NOTE ON THE BESSEL POLYNOMIALS

BY

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1. This note can be considered as an addendum to the comprehensive study of the class of Bessel polynomials carried on by H. L. Krall and O. Frink [1]. In fact I study here the expansion of particular functions in terms of Bessel polynomials as well as the location of the zeros of these polynomials. Write $p_n(z)=\sum_k p_{nk}z^k$, so that [1, p. 101]

$$p_{nk} = 2^{-k}(n+k)!/(k!(n-k)!), \qquad 0 \leqq k \leqq n;\ n \geqq 0. \tag{1}$$

The first result is the following

THEOREM A. *Let $f(z)$ be a function regular in $|z-a| \leqq R$, where $R>0$ and a is any point of the plane. Then $f(z)$ can be expanded in a series of Bessel polynomials of the form $f(z) \sim \sum c_n\, p_n(z-a)$, where*

$$c_n = 2^n(2n+1)\sum_{\nu=0}^{\infty}(-2)^\nu f^{(n+\nu)}(a)/(\nu!(2n+\nu+1)!),$$

and the series is convergent uniformly in $|z-a| \leqq R$.

We first suppose that $f(z)$ is regular in $|z| \leqq R$ and prove that $f(z)$ can be expanded in a series $\sum \gamma_n\, p_n(z)$ where

$$\gamma_n = 2^n(2n+1)\sum_{\nu=0}^{\infty}(-2)^\nu f^{(n+\nu)}(0)/(\nu!(2n+\nu+1)!), \tag{2}$$

and that the series is uniformly convergent in $|z| \leqq R$. Theorem A follows readily when $z-a$ is written for z.

Since the set $\{p_n(z)\}$ of Bessel polynomials is *basic* in the sense of J. M. Whittaker [3, Chap. II], we shall appeal in the proof to the theory of basic series of polynomials as given by J. M. Whittaker and B. Cannon. In fact suppose that z^n admits the representation

$$z^n = \sum_i \pi_{ni}p_i(z), \tag{3}$$

so that the matrix (π_{ni}) is the unique reciprocal of the matrix (p_{ni}) (see Whittaker [3, T_{31}, p. 40]). Hence

$$p_{i,i}\pi_{i,i} = 1, \tag{4}$$

and

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$$\sum_{k=i}^{n} p_{nk}\pi_{ki} = 0, \qquad n > i,\ i \geqq 0. \tag{5}$$

Setting $n=m+i$ in (5) and inserting the values of (p_{nk}) from (1) we obtain

$$\pi_{m+i,i} = -\sum_{\nu=0}^{m-1} \frac{2^{m-\nu}(2i+m+\nu)!(m+i)!}{(i+\nu)!(m-\nu)!(2i+2m)!}\pi_{i+\nu,i}. \tag{6}$$

Applying (6) and the identity

$$\sum_{\nu=0}^{m}(-1)^\nu\binom{m}{\nu}\binom{k+m+\nu}{m-1} = 0$$

(this is in fact the coefficient of x^{k+m+1} in the expansion of $(1+x)^m(1+x)^{-m} \equiv 1$), we can easily deduce by induction that

$$\pi_{m+i,i} = \frac{(-2)^m(m+i)!(2i+1)!}{m!i!(2i+m+1)!}\pi_{i,i}; \qquad m,\ i \geqq 0.$$

Substituting for $\pi_{i,i}$ from (4) it follows that

$$\pi_{n,i} = (-1)^{n-i}2^n n!(2i+1)/((n-i)!(n+i+1)!). \tag{7}$$

We now substitute for z^n from (3) in the Taylor expansion $\sum z^n f^{(n)}(0)/n!$ of $f(z)$ about the origin to get formally the series $\sum \gamma_n p_n(z)$, where

$$\gamma_n = \sum_{\nu=0}^{\infty} \pi_{n+\nu,n} f^{(n+\nu)}(0)/(n+\nu)!.$$

Hence inserting the value of $\pi_{n+\nu,n}$ from (7), (2) follows at once.

In order to prove that the series $\sum \gamma_n p_n(z)$ is convergent in $|z| \leqq R$ we form the sum (see [3, Chap. II, III])

$$\omega_n(R) = \sum_i |\pi_{ni}|\, M_i(R),$$

where $M_i(R) \equiv \max_{|z|=R} |p_i(z)| = \sum_{k=0}^{i} p_{ik}R^k$. Applying (1) and (7) we obtain after simple reduction

$$\begin{aligned}\omega_n(R) &= 2^n\sum_{k=0}^{n}\binom{n}{k}(R/2)^k\sum_{j=0}^{n-k}\binom{n-k}{j}\frac{(2k+2j+1)(2k+j)!}{(n+k+j+1)!} \\ &< (2n+1)R^n\sum_{k=0}^{n}\binom{n}{k}(4/R)^k/k! = (2n+1)B_nR^n,\end{aligned} \tag{8}$$

say. Effecting the transformation $y=x(1+x)^{-1}$ on the function $x \exp(4x/R) = \sum_{n=0}^{\infty}(4/R)^n x^{n+1}/n!$ it follows that

$$F(y) \equiv (y/(1-y)) \exp\{4y/(R(1-y))\} = \sum_{n=0}^{\infty} B_n y^{n+1}.$$

This function is regular in $|y|<1$; hence by Cauchy's inequality we have

$$B_n < K/\alpha^{n+1} \qquad (0<\alpha<1),$$

where $K=\max_{|y|=\alpha}|F(y)|<\infty$. Inserting this in (8) and making n tend to infinity we obtain

$$\lambda(R) \equiv \limsup_{n\to\infty} \{\omega_n(R)\}^{1/n} \leqq R/\alpha,$$

and since α can be taken as near to 1 as we please we conclude that $\lambda(R)=R$. According to Cannon [3, T_5, p. 11] we infer that the series $\sum\gamma_n p_n(z)$ is uniformly convergent in $|z|\leqq R$, as required.

2. As for the location of the zeros of Bessel polynomials the following result is established.

THEOREM B. *All the zeros of the Bessel polynomial $p_n(z)$, for $n>1$, lie within or on the circle $|z|=((n-1)/(2n-1))^{1/2}$.*

We shall suppose that $n\geqq 3$, as the zeros of $p_2(z)$ are of modulus $(1/3)^{1/2}$. Write $r=((2n-1)/(n-1))^{1/2}$. We shall first show that the real zeros of $p_n(z)$, if any, are of modulus less than $1/r$. Let x be any real number not less than $1/r$, then the formula (1) for the coefficients (p_{nk}) yields

$$p_{nk}x^k < p_{n,k+1}x^{k+1}, \qquad 0\leqq k\leqq n-4, \tag{9}$$

and furthermore

$$\begin{aligned} R_n(x) &\equiv p_{nn}x^n - p_{n,n-1}x^{n-1} + p_{n,n-2}x^{n-2} - p_{n,n-3}x^{n-3} \\ &= (n-1)^{-1}p_{n,n-2}x^{n-3}\{(2n-1)(x^3-x^2)+(n-1)x-(n-2)/3\}. \end{aligned}$$

It can be easily shown that the cubic polynomial inside the brackets has, for $n\geqq 3$, one real positive zero less than $1/r$. Hence for $x\geqq 1/r$ we have

$$R_n(x) > 0, \qquad (n\geqq 3). \tag{10}$$

Since the coefficients of $p_n(z)$ are all positive we need only consider $p_n(-x)$ where $x\geqq 1/r$. Thus (9) and (10) yield

$$p_n(-x) = 1 + \sum_{k=0}^{n/2-3}(-p_{n,2k+1}x^{2k+1} + p_{n,2k+2}x^{2k+2}) + R_n(x) > 0 \qquad (n \text{ even})$$

and

$$p_n(-x) = \sum_{k=0}^{(n-1)/2-2}(p_{n,2k}x^{2k} - p_{n,2k+1}x^{2k+1}) - R_n(x) < 0 \qquad (n \text{ odd}).$$

Hence all the real roots of $p_n(z)$ lie in $-1/r<x<0$.

Now write $q_n(z)=z^np_n(1/z)$ and consider the polynomial $q_n(rz)$. Since we shall always be concerned with this particular polynomial we may suppress the suffix n and write

$$q_n(rz) \equiv g(z) = \sum_{k=0}^{n} a_k z^k,$$

so that $a_k = p_{n,n-k}r^k$. Inserting the values of the coefficients (p_{nk}) from (1) we can easily observe that

$$\begin{aligned} &0 < a_1 = ra_0, \\ &a_k > ra_{k+1} > 0, \qquad 1 \leqq k \leqq n-1. \end{aligned} \tag{11}$$

We form, with M. Marden [2, p. 148], the successively derived coefficients $a_k^{(j)}$ given by

$$a_k^{(0)} = a_k,\ a_k^{(j+1)} = a_0^{(j)} a_k^{(j)} - a_{n-j}^{(j)} a_{n-j-k}^{(j)}; \quad 0 \leqq k < n-j;\ 0 \leqq j \leqq n-1. \tag{12}$$

As for the first derived coefficients $a_k^{(1)}$, the relations (11) yield the following: for $k = n-1$,

$$a_{n-1}^{(1)} = a_0 a_{n-1} - a_n a_1 > 0,$$

and for $k > 0$,

$$a_k^{(1)} - a_{k+1}^{(1)} = a_0(a_k - a_{k+1}) + a_n(a_{n-k-1} - a_{n-k}) > (1 - 1/r)a_k^{(1)},$$

so that

$$a_k^{(1)} > ra_{k+1}^{(1)} > r^{n-k-1} a_{n-1}^{(1)} > 0, \qquad 1 \leqq k \leqq n-2. \tag{13}$$

Finally, for $k = 0$,

$$a_1^{(1)} - a_0^{(1)} = a_0(a_1 - a_0) - a_n(a_{n-1} - a_n) < (r-1)a_0^{(1)},$$

so that

$$0 < a_1^{(1)} < ra_0^{(1)}. \tag{14}$$

A comparison between the relations (13) and (14) for the case $j = 1$, on the one hand, and the relations (11) for $j = 0$, on the other hand, suggests that for the jth derived coefficients we should have

$$\begin{aligned} &0 < a_1^{(j)} < ra_0^{(j)}, \\ &a_k^{(j)} > ra_{k+1}^{(j)} > 0, \qquad 1 \leqq k \leqq n-j-1. \end{aligned} \tag{15}$$

In fact (15) is valid for $j = 1$, in view of (13) and (14). Moreover supposing (15) to be true for some $j < n-2$, then the method used in deriving (13) and (14) from (11) can be similarly applied to (15) to derive easily the following relations

$$\begin{aligned} &0 < a_1^{(j+1)} < ra_0^{(j+1)}, \\ &a_k^{(j+1)} > ra_{k+1}^{(j+1)} > 0, \qquad 1 \leqq k < n-j-1, \end{aligned}$$

so that (15) is true for $1 \leqq j \leqq n-2$. We may apply (15) with $j=n-2$ to deduce that $a_0^{(n-1)} > 0$, so that

$$a_0^{(j)} > 0, \qquad 1 \leqq j \leqq n - 1.$$

Applying Marden's theorem concerning the number of zeros of a polynomial inside the unit circle in terms of the coefficients $a_0^{(j)}$ [2, Theorem (42.1), p. 150] we infer that the polynomial $g(z)$ will have at most one zero inside the unit circle. Consequently the polynomial $p_n(z)$ will have at least $n-1$ of its zeros within or on the circle $|z| = 1/r$. Suppose that the number of zeros of $p_n(z)$ within or on the circle $|z| = 1/r$ is exactly $n-1$ and that the remaining zero β is outside the circle. By the first part of the proof β cannot be real and hence its conjugate $\bar{\beta}$, which is also a zero of $p_n(z)$, lies outside the circle $|z| = 1/r$. We conclude from this contradiction that all the zeros of $p_n(z)$ lie within or on the circle $|z| = 1/r$, and this completes the proof of Theorem B.

References

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