# ON A CERTAIN CLASS OF IDEALS IN THE $L^1$-ALGEBRA OF A LOCALLY COMPACT ABELIAN GROUP

BY

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This note is an addition to an earlier paper [5] where all theorems and mathematical terms used here without further reference will be found.

If $I$ is a closed ideal in $L^1(G)$ consisting of all functions whose Fourier transforms vanish on a closed *subgroup* $G' \subset G$, then the quotient-algebra $L^1(G)/I$ is algebraically isomorphic and isometric with $L^1(\mathbf{G}')$, where $\mathbf{G}'$ is the dual group of $G'$ (cf. Theorem 1.3). It is the purpose of the present note to consider another class of closed ideals in $L^1$ for which it is similarly possible to determine explicitly the structure of the quotient-algebra $L^1/I$, namely those ideals $I$ which have a *denumerable* co-spectrum $Z_I$ consisting of *independent* elements (i.e. if $x, y, \cdots, z$ are distinct elements of $Z_I$, then no relation $x^i y^j \cdots z^k = e$ subsists unless all the integers $i, j, \cdots, k$ are zero). This case is the opposite extreme, in comparison with the previous one. The discussion is based on Carleman's little book [3, p. 79].

LEMMA. *If the co-spectrum $Z_I$ of the closed ideal $I \subset L^1$ is denumerable and if the elements of $Z_I$ are independent, then any $\phi \perp I$ is of the form*

$$\phi(\mathbf{x}) = \sum_{x \in Z_I} a_x(\mathbf{x}, x)$$

*where* $\sum_{x \in Z_I} |a_x| < \infty$ [1].

Let $x_1, x_2, \cdots, x_N$ be any *isolated* points of $Z_I$ and consider the Fejér kernels

$$k_{x_n}(\mathbf{x}) = 1 + \frac{1}{2}\left\{(\mathbf{x}, x_n) + (\mathbf{x}, x_n^{-1})\right\} \qquad (1 \leqq n \leqq N)$$

which are real, non-negative functions of $\mathbf{x} \in \mathbf{G}$. We have

$$\prod_{n=1}^{N} k_{x_n}(\mathbf{x}) = 1 + \frac{1}{2}\sum_{n=1}^{N}(\mathbf{x}, x_n) + T(\mathbf{x})$$

where $T(\mathbf{x})$ is a trigonometric polynomial (without constant term) whose frequencies are not in $Z_I$ since $Z_I$ consists, by hypothesis, of independent elements.

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(1) In connection with the proof that follows, see pp. 133–134 of Bochner's paper [1], especially the footnote on p. 134.

Thus, for $S$ sufficiently small, we have, by Theorem 2.1,

$$(*)\qquad m^{-1}(S)\left[S(\mathbf{x})^2\prod_{n=1}^{N} k_{x_n}(\mathbf{x})\right]*\phi = \frac{1}{2}\sum_{n=1}^{N} a_{x_n}(\mathbf{x}, x_n)$$

where $a_{x_n}=M\{\phi(\mathbf{x})(\mathbf{x}, x_n)^\cdot\}$.

Moreover, since $S(\mathbf{x})^2\prod_{n=1}^{N} k_{x_n}(\mathbf{x})\geqq 0$ and also $m^{-1}(S)\int S(\mathbf{x})^2\prod_{n=1}^{N} k_{x_n}(\mathbf{x})dx=1$ ($S$ being sufficiently small), it follows from $(*)$ that

$$\frac{1}{2}\sup_{\mathbf{x}\in G}\left|\sum_{n=1}^{N} a_{x_n}(\mathbf{x}, x_n)\right| \leqq \|\phi\|_\infty$$

and thus, by Kronecker's theorem([2]),

$$\frac{1}{2}\sum_{n=1}^{N}|a_{x_n}| \leqq \|\phi\|_\infty.$$

Since $x_1, x_2, \cdots, x_N$ are any isolated points of $Z_I$, we have

$$\sum_x |a_x| \leqq 2\|\phi\|_\infty \qquad (x\in Z_I - Z_I^1)$$

where $Z_I^1$ denotes the first derivative of $Z_I$. We use now transfinite induction. Let $Z_I^\alpha$ be the derivative of order $\alpha$ of $Z_I$ where $\alpha$ is an ordinal number of the first or second class. Consider an ordinal $\beta$ and suppose that

(i) $a_x = M\{\phi(\mathbf{x})(\mathbf{x}, x)^\cdot\}$ exists for all $x\in Z_I - Z_I^{\alpha+1}$ $\qquad (0\leqq\alpha<\beta)$;

(ii) $\displaystyle\sum_x |a_x| \leqq 2\|\phi\|_\infty \qquad \left(x\in\bigcup_{0\leqq\alpha<\beta}(Z_I - Z_I^{\alpha+1})\right)$.

We shall prove that this still holds if "$0\leqq\alpha<\beta$" is replaced by "$0\leqq\alpha\leqq\beta$." Define

$$\phi_\beta(\mathbf{x}) = \phi(\mathbf{x}) - \sum_x a_x(\mathbf{x}, x) \qquad \left(x\in\bigcup_{0\leqq\alpha<\beta}(Z_I - Z_I^{\alpha+1})\right)$$

and let $I_\beta$ be the closed ideal of all functions $f\in L^1$ such that $f*\phi_\beta^*=0$; denote by $Z_\beta$ its co-spectrum. We have $Z_\beta\subset Z_I^0=Z_I$.

Suppose now that $Z_\beta\subset Z_I^\alpha$ for all $\alpha$, $0\leqq\alpha<\beta$. If $\beta$ is a limiting number, we have $Z_I^\beta=\bigcap_{0\leqq\alpha<\beta}Z_I^\alpha$ and thus $Z_\beta\subset Z_I^\beta$. Otherwise consider the set $Z_I^{\beta-1}-Z_I^\beta$. If this set has any points at all in common with $Z_\beta$, they are isolated points of $Z_\beta$. Suppose $x_0$ is such a point; then, by Theorem 2.1, for small $S$,

([2]) For a simple proof of Kronecker's theorem, in the general case of an abelian group, see [2, Satz 1].

$$m^{-1}(S)\left[S(\mathbf{x})^2(\mathbf{x}, x_0)\right] * \phi_\beta = c(\mathbf{x}, x_0) \qquad (c = \text{constant})$$

or

$$m^{-1}(S)\left[S(\mathbf{x})^2(\mathbf{x}, x_0)\right] * \phi(\mathbf{x}) - m^{-1}(S)\left[S(\mathbf{x})^2(\mathbf{x}, x_0)\right] * \sum_x a_x(\mathbf{x}, x) = c(\mathbf{x}, x_0),$$

the summation being extended over all $x \in \bigcup_{0 \leqq \alpha < \beta}(Z_I - Z_I^{\alpha+1}) = Z_I - Z_I^\beta$. Taking the limit by letting $S$ shrink to $\{e\}$ (3), we have

$$a_{x_0}(\mathbf{x}, x_0) - a_{x_0}(\mathbf{x}, x_0) = c(\mathbf{x}, x_0).$$

Thus $c=0$ and therefore $m^{-1}(S)S(\mathbf{x})^2(\mathbf{x}, x_0) \in I_\beta$, i.e. $x_0 \notin Z_\beta$, and we have proved that $Z_\beta \subset Z_I^\beta$.

It follows, by Theorem 2.1, that $M\{\phi_\beta(\mathbf{x})(\mathbf{x}, x)^\cdot\}$ exists for all $x \in Z_I^\beta - Z_I^{\beta+1}$. Since $\phi(\mathbf{x}) = \phi_\beta(\mathbf{x}) + \sum_x a_x(\mathbf{x}, x)$ $(x \in \bigcup_{0 \leqq \alpha < \beta}(Z_I - Z_I^{\alpha+1}))$, it results that $\lim_{S \to \{e\}} m^{-1}(S)S(\mathbf{x})^2 * [\phi(\mathbf{x})(\mathbf{x}, x)^\cdot] = a_x$ exists for all $x \in Z_I^\beta - Z_I^{\beta+1}$. Thus (i) holds with $0 \leqq \alpha \leqq \beta$ instead of $0 \leqq \alpha < \beta$.

Once the existence of $a_x$ is established, we may show in the same way as before(4) that

$$\sum_x |a_x| \leqq 2\|\phi\|_\infty \qquad \left(x \in \bigcup_{0 \leqq \alpha \leqq \beta} (Z_I - Z_I^{\alpha+1})\right)$$

which completes the induction.

Take now $\beta = \beta_0$, where $\beta_0$ is the first ordinal number such that $Z_I^{\beta_0} = \emptyset$. Then the co-spectrum of $I_{\beta_0}$ is empty and hence, by Wiener's theorem, $\phi_{\beta_0} = 0$ almost everywhere, i.e. we may write

$$\phi(\mathbf{x}) = \sum_{x \in Z_I} a_x(\mathbf{x}, x), \qquad \sum_{x \in Z_I} |a_x| < \infty,$$

and the lemma is proved.

**Remark.** For groups $G$ such that the dual group $G$ satisfies the first axiom of countability, the proof may be given by means of the familiar diagonal process, avoiding transfinite induction, as follows: by Lemma 2.1.1 and the diagonal process, there is a subsequence $(n_r)$, $r \geqq 1$, such that $\lim_{r \to \infty} m^{-1}(S_{n_r})S_{n_r}(\mathbf{x})^2 * [\phi(\mathbf{x})(\mathbf{x}, x)^\cdot] = a_x$ exists for *all* $x \in Z_I$. The convergence of $\sum_x |a_x|$ $(x \in Z_I)$ is then immediately established, as before. Since $\phi(\mathbf{x}) - \sum_x a_x(\mathbf{x}, x)$ $(x \in Z_I)$ is orthogonal to an ideal whose co-spectrum is in $Z_I$ and cannot have isolated points, it follows that this difference vanishes almost everywhere.

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(3) In part II of [5] the notation should be changed so as to avoid an appeal to countability. The existence of the limit of the first term is part (i) of the hypothesis of the induction; the existence of the limit of the second term may be shown by means of the absolute convergence of the series.

(4) It should be observed, though, that we must now take the limit, as $S \to \{e\}$, in equation (*).

If $X$ is a locally compact Hausdorff space, denote by $C(X)$ the Banach algebra of all complex-valued continuous functions $c(x)$ on $X$ which "vanish at infinity," with norm $\|c(x)\| = \max_{x\in X} |c(x)|$ and the usual (point-wise) multiplication. We shall be concerned here with the case where $X = Z_I$, with the topology induced by that of $G$.

We are now ready to prove the following:

*If the co-spectrum $Z_I$ of the closed ideal $I \subset L^1$ is denumerable and the elements of $Z_I$ are independent, then the quotient-algebra $L^1/I$ is algebraically isomorphic and isometric with the Banach algebra $C(Z_I)$ defined above.*

Define a homomorphic mapping of $L^1$ into $C(Z_I)$ by making correspond to each $f \in L^1$ the function $f(x)$, $x \in Z_I$. The kernel of this homomorphism consists of all those functions in $L^1$ whose Fourier transforms vanish on $Z_I$; it coincides with $I$ (this follows from Theorem 2.2 (cf. also Theorem 2 in [4]), or directly from the lemma proved above).

By the "distance theorem" we have for any $f \in L^1$ which is not in $I$

$$\text{dist}\,\{f, I\} = 1/\min \|\phi\|_\infty$$

where $\phi$ ranges over all functions satisfying

$$\int f(\mathbf{x})\phi^*(\mathbf{x})d\mathbf{x} = 1, \phi \perp I.$$

By the lemma proved above, we have

$$\phi(\mathbf{x}) = \sum_{x\in Z_I} a_x^*(\mathbf{x}, x), \qquad \sum_{x\in Z_I} |a_x| < \infty.$$

By Kronecker's theorem

$$\|\phi\|_\infty = \sum_{x\in Z_I} |a_x|.$$

Now the smallest value that $\sum_{x\in Z_I} |a_x|$ can have, under the condition

$$\sum_{x\in Z_I} a_x f(x) = 1,$$

is just $1/\max_{x\in Z_I} |f(x)|$, since

$$\sum_{x\in Z_I} |a_x| \max_{x\in Z_I} |f(x)| \geqq 1.$$

Thus $\text{dist}\,\{f, I\} = \max_{x\in Z_I} |f(x)|$.

Hence the quotient-algebra $L^1/I$ is isomorphic and isometric with the image of $L^1$ in $C(Z_I)$ under the homomorphism defined above. To prove that this image is $C(Z_I)$ itself it suffices to show that it is a dense subset of $C(Z_I)$; this will imply the desired result since $L^1/I$ is a complete space and the isomorphism is isometric (cf. the end of the proof of Theorem 1.3).

The fact that the Fourier transforms $f(x)$, $x \in Z_I$, are dense in $C(Z_I)$ follows from theorems proved by M. H. Stone (cf. [6, Theorem 5, Corollary 2, and Theorem 12, Corollary 1, and the remark at the end of §6])(5). This completes the proof.

As a consequence of the result just proved, we have: *if $Z$ is any closed denumerable subset of $G$ consisting of independent elements, and if $c(x)$ is any complex-valued function defined and continuous on $Z$ and "vanishing at infinity" in case $Z$ is not compact, then there is a function $f \in L^1(G)$ such that the Fourier transform $\hat{f}(x)$ coincides with $c(x)$ for all $x$ in $Z$.*

## References


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(5) The author is obliged to Professor G. W. Mackey for pointing out the use of Stone's theorem in this connection; the original procedure was based on an adaptation of the methods used in [5] for analytic functions of Fourier transforms.