin with the solution $\phi_0(x, y)$ of the linear equation which assumed the values of $\phi(x, y)$ on the boundary. Thus the equation considered furnishes an example of an equation which is elliptic except at one point—where the coefficients approach zero as the distance to the point—and for which the first boundary value problem cannot in general be solved for any domain containing the point.

2.3. **Solutions in several variables.** The proof of Theorem II does not carry over to equations in n independent variables, since it is not known whether or not the first boundary value problem admits a solution in this case. It has already been observed that Theorem I extends without essential change and it is not hard to see that the same is true for Theorem III.

III.3. *Let* $\Theta_i = f(p_1, \cdots, p_n)$, $i = 1, \cdots, n$, *where* $p_j = \phi_{x_j}$. *Suppose that the domain of values* $(p_1, \cdots, p_n)$ *for which the form* $\sum_{i,j=1}^n (\Theta_i)_{p_j}\xi_i\xi_j$ *is definite is convex.*

Let $\phi(x_1, \cdots, x_n)$ *be a solution of* (7) $\sum_{i=1}^n (\Theta_i)_{x_i} = 0$ *which is singular at an isolated point X. Suppose that the above form is definite for this solution in some neighborhood of X. Then, if at X,* $(\sum_{i=1}^n \Theta_i^2)^{1/2} = o(1/r^{n-1})$, *the conclusions of Theorem* I *are valid for each directional derivative* ϕ_α *of* $\phi(x_1, \cdots, x_n)$. *In particular,* ϕ *is continuous at X.*

Removability is reduced to solvability of the first boundary value problem for a sufficiently small sphere.

3. Multi-valued solutions

Theorems I, II, and III have been proved under the assumption that the solution $\phi(x, y)$ is single-valued in a neighborhood of the singular point. These theorems do not in general apply as stated to multi-valued solutions. For example, if $\rho=f(\phi_x^2+\phi_y^2)$, then the infinitely multi-valued function $\phi(x, y)=\tan^{-1(y/x)}$ is a solution of (1). This solution is unbounded at the origin. Solutions which are n-valued and bounded in a neighborhood of an isolated singular point may readily be constructed by use of the hodograph variables and the method of correspondence [20]. However, in certain cases the behavior of a multi-valued solution near an isolated singular point is severely restricted.

I.6. *Let $\phi(x, y)$ be an admissible solution of*

$$\Theta_x + \Lambda_y = 0 \tag{3}$$

which is n-valued in a neighborhood of an isolated singular point X. Then if, near X, $(\Theta^2+\Lambda^2)^{1/2}=o(1/r)$, the conclusion of Theorem I *is valid for $\phi(x, y)$.*

Proof. It is necessary only to replace the (x, y)-plane near X by the n-sheeted covering surface on which $\phi(x, y)$ becomes single-valued. Domains and curves on this surface may be represented by the corresponding simply-covered domains and curves of the (ξ, η)-plane, $(\xi+i\eta)^n=(x+iy)$. Since compactness arguments are unchanged under this transformation, the proof of Theorem I may be repeated without essential change.

Theorem IV. *Let $\rho=f(\phi_x^2+\phi_y^2)$ and let* (1) *$(\rho\phi_x)_x+(\rho\phi_y)_y=0$ be of elliptic type. Let $\phi(x, y)$ be a solution of* (1) *having a single-valued gradient in a neighborhood N of an isolated singular point X and for which $\rho|\nabla\phi|=o(1/r)$. Then after a translation of X to the origin*

$$\phi(x, y) = k \text{ arc tan } \frac{y}{x} + \chi(x, y) \tag{6}$$

where $\chi(x, y)$ is single-valued in N and satisfies the conclusions of Theorem I.

Proof. The form of the representation (6) is clear from the single-valuedness of $\nabla\phi$ in N. If $k=0$, Theorem 1 may be applied directly. Otherwise, observe that $\theta(x, y)=k$ arc tan (y/x) is a solution of (3) and verify the identity

$$\begin{aligned}\iint_D \rho\,|\,\nabla\phi\,|^2 + \rho^*\,|\,\nabla\theta\,|^2 - (\rho+\rho^*)(\nabla\phi\cdot\nabla\phi)dxdy \\ = \oint_\Gamma \chi(x, y)\left[\rho\,\frac{\partial\phi}{\partial n} - \rho^*\,\frac{\partial\theta}{\partial n}\right]ds\end{aligned} \tag{11}$$

where D is a domain bounded by a piecewise smooth curve Γ, interior to which ϕ and θ are regular. Here $\rho=f(\phi_x^2+\phi_y^2)$, $\rho^*=f(\theta_x^2+\theta_y^2)$.

Since only the derivatives of $\phi(x, y)$—which are assumed single-valued—enter into the proof of Lemmas 1 and 2, it is a consequence of these lemmas that

$$\oint_\Gamma \rho \frac{\partial \phi}{\partial n} ds = 0$$

even if the curve Γ contains the singular point. Further, on the circle C_r considered in Lemma 2, $\partial\theta/\partial n = \partial\theta/\partial r = 0$. Thus

$$\oint_\Gamma \rho^* \frac{\partial \theta}{\partial n} ds = 0.$$

Exactly as in the proof of Theorem 2, it may be shown that the integrand in the left-hand term of (11) is a positive definite functional of $\chi_x^2 + \chi_y^2$. It follows that the proof of Theorem I may be repeated directly, with the identity (5) replaced by the identity (11).

IV.1. *Let $\rho = f(\phi_x^2 + \phi_y^2)$ and let* (1) *be of elliptic type. Let $\phi(x, y)$ be a solution of* (3) *having an n-valued gradient in a neighborhood N of an isolated singular point X, near which $\rho|\nabla\phi| = o(1/r)$. Then after a translation of X to the origin*

$$\phi(x, y) = k \arc \tan \frac{y}{x} + \chi(x, y)$$

where $\chi(x, y)$ is n-valued in N and satisfies the conclusion of Theorem 1.

Proof. Equation (3) is equivalent to the system of equations

$$\rho\phi_x = \psi_y, \qquad \rho\phi_y = -\psi_x$$

where $\psi(x, y) = \int(\rho\phi_x dy - \rho\phi_y dx)$. In polar coordinates (r, θ) these equations become

$$\rho\phi_r = \frac{1}{r}\psi_\theta, \qquad \rho\phi_\theta = r\psi_r.$$

This system is invariant under the transformation $r = t^n$, $\theta = n\alpha$. It follows that if $x(\xi, \eta)$, $y(\xi, \eta)$ are defined by the relation $(x+iy) = (\xi + i\eta)^n$, then $\phi(x(\xi, \eta), y(\xi, \eta))$ will be a solution of

$$(\rho\phi_\xi)_\xi + (\rho\phi_\eta)_\eta = 0$$

with single-valued gradient in N. Further, since

$$\rho(\phi_x^2 + \phi_y^2)^{1/2} = \epsilon(r)/r$$

where $\epsilon(r) \to 0$ as $r \to 0$,

$$\rho(\phi_\xi^2 + \phi_\eta^2)^{1/2} \cdot \frac{1}{nt^{n-1}} = \frac{\epsilon(t^n)}{t^n}$$

or

$$\rho(\phi_\xi^2 + \phi_\eta^2)^{1/2} = \frac{n\epsilon(t^n)}{t} = o(1/t).$$

The result thus follows from Theorem IV.

4. The equation of gas dynamics

Consider the case

$$\rho = \left[1 - \frac{\gamma - 1}{2}(\phi_x^2 + \phi_y^2)\right]^{1/(\gamma-1)}, \qquad 1 < \gamma < 2.$$

Equation (1) is then elliptic for $\phi_x^2+\phi_y^2<2/(1+\gamma)$, hyperbolic for $2/(1+\gamma)<\phi_x^2+\phi_y^2<2/(\gamma-1)$, and in general not defined for $\phi_x^2+\phi_y^2>2/(\gamma-1)$. In this case the gradient of any solution $\phi(x, y)$ is bounded, so that $\phi(x, y)$ is in fact continuous at any isolated singular point X.

It follows immediately from Theorems I and I.6 (§3) that a finitely multi-valued solution cannot attain a maximum or minimum at X whether the solution be of elliptic, hyperbolic, or mixed nature. If the solution is elliptic near X and has an n-valued gradient, then by IV.1

$$\phi(x, y) = k \arctan \frac{y}{x} + \chi(x, y)$$

where $\chi(x, y)$ is bounded and n-valued at X. But $|\nabla\phi|$ is bounded, hence $\chi_r=O(1)$, $(1/r^2)\chi_\theta=-k/r^2+O(1)$, $\chi(x, y)=-k \arctan (y/x)+O(r^2) \arctan (y/x)+O(r)+\text{const}$. It follows that $k=O(r^2)=0$, and we thus obtain

IV.2. *A solution of the gas dynamic equation, elliptic and having an n-valued gradient in a neighborhood of an isolated singular point X, is itself n-valued and continuous and satisfies the maximum principle at X.*

III.4. *Let $\phi(x, y)$ be a solution of the gas-dynamic equation, singular at an isolated point X and single-valued in a neighborhood N of X. Then if in N, $\phi_x^2+\phi_y^2<2/(1+\gamma)$, the singularity at X is removable.*

This is an immediate consequence of III.2. The equation arises from the variational problem

$$\delta \iint \left\{\int \rho d[\phi_x^2 + \phi_y^2]\right\} dxdy = 0.$$

Added in proof. In §2.2 of this paper, a proof of a strengthened form of Theorem III is made to depend on a technique developed by M. Shiffman in connection with two-dimensional variational problems. The author wishes to point out that Theorem III of the present paper can be used in conjunction with Shiffman's technique to prove continuity of the first derivatives of solutions in a much more general case. Continuity of the second derivatives then follows as before from results of C. B. Morrey and E. Hopf. The details of the argument are not difficult and are omitted. A precise statement follows.

THEOREM III a. *Let* $\Theta = f(\phi_x, \phi_y)$, $\Lambda = g(\phi_x, \phi_y)$ *be such that the domain of values* (ϕ_x, ϕ_y) *for which*

$$\Theta_x + \Lambda_y = 0 \tag{3}$$

is of elliptic type is convex. Let $\phi(x, y)$ *be a solution of* (3) *singular at an isolated point X and single-valued in a neighborhood N of X. Suppose that for this solution* (3) *is of elliptic type in N. Then if at X,* $(\Theta^2+\Lambda^2)^{1/2} = o(1/r)$, *the singularity at X is removable.*

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SYRACUSE UNIVERSITY,
SYRACUSE, N. Y.
THE INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.