SETS OF "POSITIVE" FUNCTIONS IN H-SYSTEMS

BY
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Introduction. In [1][1] Ambrose has defined H-systems to be Hilbert spaces in which multiplication is "partially defined." If H is such a system and a is in H, then L_a and R_a are the (not necessarily everywhere defined) operators of left and right multiplication by a and the *bounded algebra* of H, written $A(H)$, is $[a \mid L_a$ and R_a are everywhere defined$]$[2]. We define the *associated ring of operators* of H, written $W(H)$, to be the weak closure of $[L_a \mid a$ is in $A(H)]$.

If G is a separable, locally compact, unimodular group and $H(G)$ is the L_2 space of G under Haar measure with multiplication "partially defined" by convolution as in [1], then $H(G)$ is an H-system[3]. The left regular representation represents G faithfully as a group of unitary operators on $H(G)$ each of which commutes with every element of $[R_f \mid f \in A(H(G))]$. However, it is known [6 or 7] that $W(H)$ is the commutant of $[R_f \mid f \in A(H)]$ so that $l(G) \subset W(H(G))$. If we define $P(G) = [f \in H(G) \mid f$ is almost everywhere positive on $G]$, then the elements of $l(G)$ have the further property that $l(x)P(G) \subset P(G)$. The main result of §1 is that these properties completely characterize $l(G)$, i.e., the only unitary operators in $W(H(G))$ which take $P(G)$ into itself are the elements of $l(G)$. Using this result we prove that groups whose H-systems are isomorphic in a manner preserving positivity are themselves isomorphic. Similar results for the L_1 algebra of a group have been obtained by Kawada [8] and Wendel [9].

The question now arises: given an H-system H and a subset P of H, when is H the H-system of the group of unitary operators in $W(H)$ which take P into itself? In §2 a set of necessary and sufficient conditions is found and by means of these it is shown that any homomorphism of $H(G_1)$ onto a left ideal in $H(G_2)$ which preserves positivity arises in a natural way from a homomorphism of G_2 onto G_1.

1. **Characterization of $l(G)$.** Throughout this section we assume that G is a fixed separable, locally compact, unimodular group.

LEMMA 1.1. *If $G' = [U \in W(H(G)) \mid UP(G) \subset P(G)$ and U is unitary$]$, then G' is a topological group in the strong operator topology. $G \subset G'$ and the topology of G is that induced from the strong topology on G'.*

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

(2) If P is a property of some elements of a set S, then we write $[s \mid P(s)]$ for the subset consisting of these elements. In general, we use the notation of [2] for the elementary operations on sets. We write $c(A)$ for the characteristic function of the set A.

(3) The proof of this in [1] is incorrect; see [3].

Proof. If U and V are in G', then trivially UV is, and for any f and g in $P(G)$, $(U^*f, g) = (f, Ug) \geqq 0$, i.e., the inner product of U^*f by any element of $P(G)$ is positive, so U^*f is in $P(G)$, so $U^* \in G'$. G' is strongly closed in the group of all unitaries in $W(G)$ which is known to be a topological group, so G' is a topological group.

Since continuous functions of compact support are dense in $H(G)$, sets of the form[4] $[U\in G' \mid \|Uf-f\| < a]$ for such f form a base for the strong topology on G'. If M is the measure of the support of f, then there is a neighborhood A of the identity in G such that if $x \in A$, then $|(l(x)f)(y) - f(y)| < aM^{-1/2}$ for all y so that $\|l(x)f - f\| < a$, i.e., $l(A) \subset [U \in G' \mid \|Uf-f\| < a]$. Hence every strong open set of G is open. Conversely, if A is a neighborhood of the identity in G, we can find a neighborhood B of the identity satisfying $BB^{-1} \subset A$, and if xB and B are not disjoint, then $xb_1 = b_2$ for some b_1 and b_2 in B so that $x = b_2 b_1^{-1} \in BB^{-1} \subset A$. Hence, if $x \in C(A)$, we have $\|l(x)c(B) - c(B)\| = \|c(xB) - c(B)\| = 2^{1/2}\|c(B)\|$ so that $G \cap [U \mid \|Uc(B) - c(B)\| < 2^{1/2}\|c(B)\|] \subset A$. This shows that open sets in G are strongly open and completes the proof of the lemma.

LEMMA 1.2. *G' as above. If $U \in G'$ and S is any set in G of positive finite measure, then for some positive number $a(S)$ and measurable set $\overline{U}(S)$, $U(c(S)) = a(S)c(\overline{U}(S))$.*

Proof. For any $a > 0$ define f_a and g_a by $f_a(x) = U(c(S))(x)$ if this is greater than a, $f_a(x) = 0$ otherwise, and $g_a = U(c(S)) - f_a$. It will be sufficient to show that, for every a, either f_a or g_a is zero. Now, $c(S) = U^*f_a + U^*g_a$, but U^*f_a and U^*g_a are a.e. positive functions satisfying $(U^*f_a, U^*g_a) = (f_a, g_a) = 0$; hence, for some measurable sets S_a' and S_a'' whose union is S and whose intersection is of zero measure, $U^*f_a = c(S_a')$ and $U^*g_a = c(S_a'')$. If neither f_a nor g_a is zero, we can find an x in G for which the Haar measure of $S_a' x^{-1} \cap S_a''$ is not zero. Define $T_a' = S_a' \cap S_a'' x$ and $T_a'' = S_a' x^{-1} \cap S_a'' = T_a' x^{-1}$. Since $c(T_a')$ and $c(S_a') - c(T_a')$ are orthogonal functions in $P(G)$, so are $U(c(T_a'))$ and $U(c(S_a') - c(T_a'))$, so they must be restrictions of $U(c(S_a'))$ to subsets of its support. Similarly, $U(c(T_a''))$ is a restriction of g_a so that for a.a. x in G, $U(c(T_a''))(x) \leqq a$ and $U(c(T_a'))(x)$ is either 0 or $> a$. But, if r is the right regular representation, $U(c(T_a'')) = U(r(x)c(T_a')) = r(x)U(c(T_a'))$, which is impossible.

LEMMA 1.3. *In the above lemma, $a(S) = 1$.*

Proof. If $S_1 \subset S_2$, then $U(c(S_1))$ is a restriction of $U(c(S_2))$ so $a(S_1) = a(S_2)$. If S_1 and S_2 are arbitrary, choose an x in G so that $S_1 \cap (S_2 x) = T$ has nonzero measure, then $a(S_2)c(\overline{U}(Tx^{-1})) = U(r(x)c(T)) = r(x)a(S_1)c(\overline{U}(T))$, so $a(S_1) = a(S_2) = a$.

By a *basic sequence* we shall mean a countable set (S_n) of neighborhoods

(4) We write $\| \;\|$ for L_2 norm, $\| \;\|_p$ for L_p norm if $p \neq 2$, and $|||\; |||$ for operator norm.

of the identity having the property that if S is any neighborhood of the identity, then $S_n \subset S$ for large enough n. If (S_n) is a basic sequence, then $(1/\|c(S_n)\|_1 L_{c(S_n)}$ approaches the identity operator strongly.

Now let (S_n) be a basic sequence so that

$$\begin{aligned} 1 &\leqq \liminf \left|\left|\left|(1/\|c(S_n)\|_1) L_{U(c(S_n))}\right|\right|\right| \\ &\leqq \liminf (a/\|c(S_n)\|_1)\|c(\overline{U}(S_n))\|_1. \end{aligned}$$

But $\|c(S_n)\|_1 = (c(S_n),\ c(S_n)) = a^2\|c(U(S_n))\|_1$, and substituting this in the above gives $1 \leqq 1/a$. Applying this to U^* which multiplies characteristic functions by $1/a$ gives the opposite inequality and completes the proof.

For the basic sequence (S_n) let $F_n = (1/\|c(S_n)\|_1)c(S_n)$.

Lemma 1.4. *If F_n and S_n are as above and m is Haar measure on G, then for every integer n there is an x in G and an integer k for which $m(\overline{U}(S_k) \cap xS_n) \geqq (n/(n+1))m(S_k)$.*

Proof. $L_{UF_k}c(S_n)(x) = (1/m(S_k))(c(\overline{U}(S_k))c(S_u))(x) = (1/m(S_k))m(S_n \cap \overline{U}(S_k)^{-1}x) = (1/m(S_k))m(xS_n \cap \overline{U}(S_k))$, so that if the lemma is false, $L_{UF_k}c(S_n)(x) \leqq (n/(n+1))$ for all x and k. However, L_{UF_k} approaches U strongly, so this is impossible.

Theorem 1.1. $G = G'$.

Proof[5]. If $U \in G'$ we can choose, for some sequence S_n, integers $k(n)$ and elements x_n in G to satisfy Lemma 1.4. We wish to show that $l(x_n^{-1})L_{UF_{k(n)}}$ approaches the identity strongly. $l(x_n^{-1})L_{UF_{k(n)}} = L_{f_n}$ where $f_n = (1/m(S_{k(n)}))c(x_n^{-1}\overline{U}(S_{k(n)}))$. If we define $T_n = x_{k(n)}^{-1}\overline{U}(S_{k(n)} \cap S_n)$ then, since the T_n have nonzero measure and get arbitrarily small, the sequence $(1/m(T_n))L_{c(T_n)}$ approaches the identity strongly. However, $\left|\left|\left| L_{f_n} - (1/m(T_n))L_{c(T_n)}\right|\right|\right| \leqq \|f_n - (1/m(T_n))c(T_n)\|_1 = 2(1 - m(T_n)/m(S_{k(n)})) \to 0$ since $m(T_n) \geqq (n/(n+1))m(S_{k(n)})$. Hence, $l(x_n^{-1})L_{UF_{k(n)}}$ approaches the identity strongly so $l(x_n)$ approaches U strongly. The strong convergence of $l(x_n)$ implies that (x_n) is a Cauchy sequence and $U = l(\lim x_n)$.

If H is any H-system with elements a and b, then we write ab for their product when it is defined. Consistent with this notation, if f and g are functions in L_2 of G, we write fg for their convolution and not their pointwise product.

Lemma 1.5. *If G_1 and G_2 are separable, locally compact, unimodular groups and w is a linear transformation of $H(G_1)$ into $H(G_2)$ satisfying:*

(1) *$w(H(G_1))$ is a left ideal in $H(G_2)$,*

(2) *$w(P(G_1)) \subset P(G_2)$,*

(3) *for any f and g in $H(G_1)$, $(w(f), w(g)) = (f, g)$,*

[5] The referee has outlined a different proof of this theorem which does not require separability.

(4) *if* f *and* g *are in* $H(G_1)$ *and* fg *is defined then* $w(f)w(g)$ *is defined and* $w(fg)=w(f)w(g)$,

then there is a homomorphism $\bar{w}$ *of* G_2 *into* G_1 *such that* $l(x)w(f)=wl(\bar{w}(x))f$ *for any* x *in* G_2 *and* f *in* $H(G_1)$.

Proof. If f is in $H(G_1)$ and x is in G_2, then $l(x)w(f)$ is in $w(H(G_1))$, so there is a unique element $T(x)f$ in $H(G_1)$ satisfying $wT(x)(f)=l(x)w(f)$. Clearly $T(x)$ is an isometric linear transformation. If f and g are in $P(G_1)$, then $(T(x)f,\ g)=(wT(x)(f),\ w(g))=(l(x)w(f),\ w(g))\geqq 0$, so $T(x)P(G_1)\subset P(G_1)$. Also, $T(x)$ is in $W(G_1)$ since $W(G_1)$ is the commutant of $[R_f \mid f \text{ is in } A(G_1)]$ and for any $f\in A(G_1)$, $g\in H(G_1)$, $wT(x)R_f(g)=wT(x)(gf)=l(x)(w(g)w(f))$ $=w(T(x)g)w(f)=wR_fT(x)(g)$, i.e., $T(x)R_f=R_fT(x)$.

The map $T:G_2\rightarrow W(G_1)$ satisfies (i), $T(x)T(y)=T(xy)$, and (ii), $T(x)^*$ $=T(x^{-1})$. These follow from $wT(x)T(y)=l(x)wT(y)=l(x)l(y)w=l(xy)w$ $=wT(xy)$ and $(T(x)f,\ g)=(wT(x)f,\ w(g))=(w(f),\ l(x^{-1})w(g))=(w(f),$ $wT(x^{-1})g)=(f,\ T(x^{-1})g)$ respectively. Equation (ii), plus the fact that $T(e)$ $=I$, implies that $T(x)$ is unitary; hence, $T(x)=l(\bar{w}(x))$ for some $\bar{w}(x)$ in G_1 and equation (i) implies that $\bar{w}$ is a homomorphism.

To show the continuity of $\bar{w}$, let f be an element of $H(G_1)$ and $S=[x \text{ in } G_1 \mid \|l(x)f-f\|<a]$; then $\bar{w}^{-1}(S)=[y \text{ in } G_2 \mid \|l(\bar{w}(y))f-f\|<a]=[y \text{ in } G_2 \mid \|l(y)w(f)-w(f)\|<a]$, which is open. Since sets of this form are a sub-basis for the topology of G_1, this completes the proof.

THEOREM 1.2. *If* G_1 *and* G_2 *are locally compact, separable, unimodular groups, and* w *is a linear map of* $H(G_1)$ *onto* $H(G_2)$ *satisfying the conditions of Lemma* 1.5, *then* $\bar{w}$ *is an isomorphism onto.*

Proof. Trivially w^{-1} satisfies conditions (1) and (3) of Lemma 1.5. If f is in $P(G_2)$ and g is in $P(G_1)$, then $(w^{-1}(f),\ g)=(f,\ w(g))\geqq 0$, so $w^{-1}(f)$ is in $P(G_1)$, i.e., condition (2) is satisfied. To prove (4) it will be sufficient [1] to show that if gf is defined in $H(G_2)$ and h is in $A(G_1)$, then $(w^{-1}(g),\ zw^{-1}(f)^*)$ $=(w^{-1}(gf),\ z)$. Trivially $w^{-1}(f)^*=w^{-1}(f^*)$ so $(w^{-1}(g),\ zw^{-1}(f)^*)=(g,\ w(z)f^*)$ $=(gf,\ w(z))=(w^{-1}(gf),\ z)$. Hence Lemma 1.5 gives a homomorphism $\bar{w}^{-1}$ of G_1 into G_2 and $l(\bar{w}(\bar{w}^{-1}(x)))=wl(\bar{w}^{-1}(x))=ww^{-1}l(x)=l(x)$ so $\bar{w}\bar{w}^{-1}(x)=x$ and similarly $\bar{w}^{-1}\bar{w}(x)=x$, which completes the proof.

The assertion of Theorem 1.2 is not true if the assumption of positivity of W is dropped. Ambrose proved [1, Theorem 10] that all Abelian H-systems are essentially the same algebraically except for dimension and it is an immediate corollary of this that any two finite Abelian groups of the same order have isomorphic H-systems.

2. *HP* **systems.** We shall say that a subset P of a Hilbert space H is a *set of non-negative functions* in H if there is a representation ϕ of H as the L_2 of some measure space such that $\phi(P)$ is the set of almost everywhere non-

negative functions in this L_2(6). We write $x \leqq y$ to mean that $y-x$ is in P, and $x \leqq S$ to mean that $[s-x \mid s \in S] \subset P$. For any countable set $Q \subset P$ there is defined an element inf Q in P and if, for some y, $x \leqq y$ for all x in Q, there is also defined an element sup $Q \leqq y$ in P having all the usual properties. If Q is a convex subset of P we write inf Q for the unique element of minimal norm in the uniform closure of Q and if, for some y, $Q \leqq y$ we write sup Q for inf $[x \mid Q \leqq x]$. These definitions are consistent with one another.

If H is a proper H-system let $C(H)$ be the dense subset consisting of all finite sums of products. We shall be concerned with the linear map [ ] from $C(H)$ to the set of weakly continuous functions on $W(H)$ defined by $[\sum f_i g_i](T) = \sum (f_i, T(g^*))$. (Note that this map is well defined for by [10, p. 76] we can find a set (x_α) of approximate left identities in H and since H is separable we can choose a countable subset (x_i) which is still a set of approximate left identities and then $[x](T) = \lim (x, Tx_i)$.)

Definition. A pair (H, P) is an HP system if H is a proper H system, P is a set of non-negative functions in H, and the following conditions are satisfied; when G is the group of unitaries in $W(H)$ which carry P inside itself:

(1) $C(H) \cap P$ is dense in P.

(2) If (f_i) is a countable subset of $C(H)$ whose sup exists and sup $([f_i]) \geqq [f]$ for some f in $C(H) \cap P$, then sup $(f_i) \geqq f$.

(3) If N is any strong neighborhood of I in G there is a nonzero f in $C(H) \cap P$ with $[f]$ vanishing outside N.

If G is a separable, locally compact, unimodular group, H its H-system, and P the almost everywhere non-negative functions in H, then, by Theorem 1.1, (H, P) is an HP system. The main result of this section is that the converse is also true.

We assume until further notice that (H, P) is a fixed HP system, and write C for $C(H) \cap P$.

Lemma 2.1. *$C = [f \mid f \in C(H)$ and $[f] \geqq 0]$, $P = P^*$, and if p and q are in P and pq is defined, then pq is in P.*

Proof. If f is in $C(H)$ and $[f] \geqq 0$, then f is in C by condition 2. If f is in C and $[f] \leqq -e < 0$ on some open set N, choose h in C with $|[h](U)| \leqq e$ and $[h]$ vanishing outside N, then sup $([h], [f]) \geqq [f+2h]$ so by condition 2, $f+h \geqq$ sup $(f, h) \geqq f+2h$ which is impossible.

If f is in C then $[f^*]$ is the complex conjugate of $[f]$, hence f^* is in C and by condition 1 this implies $P = P^*$.

Finally $[pq](U) = (p, Uq^*) \geqq 0$ so pq is in C.

(6) Nagy, in [4], proves that P is a set of non-negative functions in H if and only if the following conditions are satisfied: $(u, v) \geqq 0$ for every u and v in P, if $(u, v) \geqq 0$ for every v in P then u is in P, and if u_1, u_2, v_1, and v_2 are in P and $u_1+u_2 = v_1+v_2$, then there are elements $w_{11}, w_{12}, w_{21}, w_{22}$ in P such that $u_i = w_{i1}+w_{i2}$ and $v_i = w_{1i}+w_{2i}$ for $i = 1, 2$.

If f is in C and A is a subset of G, we say that f *covers* A if $[f](U)\geqq 1$ for all U in A, and we say that A is *bounded* if there is an element of C which covers it. If A is bounded, $\Gamma(A)$ is to be the (nonempty) set $[\sup F \mid F\subset C$, $F\leqq f$ for some f, and there exists an enumerable set of sets $X_i\subset A$ and elements f_i in F such that f_i covers X_i and $\sum X_i=A]$. $\Gamma(A)$ is convex since if F_1 and F_2 are subsets of C satisfying the above conditions, then so does the set $F=[(1/2)(f_1+f_2) \mid f_i \text{ is in } F_i]$ and $\sup F=(1/2)$ $(\sup F_1+\sup F_2)$. We define, for bounded A, $d(A)=\inf \Gamma(A)$.

LEMMA 2.2. *If the sets A, B, and A_i are bounded, $A\subset B$, and U an element of G, then*

(i) $d(A)\leqq d(B)$,

(ii) $d(A_i)\leqq \inf\ (d(A_i))$,

(iii) A^{-1} *is bounded and* $d(A^{-1})=d(A)^*$,

(iv) UA *is bounded and* $d(UA)=Ud(A)$,

(v) *if* $A=\sum A_i$ *then* $d(A)=\sup\ (d(A_i))$.

Proof. The first four assertions are trivial and in the fifth it is clear that $d(A)\geqq \sup\ (d(A_i))$. Choose subsets F_i of C so that $\|\sup F_i-d(A_i)\|^2\leqq \epsilon 2^{-i}$, then $\sup\ (\sup\ F_i)=\sup\ (\sum F_i)\geqq d(A)$ and $\|d(A)-\sup\ (d(A_i))\|^2 \leqq \|\sup\ (\sup\ (F_i))-\sup d(A_i)\|^2\leqq \sum\|\sup\ (F_i)-d(A_i)\|^2\leqq\epsilon$.

LEMMA 2.3. *If A and B are closed and bounded, then* $d(A\cap B)=\inf\ (d(A), d(B))$. *If further* $A\subset B$, *then* $d(B-A)=d(B)-d(A)$.

Proof. Suppose A and B are disjoint. For any V in B there is some neighborhood N of the identity for which VN does not intersect A. Choose f_0 acording to assumption 3 for this N and let $f=2f_0/\max\ [f_0]$ so that $\inf\ ([Uf], [Vf])=0$ if U is not in A. In this case $[Uf+Vf]=\sup\ ([Uf], [Vf])$ so that $Uf+Vf\leqq \sup\ (Uf, Vf)$ by assumption 2 and this implies that $\inf\ (Uf, Vf)=0$ so we must have $(Uf, Vf)=0$. For each U, Uf covers some neighborhood of U and we can choose a countable subcovering (U_if) of A. Then $(d(A), Vf)\leqq (\sup\ (U_if), Vf)=0$. Again we can choose a countable subcovering of B from among all such Vf's so $(d(A), d(B))=0$, and hence $\inf\ (d(A), d(B))=0$.

We can now prove the second assertion. If (N_i) is a basic sequence, then by the previous lemma $d(B-A)+d(A)=\lim\ (d(B-AN_i)+d(A)) =\lim\ d(B-AN_i+A)\leqq d(B)$. The opposite inequality is trivially true for any bounded sets $B-A$ and A.

The first assertion now follows from $\inf\ (d(A), d(B))=\inf\ (d(A-A\cap B), d(B-A\cap B))+d(A\cap B)=\lim\ \inf(d(A-(A\cap B)N_i), d(B-(A\cap B)N_i)) +d(A\cap B)=d(A\cap B)$.

The set $R_0=[\sum^n(B_i-A_i) \mid A_i$ and B_i are closed and bounded, $B_i\supset A_i$, and the summands are mutually disjoint$]$ is a ring.

LEMMA 2.4. *If X_1 and X_2 are in R_0 and are disjoint, then* $\inf\ (d(X_1), d(X_2))$

$=0$, *and* $d(X_1 \cup X_2) = d(X_1) + d(X_2)$. *If* X_i *are mutually disjoint and* $\sum_1^\infty X_i = X$ *is in* R_0, *then* $d(X) = \sum_1^\infty d(X_i)$.

Proof. If $X = \sum (A_i - B_i)$ and (N_i) is a basic sequence, then X is the limit (on k) of the closed sets $\sum (A_i - B_i N_k)$ and this by the previous lemma implies the first assertion. The other two are immediate consequences of this one.

The above lemma says that the measure m on R_0 defined by: $m(X) = \|d(X)\|^2$ is countably additive, hence can be extended to the σ-ring R generated by R_0.

LEMMA 2.5. *The measure* m *is both left and right invariant and* (G, R, m) *is a measurable group* [2, *p.* 257].

Proof. Since d is left invariant on R_0 so is m, and if X is in R_0,

$$\begin{aligned} m(XU) &= \|d(XU)\|^2 = \|d(XU)\|^2 = \|d(U^{-1}X^{-1})\|^2 = \|d(X^{-1})\|^2 \\ &= \|d(X)\|^2 = m(X). \end{aligned}$$

This extends trivially to R. To complete the proof we must show that the shearing transformation T: $(U, V) \rightarrow (U, UV)$ of $G \times G$ onto itself preserves measurability. Since R is generated by the open bounded sets which it contains, it will be sufficient to show that $T(A \times B)$ is measurable if A and B are open and bounded. But if (U, V) is in $T(A \times B)$, that is, U is in A and V is in UB, and N is a bounded neighborhood of the identity with $NU \subset A$ and $N^{-1}N \subset UBV^{-1}$, then $NU \times NV \subset T(A \times B)$, and if $(N_i U_i \times N_i V_i)$ is a countable subcovering, $T(A \times B) = \sum (N_i U_i \times N_i V_i)$.

LEMMA 2.6. *The Weil topology with respect to the measure* m *coincides with the strong topology.*

Proof. A base for the Weil topology is given by sets of the form $[U \mid m(\rho(S, US)) < e]$ (for S in R and $e > 0$ where ρ is the symmetric difference). If $S = \sum S_i$ where the S_i are mutually disjoint elements of R_0 and V is in the strongly open set $\prod_1^n [U \mid \|Ud(S_i) - d(S_i)\|^2 < e2^{-i}]$, then $m(\rho(S, VS)) \leqq \sum_1^\infty m(\rho(S_i, VS_i)) = \sum_1^n \|Vd(S_i) - d(S_i)\|^2 + \sum_{n+1}^\infty m(\rho(S_i, VS_i)) < e$ if n is chosen large enough. Hence every Weil open set is strongly open. Conversely if N is a strong neighborhood of I, choose a neighborhood S satisfying $SS^{-1} \subset N$. Then if U is not in N, $S \cap US = 0$ so inf $(d(S), Ud(S)) = 0$ so $(d(S), Ud(S)) = 0$ and hence $[U \mid (d(S), Ud(S)) > 0] \subset N$. It only remains to show that $d(S) \neq 0$, but this is a trivial consequence of assumptions 2 and 3.

The above lemma implies that G is complete in the Weil topology, hence by Weil's theorem [2, p. 275] G is a locally compact group in this topology and m is its Haar measure.

Let S be the linear transformation of $H(G)$ into H which takes $c(X)$ into $d(X)$ for X in R. S takes positive elements into positive elements and (Sx, Sy)

$=(x, y)$. If we define $Tx=[x]$ for x in $C(H)$, then for x and y in P, (Tx, Ty) $=\sup (a, b)$, a and b take on only a finite number of values, all non-negative, $a\leqq [x]$ and $b\leqq [y]\leqq (x, y)$ since $(a, b)=(Sa, Sb)$ and $Sa\leqq x$, $Sb\leqq y$. Hence T can be extended to a transformation of H onto $H(G)$ which preserves positivity.

THEOREM 2.1. *ST and TS are the identity operators, S and T preserve positivity and take adjoints into adjoints. For every U in G we have $TU=l(U)T$ and $US=Sl(U)$. If ab is defined in H, then $TaTb$ is defined in $H(G)$ and $TaTb$ $=T(ab)$; if xy is defined in $H(G)$, then $SxSy$ is defined in H and $SxSy=S(xy)$.*

Proof. To show that TS is the identity it will be sufficient to show that $TSc(X)=c(X)$. Choose $(f_i^n)\subset C$ so that, for fixed n, (f_i^n) gives a covering of X, $d(X)=\lim_n \sup_i (f_i^n)$, and $c(X)=\lim_n \sup_i ([f_i^n])$. Then

$$T(d(X)) = \lim_n T\left(\sup_i f_1^n\right) \geqq \limsup_n \sup_i [f_1^n] \geqq c(X),$$

but since $\|T(d(X))\|\leqq\|c(x)\|$ this proves the assertion.

If $E=ST$, then $E(H)=S(H(G))$, E preserves positivity, $E^2=E$, and $(E^*Ex, y)=(Ex, Ey)\leqq (x, y)$ for all x and y in P, which implies that $E^*Ex\leqq x$ for all x in P. If x is in P, then so is $p=Ex-E^*E(Ex)=Ex-E^*Ex$ and, for any y, $(p, Ey)=0$. Hence if z is in $A(H)\cap P\cap S(H(G))$, for example if $z=d(X)$ for small enough X, then $[pz(U)]=(p, Uz^*)=0$ since $US(H(G))$ $=S(H(G))=S(H(G))^*$ and hence $pz=0$. But we can choose a q in $P\cap A(H)$ with $0<q\leqq p$ and by assumption 3 we can find $\lambda>0$ in $C(H)\cap P$ with $[\lambda]$ $\leqq\inf (\|z\|^2, \|q\|^2)/2$ and support contained in

$$[U \mid \|Uz - z\| < \|z\|/2, \|Up - p\| < \|q\|/2]$$

so that $\lambda<zz^*$ and $\lambda<q^*p$, which implies $0<\|\lambda\|^2<(zz^*, q^*p)=(qz, pz)=0$, so $p=0$. Thus $E=E^*E=E^*$ and, if x is in P, then $x-Ex=x-E^*Ex\geqq 0$ and, for any y, $(x-Ex, Ey)=0$ so as before $x-Ex=0$, that is, $E=I$.

T and S trivially preserve positivity and adjoints on the sets $C(H)$ and $[d(X)\mid X \text{ in } R]$ respectively, hence everywhere. If f is in $C(H)$, then $[Uf](V)$ $=[f](U^{-1}V)=l(U)[f](V)$ so, by continuity, $TU=l(U)T$, and then US $=STUS=Sl(U)TS=Sl(U)$.

If Tf is continuous and has compact support and fg is defined, then $(Tf)(Tg)(U)=(Tf, l(U)(Tg)^*)=(Tf, TUg^*)=(f, Ug^*)=[fg](U)$. If gh is defined in H and f is as before, then $(Tg, TfTh^*)=(Tg, T(fh^*))=(g, fh^*)$ $=(gh, f)=(T(gh), Tf)$ so [1, p. 29] $TgTh$ is defined and equal to $T(gh)$. If Sa is in $A(H)$, then $SaSb=S(T(SaSb))=S(ab)$ and by the same argument as before this implies the general case.

THEOREM 2.2. *The homomorphism $\bar{\omega}$ whose existence is proved in Lemma* 1.5 *carries G_2 onto G_1.*

Proof. If f is in $A(H(G_1))$, $\omega(g)$ is in $H(G_2)$, and $\omega(h)$ is the projection of z into $\omega(H(G_1))$, then $((\omega(f)-\omega(f^*)^*)\omega(g),\ z)=((\omega(f)-\omega(f^*)^*)\omega(g),\ \omega(h))$ $=(\omega(fg),\omega(h))-(\omega(g),\omega(f^*h))=(fg,h)-(g,f^*h)=0$. Hence $(\omega(f)-\omega(f^*)^*)\omega(g)$ $=0$ and if (e_n) are a set of approximate identities in $H(G_2)$, then $(\omega(g),\omega(f)^*$ $-\omega(f^*))=\lim\ ((\omega(f)-\omega(f^*)^*)\ \omega(g),\ e_n)=0$ so $\omega(f))^*-\omega(f^*)$ is orthogonal to everything in $\omega(H(G_1))\cap A(H(G_2))$ which is dense in $\omega(H(G_1))$ [1, p. 41] so $\|\omega(f^*)\|^2=(\omega(f^*),\ \omega(f)^*)$, that is, $\omega(f)^*=\omega(f^*)$.

Suppose $p=\sum f_ig_i$ is in $C(H(G_1))$ and $[p]\geqq 0$ on $l(\omega(G_2))$, then if x is in G_2, $[\omega(p)](l(x))=\sum\ (\omega(f_i),\ l(x)\omega(g_i^*))=\sum\ (\omega(f_i),\ \omega l(\bar{\omega}(x))(g_i^*))=\sum\ (f_i,$ $l(\bar{\omega}(x))(g_i^*))=[p](l(\bar{\omega}(x)))\geqq 0$. Thus $\omega(p)$ is in $P(G_2)$ and if q is in $P(G_1)$, $(p,\ q)=(\omega(p),\ \omega(q))\geqq 0$ so p is in $P(G_1)$. Now all the requirements of the definition of an HP system are satisfied for $H(G_1)$, $P(G_1)$ with the group G replaced by $l(\bar{\omega}(G_2))$ and the proof of Theorem 2.1 goes through as before, $\bar{\omega}(G_2)$ being complete, to give $H(\bar{\omega}(G_2))$ isomorphic to $H(G_1)$ under a positivity preserving map so that, by Theorem 1.2, G_1 is isomorphic to $\bar{\omega}(G_2)$.

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