ANALYTIC FUNCTIONS OF CLASS H_p

BY

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I. Introduction

1.1. The classes H_p $(0<p<\infty)$, consisting of those functions f which are analytic in the interior U of the unit circle and for which

$$M_r^p(|f|) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \tag{1.1.1}$$

is bounded for $0<r<1$, were introduced into analysis by G. H. Hardy [10]. The principal facts concerning the behavior of these functions at the boundary were established by F. Riesz [17; 22, p. 162] with the aid of an interesting factorization theorem. Macintyre, Rogosinski, and H. S. Shapiro [11; 19] have treated linear extremum problems (for $p \geqq 1$) in great detail. Walters [21] has discussed the structure of the linear space H_p for $0<p<1$.

These "classical" H_p classes will be denoted by $H_p(U)$ in the sequel. It is the purpose of this paper to define analogous classes of functions on arbitrary domains (i.e., connected open sets in the extended complex plane) and to study their properties. Most of the results of Parts I and II carry over to Riemann surfaces, but in view of the problems discussed in Parts III and IV we shall content ourselves with the plane case.

We shall base our definition on the observation that $|f|^p$ is subharmonic if f is analytic. The easiest proof of this well-known fact is probably the following: log $|f|$ is subharmonic, $|f|^p = \exp(p \log |f|)$, and every convex increasing function of a subharmonic function is again subharmonic [16, p. 15].

1.2. Definition(1). For any domain D, and any value of p(2), we define $H_p(D)$ as the set of all functions f which are single-valued and analytic in D, and for which there exists a function u, harmonic in D, such that

$$|f(z)|^p \leqq u(z) \qquad (z \in D).$$

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(1) After the original version of this paper was sent to the editors, the author's attention was drawn to Parreau's thesis, where the same definition is introduced [14, p. 178]. Although the viewpoints of the two papers are quite different, some of Parreau's ideas reappear in Part I and the beginning of Part II. The author is indebted to the referee for the detailed references to [14] which are included in the text.

(2) Unless the contrary is explicitly stated, it will always be understood that $0<p<\infty$.

It is well known [16, p. 33; 18, p. 357] that if a subharmonic function has a harmonic majorant in D, then there exists a *least harmonic majorant*. Hence, for every $f \in H_p(D)$ there exists a function u_f, harmonic in D, such that $|f|^p \leqq u_f$, and such that $u_f \leqq u$ whenever $|f|^p \leqq u$ and u is harmonic in D.

If $D = U$, it is easy to see that our definition coincides with the classical one; in fact, the following theorem shows that even in the general case we could define $H_p(D)$ by requiring certain integrals to be bounded (compare [14, p. 135]):

1.3. THEOREM. *Fix a point $t \in D$, and let f be single-valued and analytic in D. Let Δ be any domain with smooth boundary Γ, such that $\Delta \cup \Gamma \subset D$, and $t \in \Delta$. Then $f \in H_p(D)$ if and only if there exists a constant M, independent of Δ, such that*

$$\frac{1}{2\pi}\int_{\Gamma} |f|^p \frac{\partial G}{\partial n} ds \leqq M. \tag{1.3.1}$$

Here G is the Green's function of Δ, with pole at t, and the derivative is taken along the interior normal; a "smooth" boundary is the union of a finite number of continuously differentiable curves.

Proof. Let v be harmonic in Δ, with boundary values $|f|^p$; then the left member of (1.3.1) is equal to $v(t)$. If $f \in H_p(D)$, let u be a harmonic majorant of $|f|^p$. Since $|f|^p$ is subharmonic, $v(t) \leqq u(t)$, so that (1.3.1) holds with $M = u(t)$.

Conversely, let $\{\Delta_k\}$ be an increasing sequence of domains, satisfying the conditions of the theorem, whose union is D. The associated harmonic functions v_k form an increasing sequence which is bounded at t. By a well-known theorem of Harnack, the sequence $\{v_k\}$ converges to a harmonic function u, and it is easy to see that this u is a majorant (in fact, the least harmonic majorant) of $|f|^p$ in D.

1.4. ANALYTIC INVARIANCE OF $H_p(D)$. The classes $H_p(D)$ are invariant under conformal one-to-one transformations of D. In fact, they enjoy a stronger property, which Ahlfors and Beurling [2, p. 102] have called analytic invariance, and which is as follows:

Let ψ be single-valued and meromorphic in a domain D_1 and suppose $\psi(D_1) \subset D$. For any $f \in H_p(D)$, define

$$f_1(z) = f(\psi(z)) \qquad (z \in D_1). \tag{1.4.1}$$

Then $f_1 \in H_p(D_1)$.

This is clear if we note that for any harmonic majorant u of $|f|^p$ in D, the function u_1 defined by

$$u_1(z) = u(\psi(z)) \qquad (z \in D_1)$$

is a harmonic majorant of $|f_1|^p$ in D_1.

In particular, let D be a domain in the finite plane, with at least three boundary points (since the case in which the boundary of D has capacity zero is of no interest, by Theorem 1.6, this assumption involves no loss of generality). The principal uniformization theorem [6, p. 70; 13, p. 8] shows that there is a function ψ, regular in U, whose range is precisely D. This mapping function ψ is invariant under an infinite group $\mathcal{G}$ (the so-called automorphic group of D) of linear fractional transformations of U onto U, i.e.,

$$\psi(g(z)) = \psi(z) \qquad (z \in U,\ g \in \mathcal{G}). \tag{1.4.2}$$

The transformation (1.4.1) then maps $H_p(D)$ into a subset of $H_p(U)$, consisting of functions invariant under $\mathcal{G}$. In Part II we shall define a norm in $H_p(D)$, and we shall see that (1.4.1) actually induces a norm-preserving isomorphism between $H_p(D)$ and a closed subspace of $H_p(U)$ (Theorem 2.6).

1.5. For better orientation it is interesting to consider two other classes of single-valued functions analytic in D: the class $H_\infty(D)$, consisting of all bounded functions, and the class $\text{Log}^+(D)$, consisting of those functions f for which $\log^+|f|$ has a harmonic majorant in D (Parreau considers an analogous class of meromorphic functions [14, p. 180]).

It is clear that these two classes are also analytically invariant, and that $\text{Log}^+(D)$ corresponds to a subset of the analytic functions of bounded characteristic [13, p. 157] in U, under the transformation (1.4.1). Moreover, Theorem 1.3 holds for $\text{Log}^+(D)$ if $|f|^p$ is replaced by $\log^+|f|$ in (1.3.1), since $\log^+|f|$ is subharmonic. Let us note that

$$\text{Log}^+(D) \supset H_p(D) \supset H_q(D) \supset H_\infty(D) \qquad (p < q)$$

for any domain D. We shall call any of these classes *trivial* if it contains nothing but the constant functions.

It is a well-known unsolved problem, posed by Painlevé [1], to find necessary and sufficient conditions on D under which $H_\infty(D)$ is nontrivial. If B is the boundary of D, it is necessary that the linear measure of B be positive [1, p. 2]; if B is a subset of an analytic arc, this is also sufficient [2, p. 122]; but little is known about the general case.

For $\text{Log}^+(D)$, on the other hand, a very simple criterion exists:

1.6. THEOREM. $\text{Log}^+(D)$ *is trivial if and only if* cap $B=0$.

Here cap B means the logarithmic capacity of B.

Proof. It is convenient to assume that the point at infinity is not in D (if D had no boundary point, there would be no nonconstant analytic functions on D).

If cap $B>0$, let ψ be analytic in U, with range D. A theorem of Nevanlinna [13, p. 201] shows that $\psi \in \text{Log}^+(U)$. Let v be the least harmonic majorant of $\log^+|\psi|$ in U. We wish to show that v is invariant under the automorphic group $\mathcal{G}$. Since $v(g(z))$ is the least harmonic majorant of

$\log^+ |\psi(g(z))|$, which is equal to $\log^+ |\psi(z)|$, it follows immediately that $v(g(z))=v(z)$ for every $z\in U$ and every $g\in \mathcal{G}$. Hence the formula

$$u(w) = v(\psi^{-1}(w)) \qquad (w \in D)$$

defines a single-valued harmonic function u in D which is a majorant of $\log^+ |w|$; it follows that the identity function $f(w)=w$ is a member of $\mathrm{Log}^+ (D)$.

Conversely, if cap $B=0$, suppose that $f\in \mathrm{Log}^+ (D)$. Let u be a harmonic majorant of $\log^+ |f|$. The function $v=-u$ can be defined on B so as to be subharmonic in the extended plane [5, p. 31]. Since v is upper semi-continuous and bounded above, v attains its maximum, which is impossible unless v is constant. Hence f is bounded in D, and therefore constant [13, p. 132]. Thus $\mathrm{Log}^+ (D)$ is trivial.

1.7. The following two questions may now be asked.

(Q_1) For what values of p (if any) is $H_p(D)$ nontrivial whenever cap $B>0$?

(Q_2) For what values of p (if any) is $H_p(D)$ trivial whenever $H_\infty(D)$ is trivial?

Both questions seem to be very difficult. An attempt to throw some light on (Q_2) by the methods of Ahlfors and Beurling [2] led to the study of an extremum problem, analogous to Schwarz's lemma, in $H_1(D)$. The results are summarized in Theorems 4.10 and 4.12, and show some rather unusual features, reminiscent of eigenvalue problems. But in spite of the intrinsic interest of this extremum problem, the solution seems to yield no information about (Q_2).

Although it has no a priori bearing on the analytic case, we may note at this point that a question analogous to (Q_2) may be asked about harmonic functions, and that the answer is: for all $p>1$ [14, p. 164].

1.8. Removable singularities. The problem of removable singularities is closely connected with the above questions. The following theorem seems to be the best known result in this direction [14, p. 182]:

Theorem. *If N is a compact subset of D, if* cap $N=0$, *and if $f\in H_p(D-N)$, then f can be defined on N so that $f\in H_p(D)$.*

(Since cap $N=0$, $D-N$ is a domain, so that it makes sense to speak of $H_p(D-N)$.)

For $H_\infty(D)$, a much stronger result holds, since singularities distributed over sets of linear measure zero are removable [1, p. 2]. On the other hand, the analogous proposition is false for $\mathrm{Log}^+ (D)$, where even isolated singularities may not be removable: Let N consist of the point $z=0$, let $f(z)=1/z$. Then $f\in \mathrm{Log}^+ (U-N)$, but the singularity at $z=0$ is not removable.

II. The linear spaces $H_p(D)$

2.1. The H_p-norm. Let us fix a point $t\in D$. If f is single-valued and analytic in D, we define

$$\|f\|_p = (u(t))^{1/p}, \tag{2.1.1}$$

where u is the least harmonic majorant of $|f|^p$, provided $f \in H_p(D)$; the right member of (2.1.1) is interpreted as $+\infty$ if $f \notin H_p(D)$. Thus $H_p(D)$ is characterized by the inequality $\|f\|_p < +\infty$. If $D = U$, we shall take $t = 0$.

Let $\{D_k\}$ be an increasing sequence of domains with smooth boundaries C_k, such that $t \in D_1$ and $D = \bigcup D_k$. Let G_k be Green's function of D_k, with pole at t. A glance at the proof of Theorem 1.3 will show that

$$\|f\|_p = \lim_{k\to\infty} \left\{\frac{1}{2\pi}\int_{C_k} |f|^p \frac{\partial G_k}{\partial n} ds\right\}^{1/p}, \tag{2.1.2}$$

the limit being independent of the choice of $\{D_k\}$ (compare [14, pp. 178, 137]).

If $p \geqq 1$, Minkowski's inequality may be applied to the integrals in (2.1.2), and shows that $\|f\|_p$ is a genuine norm, i.e., the triangle inequality holds:

$$\|f + g\|_p \leqq \|f\|_p + \|g\|_p \qquad (p \geqq 1). \tag{2.1.3}$$

This ceases to be true if $0 < p < 1$; in that case we have, however [22, p. 67],

$$\|f + g\|_p^p \leqq \|f\|_p^p + \|g\|_p^p \qquad (0 < p < 1). \tag{2.1.4}$$

In any case, the above inequalities, combined with the obvious homogeneity of the norm, show that $H_p(D)$ is a complex linear space.

2.2. THE RELATION BETWEEN $H_p(D)$ AND $H_p(U)$. Let ψ be analytic in U, with range D, such that $\psi(0) = t$, and let $\mathcal{G}$ be the group under which ψ is invariant (compare 1.4). If $f \in H_p(D)$, define

$$f_1(z) = f(\psi(z)) \qquad (z \in U). \tag{2.2.1}$$

Then $f_1 \in H_p(U)$, and the same transformation carries the least harmonic majorant of $|f|^p$ into a harmonic majorant of $|f_1|^p$. Thus

$$\|f_1\|_p \leqq \|f\|_p \qquad (f \in H_p(D)). \tag{2.2.2}$$

Moreover, f_1 is invariant under $\mathcal{G}$.

Conversely, suppose $f_1 \in H_p(U)$, and f_1 is invariant under $\mathcal{G}$. The argument used in the proof of Theorem 1.6 shows that the least harmonic majorant of $|f_1|^p$ is invariant under $\mathcal{G}$. Thus the transformation (2.2.1) defines a single-valued function f in D which satisfies

$$\|f\|_p \leqq \|f_1\|_p \qquad (f_1 \in H_p(U)). \tag{2.2.3}$$

Combining these two inequalities, we see that *there is a natural norm-preserving isomorphism between $H_p(D)$ and the subspace of $H_p(U)$ consisting of those functions which are invariant under $\mathcal{G}$*, a relationship which is also indicated in Parreau's paper [14, p. 179]. Theorem 2.6 will describe the situation more explicitly.

2.3. Lemma. *Let K be a compact subset of D, and let u be positive and harmonic in D. There exists a constant M, depending only on K, D, and t, such that $u(z) \leqq Mu(t)$ for all $z \in K$.*

Proof. Although the complete inverse image of K under ψ is not compact (unless D is simply connected), there is a compact subset K_1 of U such that $\psi(K_1) \supset K$. It is therefore enough to prove the lemma for the case $D=U$; in that case we can use Poisson's formula, and conclude that

$$u(z) \leqq \frac{1+r}{1-r} u(0) \qquad (|z| = r < 1).$$

The lemma follows. A direct proof, not depending on the uniformization theorem, is of course also quite easy.

2.4. Corollary. (a) *Let K be a compact subset of D. There exists a constant M, depending on D, K, t, p, but not on f, such that*

$$|f(z)| \leqq M\|f\|_p \qquad (f \in H_p(D),\ z \in K). \tag{2.4.1}$$

(b) *If $\|f_n - f\|_p \to 0$ as $n \to \infty$, then $f_n(z) \to f(z)$ uniformly on every compact subset of D.*

2.5. The strong topology of $H_p(D)$. We define a subset S of $H_p(D)$ to be open if for every $f_0 \in S$ there is an $r>0$ such that $f \in S$ whenever $\|f-f_0\|_p < r$.

If $p \geqq 1$, this gives the usual topology induced by the metric. If $0<p<1$, $H_p(D)$ is not a metric space, but the above topology nevertheless makes it into a linear Hausdorff space, i.e., a linear space which satisfies the Hausdorff separation axiom (distinct points have disjoint neighborhoods) and in which addition and scalar multiplication are continuous operations. For $H_p(U)$ this was pointed out by Walters [21], and the general case follows from 2.2.

If X is a linear topological space, a sequence $\{x_n\}$ of elements in X is said to be a Cauchy sequence if $\lim (x_n - x_m) = 0$ as $n, m \to \infty$. If every Cauchy sequence converges, then X is said to be complete. This definition is inserted here since $H_p(D)$ is not metric for $p<1$.

Let C be the boundary of U. By $L_p(C)$ we mean the space of measurable complex-valued functions on C, normed by

$$\|f\|_p = \left| \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right|^{1/p}.$$

2.6. Theorem. (a) *There is a natural norm-preserving isomorphism between $H_p(D)$ and a closed subspace of $H_p(U)$.*

(b) *There is a natural norm-preserving isomorphism between $H_p(U)$ and a closed subspace of $L_p(C)$.*

(c) *$H_p(D)$ is a complete separable linear Hausdorff space.*

(d) *If $p \geqq 1$, $H_p(D)$ is a Banach space; if $p>1$, $H_p(D)$ is uniformly convex.*

(e) *Although the norm $\|f\|_p$ depends on the choice of the point $t \in D$, the induced topology does not.*

Proof. (a) follows from 2.2 if we can show that the set of all functions $f \in H_p(U)$ which are invariant under $\mathcal{G}$ forms a *closed* subspace of $H_p(U)$; but this is an immediate consequence of 2.4(b).

(b) is well known: if $f \in H_p(U)$, then f has radial boundary values $f^*(e^{i\theta})$ p.p. on C, $f^* \in L_p(C)$, $\|f^*\|_p = \|f\|_p$, and the isomorphism is the correspondence $f \leftrightarrow f^*$ [17; 21].

Part (c) of the theorem follows from (a) and (b).

The first assertion of (d) is a trivial consequence of (c) and (2.1.3). The second follows from the fact that $H_p(D)$ is isometrically isomorphic to a closed subspace of $L_p(C)$, and the latter space is known to be uniformly convex if $p>1$ [7].

Finally, Lemma 2.3 shows that if we define two norms, using two different points of reference, the ratio of these two norms will have finite and positive upper and lower bounds; (e) follows.

2.7. In view of the preceding theorem, one is naturally led to ask: for what domains D, and for what values of p, does $H_p(U)$ contain a nontrivial subspace invariant under $\mathcal{G}$? This suggests that a study of the automorphic groups may yield some information on the problems mentioned in 1.7.

2.8. Linear functionals. If T is a linear functional on $H_p(D)$, we define, as usual

$$\|T\| = \sup \left(|Tf| / \|f\|_p \right) \qquad (f \in H_p(D)). \tag{2.8.1}$$

T is bounded if $\|T\|$ is finite. It is then an easy matter to verify that $H_p(D)^*$, the space of all bounded linear functionals on $H_p(D)$, is a Banach space, even if $p<1$.

Although $L_p(C)$ has no nonzero bounded linear functionals if $p<1$, $H_p(D)$ does admit such functionals [21]:

2.9. Theorem. *The functionals $T_{m,z}$ defined on $H_p(D)$ by*

$$T_{m,z}f = f^{(m)}(z) \qquad (m = 0, 1, 2, \cdots ; z \in D)$$

are bounded.

Proof. Cover z with a closed circular disc of radius r which lies in D. Then

$$2\pi T_{m,z}f = m! \int_0^{2\pi} f(z + re^{i\theta})(re^{i\theta})^{-m} d\theta.$$

By 2.4, we can choose M such that $|f(z+re^{i\theta})| \leqq M\|f\|_p$ for $0 \leqq \theta < 2\pi$. Hence $\|T_{m,z}\| \leqq Mm! r^{-m}$.

Corollary. *There exists a countable set of bounded linear functionals on*

$H_p(D)$ which separates elements of $H_p(D)$.

For instance, the sequence $\{T_{m,z}\}$, $m=0, 1, 2, \cdots$, for fixed $z \in D$, has the property that $T_{m,z}f=0$ for all m if and only if $f=0$.

2.10. WEAK CONVERGENCE. If $f_n \in H_p(D)$ and $f \in H_p(D)$, we say that $f_n \to f$ weakly provided that $Tf_n \to Tf$, as $n \to \infty$, for every $T \in H_p(D)^*$.

We saw in 2.4 that strong convergence implies uniform convergence on all compact subsets of D. That the converse is not true, even if $\{\|f_n\|_p\}$ is bounded, is shown by the sequence $f_n(z)=z^n$ $(z \in U)$.

Walters [21, p. 804] has proved that weak convergence in $H_p(U)$ $(0<p<1)$ implies uniform convergence on every compact subset of U. His proof, which relies on a category argument in the Banach space $H_p(U)^*$, applies without change in any domain D. For $p \geqq 1$, we find the following relations between weak and uniform convergence:

2.11. THEOREM. *If $f_n \to f$ weakly in $H_p(D)$ $(p \geqq 1)$, then $\{\|f_n\|_p\}$ is bounded, and $f_n(z) \to f(z)$ uniformly on every compact subset of D. The converse is true if $p>1$, but is false for $H_1(U)$.*

For general domains D, the analogous negative conclusion concerning $H_1(D)$ is not yet established.

Proof. In any Banach space, the norms of the elements of a weakly convergent sequence are bounded. Thus $\{f_n(z)\}$ is uniformly bounded on every compact subset of D, by 2.4(a). Applying Theorem 2.9, with $m=0$, we see that $f_n(z) \to f(z)$ for every $z \in D$. The direct part of the theorem now follows from Vitali's theorem.

Since $H_p(D)$ can be regarded as a subspace of $H_p(U)$, it is sufficient to prove the converse for the case $D=U$. Let us suppose then that $\|f_n\|_p<1$, and that $f_n(z) \to 0$ for every $z \in U$; uniform convergence on compact subsets follows, as above. If

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k \qquad (n = 1, 2, 3, \cdots; z \in U),$$

Cauchy's formula shows that

$$a_{n,k} \to 0 \quad \text{as} \quad n \to \infty \qquad (k = 0, 1, 2, \cdots). \tag{2.11.1}$$

We imbed $H_p(U)$ in $L_p(C)$, whose conjugate space is $L_q(C)$, with $1/p+1/q=1$, noting that every functional on $H_p(U)$ can be extended to $L_p(C)$. If P is a polynomial in z and $1/z$, (2.11.1) shows that

$$\int_0^{2\pi} f_n(e^{i\theta})P(e^{i\theta})d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

Since these polynomials are dense in $L_q(C)$ (here we use the fact that $q<\infty$), it follows that $Tf_n \to 0$ as $n \to \infty$, for every $T \in H_p(U)$.

We shall now construct an example which shows that the converse is false in $H_1(U)$. Let F_n map the upper half of U onto a rectangle with vertices at $1/2$, i/n, 0, such that

$$F_n(1) = 1/2, \qquad F_n(e^{\pi i/n}) = i/n, \qquad F_n(-1) = 0.$$

Extend F_n to U, by reflection in the real axis, and put $f_n(z) = F_n'(z)$. F_n maps U onto a rectangle R_n whose circumference tends to 1 as $n \to \infty$. Hence

$$\|f_n\|_1 = (1/2\pi) \int_0^{2\pi} |f_n(e^{i\theta})| \, d\theta \to 1/2\pi.$$

Since $F_n(e^{i\theta}) \to 0$ boundedly p.p. on C, Cauchy's formula shows that $F_n(z) \to 0$ uniformly in $|z| < r < 1$; hence $f_n(z) \to 0$ uniformly in $|z| < r < 1$, for every $r < 1$.

Thus, if $\{f_n\}$ did converge weakly, the weak limit would be zero, and Tf_n would tend to zero for every $T \in H_1(U)^*$. But if

$$Tf = \int_{-1}^{1} f(z)dz = -i \int_0^{\pi} f(e^{i\theta})e^{i\theta}d\theta,$$

then

$$Tf_n = F_n(1) - F_n(-1) = 1/2 \qquad (n = 1, 2, 3, \cdots),$$

which is a contradiction.

Actually, $\{f_n\}$ does not even contain a weakly convergent subsequence.

2.12. Linear extremum problems. If $T \in H_p(D)^*$, let us consider the problem of maximizing $|Tf|$ under the restriction $\|f\|_p \leqq 1$.

If $p > 1$, the uniform convexity of $H_p(D)$ implies that there always exists a unique $f_0 \in H_p(D)$ such that $Tf_0 = \|T\|$ and $\|f_0\|_p = 1$ (for the uniqueness we must of course eliminate the trivial case $T = 0$) [15, p. 249].

If $p = 1$, there may not be an extremal function, and if there is one it need not be unique. Examples of this, for $H_1(U)$, may be found in [11] and [19]. Part IV is devoted to a detailed study of a particular problem of this sort in $H_1(D)$, where D has finite connectivity.

If $p > 1$, $H_p(D)$ is reflexive, since it is uniformly convex [15]. The example given in 2.11 shows that the unit sphere of $H_1(U)$ is not weakly compact, so that $H_1(U)$ is not reflexive.

2.13. The case $p = 2$. For f and $g \in H_2(D)$, we define

$$2\pi(f, g) = \lim_{k \to \infty} \int_{C_k} f\bar{g}\left(\frac{\partial G_k}{\partial n}\right) ds, \tag{2.13.1}$$

with the same notation as was used in (2.1.2). To prove that this limit exists, we multiply the identity

$$4f\bar{g} = |f + g|^2 - |f - g|^2 + i|f + ig|^2 - i|f - ig|^2$$

by $\partial G_k/\partial n$, integrate over C_k, and apply (2.1.2).

It is clear that $H_2(D)$ is now a Hilbert space, with inner product (f, g); we note that (f, g) has been defined without the use of Theorem 2.6.

For fixed $x \in D$, $f(x)$ is a bounded linear functional on $H_2(D)$. Hence there is a unique function $R_x \in H_2(D)$, which we call the H_2-kernel function of D [2, p. 40], with the reproducing property

$$f(x) = (f, R_x). \tag{2.13.2}$$

If $\{\phi_n\}$ is a complete orthonormal set of functions in $H_2(D)$, we have

$$R_x(z) = \sum_{n=1}^{\infty} \bar{\phi}_n(x)\phi_n(z) \qquad (z \in D), \tag{2.13.3}$$

the series converging uniformly on every compact subset K of D. To prove this, we note that $(R_x, \phi_n) = \bar{\phi}_n(x)$, by (2.13.2), so that $\sum \bar{\phi}_n(x)\phi_n$ converges to R_x, in the H_2 norm. By 2.4(b) the convergence is uniform on K.

A more detailed study of this conformally invariant kernel function might well be interesting. At the present, let us merely note the identity

$$R_t(z) = \sum_{n=1}^{\infty} \bar{\phi}_n(t)\phi_n(z) = 1 \qquad (z \in D) \tag{2.13.4}$$

where t is the distinguished point used in the definition of the norm. (2.13.4) follows from the uniqueness of the kernel function, and from the fact that $(f, 1) = f(t)$.

III. Domains with analytic boundary

3.1. Preliminaries. Let D be a bounded domain, whose boundary B consists of k analytic simple closed curves. For fixed $t \in D$, let $G(z) = G(z, t)$ be Green's function for D, with pole at t, and put

$$P(z) = P(z, t) = G(z) + iH(z),$$

where H is the harmonic conjugate of G. The function P is multiple-valued, but has a single-valued derivative $P'(z)$ which is analytic in D, except for a pole with principal part $-(z-t)^{-1}$.

Since $G(z) = 0$ on B, the reflection principle shows that P' is analytic on B; we have the important differential identity

$$iP'(z)dz = (\partial G/\partial n)ds > 0 \qquad (z \in B), \tag{3.1.1}$$

which, combined with the argument principle, implies that P' has precisely $k-1$ zeros in D. These are called the critical points of G [13, p. 31].

We let $A(D)$ denote the set of all functions which are single-valued and analytic on the closure of D. By $L_p(B)$ we mean the space of complex-valued measurable functions f^* on B, normed by

$$\|f^*\|_p = \left\{ \frac{1}{2\pi} \int_B |f^*|^p \frac{\partial G}{\partial n} ds \right\}^{1/p}. \tag{3.1.2}$$

If $p \geqq 1$, we let $H_p(B)$ denote the (evidently closed) subspace of $L_p(B)$ consisting of those f^* for which

$$\int_B f^*(z)\phi(z)dz = 0 \tag{3.1.3}$$

whenever $\phi \in A(D)$.

The following theorem describes the behavior of functions of class $H_p(D)$ near the boundary.

3.2. THEOREM. (a) *If $f \in H_p(D)$, there is a function f^*, defined on B, such that f has nontangential boundary values f^* almost everywhere on B.*

(b) *The mapping $\gamma: f \to f^*$ is a norm-preserving isomorphism from $H_p(D)$ into $L_p(B)$.*

(c) *If $p \geqq 1$, the range of γ is $H_p(B)$, and the inverse of γ is given by*

$$f(z) = \frac{1}{2\pi i} \int_B \frac{f^*(w)}{w - z} dw \qquad (z \in D) \tag{3.2.1}$$

and also by

$$f(z) = \frac{1}{2\pi} \int_B f^* \frac{\partial G}{\partial n} ds \tag{3.2.2}$$

where G is Green's function of D, with pole at z.

We shall call f^* the boundary function of f; the term "boundary value" will always refer to arbitrary nontangential approach.

The proof can be reduced to the simply-connected case by means of the following decomposition theorem:

3.3. THEOREM. *Let $B_1, \cdots, B_k$ be the components of the boundary of D, with B_1 forming the outer boundary. Let D_1 be the interior of B_1, and $D_2, \cdots, D_k$ the exteriors of $B_2, \cdots, B_k$, including the point at infinity.*

For every $f \in H_p(D)$ there exists a decomposition

$$f(z) = f_1(z) + \cdots + f_k(z) \qquad (z \in D), \tag{3.3.1}$$

such that $f_n \in H_p(D_n)$ $(n = 1, \cdots, k)$.

Proof. Let $C_1, \cdots, C_k$ be nonintersecting smooth curves in D, which bound a domain Δ, such that C_n is close to B_n, and put

$$f_n(z) = \frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w - z} dw \qquad (n = 1, \cdots, k; z \in \Delta). \tag{3.3.2}$$

Then (3.3.1) holds in Δ. Since the distances between C_n and B_n can be made arbitrarily small, f_n is analytic in D_n, and (3.3.1) holds in D.

To prove that $f_n \in H_p(D_n)$, suppose $n=1$ (the proof being the same in the other cases) and assume, without loss of generality, that B_1 is the unit circle. Let v be a harmonic majorant of $|f|^p$ in D. Choose $r_0<1$ so that the ring R: $r_0 \leqq |z| <1$ is in D. Since $f_2, \cdots, f_k$ are bounded in R, there exists a positive constant b such that the function $u=v+b$ is a harmonic majorant of $|f_1|^p$ in R. Then

$$u(re^{i\theta}) = u_1(re^{i\theta}) + u_2(re^{i\theta}) \qquad (r_0 < r < 1) \tag{3.3.3}$$

where u_1 is harmonic in $r<1$, u_2 is harmonic in $r_0<r$, and u_2 is bounded in $r_0<r<1$. Since $|f_1|^p$ is bounded in $r \leqq r_0$, it is clear that the function $c+u_1$, where c is a sufficiently large constant, is a harmonic majorant of $|f_1|^p$ in D_1.

3.4. LEMMA. *$A(D)$ is dense in $H_p(D)$.*

Proof. If $D=U$ and $f \in H_p(U)$, define

$$f_r(z) = f(rz) \qquad \text{for } r < 1 \text{ and } z \in U.$$

Then $\|f_r - f\|_p \to 0$ as $r \to 1$ [22, p. 162]. Thus $A(U)$ is dense in $H_p(U)$; the same is true for any simply-connected domain, by the conformal invariance of the norm. The general case follows from Theorem 3.3, noting that the norm in $H_p(D)$ does not exceed the norm in $H_p(D_n)$, if both are defined with respect to the same point $t \in D$.

3.5. **Proof of Theorem 3.2.** The existence of nontangential boundary values almost everywhere on B is a local property and follows immediately from the simply connected case [13, p. 197], even for the class $\text{Log}^+(D)$. If we apply Fatou's lemma to (2.1.2) we find that $\|f^*\|_p \leqq \|f\|_p$. Thus γ is a bounded linear transformation from $H_p(D)$ into $L_p(B)$. Since γ is clearly norm-preserving on $A(D)$, and $A(D)$ is dense in $H_p(D)$, parts (a) and (b) of the theorem are proved.

Now let $p \geqq 1$. For any $\phi \in A(D)$, the integral

$$I(f) = \int_B f^*(z)\phi(z)dz \qquad (f \in H_p(D))$$

is a bounded linear functional which is zero for $f \in A(D)$, hence for all $f \in H_p(D)$. It follows that $f^* \in H_p(B)$. The Cauchy formula (3.2.1) is now proved in the usual manner, by deleting a small circular disc with center at z from D, and applying the results established so far to the domain of connectivity $k+1$ thus obtained.

By (3.1.1) we see that the integrals in (3.2.1) and (3.2.2) are equal whenever $f^* \in H_p(B)$.

We still have to prove that the range of γ includes all of $H_p(B)$. Choose $g^* \in H_p(B)$, and define

$$f(z) = \frac{1}{2\pi}\int_B g^* \cdot \frac{\partial G}{\partial n} ds = \frac{1}{2\pi i}\int_B \frac{g^*(w)}{w - z} dz \qquad (z \in D)$$

where G has its pole at z. Then f is single-valued and analytic in D, and

$$| f(z) |^p \leqq \frac{1}{2\pi}\int_B | g^* |^p \frac{\partial G}{\partial n} ds \qquad (z \in D).$$

The integral on the right is a harmonic function of z, so that $f \in H_p(D)$. Let f^* be the corresponding boundary function, and put $h^* = f^* - g^*$. The proof will be complete if we can show that $h^* = 0$ p.p. on B.

It is clear that $h^* \in H_p(B)$, and that

$$\int_B \frac{h^*(w)}{w - z} dw = 0 \qquad (z \in D).$$

Differentiating the last equation, we obtain

$$\int_B \frac{h^*(w)}{(w - z)^n} dw = 0 \qquad (z \in D; n = 1, 2, 3, \cdots).$$

Since every continuous function on B is the uniform limit of a sequence of rational functions with poles in the complement of B, we have

$$\int_B h^*(w) K(w) dw = 0$$

for every function K which is continuous on B; thus $h^*(w) = 0$ p.p. on B.

3.6. Corollary. *If g^* is bounded and measurable on B, and if*

$$\int_B g^*(w)\phi(w) dw = 0$$

for all $\phi \in A(D)$, then there exists a unique function $g \subset H_\infty(D)$ which has non-tangential boundary values g^* almost everywhere on B.

IV. Schwarz's lemma in $H_1(D)$

4.1. We again consider a bounded domain D whose boundary consists of k analytic simple closed curves $B_1, \cdots, B_k$, and we let t be a distinguished point of D with respect to which the norm $\|f\|_1$ is defined. We are concerned with the problem of *maximizing* $|f'(t)|$, *under the restrictions* $f(t) = 0$, $\|f\|_1 \leqq 1$.

The principal difference between this problem and Garabedian's work in [9] lies in the definition of the norm. Garabedian used essentially

$$\left\{ \frac{1}{2\pi}\int_B | f^* |^p ds \right\}^{1/p} \quad \text{instead of} \quad \left\{ \frac{1}{2\pi}\int_B | f^* |^p \frac{\partial G}{\partial n} ds \right\}^{1/p}.$$

This led to his problem not being conformally invariant. The present norm seems better suited to the discussion of function theoretic problems.

We shall let X stand for the subspace of $H_1(D)$ consisting of those functions f for which $f(t)=0$, and we put

$$\alpha = \alpha(t) = \sup_{f\in X_0} |f'(t)|, \tag{4.1.1}$$

where X_0 is the intersection of X with the unit sphere of $H_1(D)$.

4.2. The corresponding problem in $H_\infty(D)$ has been the subject of investigations by Ahlfors [1], Garabedian [8], and Nehari [12]. The solution is as follows: if

$$\beta = \beta(t) = \sup |f'(t)| \qquad (f \in H_\infty(D),\ |f(z)| < 1 \text{ in } D), \tag{4.2.1}$$

there exists a unique function $F(z)=F(z, t)$ such that

$$F'(t) = \beta, \qquad |F(z)| < 1 \text{ in } D, \qquad F \in H_\infty(D). \tag{4.2.2}$$

This function F is analytic and single-valued on the closure of D, $|F(z)|=1$ for all $z\in B$, $F(t)=0$, and F has precisely $k-1$ other zeros at points $z_1, \cdots, z_{k-1}$ in D. These points are the zeros of the Szegö kernel function $K(z, t)$ of D, and F maps D in a k-to-1 manner onto U [3, Chap. VII]. The relation $\beta=2\pi K(t, t)$ makes explicit computation of β possible.

It is evident that $\beta \leqq \alpha$.

We begin with three lemmas.

4.3. Lemma. *Suppose $\phi\in A(D)$ and ϕ is not constant. If $|\phi(z)|=1$ for all $z\in B$, then ϕ has at least k zeros in D.*

Proof. Since $|\phi(z)|<1$ in D (by the maximum modulus theorem) a glance at the Cauchy-Riemann equations of the function $\log \phi$ shows that $\arg \phi(z)$ increases as z traverses B in the positive sense. Hence $\arg \phi$ increases by an integral multiple of 2π on each component of B, so that the total increase is $2m\pi$, where the integer m is not less than k. The lemma now follows from the argument principle.

4.4. Lemma. *Suppose $f\in H_1(D)$ and f has real boundary values almost everywhere on an open arc Γ of B. Then f can be continued analytically across Γ.*

Proof. If we take a simply-connected subdomain of D which is bounded in part by Γ, the conformal invariance of the problem shows that it is enough to consider the case $D=U$.

By (3.2.2), f and therefore its imaginary part v can be written as the Poisson integral of a summable function. Since the boundary values of v are zero p.p. on Γ, $v(z)\to 0$ as $z\to\Gamma$ along any path, so that v can be extended harmonically across Γ; the same is then evidently true for the real part of f, and hence for f itself.

That the hypothesis of $f \in H_1(D)$ is not superfluous is shown by the function

$$f(z) = \frac{i(1+z)}{1-z} \qquad (|z| < 1)$$

which has real boundary values on $|z| = 1$, except at $z = 1$, where there is a singularity.

4.5. LEMMA. *Let* Γ *be an open arc on the circumference of* U. *Suppose* $f \in H_1(U)$, $g \in H_\infty(U)$, $h(z) = f(z)g(z)$ *in* U, h *can be continued analytically across* Γ, *and* g *has boundary values of absolute value* 1 *almost everywhere on* Γ. *Then* f *and* g *can be continued analytically across* Γ.

Proof. If Γ_1 is any closed subarc of Γ, then the zeros of h have no limit point on Γ_1. Let Γ_2 be an analytic arc connecting the end points of Γ_1, which lies in U except for its end points, such that h has no zeros on Γ_2 and in the domain D bounded by Γ_1 and Γ_2. Mapping D conformally onto U, we see that we may add the following assumptions to the original hypotheses without loss of generality: h is continuous on the closure of U and has no zeros there, except possibly a finite number of zeros on Γ.

It is enough to show that $\log |g|$ is the Poisson integral of a summable function; this will show that $\log |g|$ can be continued harmonically across Γ, since its boundary values are zero, so that g is analytic on Γ. The analyticity of f follows from $f(z) = h(z)/g(z)$.

Using the notation of (1.1.1) we see that $M_r^2(\log^+ |h|)$ and $M_r^2(\log^+ |1/h|)$ are bounded as $r \to 1$. Since $f \in H_1(U)$, $M_r^2(\log^+ |f|)$ is also bounded, and the formula $1/f = g/h$, together with the boundedness of g, shows that the same is true of $M_r^2(\log^+ |1/f|)$. Hence [22, p. 87] $\log |h|$ and $\log |f|$ are Poisson integrals of summable functions; the same is true of $\log |g|$ since $\log |g| = \log |h| - \log |f|$.

We now take up our maximum problem.

4.6. Our first aim is to show that there always exists a function $f_0 \in X_0$ such that $f_0'(t) = \alpha$. We shall call f_0 an *extremal function*.

Choose a sequence $\{f_n\}$ $(n = 1, 2, 3, \cdots)$ of members of X_0 such that $f_n'(t) \to \alpha$. The inequality (2.4.1) shows that $\{f_n\}$ is a normal family, so that there exists a subsequence, again denoted by $\{f_n\}$, which converges uniformly on every compact subset of D, to a limit function f_0. If C is a small circle about t, we therefore have

$$f_0'(t) = \frac{1}{2\pi i}\int_C \frac{f_0(z)}{(z-t)^2}\,dz = \lim \frac{1}{2\pi i}\int_C \frac{f_n(z)}{(z-t)^2}\,dz = \lim f_n'(t) = \alpha.$$

A similar passage to the limit, applied to the integrals in (2.1.2), shows that $\|f_0\|_1 \leqq 1$. Since f_0 is extremal, $\|f_0\|_1 = 1$.

4.7. Let T be the linear functional defined on X by $Tf = f'(t)$. Since the

Cauchy integral formula is valid with B as path of integration (Theorem 3.2) we can write

$$Tf = f'(t) = \frac{i}{2\pi}\int_B \frac{f^*(z)}{z-t} P'(z)dz \qquad (f \in X). \tag{4.7.1}$$

We continue to use the notation introduced in III: f^* stands for the boundary function of f.

By Theorem 3.2, X is (isometrically isomorphic to) a closed subspace of $L_1(B)$. We apply the Hahn-Banach theorem [4] to extend T from X to $L_1(B)$, preserving its norm α, so that T is of the form

$$Tf^* = \frac{i}{2\pi}\int_B f^*(z)g^*(z)\, P'(z)dz \qquad (f^* \in L_1(B)), \tag{4.7.2}$$

where g^* is measurable on B, and $|g^*(z)| \leqq \alpha$. Comparison of (4.7.1) and (4.7.2) shows that

$$\int_B f^*(z)\left\{g^*(z) - \frac{1}{z-t}\right\} P'(z)dz = 0 \qquad (f \in X). \tag{4.7.3}$$

In particular,

$$\int_B \phi(z)\left\{g^*(z) - \frac{1}{z-t}\right\}(z-t)P'(z)dz = 0 \qquad (\phi \in A(D)). \tag{4.7.4}$$

If we write the integrand of (4.7.4) in the form $\phi(z)w^*(z)$, it follows from 3.6 that there exists a function $w \in H_\infty(D)$ whose boundary values are w^* p.p. Thus there exists a function g of the form

$$g(z) = \frac{1}{z-t} + \sum_{m=1}^{k-1} \frac{c_m}{z-t_m} + b(z) \qquad (z \in D), \tag{4.7.5}$$

with boundary values g^* p.p. on B; here $b \in H_\infty(D)$, $t_1, \cdots, t_{k-1}$ are the critical points of the Green's function of D with pole at t, and $c_1, \cdots, c_{k-1}$ are constants, some of which may conceivably be zero. If two or more of the critical points coincide, (4.7.5) must be modified so as to contain poles of the appropriate orders; nothing in our proof is affected by such a change.

Substitution of an extremal function f_0 into (4.7.2) yields

$$\begin{aligned} \alpha = Tf_0 &= \frac{i}{2\pi}\int_B f_0^*(z)g^*(z)P'(z)dz \\ &\leqq \frac{i}{2\pi}\int_B |f_0^*(z)g^*(z)|\, P'(z)dz \leqq \alpha\|f_0\|_1 = \alpha. \end{aligned}$$

Hence the equality signs must hold in both inequalities; we conclude first that

$$f_0^*(z)g^*(z) \geqq 0 \qquad \text{(p.p. on } B), \tag{4.7.6}$$

and secondly that

$$|g^*(z)| = \alpha \qquad \text{(p.p. on } B), \tag{4.7.7}$$

since $f_0^*(z) \neq 0$ p.p. on B [13, pp. 197–198].

Let us put

$$h(z) = f_0(z)g(z) \qquad (z \in D). \tag{4.7.8}$$

Restricting ourselves to a neighborhood of B in which g has no poles, 4.4 shows that h is analytic on B. Letting Δ be a simply-connected subdomain of D, partly bounded by an arc Γ of B, 4.5 shows *that f_0 and g are analytic on B.*

The relations (4.7.6) and (4.7.7) determine arg g and $|g|$ on B, so that g is unique; this means that there exists only one norm-preserving extension of T from X to $L_1(B)$.

On the other hand, (4.7.6) determines arg f_0 on B, but this is not enough to give uniqueness of f_0.

4.8. Let us define $\gamma = \inf \sup_{z \in B} |g^*(z)|$, the inf being taken over all functions of the form (4.7.5). Our preceding work shows that $\gamma \leqq \alpha$. Actually, $\gamma = \alpha$, since $\gamma < \alpha$ would lead to a norm-decreasing extension of T from X to $L_p(B)$, which is absurd.

This maximum-minimum duality is a well known phenomenon [9; 11; 19]. It may be formulated abstractly as follows: *Let X be a closed linear subspace of a Banach space E. Let X be the subspace of E^* which annihilates X. For any $T \in E^*$, let*

$$\delta = \sup |Tx| \qquad (x \in X, \|x\| \leqq 1).$$

Then

$$\delta = \inf \|T - S\| \qquad (S \in X^{\perp}),$$

and the minimum is attained.

This can be proved quite easily by applying the Hahn-Banach theorem to the restriction of T to X [19].

4.9. We return to our maximum problem. By (4.7.6) and the argument principle, h has as many zeros as poles in D, if zeros on B (necessarily of even order) are counted with half their multiplicities. Since $f_0(t) = 0$ and f_0 has no poles in D, g has fewer zeros than poles in D.

Two possibilities can now be distinguished.

Case (a). The zeros $z_1, \cdots, z_{k-1}$ of the Szegö kernel function (see 4.2) coincide with the critical points $t_1, \cdots, t_{k-1}$ of $G(z)$ (see 3.1).

Case (b). The sets $(z_1, \cdots, z_{k-1})$ and $(t_1, \cdots, t_{k-1})$ are not identical.

If (a) holds, set $\phi(z) = F(z)g(z)$, where F is described in 4.2. We see that $\phi \in A(D)$ and that $|\phi| = \alpha$ on B. The function F has precisely k zeros, and we saw above that g has fewer zeros than poles, so that ϕ has at most $k-1$ zeros in D. By Lemma 4.3, ϕ is constant. Since $\phi(t)=\beta$, by (4.2.2) and (4.7.5), we have

$$g(z) = \beta/F(z). \tag{4.9.1}$$

Thus g has k poles and no zeros in D, so that f_0 has k zeros. Also, $\alpha=\beta$ since $|\phi|=\alpha$ on B; in particular, F is an extremal function of 4.1 as well as of 4.2.

Suppose next that (b) holds, and that g has no zero in D. Then $|g(z)|>\alpha$ in D. Put $\phi(z)=\alpha/g(z)$. Then $|\phi(z)|<1$ in D, and $\phi'(t)=\alpha$. Thus $\beta=\alpha$, and the uniqueness of F shows that $\phi=F$, so that the poles of g do coincide with the zeros of F, which takes us back to Case (a).

It follows that g has at least one zero in D if (b) holds, so that any extremal function f_0 has fewer than k zeros in D. Also, $\alpha>\beta$, for otherwise F would be an extremal function with k zeros.

4.10. We now summarize the information we have obtained:

THEOREM. (i) *The extremum problem* 4.1 *always has at least one solution* $f_0\in X_0$ *such that* $f_0'(t)=\alpha$. *Every such* f_0 *is analytic on the closure of* D.

(ii) *There exists a unique function* g *of the form* (4.7.5), *analytic on* B, *such that* $|g(z)|=\alpha$ *on* B; *for every* f_0, $f_0(z)g(z)\geqq 0$ *on* B.

(iii) *Depending on the choice of* t *and* D, *there exist integers* N, M, $1\leqq M\leqq N\leqq k$, *such that* g *has* N *poles and* $N-M$ *zeros in* D, *and every* f_0 *has* M *zeros in* D. *Any zeros of* f_0 *on* B *are of even order, and are to be counted with half their multiplicities in this enumeration.*

(iv) *In Case* (a) *of* 4.9, $M=N=k$, *and* $\alpha=\beta$. *In Case* (b), $M<N$, *and* $\alpha>\beta$.

(v) *If two extremal functions* f_0 *have the same zeros, they are identical.*

The only assertion not yet proved is (v); it follows from the fact that the ratio of any two such functions, which is equal to 1 at t, is regular in D, and has real boundary values, so that its imaginary part must vanish in D.

4.11. REMARKS. Despite the discontinuous manner in which the zeros of f_0 depend on the location of t (4.10(iv) and 4.12), the maximum $\alpha(t)$ is a continuous function of t. This follows from Theorem 1, p. 103, of [2], since X_0 satisfies the hypotheses of that theorem.

If $k\geqq 3$, we do not know whether for every D there is a $t\in D$ such that Case (a) occurs, or indeed whether Case (a) can occur for any domain of connectivity greater than two. If $k=2$, we can describe the situation more completely:

4.12. DOMAINS OF CONNECTIVITY TWO. Since every domain bounded by two curves can be mapped conformally onto an annulus $0<r<|z|<1$ [3, p. 61], we may restrict ourselves to that case. We also lose no generality by assuming that $r<t<1$.

THEOREM. *If* $t \neq r^{1/2}$, *Case* (b) *occurs*; $N=2$, $M=1$, *and there is a unique extremal function.*

If $t=r^{1/2}$, *Case* (a) *occurs. For every* $x \in [-1, -r]$ *there is an extremal function* f_0 *such that* $f_0(x)=0$, *and every extremal function is obtained in this way. The ratio of any two of these is an elliptic function of* $\log z$.

Proof. The critical point t_1 of $G(z, t)$ satisfies the inequality

$$\begin{aligned} -t \leqq t_1 \leqq -r^{1/2} \qquad & (r^{1/2} \leqq t), \\ -r^{1/2} \leqq t_1 \leqq -t \qquad & (t < r^{1/2}). \end{aligned} \tag{4.12.1}$$

This follows from the work of Walsh ([20, p. 266, Theorem 2], plus the concluding remarks on p. 268 of [20]).

The Szegö kernel function can be computed explicitly for the annulus by normalizing the complete orthogonal set $\{z^n\}$, $n=0, \pm 1, \pm 2, \cdots$, over the boundary [3, p. 77]. The result is

$$K(z, t) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{(z\bar{t})^n}{1 + r^{2n+1}}. \tag{4.12.2}$$

(The series also converges outside D, for $r^2 < |tz| < 1$.) It is easily verified that $K(z, t)=0$ if

$$z = z_1 = -r/t \qquad (r < t < 1). \tag{4.12.3}$$

A comparison of (4.12.1) and (4.12.3) shows that Case (a) occurs if and only if $t=r^{1/2}$.

If $t \neq r^{1/2}$, our assertions therefore follow from Theorem 4.10 (iv), (v).

Suppose now that $t=r^{1/2}$, so that $t_1=z_1=-t$. In this case F (see 4.2) is an extremal function. If f_0 is another extremal, with a zero at x, where $x \neq t$, $x \neq -t$ (if $x \in B$, f_0 must have a double zero at x), set

$$\phi(z) = f_0(z)/F(z). \tag{4.12.4}$$

Then ϕ has the following properties: $\phi(t)=1$, ϕ has a simple pole at $-t$, $\phi(x)=0$, and $\phi(z) \geqq 0$ on B, by (4.7.6).

Conversely, if ϕ is any function with these properties, we obtain an extremal f_0 by putting $f_0(z)=\phi(z)F(z)$; for $f_0'(t)=F'(t)=\alpha$, and since $|F(z)| = 1$ for all $z \in B$ we have

$$\|f_0\|_1 = \|\phi\|_1 = \frac{i}{2\pi} \int_B \phi(z) P'(z) dz = \phi(t) = 1,$$

the pole of ϕ being cancelled by the zero of P' at $-t$.

Let

$$\wp(s) = s^{-2} + \sum_{n,m}{}' \{(s - mw - nw')^{-2} - (mw + nw')^{-2}\}$$

be Weierstrass' elliptic function, with periods $w=2 \log r$, $w'=2\pi i$, and put

$q(z)=\wp(\log z)$. Since $\wp(s)$ is even, real for real s, and attains every value twice in the period parallelogram, we see that q is schlicht in D (D corresponds to one half of the parallelogram). Also, $q(z)$ is real on B and on the real axis. Put

$$\phi_1(z) = [q(z) - q(-t)]^{-1} - [q(x) - q(-t)]^{-1} \tag{4.12.5}$$

and

$$\phi(z) = \phi_1(z)/\phi_1(t). \tag{4.12.6}$$

If x is real, $\phi(z)$ is real wherever $q(z)$ is. If $-1 \leqq x \leqq -r$, ϕ is negative between x and $-t$, and non-negative on B; it is clear that ϕ also has the other desired properties.

On the other hand, if the problem can be solved for a given x, we can use the mapping $s=\log z$, extend ϕ to an elliptic function in the s-plane, use the theorem that an elliptic function is determined (up to a multiplicative constant) by the location and multiplicity of its zeros and poles, and conclude that ϕ must be of the form (4.12.6). But if $x \in D$ is imaginary, so is $q(x)$, and $\phi(z)$ cannot have real boundary values on all of B. If $x \in [r, 1]$, $\phi(z)$ is positive on one component of B and negative on the other. If $x \in B$, ϕ has a double zero at x; but $\phi'(z)=0$ at $z=1, -1, r, -r$ only.

4.13. If $t \neq r^{1/2}$, then $g(w)=0$ for some $w \in D$, whereas g has no zero in the closure of D if $t=r^{1/2}$. We wish to conclude by showing what happens to this zero as $t \to r^{1/2}$.

Suppose $t \neq r^{1/2}$. First, $g(z)$ is real for real z; for otherwise the function $\overline{g(\bar{z})}$ would also be of the form (4.7.5), which would contradict the uniqueness of g. Thus w is real. Similarly, $f_0(z)$ is real for real z.

We claim that $-1 < w < -r$; for if $r < w < 1$, the function $h(z) = f_0(z)g(z)$ would change sign in $(r, 1)$, so that $h(z)$ could not be positive on both components of B. Let

$$u(z) = \log |g(z)| - \log \alpha = G(z, w) - G(z, t) - G(z, t_1),$$

so that

$$\frac{1}{2\pi}\int_C \frac{\partial u}{\partial n} ds = \frac{\log |w| - \log t - \log |t_1|}{\log r}, \tag{4.13.1}$$

where C is the circle $|z| = r$. Since $\exp(u+iv)$, where v is the conjugate of u, is single-valued, the left member of (4.13.1) must be an integer m, so that

$$|w| = |tt_1| r^m.$$

Using (4.12.1) it is seen that $m=0$ if $t > r^{1/2}$, and that $m=-1$ if $t < r^{1/2}$.

It follows that $w \to -r$ as $t \to r^{1/2}$ from the right, and that $w \to -1$ as $t \to r^{1/2}$ from the left.

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