SYMPLECTIC MODULAR COMPLEMENTS

BY

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Introduction. Let Ω_n denote the group of $n\times n$ integral matrices of determinant ± 1 (the unimodular group), and let $I^{(n)}$ be the identity matrix in Ω_n. We use X' to represent the transpose of X, and $X \dotplus Y$ for the direct sum of X and Y.

The symplectic modular group(1) Γ_{2n} is the group of $2n\times 2n$ integral matrices $\mathfrak{M}$ such that

$$\mathfrak{M}\begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}\mathfrak{M}' = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}.$$

A primitive integral $(j+k)\times 2n$ matrix

$$\begin{Bmatrix} A_1^{(j,n)} & B_1^{(j,n)} \\ C_1^{(k,n)} & D_1^{(k,n)} \end{Bmatrix} \qquad (j,\ k \leqq n) \tag{1}$$

in which

$$A_1B_1' \text{ and } C_1D_1' \text{ are symmetric} \tag{2}$$

and

$$A_1D_1' - B_1C_1' = (I^{(j)}\ 0) \quad \text{or} \quad \begin{pmatrix} I^{(k)} \\ 0 \end{pmatrix} \tag{3}$$

(depending on whether $j \leqq k$ or $j \geqq k$) will be called a *normal* $(j,\ k)$ *array*. A normal $(j,\ 0)$ array will be called a *normal pair*. Then Γ_{2n} is known to consist of all normal $(n,\ n)$ arrays.

In this paper we shall consider the problem of completing a normal $(j,\ k)$ array to an element of Γ_{2n} by placing $(n-j)$ rows after the first j rows, and $(n-k)$ rows after the last k rows. Since a sub-array of a normal array is normal, it is clear that an array cannot be completed unless it is normal. It will be shown that every normal array may be so completed, and a parametrization of the general completion will be obtained. These results will generalize those due to C. L. Siegel(2) for the special case $j=n$, $k=0$, but the proofs given here will not depend on his results.

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(1) It is sometimes more convenient to define the symplectic modular group as the factor group of Γ_{2n} over its centrum. See C. L. Siegel, Math. Ann. vol. 116 (1939) pp. 617–657; L. K. Hua, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 441–490; L. K. Hua and I. Reiner, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 415–426.

(2) C. L. Siegel, Ann. of Math. vol. 36 (1935) p. 592.

1. Let $\mathfrak{X}_1$, $\mathfrak{X}_2$ be arrays of the type given in (1); we write $\mathfrak{X}_1 \sim \mathfrak{X}_2$ if there exists $\mathfrak{Y} \in \Gamma_{2n}$ such that $\mathfrak{X}_1 = \mathfrak{X}_2 \mathfrak{Y}$. This relationship is an equivalence relationship, and we have:

LEMMA 1. *Let $\mathfrak{X}_1$ be a normal array and $\mathfrak{X}_2 \sim \mathfrak{X}_1$. Then $\mathfrak{X}_2$ is also a normal array, and $\mathfrak{X}_2$ can be completed if and only if $\mathfrak{X}_1$ can be completed.*

Proof. Clear.

Before proceeding to the next lemma, it will be convenient to single out certain elements of Γ_{2n} which play the same role in Γ_{2n} as do the elementary transformations in Ω_n. Specifically, we define three types of elements of Γ_{2n}:

(I) Translations

$$\mathfrak{T}_S = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \qquad S \text{ symmetric.}$$

(II) Rotations

$$\mathfrak{R}_U = \begin{pmatrix} U & 0 \\ 0 & U'^{-1} \end{pmatrix}, \qquad U \in \Omega_n.$$

(III) Semi-involutions

$$\mathfrak{S}_J = \begin{pmatrix} J & I - J \\ J - I & J \end{pmatrix}, \qquad J \text{ diagonal with elements 0's and 1's.}$$

LEMMA 2. *If* G.C.D. $(a_1, \cdots, a_n, b_1, \cdots, b_n) = 1$, *then*

$$(a_1, \cdots, a_n, b_1, \cdots, b_n) \sim (1, 0, \cdots, 0, 0, \cdots, 0).$$

Proof. We first observe that

$$(a_1, \cdots, a_n, b_1, \cdots, b_n)\mathfrak{R}_U = ((a_1, \cdots, a_n)U, (b_1, \cdots, b_n)U'^{-1})$$

for $U \in \Omega_n$. If we set $a_0 =$ G.C.D. $(a_1, \cdots, a_n)$, by proper choice of U we obtain

$$(a_1, \cdots, a_n, b_1, \cdots, b_n) \sim (a_0, 0, \cdots, 0, \bar{b}_1, \cdots, \bar{b}_n)$$

for some integers $\bar{b}_1, \cdots, \bar{b}_n$. Let $b_0 =$ G.C.D. $(\bar{b}_2, \cdots, \bar{b}_n)$; then the above reasoning with $U = 1 \dotplus U_1$, $U_1 \in \Omega_{n-1}$, shows that

$$(a_1, \cdots, a_n, b_1, \cdots, b_n) \sim (a_0, 0, \cdots, 0, \bar{b}_1, b_0, 0, \cdots, 0),$$

and furthermore G.C.D. $(a_0, \bar{b}_1, b_0) = 1$.

We now note the formulas

$$(x_1, x_2, 0, \cdots, 0, y_1, y_2, 0, \cdots, 0)\mathfrak{T}_S = (x_1, x_2, 0, \cdots, 0, y_1 + \lambda x_1, y_2, 0, \cdots, 0),$$

where $S = \lambda \dotplus 0^{(n-1)}$, and

$$(x_1, x_2, 0, \cdots, 0, y_1, y_2, 0, \cdots, 0)\mathfrak{S}_J = (-y_1, -y_2, 0, \cdots, 0, x_1, x_2, 0, \cdots, 0),$$

where $J = 0^{(n)}$. The alternate use of these formulas has the effect of setting up a Euclidean algorithm on the elements in the first and $(n+1)$st positions. Therefore after a finite number of steps we have either

$$(a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \sim (a, 0, \cdots, 0, 0, c, 0, \cdots, 0)$$

(where c occurs in the $(n+2)$nd position) or

$$(a_1, a_2, \cdots, a_n, b_1, \cdots, b_n) \sim (a, b, 0, \cdots, 0, 0, 0, \cdots, 0),$$

for some integers a, b, and c. In the former case observe that

$$(a, 0, \cdots, 0, 0, c, 0, \cdots, 0)\mathfrak{S}_J = (a, -c, 0, \cdots, 0, 0, 0, \cdots, 0),$$

where $J = 1 \dotplus 0^{(n-1)}$. In either case, therefore,

$$(a_1, \cdots, a_n, b_1, \cdots, b_n) \sim (a, d, 0, \cdots, 0, 0, \cdots, 0),$$

where a and d are relatively prime. Now choose $V \in \Omega_2$ so that $(a, d)V = (1, 0)$, and set $U = V \dotplus I^{(n-2)}$. Then

$$(a, d, 0, \cdots, 0, 0, \cdots, 0)\mathfrak{R}_U = (1, 0, \cdots, 0, 0, \cdots, 0).$$

This proves the result.

THEOREM 1. *Let A_1 and B_1 be $j \times n$ integral matrices, $j \leqq n$. Then $(A_1\ B_1)$ can be completed to an element of Γ_{2n} by placing $2n-j$ rows below $(A_1\ B_1)$ if and only if (A_1, B_1) is a normal pair.*

Proof. If $(A_1\ B_1)$ is completable, trivially (A_1, B_1) form a normal pair. We now prove the converse by induction on n. The result for $n=1$ is an immediate consequence of Lemma 2; let $n>1$, and assume that a normal pair of $t \times (n-1)$ integral matrices can be completed to an element of $\Gamma_{2(n-1)}$ for $t \leqq n-1$.

Since $(A_1\ B_1)$ is primitive, the G.C.D. of the elements of its first row is 1. By Lemma 2 we have therefore

$$(A_1\, B_1) \sim \begin{pmatrix} 1 & \mathfrak{n}' & 0 & \mathfrak{n}' \\ \mathfrak{x} & A_2 & \mathfrak{y} & B_2 \end{pmatrix},$$

where $\mathfrak{x}$ and $\mathfrak{y}$ are $(j-1) \times 1$ vectors, and $\mathfrak{n}$ represents a null column vector whose size depends on the context. Since the right-hand side is a normal pair, the matrix

$$\begin{pmatrix} 1 & \mathfrak{n}' \\ \mathfrak{x} & A_2 \end{pmatrix}\begin{pmatrix} 0 & \mathfrak{y}' \\ \mathfrak{n} & B_2' \end{pmatrix} = \begin{pmatrix} 0 & \mathfrak{y}' \\ \mathfrak{n} & \mathfrak{x}\mathfrak{y}' + A_2 B_2' \end{pmatrix}$$

must be symmetric; therefore $\mathfrak{y} = \mathfrak{n}$, and consequently (A_2, B_2) form a normal pair of $(j-1)\times(n-1)$ matrices. By the induction hypothesis there exists a matrix

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \Gamma_{2(n-1)}$$

with its first $(j-1)$ rows given by $(A_2\ B_2)$. Define

$$\mathfrak{X} = \begin{pmatrix} 1 \dotplus R & 0 \dotplus S \\ 0 \dotplus T & 1 \dotplus U \end{pmatrix}.$$

Then $\mathfrak{X} \in \Gamma_{2n}$, and

$$\begin{pmatrix} 1 & \mathfrak{n}' & 0 & \mathfrak{n}' \\ \mathfrak{x} & A_2 & \mathfrak{n} & B_2 \end{pmatrix} \mathfrak{X}^{-1} = \begin{pmatrix} 1 & \mathfrak{n}' & & 0 & \mathfrak{n}' \\ \mathfrak{x} & I^{(j-1)} & 0 & \mathfrak{n} & 0 \end{pmatrix}.$$

But the right-hand matrix consists of the first j rows of $\mathfrak{N}_V$, where

$$V = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \mathfrak{x} & I^{(j-1)} & & 0 \\ \mathfrak{n} & 0 & & I^{(n-j)} \end{bmatrix}.$$

The theorem now follows by the use of Lemma 1.

THEOREM 2. *The array given by* (1) *is completable if and only if it is normal.*

Proof. As observed before, a completable array is obviously normal. Assume hereafter that the array given in (1) is normal, and (without loss of generality) that $j \geqq k$.

For, if the given array is a normal (j, k) array with $j<k$, then

$$\begin{pmatrix} C_1 & D_1 \\ -A_1 & -B_1 \end{pmatrix}$$

is a normal (k, j) array; if this latter array is completed to an element $Y \in \Gamma_{2n}$, then the original array is completable to

$$\begin{pmatrix} 0 & -I^{(n)} \\ I^{(n)} & 0 \end{pmatrix} \cdot Y \in \Gamma_{2n}.$$

By Theorem 1 we have

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \sim \begin{pmatrix} I^{(j)}\ 0 & 0 \\ C_2 & D_2 \end{pmatrix}$$

for some C_2, D_2. By Lemma 1 we see that (C_2, D_2) form a normal pair, and furthermore

$$(I^{(j)}\ 0)D_2' = (I^{(k)}\ 0)',$$

so that we have

$$D_2 = (I^{(k)}\ 0^{(k,j-k)}\ X^{(k,n-j)})$$

for some X. Now set

$$U = \begin{bmatrix} I^{(k)} & 0 & 0 \\ 0 & I^{(j-k)} & 0 \\ X' & 0 & I^{(n-j)} \end{bmatrix},$$

and observe that

$$\begin{pmatrix} I^{(j)}\ 0 & 0 & 0 & 0 \\ C_2 & I^{(k)} & 0 & X \end{pmatrix} \mathfrak{R}_U = \begin{pmatrix} I^{(j)} & 0 & 0 & 0 & 0 \\ C_3 & C_4 & I^{(k)} & 0 & 0 \end{pmatrix}$$

where C_3 is a $k \times j$ matrix, and C_4 a $k \times (n-j)$ matrix. Again using Lemma 1, the matrix

$$(C_3\ C_4)(I^{(k)}\ 0\ 0)'$$

must be symmetric, so that

$$C_3 = (C_{31}^{(k,k)}\ C_{32})$$

with symmetric C_{31}. But now

$$\begin{pmatrix} & I^{(j)} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_4 & I^{(k)} & 0 & 0 \end{pmatrix}$$

consists of the first j rows and the $(n+1)$st, $\cdots$, $(n+k)$th rows of

$$\begin{bmatrix} I^{(j)} & & 0 & & & \\ & 0 & I^{(n-j)} & & 0 & \\ C_{31} & C_{32} & C_4 & I^{(k)} & 0 & 0 \\ C_{32}' & 0 & 0 & 0 & I^{(j-k)} & 0 \\ C_4' & 0 & 0 & 0 & 0 & I^{(n-j)} \end{bmatrix} \in \Gamma_{2n}.$$

This proves the result.

2. We now turn to the problem of finding an expression for the general completion $\mathfrak{C}$ of a given normal (j, k) array, and we again assume without loss of generality that $j \geqq k$. If $\mathfrak{C}_0$ is a specific completion of the given array, then $\mathfrak{C}$ is a completion if and only if $\mathfrak{C}\mathfrak{C}_0^{-1}$ is an element of Γ_{2n} whose first j rows are given by

$$(I^{(j)}\ 0^{(j,2n-j)})$$

and its $(n+1)$st, $\cdots$, $(n+k)$th rows by

$$(0^{(k,n)}\ I^{(k)}\ 0^{(k,n-k)}).$$

Thus if $\mathfrak{X}$ represents the general such element of Γ_{2n}, then $\mathfrak{X}\mathfrak{C}_0$ is the general completion of the given array.

Let us write

$$\mathfrak{X} = \begin{array}{c@{}c} \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ E_1 & E_2 & E_3 & F_1 & F_2 & F_3 \\ 0 & 0 & 0 & I & 0 & 0 \\ G_4 & G_5 & G_6 & H_4 & H_5 & H_6 \\ G_1 & G_2 & G_3 & H_1 & H_2 & H_3 \end{pmatrix} & \begin{array}{l} k \\ j-k \\ n-j \\ k \\ j-k \\ n-j \end{array} \\ \begin{array}{cccccc} k & j-k & n-j & k & j-k & n-j \end{array} & \end{array}$$

where we have indicated the numbers of rows and columns in the various submatrices. Then $\mathfrak{X}\in\Gamma_{2n}$ if and only if $\mathfrak{X}$ is an integral matrix for which

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ E_1 & E_2 & E_3 \end{pmatrix}\begin{pmatrix} 0 & 0 & F_1' \\ 0 & 0 & F_2' \\ 0 & 0 & F_3' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ G_4 & G_5 & G_6 \\ G_1 & G_2 & G_3 \end{pmatrix}\begin{pmatrix} I & H_4' & H_1' \\ 0 & H_5' & H_2' \\ 0 & H_6' & H_3' \end{pmatrix}$$

are symmetric, and

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ E_1 & E_2 & E_3 \end{pmatrix}\begin{pmatrix} I & H_4' & H_1' \\ 0 & H_5' & H_2' \\ 0 & H_6' & H_3' \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_1 & F_2 & F_3 \end{pmatrix}\begin{pmatrix} 0 & G_4' & G_1' \\ 0 & G_5' & G_2' \\ 0 & G_6' & G_3' \end{pmatrix} = I.$$

These conditions give:

$F_1 = 0$, $F_2 = 0$, E_3F_3' symmetric, $G_4 = 0$, $G_1 = 0$, $G_5H_5' + G_6H_6'$ symmetric, $G_2H_2' + G_3H_3'$ symmetric, $G_5H_2' + G_6H_3' = H_5G_2' + H_6G_3'$,
$E_1 = 0$, $H_4 = 0$, $H_1 = 0$, $H_5 = I$, $H_2 = 0$,
$E_2H_5' + E_3H_6' - F_3G_6' = 0$, $E_2H_2' + E_3H_3' - F_3G_3' = I$.

Hence we have

$$\mathfrak{X} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & E_2 & E_3 & 0 & 0 & F_3 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & G_5 & G_6 & 0 & I & H_6 \\ 0 & G_2 & G_3 & 0 & 0 & H_3 \end{pmatrix},$$

with symmetric E_3F_3', $G_5+G_6H_6'$, G_3H_3', where

$$E_2 = F_3G_6' - E_3H_6', \qquad G_2 = H_3G_6' - G_3H_6', \quad \text{and} \quad E_3H_3' - F_3G_3' = I.$$

Therefore

$$\begin{pmatrix} E_3 & F_3 \\ G_3 & H_3 \end{pmatrix} \in \Gamma_{2(n-j)},$$

and so

$$\begin{pmatrix} E_2 \\ G_2 \end{pmatrix} = \begin{pmatrix} E_3 & F_3 \\ G_3 & H_3 \end{pmatrix} \begin{pmatrix} -H_6' \\ G_6' \end{pmatrix}$$

is true if and only if

$$\begin{pmatrix} -H_6' \\ G_6' \end{pmatrix} = \begin{pmatrix} H_3' & -F_3' \\ -G_3' & E_3' \end{pmatrix} \begin{pmatrix} E_2 \\ G_2 \end{pmatrix}.$$

We find easily that

$$\begin{aligned} G_5 + G_6H_6' &= G_5 + (-E_2'G_3 + G_2'E_3)(-H_3'E_2 + F_3'G_2) \\ &= G_5 + E_2'G_3H_3'E_2 + G_2'E_3F_3'G_2 - G_2'E_3H_3'E_2 - E_2'G_3F_3'G_2. \end{aligned}$$

But $E_3H_3' = I + F_3G_3'$, so

$$G_5 + G_6H_6' = G_5 - G_2'E_2 + \text{symmetric matrix},$$

and therefore $G_5+G_6H_6'$ is symmetric if and only if $G_5 - G_2'E_2 = S$ is symmetric. We now observe that

$$\mathfrak{X} = \begin{pmatrix} & I & & 0 \\ 0 & 0 & 0 & \\ 0 & S & G_2' & I \\ 0 & G_2 & 0 & \end{pmatrix} \begin{pmatrix} I & 0 & 0 & & & \\ 0 & I & 0 & & 0 & \\ 0 & E_2 & I & & & \\ & & & I & 0 & 0 \\ & 0 & & 0 & I & -E_2' \\ & & & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I \dotplus E_3 & 0 \dotplus F_3 \\ 0 \dotplus G_3 & I \dotplus H_3 \end{pmatrix}.$$

Hence, if $\mathfrak{C}_0$ is a specific completion of a given normal (j, k) array, the general completion equals $\mathfrak{C}_0$ multiplied on the left by the above expression for $\mathfrak{X}$, where

$$\begin{pmatrix} E_3 & F_3 \\ G_3 & H_3 \end{pmatrix}$$

is an arbitrary element of $\Gamma_{2(n-j)}$, where S is an arbitrary symmetric $(j-k) \times (j-k)$ matrix, and where G_2 and E_2 are arbitrary $(n-j) \times (j-k)$ matrices.

For the special case $j=n$, $k=0$ we obtain from the above Siegel's result that

$$\mathfrak{C} = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \mathfrak{C}_0,$$

with symmetric S.

We finally note that the above reasoning holds true for any Euclidean ring.

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